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Exit problem of McKean-Vlasov diffusions in convex landscapes*

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Abstract

The exit time and the exit location of a non-Markovian diffusion is analyzed. More particularly, we focus on the so-called self-stabilizing process. The question has been studied by Herrmann, Imkeller and Peithmann in [6] with results similar to those by Freidlin and Wentzell. We aim to provide the same results by a more intuitive approach and without reconstructing the proofs of Freidlin and Wentzell. Our arguments are as follows. In one hand, we establish a strong version of the propagation of chaos which allows to link the exit time of the McKean-Vlasov diffusion and the one of a particle in a mean-field system. In the other hand, we apply the Freidlin-Wentzell theory to the associated mean-field system, which is a Markovian diffusion.

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Introduction

The questions that we address in this paper are about the pathwise asymptotic behavior of a particular class of inhomogeneous diffusions:

$$X_t^\epsilon = X_0 + \sqrt{\epsilon}B_t - \int_0^t b^\epsilon\left(s, X_s^\epsilon\right) ds\,.$$

We study here the so-called self-stabilizing process. The term “self-stabilizing” comes from the fact that each trajectory is attracted by the whole set of trajectories in the following sense:

$$b^\epsilon(t,x) := \nabla V(x) + \mathbb{E}\left\{\nabla F\left(x - X_t^\epsilon\right)\right\}\,.$$

The model is detailed subsequently. Let us present what we denote by exit problem. We consider a domain $\mathcal{D} \subset \mathbb{R}^d$ and we introduce

$$S(\epsilon) := \inf\left\{t \geq 0 \;\middle|\; X_t^\epsilon \in \mathcal{D}\right\}$$

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the first hitting time of X^ϵ in the domain $\mathcal{D}$. Then, we define

$$\tau(\epsilon) := \inf\left\{t \geq S(\epsilon) \;\mid\; X_t^\epsilon \notin \mathcal{D}\right\}$$

the first exit time of X^ϵ from the domain $\mathcal{D}$. The exit problem consists of two questions. What is the exit time? What is the exit location?

In the small-noise limit, the questions become:

1. What is the exit time $\tau(\epsilon)$ for ϵ going to 0?
2. What is the exit location $X_{\tau(\epsilon)}^\epsilon$ for ϵ going to 0?

The subject of this article is to study these questions. They have been solved by Freidlin and Wentzell for homogeneous difffusions. See [5, 4] for a complete review. Let us briefly present their results. We study the diffusion

$$x_t^\epsilon = x_0 + \sqrt{\epsilon}\beta_t - \int_0^t \nabla U\left(x_s^\epsilon\right) ds\,.$$

U is a $\mathcal{C}^\infty$-continuous function from $\mathbb{R}^k$ ($k \geq 1$) to $\mathbb{R}$ and β is a Brownian motion in $\mathbb{R}^k$. a_0 is a minimizer of U and $\mathcal{G}$ is a domain which contains a_0. Under easy to check assumptions (which are detailed in Appendix A), for all $\delta > 0$, the following Kramers' type law holds:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H-\delta\right)\right] < \tau(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H+\delta\right)\right]\right\} = 1\,.$$

Here, the exit cost is $H := \inf_{z\in\partial\mathcal{G}} U(z) - U(a_0)$. We immediately remark that $H = \frac{1}{2}\lim_{\epsilon\to 0}\epsilon\,\log\left\{\tau(\epsilon)\right\}$. Moreover, the exit location is near the points of the boundary which minimize U. Indeed,

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{x_{\tau(\epsilon)}^\epsilon \in \mathcal{N}\right\} = 0$$

if $\mathcal{N} \subset \partial\mathcal{D}$ is such that $\inf_{z\in\mathcal{N}} U(z) > H$.

Let us note that we also have results if we replace Brownian motion by a Lévy process, see [11, 10].

Let us present more precisely the model studied in the article. Let X_0 be an element of $\mathbb{R}^d$, $d \geq 1$. We consider the McKean-Vlasov diffusion

$$\left\{\begin{array}{l} X_t^\epsilon = X_0 + \sqrt{\epsilon}B_t - \int_0^t \nabla W_s^\epsilon\left(X_s^\epsilon\right) ds \\ W_t^\epsilon := V + F * u_t^\epsilon := V + F * \mathcal{L}\left(X_t^\epsilon\right) \end{array}\right. . \tag{0.1}$$

The star in the previous line corresponds to a convolution and u_t^ϵ is the own law of the diffusion X^ϵ at time t. Let us point out that $\mathbb{E}\left\{\nabla W_t^\epsilon\left(X_t^\epsilon\right)\right\}$ is not equal to $\mathbb{E}\left\{\nabla V\left(X_t^\epsilon\right)\right\}$. It is equal to $\mathbb{E}\left\{\nabla V\left(X_t^\epsilon\right) + \nabla F\left(X_t^\epsilon - Y_t^\epsilon\right)\right\}$ where Y^ϵ is an independent version of X^ϵ.

Since the own law of the process intervenes in the drift, this equation is nonlinear, in the sense of McKean. Three terms generate the dynamic. The first one is a Brownian motion B in $\mathbb{R}^d$ with intensity $\frac{\epsilon}{2}\,d$. It allows X^ϵ to visit the whole space. The second force describes the attraction between one trajectory $t \mapsto X_t^\epsilon(\omega_0)$ and the whole set of trajectories. Indeed, we notice: $\nabla F * u_t^\epsilon\left(X_t^\epsilon(\omega_0)\right) = \int_{\omega\in\Omega}\nabla F\left(X_t^\epsilon(\omega_0) - X_t^\epsilon(\omega)\right) d\mathbb{P}\left(\omega\right)$ where $(\Omega, \mathcal{F}, \mathbb{P})$ is the underlying measurable space. Consequently, we say that F is the interaction potential. The last term is V, the so-called confining potential. It forces the diffusion to move to the minimizers of V. These three forces are concurrent.

As a first observation, we note that the future of the couple $(X^\epsilon\,;\,u^\epsilon)$ is independent of its past if its present is known. However, the diffusion X^ϵ is not Markovian since the past intervenes in the drift ∇W_t^ϵ through the law u_t^ϵ. This kind of processes were introduced by McKean. The reader is referred to [14]. X^ϵ corresponds to the hydrodynamic limit of the interacting particle system:

$$Z_t^{\epsilon,i,N} = X_0 + \sqrt{\epsilon} B_t^i - \int_0^t \nabla V\left(Z_s^{\epsilon,i,N}\right) ds - \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla F\left(Z_s^{\epsilon,i,N} - Z_s^{\epsilon,j,N}\right) ds$$

for all $1 \leq i \leq N$. The N Brownian motions are supposed independent and $B^1 = B$. Each particle is attracted by the whole set of particles. We call this a *mean-field system*. The drift which intervenes in each diffusion $Z^{\epsilon,i,N}$ can be written similarly to the one of the self-stabilizing diffusion (0.1):

$$W_t^{\epsilon,N} := V + F * \left(\frac{1}{N}\sum_{i=1}^N \delta_{Z_t^{\epsilon,i,N}}\right). \tag{0.2}$$

Heuristically, the empirical law $\frac{1}{N}\sum_{i=1}^N \delta_{Z_t^{\epsilon,i,N}}$ of the system converges to u_t^ϵ as N tends to 0. This phenomenon is called propagation of chaos.

Under some hypotheses on V and F, the self-stabilizing diffusion X^ϵ corresponds to the limit for large N of the first particle $Z^{\epsilon,1,N}$ in the following sense:

$$\lim_{N\to\infty} \mathbb{E}\left\{\sup_{t\in[0;T]} \left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|^2\right\} = 0$$

for all $T \in \mathbb{R}_+$. See [15, 1, 12, 13, 3]. Proofs of the classical results on propagation of chaos are in Appendix B. The mean-field system is Markovian. Indeed, by denoting $\mathcal{Z}^{\epsilon,N} := \left(Z^{\epsilon,1,N}, \cdots, Z^{\epsilon,N,N}\right)$, $\mathcal{B}^N := \left(B^1, \cdots, B^N\right)$ and $\mathcal{Z}_0^N := (X_0, \cdots, X_0)$, equation (0.2) can be rewritten

$$\mathcal{Z}_t^{\epsilon,N} = \mathcal{Z}_0^N + \sqrt{\epsilon}\mathcal{B}_t^N - N\int_0^t \nabla \Upsilon^N\left(\mathcal{Z}_s^{\epsilon,N}\right) ds \tag{0.3}$$

where the potential Υ^N is defined by

$$\Upsilon^N(\mathcal{Z}) := \frac{1}{N}\sum_{j=1}^N V(Z_j) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F(Z_i - Z_j) \tag{0.4}$$

for all $\mathcal{Z} := (Z_1, \cdots, Z_N) \in \left(\mathbb{R}^d\right)^N$. Then we can apply Freidlin-Wentzell results to the homogeneous diffusion $\mathcal{Z}^{\epsilon,N}$. We note that the exit problem of $Z^{\epsilon,1,N}$ from $\mathcal{D}$ is equivalent to the one of $\mathcal{Z}^{\epsilon,N}$ from $\mathcal{D} \times \mathbb{R}^{d(N-1)}$. A strong version of the propagation of chaos allows then to link the exit time of X^ϵ from $\mathcal{D}$ and the one of $Z^{\epsilon,1,N}$ from $\mathcal{D}$.

Let us briefly recall some previous results on McKean-Vlasov diffusions. The existence and the uniqueness of a strong solution X^ϵ on $\mathbb{R}_+$ for equation (0.1) has been proved in [6] (Theorem 2.13). The asymptotic behavior of the law has been studied in [3, 2] (for the convex case) and in [16, 18] in the non-convex case by using the results in [7, 8, 9] about the non-uniqueness of the stationary measures and their small-noise behavior.

The exit problem of self-stabilizing processes has already been solved if both V and F are uniformly strictly convex, see [6]. The authors follow and extend the method of Freidlin and Wentzell. The difficulty is the lack of Markov property. Indeed, in inhomogeneous diffusions, the first exit time and the second exit time can not be identified up to a shift. However, if V and F are uniformly strictly convex that is to say if

$\inf_{x\in\mathbb{R}^d} \mathrm{Hess}\, V(x) \geq \theta > 0$ and $\inf_{x\in\mathbb{R}^d} \mathrm{Hess}\, F(x) \geq \alpha > 0$, they prove a Kramers' type law. The exit time $\tau(\epsilon)$ of X^ϵ from a domain $\mathcal{D}$ satisfies the limit:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\,(H-\delta)\right] \leq \tau(\epsilon) \leq \exp\left[\frac{2}{\epsilon}\,(H+\delta)\right]\right\} = 1$$

for all $\delta > 0$. Here, $H := \inf_{\partial\mathcal{D}} (V + F * \delta_{a_0}) - V(a_0)$ where a_0 is the unique minimizer of V. They also provide a result on the exit location which is similar to the one of Freidlin-Wentzell. They also give an example of the influence of self-stabilizing term on the exit location.

This paper proposes a new simpler and more intuitive approach of the problem.

The article is organized as follows. First, we present the assumptions on the potentials and the definitions. Then, the uniform boundedness of the moments is established. This justifies the assumptions on the domain $\mathcal{D}$. The main results are written in the end of Section 1. In the second section, the exit problem of the particle $Z^{\epsilon,1,N}$ is addressed by applying classical Freidlin-Wentzell theory. The third section deals with a new version of the propagation of chaos. Finally, the main results are proved.

The article contains also two appendixes. One deals with the results and the hypotheses of the Freidlin-Wentzell theory and the other one presents the classical results on the propagation of chaos, including the proofs.

1 Preliminaries and main results

First, let us denote by $||\,.\,||$ the euclidian norm on $\mathbb{R}^d$: $||x||^2 := \sum_{r=1}^d x_r^2$ for all $x = (x_1, \cdots, x_d) \in \mathbb{R}^d$. The associated distance is d.

We assume the following properties on the confining potential V:

(V-1) V is a smooth function from $\mathbb{R}^d$ to $\mathbb{R}$.

(V-2) V is uniformly strictly convex: $\mathrm{Hess}\, V \geq \theta > 0$.

(V-3) The unique minimizer of V is 0 and $V(0) = 0$.

We would like to point out that the aim of the hypothesis (V-3) is just to simplify the writing. Indeed, if the point of the global minimum is $a_0 \neq 0$, it is sufficient to consider the diffusion $\widetilde{X} := X - a_0$ and the potential $\widetilde{V} := V(.+a_0) - V(a_0)$. An immediate consequence of (V-1)–(V-3) is the following inequality:

$$\langle x\,;\,\nabla V(x)\rangle \geq \theta\,||x||^2 \quad \text{for all } x \in \mathbb{R}^d\,. \tag{1.1}$$

Let us now present the assumptions on the interaction potential F:

(F-1) There exists a function G from $\mathbb{R}_+$ to itself such that $F(x) = G\left(||x||\right)$.

(F-2) G is an even polynomial function such that $\deg(G) =: 2n \geq 2$.

(F-3) G is convex.

(F-4) $G(0) = 0$.

Let us note that (F-1)–(F-4) imply

$$\nabla F(x) = x\,\frac{G'\left(||x||\right)}{||x||} = x\sum_{k=1}^n \frac{G^{(2k)}(0)}{(2k-1)!}\,||x||^{2k-2}\,.$$

Since the initial law is a Dirac measure, we know that there exists a unique strong solution X^ϵ to the equation (0.1), see Theorem 2.13 in [6] for a proof. Moreover:

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{||X_t^\epsilon||^{2p}\right\}<\infty \tag{1.2}$$

for all $p\in\mathbb{N}^*$. We immediately deduce the tightness of the family $(u_t^\epsilon)_{t\in\mathbb{R}_+}$.

We now present some notations concerning the space $\left(\mathbb{R}^d\right)^N =: \mathbb{R}^{dN}$.

Definition 1.1. **1.** *For all $\mathcal{Z}=(Z_1,\cdots,Z_N)\in\mathbb{R}^{dN}$, we define the following norm:*

$$|||\mathcal{Z}|||:=\left\{\frac{1}{N}\sum_{i=1}^N||Z_i||^{2n}\right\}^{\frac{1}{2n}}\,.$$

2. *For all $\kappa>0$, we introduce the ball:*

$$\mathbb{B}_\kappa^N:=\left\{\mathcal{Z}\in\mathbb{R}^{dN}\;\middle|\;|||\mathcal{Z}|||<\kappa\right\}\,.$$

3. *Finally, for all $x\in\mathbb{R}^d$, the vector $(x,\cdots,x)\in\mathbb{R}^{dN}$ is denoted by $\overline{x}$.*

We remark that $|||\overline{x}|||=||x||$ for all $x\in\mathbb{R}^d$. In order to simplify the writing, we use the following terminology in the whole article:

Definition 1.2. *Let $\mathcal{G}$ be a subset of $\mathbb{R}^k$ and let U be a $\mathcal{C}^\infty$-continuous function from $\mathbb{R}^k$ to $\mathbb{R}$. For all $x\in\mathbb{R}^k$, we consider the dynamical system*

$$\psi_t(x)=x-\int_0^t\nabla U\left(\psi_s(x)\right)ds\,.$$

We say that the domain $\mathcal{G}$ is stable by $-\nabla U$ if the orbit $\{\psi_t(x)\,;\,t\in\mathbb{R}_+\}$ is included in $\mathcal{G}$ for all $x\in\mathcal{G}$.

We now establish an important result about the moments of X^ϵ. Indeed, since these moments intervene in the drift, the asymptotic behavior (deterministic) of the law u_t^ϵ is related to the asymptotic behavior (probabilistic) of the trajectories. Moreover, it allows to understand what are the relevant sets from which we shoud study the exit problem.

Proposition 1.3. **1.** *The $2n$-moment is uniformly bounded:*

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\max\left\{||X_0||^{2n}\,;\,\left(\frac{2n-1}{2\theta}\right)^n\epsilon^n\right\}\,. \tag{1.3}$$

2. *For all $\kappa>0$ and $\epsilon>0$, we introduce the* deterministic *time*

$$T_\kappa(\epsilon):=\min\left\{t\geq0\;\middle|\;\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\kappa^{2n}\right\}\,.$$

For $\epsilon<\frac{\kappa^2\theta}{2n-1}$, we have the inequality:

$$T_\kappa(\epsilon)\leq\frac{1}{n\theta\kappa^{2n}}\,||X_0||^{2n}\,. \tag{1.4}$$

3. *Moreover, for all $t\geq T_\kappa(\epsilon)$, $\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}\leq\kappa^{2n}$.*

Proof. **1.** After applying the Itô formula and integrating, we obtain

$$\begin{aligned}||X_t^\epsilon||^{2n} = &||X_0||^{2n} + 2n\sqrt{\epsilon}\int_0^t ||X_s^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\, dB_s\right\rangle\\ &- 2n\int_0^t ||X_s^\epsilon||^{2n-2}\Big\{\left\langle X_t^\epsilon\,;\,\nabla V\left(X_s^\epsilon\right)\right\rangle + \left\langle X_s^\epsilon\,;\,\nabla F * u_s^\epsilon\left(X_s^\epsilon\right)\right\rangle\Big\}ds\\ &+ n(2n-1)\epsilon\int_0^t ||X_s^\epsilon||^{2n-2}\,ds\,.\end{aligned}$$

We put $\xi_\epsilon(t) := \mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\}$. The previous equality implies:

$$\begin{aligned}\xi_\epsilon'(t) = &-2n\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\nabla V\left(X_t^\epsilon\right)\right\rangle\right\}\\ &-2n\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\nabla F * u_t^\epsilon\left(X_t^\epsilon\right)\right\rangle\right\} + n(2n-1)\epsilon\mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\right\}\\ =&: a_\epsilon(t) + b_\epsilon(t) + c_\epsilon(t)\,.\end{aligned}$$

By definition, the second term $b_\epsilon(t)$ can be written as

$$b_\epsilon(t) = \mathbb{E}\left[||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\nabla F\left(X_t^\epsilon - Y_t^\epsilon\right)\right\rangle\right]$$

where Y^ϵ is a solution of (0.1) independent from X^ϵ. We can exchange X^ϵ and Y^ϵ. Thereby, by using (F-1)–(F-4), we get:

$$\begin{aligned}b_\epsilon(t) &= \mathbb{E}\left\{\frac{G'\left(||X_t^\epsilon - Y_t^\epsilon||\right)}{||X_t^\epsilon - Y_t^\epsilon||}\left\langle ||X_t^\epsilon||^{2n-2}X_t^\epsilon\,;\,X_t^\epsilon - Y_t^\epsilon\right\rangle\right\}\\ &= \frac{1}{2}\mathbb{E}\left\{\frac{G'\left(||X_t^\epsilon - Y_t^\epsilon||\right)}{||X_t^\epsilon - Y_t^\epsilon||}\left\langle X_t^\epsilon\,||X_t^\epsilon||^{2n-2} - Y_t^\epsilon\,||Y_t^\epsilon||^{2n-2}\,;\,X_t^\epsilon - Y_t^\epsilon\right\rangle\right\}.\end{aligned}$$

This last term is nonnegative. Indeed, the Cauchy-Schwarz inequality implies

$$\left\langle x\,||x||^{2n-2} - y\,||y||^{2n-2}\,;\,x-y\right\rangle \geq \left(||x||^{2n-1} - ||y||^{2n-1}\right)\left(||x|| - ||y||\right) \geq 0$$

for all $x, y \in \mathbb{R}^d$. Therefore, we obtain $b_\epsilon(t) = \mathbb{E}\left\{||X_t^\epsilon||^{2n-1}\nabla F * u_t^\epsilon\left(X_t^\epsilon\right)\right\} \geq 0$.

Moreover, inequality (1.1) implies

$$a_\epsilon(t) = \mathbb{E}\left\{||X_t^\epsilon||^{2n-2}\left\langle X_t^\epsilon\,;\,\nabla V\left(X_t^\epsilon\right)\right\rangle\right] \geq \theta\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\} = \theta\xi_\epsilon(t)\,.$$

Hence, by using Jensen inequality, we deduce $c_\epsilon(t) \leq n(2n-1)\epsilon\xi_\epsilon(t)^{1-\frac{1}{2n}}$. By combining results on $a_\epsilon(t)$, $b_\epsilon(t)$ and $c_\epsilon(t)$, we obtain

$$\begin{aligned}\xi_\epsilon'(t) &\leq -2n\theta\xi_\epsilon(t) + n(2n-1)\epsilon\xi_\epsilon(t)^{1-\frac{1}{n}}\\ &\leq -2n\theta\xi_\epsilon(t)^{1-\frac{1}{n}}\left\{\xi_\epsilon(t)^{\frac{1}{n}} - \frac{(2n-1)\epsilon}{2\theta}\right\}.\end{aligned}\tag{1.5}$$

Inequality (1.3) is an obvious consequence of (1.5).

2. From now on, we take $\epsilon < \frac{\kappa^2\theta}{2n-1}$. This implies $\frac{\kappa^2}{2} > \frac{(2n-1)\epsilon}{2\theta}$. Consequently, for all $t < T_\kappa(\epsilon)$, we have

$$\xi_\epsilon(t)^{\frac{1}{n}} \geq \kappa^2 \geq \frac{(2n-1)\epsilon}{\theta}\,.$$

We obtain from (1.5):

$$-\xi_\epsilon'(t) \geq n\theta\kappa^{2n}\,.$$

By definition, if $\xi_\epsilon(0) = \mathbb{E}\left\{||X_0||^{2n}\right\} \geq \kappa^{2n}$:

$$\int_0^{T_\kappa(\epsilon)} n\theta\kappa^{2n}dt \leq \int_0^{T_\kappa(\epsilon)} -\xi_\epsilon'(t)dt = \xi_\epsilon(0) - \kappa^{2n} \leq ||X_0||^{2n} \,.$$

(1.4) immediately holds.

3. Finally, for all $T > 0$, (1.5) implies $\sup_{t\geq T}\xi_\epsilon(t) \leq \max\left\{\xi_\epsilon(T)\,;\,\left(\frac{(2n-1)\epsilon}{2\theta}\right)^n\right\}$. Then, for all $t \geq T_\kappa(\epsilon)$,

$$\mathbb{E}\left\{||X_t||^{2n}\right\} \leq \max\left\{\xi_\epsilon\left(T_\kappa(\epsilon)\right)\,;\,\left(\frac{(2n-1)\epsilon}{2\theta}\right)^n\right\} \leq \kappa^{2n}\,.$$

□

This means that the self-stabilizing process tends to be trapped in a ball with center 0. This result concerns the law u_t^ϵ and not the trajectories $t \mapsto X_t^\epsilon(\omega)$. But it points out the importance of δ_0 in the study. Indeed, Proposition 1.3 implies

$$\lim_{\epsilon\to 0}\limsup_{t\to\infty}\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\} = 0\,.$$

Consequently, the relevant sets for the exit problem of the McKean-Vlasov diffusions are the ones which contain the attractive point 0.

Remark 1.4. *In Proposition 1.3, we established the uniform boundedness of the moment of degree $2n$. We would like to point out that we can prove*

$$\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{||X_t^\epsilon||^{2p}\right\} \leq \max\left\{||X_0||^{2p}\,;\,\left(\frac{2p-1}{2\theta}\right)^p\epsilon^p\right\}$$

for all $p \in \mathbb{N}$.

We now give the assumptions on the domain $\mathcal{D}$.

Assumption 1.5. *We consider the dynamical system*

$$\varphi_t = X_0 - \int_0^t \nabla V\left(\varphi_s\right)ds$$

where X_0 is introduced in (0.1). *There exists $T_0 \geq 0$ such that $\{\varphi_{T_0+t}\,;\,t > 0\}$ is included in $\mathcal{D}$ and the orbit $\{\varphi_t\,;\,0 < t < T_0\}$ is included in $\mathcal{D}^c$.*

We point out that the domain $\mathcal{D}$ is not necessary stable by $-\nabla V$.

In order to heuristically understand this assumption, let us consider the dynamical system

$$\mathcal{Z}_t^N = \overline{X_0} - N\int_0^t \nabla\Upsilon^N\left(\mathcal{Z}_s^N\right)ds \tag{1.6}$$

where Υ^N is defined in (0.4). We remark that $\mathcal{Z}_t^N$ is equal to $\overline{\varphi_t}$ for all $t \geq 0$. Then, by Assumption 1.5, the orbit $\left\{\mathcal{Z}_{T_0+t}^N\,;\,t\in\mathbb{R}_+\right\}$ is included in $\mathcal{D}^N \subset \mathcal{D}\times\mathbb{R}^{d(N-1)}$.

Let us note that this assumption is weaker than Assumption 4.1.i) in [6]. We now present the other hypothesis:

Assumption 1.6. *The open domain $\mathcal{D}$ is stable by $-\nabla V - \nabla F =: -\nabla W$.*

This hypothesis is natural according to Proposition 1.3. Indeed, the law u_t^ϵ is as close as we want to δ_0. Consequently, the drift $\nabla V + \nabla F * u_t^\epsilon$ is close to $\nabla V + \nabla F * \delta_0 = \nabla V + \nabla F$.

Next, we define the exit cost.

Definition 1.7. *The exit cost of a bounded domain $\mathcal{D}$ which contains 0 is*

$$H := \inf_{z\in\partial\mathcal{D}} W(z)$$

with $W(z) := V(z) + F(z)$.

We now give an example of a domain satisfying both Assumptions 1.5–1.6.

Lemma 1.8. *For all $H > 0$, the domain $\mathcal{K}_H := \left\{x \in \mathbb{R}^d \;\mid\; V(x) + F(x) < H\right\}$ satisfies Assumptions 1.5–1.6. Moreover, its exit cost is H.*

Proof. Assumption 1.6 is obviously verified since $\mathcal{K}_H$ is a level set of the potential $V+F$ and its exit cost is H by definition.

Let us prove the first hypothesis. We take any $x \in \mathbb{R}^d$ and we consider the dynamical system

$$\varphi_t(x) = x - \int_0^t \nabla V\left(\varphi_s(x)\right) ds\,.$$

Since V is convex, $\varphi_t(x)$ converges to 0 so there exists $T_0 \geq 0$ such that the orbit $\{\varphi_t(x)\;;\; t < T_0\}$ is included in $\mathcal{K}_H^c$. Let us show $\{\varphi_t(x)\;;\; t > T_0\} \subset \mathcal{K}_H$. For this, it is now sufficient to establish that $\mathcal{K}_H$ is stable by $-\nabla V$:

$$\begin{aligned}
\frac{d}{dt}W(\varphi_t(x)) &= -\left\langle \nabla V(\varphi_t(x))\,;\, \nabla V(\varphi_t(x))\right\rangle - \left\langle \nabla V(\varphi_t(x))\,;\, \nabla F(\varphi_t(x))\right\rangle \\
&= -\left|\left|\nabla V(\varphi_t(x))\right|\right|^2 - \left\langle \nabla V(\varphi_t(x))\,;\, \varphi_t(x)\frac{G'\left(||\varphi_t(x)||\right)}{||\varphi_t(x)||}\right\rangle \\
&\leq -\left|\left|\nabla V(\varphi_t(x))\right|\right|^2 - \theta G'\left(||\varphi_t(x)||\right)||\varphi_t(x)|| < 0\,.
\end{aligned}$$

This finishes the proof. □

Before giving the main results of the paper, we recall a simple fact.

Lemma 1.9. *Υ^N admits exactly one critical point: $\overline{0}$. Moreover, it is the point of the global minimum.*

The proof is similar - up to some details due to the dimension d - to the one of Proposition 2.1 in [19]. Thereby, it is left to the reader.

Let us now provide the two main results.

Theorem: *We consider a function V which satisfies (V-1)–(V-3), a function F which satisfies (F-1)–(F-4). Under Assumptions 1.5–1.6, for all $\xi > 0$, we have the limit:*

$$\lim_{\epsilon\to0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H-\xi\right)\right] < \tau(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H+\xi\right)\right]\right\} = 1$$

with $H := \inf_{z\in\partial\mathcal{D}} W(z)$ where the potential W is defined as $W(z) := V(z) + F(z)$. Let $\mathcal{N}$ be a subset of $\partial\mathcal{D}$ such that $\inf_{z\in\mathcal{N}} W(z) > \inf_{z\in\partial\mathcal{D}} W(z)$. Then:

$$\lim_{\epsilon\to0}\mathbb{P}\left\{X^\epsilon_{\tau(\epsilon)} \in \mathcal{N}\right\} = 0\,.$$

Let us note that this result is stronger than the one in [6] since we do not assume that the domain $\mathcal{D}$ is stable by $-\nabla V$.

Theorem: *We consider a function V which satisfies (V-1)–(V-3), a function F which satisfies (F-1)–(F-4). Let H and ρ be two positive real numbers. For all $\delta > 0$, there exist $N_\delta \in \mathbb{N}^*$ and $\epsilon_\delta > 0$ such that:*

$$\sup_{N\geq N_\delta}\sup_{\epsilon<\epsilon_\delta}\mathbb{P}\left\{\sup_{0\leq t\leq\exp\left[\frac{H}{\epsilon}\right]}\left|\left|X^\epsilon_t - Z^{\epsilon,1,N}_t\right|\right| \geq \rho\right\} \leq \delta\,.$$

This result establishes that - in the small-noise limit - the particle $Z^{\epsilon,1,N}$ is a good approximation of the McKean-Vlasov diffusion, even in the long-time.

2 Exit problem of the first particle

In this section, we study the exit problem of the diffusion $Z^{\epsilon,1,N}$ from the domain $\mathcal{D}$ with large N and small ϵ. We recall the equation satisfied by each particle

$$\begin{aligned}Z_t^{\epsilon,i,N} =& X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla V\left(Z_s^{\epsilon,i,N}\right)ds\\ &- \frac{1}{N}\sum_{j=1}^N \int_0^t \nabla F\left(Z_s^{\epsilon,i,N} - Z_s^{\epsilon,j,N}\right)ds\,.\end{aligned}$$

And the whole system $\mathcal{Z}^{\epsilon,N} := \left(Z^{\epsilon,1,N}, \cdots, Z^{\epsilon,N,N}\right)$ verifies

$$\mathcal{Z}_t^{\epsilon,N} \quad = \quad \overline{X_0} + \sqrt{\epsilon}\mathcal{B}_t^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{Z}_s^{\epsilon,N}\right)ds$$

where the potential Υ^N is defined in (0.4). We observe that the exit problem of $Z^{\epsilon,1,N}$ from $\mathcal{D}$ is equivalent to the one of $\mathcal{Z}^{\epsilon,N}$ from $\mathcal{D}\times\mathbb{R}^{d(N-1)}$. Furthermore, the diffusion $\mathcal{Z}^{\epsilon,N}$ is homogeneous.

The domain $\mathcal{D}$ satisfies Assumptions 1.5–1.6. However, *nothing ensures us that the domain $\mathcal{D}\times\mathbb{R}^{d(N-1)}$ satisfies Assumption A.1*, described in the appendix. Assumption A.2 is obvious since the potential Υ^N is convex due to the convexity of both V and F. It is then necessary and sufficient to prove the stability of $\mathcal{D}\times\mathbb{R}^{d(N-1)}$ by $-N\nabla\Upsilon^N$ for applying the Freidlin-Wentzell theory. We recall that the notion of "stable by" has been introduced in Definition 1.2.

As remarked previously, the drift term $-\nabla V - \nabla F * u_s^\epsilon$ is close to $-\nabla V - \nabla F * \delta_0$ for s sufficiently large. The propagation of chaos implies that $-\nabla V - \nabla F * \left(\frac{1}{N}\sum_{j=1}^N \delta_{Z_s^{\epsilon,j,N}}\right)$ tends also to $-\nabla V - \nabla F * \delta_0$. Heuristically, since $\mathcal{D}$ is stable by $-\nabla V - \nabla F$, we can imagine that it is stable by $-\nabla V - \nabla F * \nu$ for all the measures ν sufficiently close to δ_0. This would imply that $\left(\mathcal{D}\times\mathbb{R}^{d(N-1)}\right)\bigcap\mathbb{B}_\kappa^N$ is stable by $-N\nabla\Upsilon^N$ for κ sufficiently small.

Of course, this does not have any reason to be true. Consequently, we consider two sequences of sets which frame the domain and which satisfy Assumption 1.6. Let us consider $\kappa > 0$. We recall that $2n = \deg(G)$, see (F-1)–(F-2).

Definition 2.1. **1.** $\mathbb{B}_\kappa^\infty$ *denotes the set of all the probability measures μ on $\mathbb{R}^d$ satisfying $\int_{\mathbb{R}^d}||x||^{2n}\mu(dx) \leq \kappa^{2n}$.*
2. *For all the measures μ, W_μ is equal to $V + F * \mu$.*
3. *For all $\nu \in \left(\mathbb{B}_\kappa^\infty\right)^{\mathbb{R}_+} =: \mathbb{M}_\kappa^\infty$ and for all $x \in \mathbb{R}^d$, we also introduce the dynamical system:*

$$\psi_t^\nu(x) = x - \int_0^t \nabla W_{\nu_s}\left(\psi_s^\nu(x)\right)ds\,.$$

4. *Let r be an increasing function from $\mathbb{R}_+$ to itself such that $r(0) = 0$. This function is chosen subsequently, see Section 3. For all $\kappa > 0$, we introduce the following two domains:*

$$\mathcal{D}_{i,\kappa} := \left\{x\in\mathcal{D}\ \mid\ \inf_{\nu\in\mathbb{M}_\kappa^\infty}\inf_{t\in\mathbb{R}_+} d\left(\psi_t^\nu(x)\,;\mathcal{D}^c\right) > r(\kappa)\right\} \tag{2.1}$$

$$\textit{and}\quad \mathcal{D}_{e,\kappa} := \left\{\psi_t^\nu(x)\ \mid\ t\geq 0,\, \nu\in\mathbb{M}_\kappa^\infty,\, d\left(x\,,\mathcal{D}\right) < r(\kappa)\right\}. \tag{2.2}$$

Obviously, for all $\kappa > 0$, and for all $\mu \in \mathbb{B}_\kappa^\infty$, the two sets $\mathcal{D}_{i,\kappa}$ and $\mathcal{D}_{e,\kappa}$ are stable by $-\nabla W_\mu = -\nabla V - \nabla F * \mu$. Moreover, we have the inclusions

$$\mathcal{D}_{i,\kappa_2} \subset \mathcal{D}_{i,\kappa_1} \subset \mathcal{D} \subset \mathcal{D}_{e,\kappa_1} \subset \mathcal{D}_{e,\kappa_2}\,,$$

for all $0 < \kappa_1 < \kappa_2$. More precisely:

$$\mathrm{d}\left(\mathcal{D}_{i,\kappa}\,;\,\mathcal{D}^c\right) \geq r(\kappa) \quad \text{and} \quad \mathrm{d}\left(\mathcal{D}\,;\,\mathcal{D}^c_{e,\kappa}\right) \geq r(\kappa)\,.$$

Now we justify why the two sets frame the open $\mathcal{D}$.

Proposition 2.2. *The following limits hold:*

$$\lim_{\kappa\to 0}\sup_{z\in\partial\mathcal{D}_{i,\kappa}}\mathrm{d}\left(z\,;\,\mathcal{D}^c\right) = \lim_{\kappa\to 0}\sup_{z\in\partial\mathcal{D}_{e,\kappa}}\mathrm{d}\left(z\,;\,\mathcal{D}\right) = 0\,.$$

Proof. **Step 1.** Let μ be a measure in $\mathbb{B}^\infty_\kappa$. We note that, by applying Lemma 1.1 in [17], the drift $\nabla F * \mu$ is the product of x with a polynomial function of degree $2n-2$ of $||x||$ and with a finite number of parameters of the form:

$$C(l_0, l_1, \cdots, l_d) := \int_{\mathbb{R}^d}\prod_{i=1}^{d}\langle x\,;\,e_i\rangle^{l_i}\,||x||^{l_0}\,\mu(dx)$$

where $l_0 + \sum_{i=1}^d l_i \leq 2n$. The definition of $\mathbb{B}^\infty_\kappa$ implies $C(l_0, l_1, \cdots, l_d) \leq \kappa^{2n}$ for all $l_0, \cdots, l_d \geq 0$ such that $l_0 + \cdots + l_d \leq 2n$. Thereby, for any compact set K which contains $\mathcal{D}$, there exists $f(\kappa)$ which tends to 0 when κ goes to 0 such that

$$\sup_{\mu\in\mathbb{B}^\infty_\kappa}\sup_{x\in K}||\nabla F*\mu(x) - \nabla F(x)|| \leq f(\kappa)\,.$$

Moreover, (V-2) and (F-3) imply $\inf\limits_{x\in K}\inf\limits_{\mu\in\mathbb{B}^\infty_\kappa}\mathrm{Hess}\,W_\mu(x) \geq \theta$ for any compact set K as above.

Step 2. Let x_0 be an element of $\mathcal{D}$. Let us prove that $x_0\in\mathcal{D}_{i,\kappa}$ when κ is small enough. We introduce the dynamical system $\psi(x_0)$:

$$\psi_t(x_0) = x_0 - \int_0^t\nabla V(\psi_s(x_0))ds - \int_0^t\nabla F(\psi_s(x_0))ds\,.$$

We remark that $\psi_t(x_0)\in K$ for all $t\geq 0$. We recall

$$\psi^\nu_t(x_0) = x_0 - \int_0^t\nabla V(\psi^\nu_s(x_0))ds - \int_0^t\nabla F*\nu_s(\psi^\nu_s(x_0))ds\,.$$

Assumption 1.6 implies that $\delta(x_0) := \inf\limits_{t\geq 0} d\left(\psi_t(x_0)\,;\,\mathcal{D}^c\right) > 0$. From now on, we take $r(\kappa) < \frac{\delta(x_0)}{4}$. We introduce $\xi_t(x_0) := ||\psi^\nu_t(x_0) - \psi_t(x_0)||$. Then, for all $\nu\in\mathbb{M}^\infty_\kappa$, if $\psi^\nu_t(x_0)\in K$, we get

$$\begin{aligned}
\frac{d}{dt}\xi_t(x_0)^2 =& -2\left\langle\nabla W_{\nu_t}(\psi^\nu_t(x_0)) - \nabla W(\psi_t(x_0))\,;\,\psi^\nu_t(x_0) - \psi_t(x_0)\right\rangle\\
=& -2\left\langle\nabla W_{\nu_t}(\psi^\nu_t(x_0)) - \nabla W_{\nu_t}(\psi_t(x_0))\,;\,\psi^\nu_t(x_0) - \psi_t(x_0)\right\rangle\\
& -2\left\langle\nabla W_{\nu_t}(\psi_t(x_0)) - \nabla W(\psi_t(x_0))\,;\,\psi^\nu_t(x_0) - \psi_t(x_0)\right\rangle\\
\leq& -2\theta\xi_t(x_0)^2 + 2\xi_t(x_0)\sup_{\mu\in\mathbb{B}^\infty_\kappa}\sup_{x\in K}||\nabla F*\mu(x) - \nabla F(x)||\\
\leq& 2\xi_t(x_0)\left\{f(\kappa) - \theta\xi_t(x_0)\right\}\,.
\end{aligned} \tag{2.3}$$

By taking κ sufficiently small, $d\left(\psi^\nu_t(x_0)\,;\,\psi_t(x_0)\right) \leq \frac{\delta(x_0)}{2}$ for all $0\leq t\leq\tau_K$ with $\tau_K := \inf\left\{t\geq 0 \;|\; \psi^\nu_t(x_0)\notin K\right\}$. We deduce: $\inf\limits_{t\geq 0} d\left(\psi^\nu_t(x_0)\,;\,\mathcal{D}^c\right) \geq \dfrac{\delta(x_0)}{2}$ for all $t\leq\tau_K$. This implies $\tau_K = \infty$ and $\inf\limits_{t\geq 0} d\left(\psi^\nu_t(x_0)\,;\,\mathcal{D}^c\right) \geq r(2\kappa)$ for all $t\geq 0$ and for all $\nu\in\mathbb{M}^\infty_\kappa$. This means that $x_0\in\mathcal{D}_{i,\kappa}$ for κ small enough.

Step 3. We now prove $\lim_{\kappa\to 0}\sup_{z\in\mathcal{D}_{e,\kappa}} \mathrm{d}\,(z\,;\,\mathcal{D}) = 0$. Let x_0 be a point in $\mathbb{R}^d$ satisfying $\mathrm{d}\,(x_0\,;\,\mathcal{D}) \le r(\kappa)$. There exists $y_0 \in \mathcal{D}$ such that $d(x_0, y_0) \le r(2\kappa)$. We study the two dynamical systems:

$$\psi_t(x_0) = x_0 - \int_0^t \nabla W(\psi_s(x_0))ds \quad \text{and} \quad \psi_t(y_0) = y_0 - \int_0^t \nabla W(\psi_s(y_0))ds\,.$$

Since $\mathrm{Hess}\, W \ge \theta$, the function $t \mapsto \mathrm{d}(\psi_t(x_0), \psi_t(y_0))$ is nonincreasing. This means $d(\psi_t(x_0), \psi_t(y_0)) \le r(2\kappa)$ for all $t \ge 0$. By proceeding like in Step 2, the distance $d(\psi_t(x_0), \psi_t^\nu(x_0))$ is less than $\frac{f(\kappa)}{\theta}$. Hence:

$$\mathrm{d}\left(\psi_t^\nu(x_0), \psi_t(y_0)\right) \le \mathrm{d}\left(\psi_t^\nu(x_0), \psi_t(x_0)\right) + \mathrm{d}\left(\psi_t(x_0), \psi_t(y_0)\right) \le \frac{f(\kappa)}{\theta} + r(2\kappa)\,.$$

We deduce that $\sup_{\mathrm{d}(x\,;\,\mathcal{D})\le r(\kappa)} \sup_{t\in\mathbb{R}_+} \mathrm{d}\left(\psi_t^\nu(x_0)\,;\,\mathcal{D}\right) \to 0$ as κ goes to 0. It implies the convergence of $\sup_{z\in\mathcal{D}_{e,\kappa}} \mathrm{d}\,(z\,;\,\mathcal{D})$ to 0 when κ tends to 0. □

We define the two domains to which we will apply Freidlin-Wentzell theory:

$$\mathcal{D}_{i,\kappa}^{(N)} := \left(\mathcal{D}_{i,\kappa} \times \mathbb{R}^{d(N-1)}\right) \bigcap \mathbb{B}_\kappa^N$$
$$\text{and} \quad \mathcal{D}_{e,\kappa}^{(N)} := \left(\mathcal{D}_{e,\kappa} \times \mathbb{R}^{d(N-1)}\right) \bigcap \mathbb{B}_\kappa^N\,.$$

First, let us prove that the ball $\mathbb{B}_\kappa^N$ is stable by $-N\nabla\Upsilon^N$. It is not an obvious consequence of the convexity of Υ^N because the norm $|||\,.\,|||$ does not derive from a scalar product.

Lemma 2.3. *The open domain $\mathbb{B}_\kappa^N$ is stable by $-N\nabla\Upsilon^N$. Moreover, its exit cost goes to infinity when N goes to infinity:*

$$\lim_{N\to+\infty} \inf_{Z\in\partial\mathbb{B}_\kappa^N} N\Upsilon^N(Z) = +\infty\,.$$

Proof. **Step 1.** We take $\mathcal{Z}_0^N := \left(Z_0^1, \cdots, Z_0^N\right) \in \mathbb{R}^{dN}$ and we consider the deterministic dynamical system already introduced in (1.6)

$$\mathcal{Z}_t^N = \mathcal{Z}_0^N - N\int_0^t \nabla\Upsilon^N\left(\mathcal{Z}_s^N\right) ds =: \left(Z_t^1, \cdots, Z_t^N\right)\,.$$

We recall that $\Upsilon^N\left(Z_1, \cdots, Z_N\right) = \frac{1}{N}\sum_{i=1}^N V(Z_i) + \frac{1}{2N^2}\sum_{i=1}^N\sum_{j=1}^N F(Z_i - Z_j)$. By definition, $\left|\left|\left|\mathcal{Z}_t^N\right|\right|\right|^{2n} := \frac{1}{N}\sum_{i=1}^N \left|\left|Z_t^i\right|\right|^{2n}$. Then:

$$\begin{aligned}
\frac{d}{dt}\left|\left|\left|\mathcal{Z}_t^N\right|\right|\right|^{2n} =& -\frac{2n}{N}\sum_{i=1}^N \left|\left|Z_t^i\right|\right|^{2n-2}\left\langle Z_t^i\,;\,\nabla V\left(Z_t^i\right)\right\rangle \\
& -\frac{2n}{N}\sum_{i=1}^N\sum_{j=1}^N \left|\left|Z_t^i\right|\right|^{2n-2}\left\langle Z_t^i\,;\,\nabla F\left(Z_t^i - Z_t^j\right)\right\rangle \\
=& -\frac{2n}{N}\sum_{i=1}^N \left|\left|Z_t^i\right|\right|^{2n-2}\left\langle Z_t^i\,;\,\nabla V\left(Z_t^i\right)\right\rangle \\
& -\frac{n}{N}\sum_{i=1}^N\sum_{j=1}^N \left\langle \left|\left|Z_t^i\right|\right|^{2n-2} Z_t^i - \left|\left|Z_t^j\right|\right|^{2n-2} Z_t^j\,;\,\nabla F\left(Z_t^i - Z_t^j\right)\right\rangle
\end{aligned}$$

Like in Step 1 in the proof of Proposition 1.3, we can prove:

$$\sum_{i=1}^{N}\sum_{j=1}^{N}\left\langle \left|\left|Z_t^i\right|\right|^{2n-2} Z_t^i - \left|\left|Z_t^j\right|\right|^{2n-2} Z_t^j \,;\, \nabla F\left(Z_t^i - Z_t^j\right)\right\rangle \geq 0\,.$$

Hypothesis (V-2) implies $\frac{d}{dt}\left|\left|\left|\mathcal{Z}_t^N\right|\right|\right|^{2n} \leq -2n\theta\left|\left|\left|\mathcal{Z}_t^N\right|\right|\right|^{2n}$. Consequently, the ball $\mathbb{B}_\kappa^N$ is stable by $-N\nabla\Upsilon^N$.

Step 2. We now compute the exit cost. Hypotheses (V-2) and (F-1) imply

$$N\Upsilon^N\left(Z_1,\cdots,Z_N\right) \geq \frac{\theta}{2}\sum_{i=1}^{N}\left|\left|Z_i\right|\right|^2 \geq \frac{\theta}{2}N^{\frac{1}{n}}\left(\frac{1}{N}\sum_{i=1}^{N}\left|\left|Z_i\right|\right|^{2n}\right)^{\frac{1}{n}}\,.$$

Consequently, $\inf_{\mathcal{Z}\in\partial\mathbb{B}_\kappa^N} N\Upsilon^N\left(\mathcal{Z}\right) \geq \frac{\theta}{2}N^{\frac{1}{n}}\kappa^2$ which converges to infinity when N goes to infinity. □

Before looking at the sets $\mathcal{D}_{i,\kappa}^{(N)}$ and $\mathcal{D}_{e,\kappa}^{(N)}$, we compute the exit cost of a set of the form $\mathcal{O}\times\mathbb{R}^{d(N-1)}$.

Lemma 2.4. *Let $\mathcal{O}$ be a bounded domain which contains 0. We have:*

$$\lim_{N\to\infty}\inf_{Z\in\partial\mathcal{O}\times\mathbb{R}^{d(N-1)}} N\Upsilon^N(Z) = \inf_{z\in\partial\mathcal{O}}\left(V(z)+F(z)\right)\,.$$

Proof. We study the function ξ_z from $\mathbb{R}^{d(N-1)}$ to $\mathbb{R}$:

$$\xi_z(x_2,\cdots,x_N) := \Upsilon^N\left(z,x_2,\cdots,x_N\right)\,.$$

ξ_z is convex on $\mathbb{R}^{d(N-1)}$ and the unique minimizer is $\left(x_0^N(z),\cdots,x_0^N(z)\right)\in\mathbb{R}^{d(N-1)}$ where $x_0^N(z)$ satisfies

$$\nabla V\left(x_0^N(z)\right) + \frac{1}{N}\nabla F\left(x_0^N(z)-z\right) = 0\,.$$

This implies the existence of a continuous function f_1^N satisfying $\lim_{N\to\infty} f_1^N(z) = 0$ for all $z\in\mathbb{R}^d$ such that

$$x_0^N(z) = \frac{1}{N}\left(\mathrm{Hess}\,V(0)\right)^{-1}\nabla F(z) + \frac{f_1^N(z)}{N}\,.$$

Simple computations imply

$$\Upsilon^N\left(z,x_0^N(z),\cdots,x_0^N(z)\right) = \frac{1}{N}\left\{V(z)+F(z\right\} + \frac{f_2^N(z)}{N}$$

where f_2^N is a continuous function satisfying $\lim_{N\to\infty} f_2^N(z) = 0$ for all $z\in\mathbb{R}^d$. Then:

$$N\Upsilon^N(z,x_0^N(z),\cdots,x_0^N(z)) = W(z) + f_2^N(z)\,.$$

Let us note that $\lim_{N\to\infty}\sup_{z\in\partial\mathcal{O}} f_2^N(z) = 0$ since $\partial\mathcal{O}$ is bounded. This ends the proof. □

Now we study the two sets $\mathcal{D}_{i,\kappa}^{(N)}$ and $\mathcal{D}_{e,\kappa}^{(N)}$.

Lemma 2.5. *The two domains $\mathcal{D}_{i,\kappa}^{(N)}$ and $\mathcal{D}_{e,\kappa}^{(N)}$ are stable by $-N\nabla\Upsilon^N$.*

Proof. Let $\left(Z_0^1,\cdots,Z_0^N\right)$ be an element of $\mathcal{D}_{i,\kappa}^{(N)}$. By definition, it is in $\mathbb{B}_\kappa^N$. The stability of the ball $\mathbb{B}_\kappa^N$ proved in Lemma 2.3 implies $\left(Z_t^1,\cdots,Z_t^N\right)\in\mathbb{B}_\kappa^N$ for all $t\geq 0$. Then, $\mu_t^N:=\frac{1}{N}\sum_{j=1}^N\delta_{Z_t^j}\in\mathbb{B}_\kappa^\infty$ for all $t\geq 0$ which means $\mu^N\in\mathbb{M}_\kappa^\infty$. However, by definition, $Z_t^1=\psi_t^{\mu^N}\left(Z_0^1\right)$. Since $\mathcal{D}_{i,\kappa}$ is stable by $-\nabla V-\nabla F*\mu_t^N$ for all $t\geq 0$, we deduce that $Z_t^1\in\mathcal{D}_{i,\kappa}$ for all $t\geq 0$. This finishes to prove the stability of $\mathcal{D}_{i,\kappa}^{(N)}$ by $-N\nabla\Upsilon^N$. We proceed in the same way with $\mathcal{D}_{e,\kappa}^{(N)}$. □

We now define the exit times that we use. We recall that Assumption 1.5 is assumed. Consequently, nothing forbides X_0 to be an element of $\mathcal{D}^c$. In this case, we introduce the first hitting time.

Definition 2.6. *By $S_{i,\kappa}^{1,N}(\epsilon)$ (resp. by $S_{e,\kappa}^{1,N}(\epsilon)$), we denote the first hitting time of the diffusion $\mathcal{Z}^{\epsilon,N}$ defined in* (0.3)–(0.4) *on the domain $\mathcal{D}_{i,\kappa}\times\mathbb{R}^{d(N-1)}$ (resp. $\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}$).*

We already know that these times are less than a deterministic time with high probability for ϵ going to 0:

Lemma 2.7. *For all $\kappa>0$, we have the limit*

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{S_{i,\kappa}^{1,N}(\epsilon)\leq T_0+1\right\}=\lim_{\epsilon\to 0}\mathbb{P}\left\{S_{e,\kappa}^{1,N}(\epsilon)\leq T_0+1\right\}=1$$

where T_0 has been defined in Assumption 1.5.

Since $0\in\mathcal{D}$, this result is an obvious consequence of Assumption 1.5, Proposition A.4 and Proposition 2.2. The proof is left to the reader.

We can now define the exit times.

Definition 2.8. *We denote by*

$$\tau_{i,\kappa}^{1,N}(\epsilon):=\inf\left\{t\geq S_{i,\kappa}^{1,N}(\epsilon)\;\mid\;Z_t^{\epsilon,N}\notin\mathcal{D}_{i,\kappa}\times\mathbb{R}^{d(N-1)}\right\}$$

the first exit time of the diffusion $\mathcal{Z}^{\epsilon,N}$ defined in (0.3)–(0.4) *from $\mathcal{D}_{i,\kappa}\times\mathbb{R}^{d(N-1)}$*

$$\text{and}\quad\tau_{e,\kappa}^{1,N}(\epsilon):=\inf\left\{t\geq S_{e,\kappa}^{1,N}(\epsilon)\;\mid\;Z_t^{\epsilon,N}\notin\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}\right\}$$

the first exit time of the diffusion $\mathcal{Z}^{\epsilon,N}$ from $\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}$.

We remark that $\tau_{i,\kappa}^{1,N}(\epsilon)$ (resp. $\tau_{e,\kappa}^{1,N}(\epsilon)$) is the exit time of the diffusion $Z^{\epsilon,1,N}$ defined in (0.2) from the domain $\mathcal{D}_{i,\kappa}$ (resp. from the domain $\mathcal{D}_{e,\kappa}$).

We recall that we can not apply Freidlin-Wentzell theory directly to the two domains $\mathcal{D}_{i,\kappa}\times\mathbb{R}^{d(N-1)}$ and $\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}$. Consequently, we introduce two other exit times.

Definition 2.9. *We denote by*

$$T_{i,\kappa}^N(\epsilon):=\inf\left\{t\geq S_{i,\kappa}^{1,N}(\epsilon)\;\mid\;Z_t^{\epsilon,N}\notin\mathcal{D}_{i,\kappa}^{(N)}\right\}$$

the first exit time of the diffusion $\mathcal{Z}^{\epsilon,N}$ from $\mathcal{D}_{i,\kappa}^{(N)}=\mathcal{D}_{i,\kappa}\times\mathbb{R}^{d(N-1)}\bigcap\mathbb{B}_\kappa^N$

$$\text{and}\quad T_{e,\kappa}^N(\epsilon):=\inf\left\{t\geq S_{e,\kappa}^{1,N}(\epsilon)\;\mid\;Z_t^{\epsilon,N}\notin\mathcal{D}_{e,\kappa}^{(N)}\right\}$$

the first exit time of the diffusion $\mathcal{Z}^{\epsilon,N}$ from $\mathcal{D}_{e,\kappa}^{(N)}=\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}\bigcap\mathbb{B}_\kappa^N$.

We have all the ingredients in order to obtain the exit times.

Proposition 2.10. *For all $\delta > 0$, there exists κ_0 such that for all $0 < \kappa < \kappa_0$ and for all N large enough, the following limit holds:*

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H-\delta\right)\right] < \tau_{e,\kappa}^{1,N}(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H+\delta\right)\right]\right\} = 1 \tag{2.4}$$
$$\textit{with}\quad H := \inf_{z\in\partial\mathcal{D}} W(z) \quad \textit{and} \quad W(z) = V(z) + F(z)\,.$$

Furthermore, we have information on the exit location. Indeed, for all $\mathcal{N} \subset \partial\mathcal{D}_{e,\kappa}$ such that $\inf\limits_{z\in\mathcal{N}} W(z) > \inf\limits_{z\in\partial\mathcal{D}_{e,\kappa}} W(z)$, we have:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{Z^{\epsilon,1,N}_{\tau_{e,\kappa}^{1,N}(\epsilon)} \in \mathcal{N}\right\} = 0 \tag{2.5}$$

for κ small enough and N large enough.

Proof. **Outline** First, we prove that the whole system $\mathcal{Z}^{\epsilon,N}$ enters with high probability before a time T_κ (finite, independent of N, independent of ϵ and deterministic) in the domain $\mathbb{B}_\kappa^N$. Next, we prove that the system does not exit from $\mathcal{D}_{e,\kappa} \times \mathbb{R}^{d(N-1)}$ before this time T_κ with probability close to 1.

The set $\mathcal{D}_{e,\kappa}^{(N)}$ is stable by $-N\nabla\Upsilon^N$. We apply Freidlin-Wentzell theory. Finally, we prove that the diffusion $\mathcal{Z}^{\epsilon,N}$ exits from the domain $\mathcal{D}_{e,\kappa} \times \mathbb{R}^{d(N-1)}$ before exiting from $\mathbb{B}_\kappa^N$.

Step 1. We recall the dynamical system introduced in (1.6):

$$\mathcal{Z}_t^N = \overline{X_0} - N\int_0^t \nabla\Upsilon^N\left(\mathcal{Z}_s^N\right)ds\,.$$

As $\mathcal{Z}_0^N = \overline{X_0}$, we deduce that for all $t \geq 0$, $\mathcal{Z}_t^N = \overline{\psi_t(X_0)}$ with

$$\psi_t(X_0) = X_0 - \int_0^t \nabla V(\psi_s(X_0))ds\,.$$

Hypotheses (V-2) and (V-3) imply the convergence of $\mathcal{Z}^N$ to $\overline{0}$ and there exists T_κ, deterministic and independent from N such that

$$\mathcal{Z}_{T_\kappa}^N \in \mathbb{B}_\kappa^N\,.$$

We assume without any loss of generality that $T_\kappa \geq T_0 + 1$ where T_0 is defined in Lemma 2.7. Proposition A.4 and Lemma 2.7 allow to obtain the following limits:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{T_{e,\kappa}^N(\epsilon) \leq T_\kappa\right\} = 0 \quad \text{and} \quad \lim_{\epsilon\to 0}\mathbb{P}\left\{\mathcal{Z}_{T_\kappa}^{\epsilon,N} \in \mathbb{B}_\kappa^N\right\} = 1\,. \tag{2.6}$$

Step 2. From now on, we consider the new exit time:

$$\eta_{e,\kappa}^N(\epsilon) := \inf\left\{t \geq T_\kappa \;\;|\;\; \mathcal{Z}_t^{\epsilon,N} \notin \mathcal{D}_{e,\kappa}^{(N)}\right\}\,.$$

The domain $\mathcal{D}_{e,\kappa}^{(N)}$ is stable by $-N\nabla\Upsilon^N$ according to Proposition 2.5. We apply Proposition A.3 to $\mathcal{D}_{e,\kappa}^{(N)}$ and we obtain

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H_\kappa^N - \frac{\delta}{2}\right)\right] < \eta_{e,\kappa}^N(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H_\kappa^N + \frac{\delta}{2}\right)\right]\right\} = 1 \tag{2.7}$$
$$\text{with}\quad H_\kappa^N := N \inf_{Z\in\partial\mathcal{D}_{e,\kappa}^{(N)}} \Upsilon^N(Z)\,.$$

The limits in (2.6) and in (2.7) imply

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H_\kappa^N-\frac{\delta}{2}\right)\right]<T_{e,\kappa}^N(\epsilon)<\exp\left[\frac{2}{\epsilon}\left(H_\kappa^N+\frac{\delta}{2}\right)\right]\right\}=1\,.$$

Step 3. We now compute the exit cost H_κ^N. By definition,

$$\begin{aligned}H_\kappa^N :=& N\inf_{Z\in\partial\mathcal{D}_{e,\kappa}^{(N)}}\Upsilon^N(Z)\\ =&\inf\left\{N\inf_{Z\in\partial\mathcal{D}_{e,\kappa}\times\mathbb{R}^{d(N-1)}}\Upsilon^N(Z)\,;\,N\inf_{Z\in\partial\mathbb{B}_\kappa^N}\Upsilon^N(Z)\right\}.\end{aligned}$$

Lemmas 2.3 and 2.4 imply that H_κ^N converges to $\inf\limits_{z\in\partial\mathcal{D}_{e,\kappa}}W(z)$ when N goes to infinity. Finally, the continuity of the function W and Proposition 2.2 imply the convergence of $\inf\limits_{z\in\partial\mathcal{D}_{e,\kappa}}W(z)$ to H when κ tends to 0. By taking κ sufficiently small, then N sufficiently large, we obtain $\left|H_\kappa^N-H\right|<\frac{\delta}{2}$ which ends the proof of (2.4).

Step 4. We now prove that the two exit times $T_{e,\kappa}^N(\epsilon)$ and $\tau_{e,\kappa}^{1,N}(\epsilon)$ are equal with probability close to 1 for N large enough and ϵ small enough. We just remark that

$$\inf_{Z\in\partial\mathcal{D}_{e,\kappa}^{(N)}}N\Upsilon^N(Z)<\inf_{Z\in\partial\mathbb{B}_\kappa^N}N\Upsilon^N(Z)$$

for N large enough, and we apply (A.3) of Proposition A.3.

Step 5. By applying Lemma 2.4, we have

$$\inf_{Z\in\mathcal{N}\times\mathbb{R}^{d(N-1)}}N\Upsilon^N(Z)>\inf_{Z\in\partial\mathcal{D}_{e,\kappa}^{(N)}}N\Upsilon^N(Z)$$

if $\mathcal{N}\subset\partial\mathcal{D}_{e,\kappa}$ such that $\inf\limits_{z\in\mathcal{N}}W(z)>\inf\limits_{z\in\partial\mathcal{D}_{e,\kappa}}W(z)$ for N large enough. Applying Proposition A.3 for N large enough leads to (2.5). □

An analogous result holds with $\mathcal{D}_{i,\kappa}$. We do not give the proof since it is similar to the previous one.

Proposition 2.11. *For all $\delta>0$, there exists κ_0 such that for all $0<\kappa<\kappa_0$ and for all N large enough, the following limit holds:*

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}(H-\delta)\right]<\tau_{i,\kappa}^{1,N}(\epsilon)<\exp\left[\frac{2}{\epsilon}(H+\delta)\right]\right\}=1$$
$$\text{with}\quad H:=\inf_{z\in\partial\mathcal{D}}W(z)\quad\text{and}\quad W(z):=V(z)+F(z)\,.$$

Furthermore, for all $\mathcal{N}\subset\partial\mathcal{D}_{i,\kappa}$ such that $\inf\limits_{z\in\mathcal{N}}W(z)>\inf\limits_{z\in\partial\mathcal{D}_{i,\kappa}}W(z)$, we have:

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{Z^{\epsilon,1,N}_{\tau_{i,\kappa}^{1,N}(\epsilon)}\in\mathcal{N}\right\}=0$$

if κ is small enough and if N is sufficiently large.

Proposition 2.10 and Proposition 2.11 allow to obtain the results on $\mathcal{D}$.

Corollary 2.12. *By $\tau^{1,N}(\epsilon)$, we denote the exit time of the diffusion $Z^{\epsilon,1,N}$ from the domain $\mathcal{D}$. For all $\rho>0$, there exists $N_0\geq 2$ such that for all $N\geq N_0$, we have the following limit:*

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}(H-\rho)\right]<\tau^{1,N}(\epsilon)<\exp\left[\frac{2}{\epsilon}(H+\rho)\right]\right\}=1\qquad(2.8)$$

where H is like in in Definition 1.7: $H := \inf_{z\in\partial\mathcal{D}} (V(z) + F(z))$.

Furthermore, for all $\mathcal{N} \subset \partial\mathcal{D}$ *such that* $\inf_{z\in\mathcal{N}} W(z) > \inf_{z\in\partial\mathcal{D}} W(z)$, *there exists* $N_1 \geq 2$ *such that for all* $N \geq N_1$, *we have:*

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{Z^{\epsilon,1,N}_{\tau^{1,N}(\epsilon)} \in \mathcal{N}\right\} = 0\,. \tag{2.9}$$

Proof. **Step 1.** For all $\kappa > 0$, $Z^{\epsilon,1,N}$ needs to exit from $\mathcal{D}_{i,\kappa}$ before exiting from $\mathcal{D}$. Consequently, for all $\rho > 0$, we have:

$$\mathbb{P}\left\{\tau^{1,N}(\epsilon) \leq \exp\left[\frac{2}{\epsilon}(H-\rho)\right]\right\} \leq \mathbb{P}\left\{\tau^{1,N}_{i,\kappa}(\epsilon) \leq \exp\left[\frac{2}{\epsilon}(H-\rho)\right]\right\}\,.$$

We apply Proposition 2.11 by taking κ sufficiently small and N large enough. This implies the convergence of $\mathbb{P}\left\{\tau^{1,N}_{i,\kappa}(\epsilon) \leq \exp\left[\frac{2}{\epsilon}(H-\rho)\right]\right\}$ to 0 when ϵ goes to 0 ; if N is large enough.

Step 2. If $Z^{\epsilon,1,N}$ does not exit from $\mathcal{D}$, it does not exit from $\mathcal{D}_{e,\kappa}$. We apply Proposition 2.10 by taking κ sufficiently small and N large enough. It implies

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau^{1,N}(\epsilon) \geq \exp\left[\frac{2}{\epsilon}(H+\rho)\right]\right\} = 0$$

for N large enough.

Step 3. By definition of $\mathcal{N}$, there exists $\xi > 0$ such that $\inf_{z\in\mathcal{N}} W(z) = H + 3\xi$. In order to prove (2.9), we introduce the set

$$\mathcal{K}_{H+2\xi} := \left\{x \in \mathbb{R}^d \;\mid\; W(x) < H + 2\xi\right\}\,.$$

By Lemma 1.8, the domain $\mathcal{K}_{H+2\xi}$ satisfies Assumptions 1.5–1.6. Then, we can apply (2.8) to $\mathcal{K}_{H+2\xi}$. We denote $\tau^{1,N}_{\xi}(\epsilon)$ the first exit time of $Z^{\epsilon,1,N}$ from $\mathcal{K}_{H+2\xi}$. We immediately have:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}(H+2\xi-\rho)\right] < \tau^{1,N}_{\xi}(\epsilon) < \exp\left[\frac{2}{\epsilon}(H+2\xi+\rho)\right]\right\} = 1 \tag{2.10}$$

for all $\rho > 0$ and for N large enough. By construction of $\mathcal{K}_{H+2\xi}$, we have $\mathcal{N} \subset \mathcal{K}^c_{H+2\xi}$. This implies:

$$\begin{aligned}
\mathbb{P}\left\{Z^{\epsilon,1,N}_{\tau^{1,N}(\epsilon)} \in \mathcal{N}\right\} \leq& \mathbb{P}\left\{Z^{\epsilon,1,N}_{\tau^{1,N}(\epsilon)} \notin \mathcal{K}_{H+2\xi}\right\} \\
\leq& \mathbb{P}\left\{\tau^{1,N}_{\xi}(\epsilon) \leq \tau^{1,N}(\epsilon)\right\} \\
\leq& \mathbb{P}\left\{\tau^{1,N}_{\xi}(\epsilon) \leq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} \\
&+ \mathbb{P}\left\{\exp\left[\frac{H+\xi}{\epsilon}\right] \leq \tau^{1,N}(\epsilon)\right\}\,.
\end{aligned}$$

The limit (2.10) with $\rho = \xi$ implies the convergence to 0 of the first term as ϵ going to 0. By applying (2.8), the second term goes to 0 when ϵ tends to 0. □

3 Strong propagation of chaos

It is well known that the two diffusions X^ϵ and $Z^{\epsilon,1,N}$, defined by (0.1) and (0.2), are close. Indeed, propagation of chaos holds: there exist $K > 0$ and $M > 0$ such that

$$\sup_{t\in\mathbb{R}_+} \mathbb{E}\left\{\left|\left|X^\epsilon_t - Z^{\epsilon,1,N}_t\right|\right|^2\right\} \leq \frac{K}{N}$$

$$\text{and}\quad \mathbb{E}\left\{\sup_{t\in[0;T]} \left|\left|X^\epsilon_t - Z^{\epsilon,1,N}_t\right|\right|^2\right\} \leq \frac{MT}{N} \quad \text{for all } T > 0\,.$$

See Appendix B for the proof of the first statement.

These two inequalities have strong restrictions. In the first one, the supremum is not under the expectation. Consequently, if τ is a (not necessary bounded) stopping time, nothing tells us that the quantity $\mathbb{E}\left\{\left|\left|X_\tau^\epsilon - Z_\tau^{\epsilon,1,N}\right|\right|^2\right\}$ tends to 0 when N goes to infinity. Note that this cannot be deduced from the second inequality since the supremum is restricted to a fixed and finite interval.

However, by Proposition A.4, we know that the exit time of X^ϵ from a domain $\mathcal{D}$ which satisfies both Assumptions 1.5–1.6 goes to infinity when ϵ tends to 0.

From now on, we consider a compact convex set $\mathcal{K} \subset \mathbb{R}^d$ which contains 0 and X_0. We introduce the following exit times.

Definition 3.1. *By $\tau(\epsilon)$ (resp. by $\tau^{1,N}(\epsilon)$), we denote the first exit time of the diffusion X^ϵ (resp. $Z^{\epsilon,1,N}$) from the compact set $\mathcal{K}$. The first exit time of the whole system $\mathcal{Z}^{\epsilon,N}$ from the ball $\mathbb{B}_\kappa^N$ is denoted by $\tau_\kappa^N(\epsilon)$, where $\kappa > 0$.*

We now introduce

$$\mathcal{T}_\kappa^N(\epsilon) := \inf\left\{\tau(\epsilon)\,;\,\tau^{1,N}(\epsilon)\,;\,\tau_\kappa^N(\epsilon)\right\}\,. \tag{3.1}$$

From now on, we use the function r:

$$r(\kappa) := \left\{\frac{2}{\theta}\sup_{\mu_1,\mu_2\in\mathbb{B}_\kappa^\infty}\sup_{x\in\mathcal{K}}||\nabla F*\mu_1(x)-\nabla F*\mu_2(x)||\right\}^{\frac{1}{3}}\,.$$

By Lemma 1.1 in [17], $\nabla F*\mu$ is the product of x with a polynomial function of degree $2n-2$ of $||x||$ and with a finite number of parameters of the form:

$$C(l_0,l_1,\cdots,l_d) := \int_{\mathbb{R}^d}\prod_{i=1}^d\langle x\,;\,e_i\rangle^{l_i}\,||x||^{l_0}\,\mu(dx)\,,$$

where $l_0+\sum_{i=1}^d l_i \leq 2n$. If μ is in $\mathbb{B}_\kappa^\infty$, $|C(l_0,l_1,\cdots,l_d)| \leq C\kappa^{2n}$ for some constant $C>0$. Consequently, the quantity $r(\kappa)$ goes to 0 when κ tends to 0.

The following result tells us that the propagation of chaos is uniform on $\left[0\,;\,\mathcal{T}_\kappa^N(\epsilon)\right]$.

Theorem 3.2. *There exists κ_0 such that for all $\kappa < \kappa_0$, there exists $N_0(\kappa)\in\mathbb{N}^*$ and $\epsilon_0(\kappa)>0$ such that*

$$\mathbb{P}\left\{\sup_{0\leq t\leq\mathcal{T}_\kappa^N(\epsilon)}\left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right| \geq r(\kappa)\right\} \leq r(\kappa)\,,$$

for all $N \geq N_0(\kappa)$ and for all $\epsilon < \epsilon_0(\kappa)$.

Proof. **Step 1.** By Proposition 1.3, there exist $\epsilon_1 > 0$ and a time T_κ which is deterministic and independent from N and ϵ such that

$$\mathbb{E}\left\{\left|\left|X_{T_\kappa+t}^\epsilon\right|\right|^{2n}\right\} < \kappa^{2n} \tag{3.2}$$

for all $t\geq 0$ and $\epsilon<\epsilon_1$. Furthermore, by Proposition B.3, there exists $\epsilon_2>0$ such that

$$\sup_{0<\epsilon<\epsilon_2}\mathbb{E}\left\{\sup_{0\leq t\leq T_\kappa}\left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq \frac{r(\kappa)^3}{2} \tag{3.3}$$

for N large enough. Note that (3.2) holds in the small-noise case, uniformly with respect to N. Also, (3.3) is true for N large enough, uniformly with respect to ϵ.

Step 2. We denote $\mu_t^{\epsilon,N} := \frac{1}{N}\sum_{i=1}^N \delta_{Z_t^{\epsilon,i,N}}$. Recall that $W_\mu := V + F * \mu$ for all the measures μ and that $\mathbb{B}_\kappa^\infty$ denotes the set of all the measures μ such that $\int ||x||^{2n}\mu(dx) < \kappa^{2n}$.

The assumptions on V and F imply $\mathrm{Hess}\, W_\mu \geq \theta > 0$. From now on, we put $\xi_\epsilon^N(t) := \left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|$. If $X_{T_\kappa}^\epsilon, Z_{T_\kappa}^{\epsilon,1,N} \in \mathcal{K}$ and $\mathcal{Z}_{T_\kappa}^{\epsilon,N} \in \mathbb{B}_\kappa^N$ then, for all $T_\kappa \leq t \leq \mathcal{T}_\kappa^N(\epsilon)$, we have:

$$\begin{aligned}\frac{d}{dt}\left(\xi_\epsilon^N(t)\right)^2 =& -2\left\langle \nabla W_{u_t^\epsilon}\left(X_t^\epsilon\right) - \nabla W_{\mu_t^{\epsilon,N}}\left(Z_t^{\epsilon,1,N}\right) ; X_t^\epsilon - Z_t^{\epsilon,1,N}\right\rangle \\ =& -2\left\langle \nabla W_{u_t^\epsilon}\left(X_t^\epsilon\right) - \nabla W_{u_t^\epsilon}\left(Z_t^{\epsilon,1,N}\right) ; X_t^\epsilon - Z_t^{\epsilon,1,N}\right\rangle \\ & -2\left\langle \nabla F * u_t^\epsilon\left(Z_t^{\epsilon,1,N}\right) - \nabla F * \mu_t^{\epsilon,N}\left(Z_t^{\epsilon,1,N}\right) ; X_t^\epsilon - Z_t^{\epsilon,1,N}\right\rangle \\ \leq & -2\theta\left(\xi_\epsilon^N(t)\right)^2 + 2\xi_\epsilon^N(t) f_{\mathcal{K}}(\kappa)\,, \end{aligned} \tag{3.4}$$

where we have set

$$f_{\mathcal{K}}(\kappa) := \sup_{\mu_1,\mu_2\in\mathbb{B}_\kappa^\infty}\sup_{x\in\mathcal{K}} ||\nabla F * \mu_1(x) - \nabla F * \mu_2(x)|| = \frac{\theta}{2} r(\kappa)^3\,.$$

Inequality (3.4) directly implies:

$$\sup_{T_\kappa\leq t\leq \mathcal{T}_\kappa^N(\epsilon)}\left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|^2 \leq \max\left\{\left|\left|X_{T_\kappa}^\epsilon - Z_{T_\kappa}^{\epsilon,1,N}\right|\right|^2 ; \frac{r(\kappa)^3}{2}\right\}\,,$$

which together with (3.3) yields

$$\mathbb{E}\left\{\sup_{T_\kappa\leq t\leq \mathcal{T}_\kappa^N(\epsilon)}\left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq \frac{r(\kappa)^3}{2}\,. \tag{3.5}$$

From (3.3), (3.5) and the inequality $\max\{a,b\} \leq a+b$ for all $a,b\in\mathbb{R}_+$, we obtain

$$\mathbb{E}\left\{\sup_{0\leq t\leq \mathcal{T}_\kappa^N(\epsilon)}\left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq r(\kappa)^3\,. \tag{3.6}$$

The claim thus follows from the Markov inequality. □

This theorem links the exit time of X^ϵ with the one of $Z^{\epsilon,1,N}$. It also shows that the McKean-Vlasov diffusion is a good approximation (even in the long time) of the first particle in a mean-field system in the small-noise limit. Let us point out that the only use of the convexity was in the inequality $\mathbb{E}\left\{||X_t^\epsilon||^{2n}\right\} \leq \kappa^{2n}$ for all $t \geq T_\kappa$.

4 Exit problem of the McKean-Vlasov diffusion

In this section, we provide our main results: the exit time and the exit location of the McKean-Vlasov diffusion.

Let us consider a domain $\mathcal{D} \subset \mathbb{R}^d$ satisfying Assumptions 1.5–1.6. By $\tau(\epsilon)$, we denote the first exit time of the diffusion (0.1) from the domain $\mathcal{D}$. Let $\mathcal{K}$ be a compact set which contains $\mathcal{D}$ and such that $d(\mathcal{D};\mathcal{K}^c) \geq 1$.

Theorem 4.1. *For all* $\xi > 0$

$$\lim_{\epsilon\to 0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}(H-\xi)\right] < \tau(\epsilon) < \exp\left[\frac{2}{\epsilon}(H+\xi)\right]\right\} = 1$$

where $H := \inf_{z\in\partial\mathcal{D}} W(z)$ *with* $W(z) := V(z) + F(z)$.

Proof. **Step 1.** Let $\kappa > 0$. According to Definition 2.1 and Proposition 2.2, there exist two families of domains $(\mathcal{D}_{i,\kappa})_{\kappa>0}$ and $(\mathcal{D}_{e,\kappa})_{\kappa>0}$ such that

- $\mathcal{D}_{i,\kappa} \subset \mathcal{D} \subset \mathcal{D}_{e,\kappa}$.
- $\mathcal{D}_{i,\kappa}$ and $\mathcal{D}_{e,\kappa}$ are stable by $-\nabla V - \nabla F * \mu$ for all $\mu \in \mathbb{B}_\kappa^\infty$. The terminology "stable by" has been introduced in Definition 1.2.
- $\sup_{z\in\partial\mathcal{D}_{i,\kappa}} \mathrm{d}\left(z\,;\,\mathcal{D}^c\right) + \sup_{z\in\partial\mathcal{D}_{e,\kappa}} \mathrm{d}\left(z\,;\,\mathcal{D}\right)$ tends to 0 when κ goes to 0.

Let us recall that $\tau_{i,\kappa}^{1,N}(\epsilon)$ (resp. $\tau_{e,\kappa}^{1,N}(\epsilon)$) is the first exit time of $Z^{\epsilon,1,N}$ from $\mathcal{D}_{i,\kappa}$ (resp. $\mathcal{D}_{e,\kappa}$). Set $\tau_\kappa^N(\epsilon)$ to be the exit time of the diffusion $\mathcal{Z}^{\epsilon,N}$ from the domain $\mathbb{B}_\kappa^N$. Finally, we denote $\mathcal{T}_\kappa^N(\epsilon) := \min\left\{\tau(\epsilon)\,;\,\tau_{e,\kappa}^{1,N}(\epsilon)\,;\,\tau_\kappa^N(\epsilon)\right\}$.

Step 2. We prove here the upper bound:

$$\begin{aligned}
&\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} = \\
&\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} \\
&+\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) < \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_\kappa^N(\epsilon) \leq \tau_{e,\kappa}^{1,N}(\epsilon)\right\} \\
&+\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) < \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_\kappa^N(\epsilon) > \tau_{e,\kappa}^{1,N}(\epsilon)\right\} \\
\leq&\mathbb{P}\left\{\tau_{e,\kappa}^N(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} + \mathbb{P}\left\{\tau_\kappa^N(\epsilon) \leq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} \\
&+\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) < \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_\kappa^N(\epsilon) > \tau_{e,\kappa}^{1,N}(\epsilon)\right\} \\
&=: a_N(\epsilon) + b_N(\epsilon) + c_N(\epsilon)\,.
\end{aligned}$$

Step 2.1. By Proposition 2.10, there exists $\kappa_1 > 0$ such that for all $0 < \kappa < \kappa_1$ and N large enough

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau_{e,\kappa}^{1,N}(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H+\xi\right)\right]\right\} = 0$$

Therefore, the first term $a_N(\epsilon)$ tends to 0 as ϵ goes to 0.

Step 2.2. Let us look at the third term $c_N(\epsilon)$. For κ sufficiently small, we have $\mathcal{D}_{e,\kappa} \subset \mathcal{K}$. On the event

$$\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) \leq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_\kappa^N(\epsilon) > \tau_{e,\kappa}^{1,N}(\epsilon)\right\}\,,$$

we have $\tau_{e,\kappa}^{1,N}(\epsilon) \leq \tau(\epsilon)$ and $\tau_{e,\kappa}^{1,N}(\epsilon) \leq \tau_\kappa^N(\epsilon)$. This implies $\tau_{e,\kappa}^{1,N}(\epsilon) \leq \mathcal{T}_\kappa^N(\epsilon)$. We deduce that

$$\begin{aligned}
&\mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) \leq \exp\left[\frac{H+\xi}{\epsilon}\right]\,;\,\tau_\kappa^N(\epsilon) > \tau_{e,\kappa}^{1,N}(\epsilon)\right\} \\
&\leq \mathbb{P}\left\{\left|\left|X_{\tau_{e,\kappa}^{1,N}(\epsilon)}^\epsilon - Z_{\tau_{e,\kappa}^{1,N}(\epsilon)}^{\epsilon,1,N}\right|\right| \geq \delta(\kappa)\,;\,\tau_{e,\kappa}^{1,N}(\epsilon) \leq \mathcal{T}_\kappa^N(\epsilon)\right\} \\
&\leq \mathbb{P}\left\{\sup_{0\leq t\leq\mathcal{T}_\kappa^N(\epsilon)} \left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right| \geq \delta(\kappa)\right\}\,,
\end{aligned}$$

where $\delta(\kappa)$ denotes the distance between $\mathcal{D}$ and $\mathcal{D}_{e,\kappa}^c$. By construction, we have $\delta(\kappa) \geq r(\kappa)$. According to Theorem 3.2, there exist $N_0 \geq 2$ and $\epsilon_0 > 0$ such that

$$\mathbb{P}\left\{\sup_{0\leq t\leq\mathcal{T}_\kappa^N(\epsilon)} \left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right| \geq r(\kappa)\right\} \leq r(\kappa)\,,$$

for all $N \geq N_0$ and $\epsilon < \epsilon_0$.

Step 2.3. Let us look at the second term $b_N(\epsilon)$. By Lemma 2.3,

$$\lim_{N\to+\infty} \inf_{Z\in\partial\mathbb{B}_\kappa^N} N\Upsilon^N(Z) = +\infty\,.$$

Consequently, for N large enough, we obtain

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau_\kappa^N(\epsilon) \leq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} = 0\,.$$

Step 2.4. Let $\xi > 0$. By taking first κ small enough and then N large enough, we obtain the upper bound

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau(\epsilon) \geq \exp\left[\frac{H+\xi}{\epsilon}\right]\right\} = 0\,.$$

Step 3. Analogous arguments with Proposition 2.11 instead of Proposition 2.10 show that

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau(\epsilon) \leq \exp\left[\frac{H-\xi}{\epsilon}\right]\right\} = 0\,.$$

□

As an immediate application of Theorem 3.2 and Theorem 4.1, we obtain a good approximation of the self-stabilizing process on unbounded family of intervals:

Corollary 4.2. *Let H and ρ be two positive real numbers. For all $\delta > 0$, there exist $N_\delta \in \mathbb{N}^*$ and $\epsilon_\delta > 0$ such that*

$$\sup_{N\geq N_\delta} \sup_{\epsilon<\epsilon_\delta} \mathbb{P}\left\{\sup_{0\leq t\leq \exp\left[\frac{H}{\epsilon}\right]} \left|\left|X_t^\epsilon - Z_t^{\epsilon,1,N}\right|\right| \geq \rho\right\} \leq \delta\,.$$

Proof. We introduce the set $\mathcal{K}_{\frac{H}{2}+1} := \left\{x\in\mathbb{R}^d \;\middle|\; V(x)+F(x) < \frac{H}{2}+1\right\}$. It satisfies Assumptions 1.5–1.6 by Lemma 1.8. For $\kappa > 0$ sufffficiently small, Inequality (3.6) gives the existence of $N_0 \in \mathbb{N}^*$ and $\epsilon_0 > 0$ such that for all $N \geq N_0$ and $\epsilon < \epsilon_0$,

$$\mathbb{E}\left\{\sup_{0\leq t\leq \mathcal{T}_\kappa^N(\epsilon)} \left|\left|X_t^\epsilon - X_t^{\epsilon,1,N}\right|\right|^2\right\} \leq \frac{\delta}{2}\,,$$

where $\mathcal{T}_\kappa^N(\epsilon) := \inf\left\{\tau(\epsilon)\,;\,\tau^{1,N}(\epsilon)\,;\,\tau_\kappa^N(\epsilon)\right\}$. Here $\tau(\epsilon)$ (resp. $\tau^{1,N}(\epsilon)$) is the first exit time of the diffusion X^ϵ (resp. $Z^{\epsilon,1,N}$) from $\mathcal{K}_{\frac{H}{2}+1}$ and $\tau_\kappa^N(\epsilon)$ is the first exit time of the whole system $\mathcal{Z}^{\epsilon,N}$ from the ball $\mathbb{B}_\kappa^N$. For N large enough, Lemma 2.3 implies

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau_\kappa^N(\epsilon) < \exp\left[\frac{H}{\epsilon}\right]\right\} = 0\,. \tag{4.1}$$

The domain $\mathcal{K}_{\frac{H}{2}+1}$ satisfies Assumptions 1.5–1.6 so we can apply Theorem 4.1 and Corollary 2.12 to deduce that for all $\xi > 0$,

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(\frac{H}{2}+1-\xi\right)\right]\right\} = 0 \tag{4.2}$$

$$\text{and}\quad \lim_{\epsilon\to 0} \mathbb{P}\left\{\tau^{1,N}(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(\frac{H}{2}+1-\xi\right)\right]\right\} = 0\,. \tag{4.3}$$

In particular, for $\xi = 1$, (4.1), (4.2) and (4.3) imply

$$\mathbb{P}\left\{\mathcal{T}_\kappa^N(\epsilon) < \exp\left[\frac{H}{\epsilon}\right]\right\} < \frac{\delta}{2}$$

for ϵ small enough. Finally,

$$\begin{aligned}&\mathbb{P}\left\{\sup_{0\le t\le\exp\left[\frac{H}{\epsilon}\right]}\left|\left|X_t^\epsilon-Z_t^{\epsilon,1,N}\right|\right|\ge\rho\right\}\\&\le\mathbb{P}\left\{\sup_{0\le t\le\mathcal{T}_\kappa^N(\epsilon)}\left|\left|X_t^\epsilon-Z_t^{\epsilon,1,N}\right|\right|\ge\rho\right\}+\mathbb{P}\left\{\mathcal{T}_\kappa^N(\epsilon)<\exp\left[\frac{H}{\epsilon}\right]\right\}\le\delta\,.\end{aligned}$$

This ends the proof. □

We now provide the result on the exit location.

Theorem 4.3. *Let $\mathcal{N}$ be a subset of $\partial\mathcal{D}$ satisfying*

$$\inf_{z\in\mathcal{N}}W(z)>\inf_{z\in\partial\mathcal{D}}W(z)\,.$$

Then

$$\lim_{\epsilon\to0}\mathbb{P}\left\{X_{\tau(\epsilon)}^\epsilon\in\mathcal{N}\right\}=0\,.$$

Proof. The proof is similar to the Step 3 of the proof of Corollary 2.12.

By definition of $\mathcal{N}$, there exists $\xi>0$ such that $\inf_{z\in\mathcal{N}}W(z)=H+3\xi$. We introduce

$$\mathcal{K}_{H+2\xi}:=\left\{x\in\mathbb{R}^d\ \mid\ W(x)<H+2\xi\right\}\,.$$

By Lemma 1.8, the domain $\mathcal{K}_{H+2\xi}$ satisfies Assumptions 1.5–1.6. Then, we can apply (2.8) to $\mathcal{K}_{H+2\xi}$. If we denote by $\tau_\xi(\epsilon)$ the first exit time of X^ϵ from $\mathcal{K}_{H+2\xi}$, then we obtain

$$\lim_{\epsilon\to0}\mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H+2\xi-\rho\right)\right]<\tau_\xi(\epsilon)<\exp\left[\frac{2}{\epsilon}\left(H+2\xi+\rho\right)\right]\right\}=1\tag{4.4}$$

for all $\rho>0$. By construction of $\mathcal{K}_{H+2\xi}$, $\mathcal{N}\subset\mathcal{K}_{H+2\xi}^c$, which implies

$$\begin{aligned}\mathbb{P}\left\{X_{\tau(\epsilon)}^\epsilon\in\mathcal{N}\right\}&\le\mathbb{P}\left\{X_{\tau(\epsilon)}^\epsilon\notin\mathcal{K}_{H+2\xi}\right\}\\&\le\mathbb{P}\left\{\tau_\xi(\epsilon)\le\tau(\epsilon)\right\}\\&\le\mathbb{P}\left\{\tau_\xi(\epsilon)\le\exp\left[\frac{H+\xi}{\epsilon}\right]\right\}+\mathbb{P}\left\{\exp\left[\frac{H+\xi}{\epsilon}\right]\le\tau(\epsilon)\right\}\,.\end{aligned}$$

Applying (4.4) with $\rho:=\xi$ to the first term and Theorem 4.1 to the second one, we obtain the result. □

Remark 4.4. *Note that we have not used convexity of V in the whole space $\mathbb{R}^d$. We have used the convexity in a compact set which contains the point of the global minimum 0 and the captivity of the law u_t^ϵ in a small ball which contains δ_0. This means that it is possible to characterize the exit time and the exit location even if V is not convex by using the new approach of this paper.*

A Freidlin-Wentzell Theory

Here we present the main results of the Freidlin-Wentzell theory. We restrict ourselves to a simple case in $\mathbb{R}^k$, $k\ge1$. We consider a homogeneous diffusion x^ϵ:

$$x_t^\epsilon=x_0+\sqrt{\epsilon}B_t-\int_0^t\nabla U\left(x_s^\epsilon\right)ds\,,\tag{A.1}$$

where $x_0 \in \mathbb{R}^k$, B is a Brownian motion and potential $U \in \mathcal{C}^\infty\left(\mathbb{R}^k\right)$. For a more general setting and the proofs, the reader is referred to [4].

Let a_0 be a minimizer of the potential U. Let $\mathcal{G}$ be an open domain which contains x_0 and a_0. $\tau(\epsilon)$ denotes the first exit time of the diffusion x_t^ϵ from the domain $\mathcal{G}$. Let us introduce the exit cost H:

$$H := \inf_{z\in\partial\mathcal{G}} U(z) - U(a_0)\,.$$

Define the deterministic dynamical system

$$\varphi_t\left(x\right) = x - \int_0^t \nabla U\left(\varphi_s\left(x\right)\right) ds\,.$$

We need two assumptions.

Assumption A.1. *The unique critical point of U in the domain $\mathcal{G}$ is a_0. Moreover, for all $x \in \mathcal{G}$, $\varphi_t\left(x\right) \in \mathcal{G}$ for all $t > 0$ and $\lim_{t\to\infty} \varphi_t\left(x\right) = a_0$.*

Note that this asssumption is about the domain $\mathcal{G}$ and it is always true if $\mathcal{G}$ is the basin of attraction of a_0.

Assumption A.2. *All the trajectories of the deterministic system $\varphi_t\left(x\right)$ with $x \in \partial\mathcal{G}$ converge to a_0 as $t \to \infty$.*

If U is convex on $\mathcal{G}$ then Asssumption A.2 is satisfied.

Assume that Assumptions A.1 and A.2 hold.

Proposition A.3. *For all $\delta > 0$, the following limit holds:*

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\exp\left[\frac{2}{\epsilon}\left(H-\delta\right)\right] < \tau(\epsilon) < \exp\left[\frac{2}{\epsilon}\left(H+\delta\right)\right]\right\} = 1\,. \tag{A.2}$$

Moreover, for each subset $\mathcal{N} \subset \partial\mathcal{G}$ satisfying $\inf_{z\in\mathcal{N}} U(z) > \inf_{z\in\partial\mathcal{G}} U(z)$, we have:

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{x^\epsilon_{\tau(\epsilon)} \in \mathcal{N}\right\} = 0\,. \tag{A.3}$$

At the end, we recall a classical result of the theory of large deviations.

Proposition A.4. *If*

$$\mathcal{F}_\delta := \left\{z \in \mathbb{R}^d \;\mid\; \inf_{t\geq 0} \mathrm{d}\left(z\,;\,\varphi_t(x_0)\right) \leq \delta\right\}$$

for $\delta > 0$ and $\tau_\delta(\epsilon)$ denotes the first exit time of the diffusion x^ϵ from the domain $\mathcal{F}_\delta$, then

$$\lim_{\epsilon\to 0} \mathbb{P}\left\{\tau_\delta(\epsilon) < T\right\} = 0\,,$$

for all $\delta > 0$ and $T > 0$.

By using Proposition A.4, we can improve the results of Proposition A.3 with domains which do not satisfy Hypotheses A.1 and A.2.

Proposition A.5. *Let us consider a domain $\mathcal{G}$ which satisfies Assumptions A.1 and A.2 and let x_0 be a point in $\mathbb{R}^k$ such that $x_0 \notin \mathcal{G}$. Also assume that $\varphi_t\left(x_0\right)$ converges to a_0 as t goes to infinity. Let T_0 be the hitting time of $\mathcal{G}$ for the dynamical system $\varphi\left(x_0\right)$. If we denote by*

$$S(\epsilon) := \inf\left\{t \geq 0 \;\mid\; x_t^\epsilon \in \mathcal{G}\right\}$$

the first hitting time in $\mathcal{G}$ of the diffusion x^ϵ and by

$$\tau(\epsilon) := \inf\left\{t \geq S(\epsilon) \;\mid\; x_t^\epsilon \notin \mathcal{G}\right\}$$

the first exit time, then (A.2) *holds for $\tau(\epsilon)$.*

The proof is left to the reader.

B Propagation of chaos

The aim of this appendix is to present the classical results of the propagation of chaos and the proofs. We recall the mean-field system (0.2):

$$Z_t^{\epsilon,i,N} = X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla V\left(Z_s^{\epsilon,i,N}\right)ds - \frac{1}{N}\sum_{j=1}^N\int_0^t \nabla F\left(Z_s^{\epsilon,i,N} - Z_s^{\epsilon,j,N}\right)ds\,. \tag{B.1}$$

Also, we consider a system of N independent self-stabilizing diffusions:

$$X_t^{\epsilon,i} = X_0 + \sqrt{\epsilon}B_t^i - \int_0^t \nabla V\left(X_s^{\epsilon,i}\right)ds - \int_0^t \nabla F * u_s^\epsilon\left(X_s^{\epsilon,i}\right)ds\,. \tag{B.2}$$

The two diffusions $X^{\epsilon,i}$ and $Z^{\epsilon,i,N}$ are close when N is large enough.

Proposition B.1. *There exists $K>0$ and $\epsilon_0>0$ such that, for all $N\geq 1$:*

$$\sup_{0<\epsilon<\epsilon_0}\sup_{t\in\mathbb{R}_+}\mathbb{E}\left\{\left|\left|X_t^{\epsilon,1} - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq \frac{K}{N}\,.$$

Proof. We apply the Itô formula to $X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}$ and the function $x\mapsto x^2$. By denoting $\xi_i^\epsilon(t) := \left|\left|X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}\right|\right|^2$, we obtain

$$d\sum_{i=1}^N \xi_i^\epsilon(t) = -2\sum_{i=1}^N \Delta_1^\epsilon(i,t)dt - \frac{2}{N}\sum_{i=1}^N\sum_{j=1}^N\left(\Delta_2^\epsilon(i,j,t) + \Delta_3^\epsilon(i,j,t)\right)dt$$

$$\text{with}\quad \Delta_1^\epsilon(i,t) := \left\langle X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}\,;\,\nabla V(X_t^{\epsilon,i}) - \nabla V\left(Z_t^{\epsilon,i,N}\right)\right\rangle\,,$$

$$\Delta_2^\epsilon(i,j,t) := \left\langle X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}\,;\,\nabla F(X_t^{\epsilon,i} - X_t^{\epsilon,j}) - \nabla F\left(Z_t^{\epsilon,i,N} - Z_t^{\epsilon,j,N}\right)\right\rangle \text{ and}$$

$$\Delta_3^\epsilon(i,j,t) := \left\langle X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}\,;\,\nabla F\left(Z_t^{\epsilon,i,N} - Z_t^{\epsilon,j,N}\right) - \nabla F * u_t^\epsilon\left(Z_t^{\epsilon,i,N}\right)\right\rangle\,.$$

The convexity of F implies $\Delta_2^\epsilon(i,j,t) + \Delta_2^\epsilon(j,i,t)\geq 0$. Indeed, by writing $\eta_t^{\epsilon,i,j} := X_t^{\epsilon,i} - X_t^{\epsilon,j}$ and $\zeta_t^{\epsilon,i,j,N} := Z_t^{\epsilon,i,N} - Z_t^{\epsilon,j,N}$, we have:

$$\begin{aligned}&\Delta_2^\epsilon(i,j,t) + \Delta_2^\epsilon(j,i,t)\\ =&\left\langle\nabla F(\eta_t^{\epsilon,i,j}) - \nabla F(\zeta_t^{\epsilon,i,j,N})\,;\,\left(X_t^{\epsilon,i} - Z_t^{\epsilon,i,N}\right) - \left(X_t^{\epsilon,j} - Z_t^{\epsilon,j,N}\right)\right\rangle\\ =&\left\langle\nabla F(\eta_t^{\epsilon,i,j}) - \nabla F(\zeta_t^{\epsilon,i,j,N})\,;\,\eta_t^{\epsilon,i,j} - \zeta_t^{\epsilon,i,j,N}\right\rangle \geq \alpha\left|\left|\eta_t^{\epsilon,i,j} - \zeta_t^{\epsilon,i,j,N}\right|\right|^2 \geq 0\,,\end{aligned}$$

where $\alpha\geq 0$ depends on F. Consequently

$$\mathbb{E}\left\{\sum_{i=1}^N\sum_{j=1}^N \Delta_2^\epsilon(i,j,t)\right\} = \mathbb{E}\left\{\sum_{1\leq i<j\leq N}\left(\Delta_2^\epsilon(i,j,t) + \Delta_2^\epsilon(j,i,t)\right)\right\} \geq 0\,. \tag{B.3}$$

Inequality (1.1) implies

$$-2\sum_{i=1}^N \Delta_1^\epsilon(i,t) \leq -2\theta\sum_{i=1}^N \xi_i^\epsilon(t)\,. \tag{B.4}$$

To deal with the last term, we apply the Cauchy-Schwarz inequality:

$$\mathbb{E}\left\{\left|\sum_{j=1}^N \Delta_3^\epsilon(i,j,t)\right|\right\} \leq \sqrt{\xi_i^\epsilon(t)} \times \left\{\sum_{j=1}^N\sum_{k=1}^N \mathbb{E}\left\{\left\langle\rho_j^\epsilon(i)\,;\,\rho_k^\epsilon(i)\right\rangle\right\}\right\}^{\frac{1}{2}}$$
$$\text{with}\quad \rho_j^\epsilon(i) := \nabla F\left(X_t^{\epsilon,i} - X_t^{\epsilon,j}\right) - \nabla F * u_t^\epsilon\left(Z_t^{\epsilon,i,N}\right).$$

Taking the conditional expectation with respect to $Z_t^{\epsilon,i,N}$ and then to $Z_t^{\epsilon,j,N}$, we obtain $\mathbb{E}\left\{\rho_j^\epsilon(i)\rho_k^\epsilon(i)\right\} = 0$ for $j \neq k$. Therefore

$$\mathbb{E}\left\{\left|\sum_{j=1}^N \Delta_3^\epsilon(i,j,t)\right|\right\} \leq \sqrt{N\mathbb{E}\left[\xi_i^\epsilon(t)\right]\mathbb{E}\left[\left|\left|\nabla F(X_t^\epsilon - Y_t^\epsilon) - \nabla F * u_t^\epsilon(X_t^\epsilon)\right|\right|^2\right]}$$

where X_t^ϵ and Y_t^ϵ are two independent random variables with law u_t^ϵ. We know by Lemma 1.1 in [17] that ∇F is the product of x with a polynomial function of degree $2n-2$ of $||x||$. By Proposition 1.3 and Remark 1.4, there exist ϵ_0 and $C > 0$ such that

$$\mathbb{E}\left\{\left|\sum_{j=1}^N \Delta_3^\epsilon(i,j,t)\right|\right\} \quad \leq C\sqrt{N\mathbb{E}\left\{\xi_i^\epsilon(t)\right\}}\,, \tag{B.5}$$

for every $\epsilon < \epsilon_0$. By combining (B.3), (B.4) and (B.5), we obtain

$$\frac{d}{dt}\sum_{i=1}^N \mathbb{E}\left\{\xi_i^\epsilon(t)\right\} \leq 2\sum_{i=1}^N\left\{-\theta\mathbb{E}\left\{\xi_i^\epsilon(t)\right\} + \frac{C}{\sqrt{N}}\sqrt{\mathbb{E}\left\{\xi_i^\epsilon(t)\right\}}\right\}.$$

The particles are exchangeable and so

$$\frac{d}{dt}\mathbb{E}\left\{\xi_1^\epsilon(t)\right\} \leq -2\theta\mathbb{E}\left\{\xi_1^\epsilon(t)\right\} + \frac{2C}{\sqrt{N}}\sqrt{\mathbb{E}\left\{\xi_1^\epsilon(t)\right\}}\,.$$

Since $\xi_i^\epsilon(0) = 0$,

$$\mathbb{E}\left\{\xi_1^\epsilon(t)\right\} \leq \frac{C^2}{\theta^2 N}\,.$$

This inequality holds uniformly with respect to $0 < \epsilon < \epsilon_0$. □

Let us note that this uniform propagation of chaos would not hold if V was not convex. But it is true even if V is not uniformly strictly convex which means that the Hessian of V is not necessary definite positive.

Remark B.2. *Instead of (V-2), let us assume that there exist $\zeta \geq 2$ and $\lambda > 0$ such that*

$$\langle\nabla V(x) - \nabla V(y)\,;\,x-y\rangle \geq \lambda\,||x-y||^\zeta\,.$$

Then, there exists $K > 0$ such that

$$\sup_{0<\epsilon<1}\sup_{t\geq 0}\mathbb{E}\left\{\left|\left|X_t^{\epsilon,1} - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq KN^{-\frac{1}{\zeta-1}}\,.$$

We can also remark that the supremum is not under the expectation. However, such a result is available on a finite interval (even if V is not convex):

Proposition B.3. *There exists $\epsilon_0 > 0$ and $M > 0$ such that for all $T > 0$ and for all $N \geq 1$,*

$$\sup_{0<\epsilon<\epsilon_0} \mathbb{E}\left\{\sup_{0\leq t\leq T}\left|\left|X_t^{\epsilon,1} - Z_t^{\epsilon,1,N}\right|\right|^2\right\} \leq \frac{MT}{N}\,.$$

By putting the supremum under the expectation, we lose the uniformity with respect to the time. However, the position of the two particles $X_t^{\epsilon,i}$ and $Z_t^{\epsilon,i,N}$ was not used in the previous proofs. By doing it, we obtained a stronger result in this paper, see Section 3.

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