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Two-sided random walks conditioned to have no intersections*

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Abstract

Let S^1, S^2 be independent simple random walks in $\mathbb{Z}^d$ ($d = 2, 3$) started at the origin. We construct two-sided random walk paths conditioned that $S^1[0,\infty) \cap S^2[1,\infty) = \emptyset$ by showing the existence of the following limit:

$$\lim_{n\to\infty} P(\cdot \mid S^1[0,\tau^1(n)] \cap S^2[1,\tau^2(n)] = \emptyset),$$

where $\tau^i(n) = \inf\{k \geq 0 : |S^i(k)| \geq n\}$. Moreover, we give upper bounds of the rate of the convergence. These are discrete analogues of results for Brownian motion obtained in [3] and [8].

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1 Introduction and Main Results

1.1 Introduction

Let $S = (S(n))$ be a simple random walk in $\mathbb{Z}^d$ ($d = 2, 3$) started at the origin. Take integers $k < n$. A time k is called cut time up to n if

$$S[0,k] \cap S[k+1,n] = \emptyset, \tag{1.1}$$

where $S[0,k] = \{S(j) : 0 \leq j \leq k\}$. We call $S(k)$ a cut point if k is a cut time. Lawler [4] has shown that there are constants $0 < c, c' < \infty$ such that for all n,

$$cn^{-\frac{\xi}{2}} \leq P(S[0,n] \cap S[n+1,2n] = \emptyset) \leq c'n^{-\frac{\xi}{2}}, \tag{1.2}$$

where $\xi = \xi_d$ is the intersection exponent (see Section 2.1 below). Lawler, Schramm and Werner [6] have proved that $\xi_2 = \frac{5}{4}$ by using the SLE techniques. The value of ξ_3 is

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not still known. Let J_k be the indicator function of the event that k is a cut time up to n and let $R_n = \sum_{k=0}^{n} J_k$. Lawler [4] also proved that there exists $c > 0$ such that

$$P(R_n \geq cn^{1-\frac{\xi}{2}}) \geq c \text{ for } d = 2,$$
$$R_n \approx n^{1-\frac{\xi}{2}} \text{ with probability one for } d = 3,$$

where $\approx$ denotes that the logarithms of both sides are asymptotic.

While the understanding of the number of cut times has been advanced, there is a few results about the geometrical structure of the path around cut points, which is the purpose of this paper. We consider the following problem. If we condition that $S[0,n] \cap S[n+1,2n] = \emptyset$, then what kind of structure does the path have around $S(n)$? Let S^1, S^2 be independent simple random walks started at the origin. Then, thanks to the translation invariance and the reversibility of the simple random walk, our problem may be deduced to clarify the structure of S^1, S^2 around the origin when we condition that $S^1[0,n] \cap S^2[1,n] = \emptyset$. Letting $n \to \infty$, we will face the following problems:

(1) Construct two-sided path conditioned that $S^1[0,\infty) \cap S^2[1,\infty) = \emptyset$.

(2) What kind of geometrical structure does such a conditioned path have?

(3) For each l, is the difference between the distribution of $S^1[0,l] \cup S^2[0,l]$ conditioned on $\{S^1[0,n] \cap S^2[1,n] = \emptyset\}$ and that conditioned on $\{S^1[0,\infty) \cap S^2[1,\infty) = \emptyset\}$ small for large n?

By (1.2), the probability that $S^1[0,\infty) \cap S^2[1,\infty) = \emptyset$ is 0 for $d = 2, 3$, so question (i) is not trivial. For Brownian motions, Lawler [3], and Lawler, Vermesi [8] have constructed Brownian paths conditioned to have no intersections. More precisely, let B^1, B^2 be Brownian motions in $\mathbb{R}^d$ ($d = 2, 3$) starting distance one apart and

$$T^i(R) = \inf\{t \geq 0 : |B^i(t)| = R\}.$$

In [3], it was proved that for $d = 2$, the limit

$$\lim_{n\to\infty} P(\cdot \mid B^1[0, T^1(e^n)] \cap B^2[0, T^2(e^n)] = \emptyset) \tag{1.3}$$

exists and the rate of convergence is bounded above by $O(e^{-\delta\sqrt{n}})$ for some $\delta > 0$. For $d = 3$, it was shown in [8] that the limit of (1.3) also exists and the rate of convergence is at most $O(e^{-\delta n})$ (see Proposition 2.9).

In this paper we will answer the question (i) and (iii). We will construct the path in (1) by proving the existence of the limit as in (1.3) for simple random walk (Theorem 1.1). Furthermore, we will derive same rates of convergence as Brownian cases. Since the speed of convergence in Theorem 1.1 is relatively fast, it would give evidence that the gap considered in (3) is small.

Even though the conditioned Brownian paths were already constructed as in (1.3), it is not straightforward to construct it for the simple random walk. Both in [3] and [8], the scaling property of Brownian motion is crucial in the construction and hence the same arguments cannot be applied for the simple random walk case. To overcome this problem, we will use the strong approximation of Brownian motion by simple random walk derived from the Skorohod embedding. By this approximation, we can define simple random walks S^1, S^2 and Brownian motions B^1, B^2 on the same probability space so that with high probability, the paths of S^i are very close to those of B^i. However, if S^1 and S^2 start from a same point, then the difference between the path of S^i and

that of B^i is too large to control the difference between $P(B^1[0,n] \cap B^2[1,n] = \emptyset)$ and $P(S^1[0,n] \cap S^2[1,n] = \emptyset)$. (See Proposition 2.4 for the difference between $S^i[0,n]$ and $B^i[0,n]$. We must admit the fact that the difference may be of order $n^{\frac{1}{4}}$.) This difficulty can be dealt with using the following ideas. Even if starting points of S^1 and S^2 are very close, they gradually have a good chance of being reasonably far apart because of the conditioning not to intersect. Once S^1 and S^2 are far apart, we can use the Skorohod embedding to control the non-intersection probability of simple random walks (see Proposition 3.19 for details).

The question (iii) will be discussed in a forthcoming paper [9]. Let $\overline{S}^1, \overline{S}^2$ be the associated two-sided random walks whose probability law is $P^\sharp$ in Theorem 1.1. In order to show that paths of $\overline{S}^i$ have different structures from those of usual simple random walk S^i, we will consider a simple random walk on $\overline{\mathcal{G}} := \overline{S}^1[0,\infty) \cup \overline{S}^2[0,\infty)$. (Here we regard $\overline{\mathcal{G}}$ as the subgraph consisting of all the vertices visited and edges traversed by either $\overline{S}^1$ or $\overline{S}^2$.) In [9], it will be shown that the simple random walk on $\overline{\mathcal{G}}$, say X, has subdiffusive behavior for $d = 2$. This is due to that $\overline{\mathcal{G}}$ has many so called bottleneck edges and it takes much longer for X to move away from its starting point compared to the simple random walk in $\mathbb{Z}^2$.

Throughout this paper, we use $c, c', c_1, c_2, \delta, \cdots$ to denote arbitrary constants that depend only on the dimension d. The values of them may change from place to place.

1.2 Framework and Main results

Let $d = 2, 3$. For $x \in \mathbb{Z}^d$, let

$$\mathcal{B}(x,n) = \{z \in \mathbb{Z}^d : |z - x| < n\}$$

and

$$\partial\mathcal{B}(x,n) = \{z \in \mathbb{Z}^d \backslash \mathcal{B}(x,n) : |z - y| = 1 \text{ for some } y \in \mathcal{B}(x,n)\}.$$

We write $\mathcal{B}(n) = \mathcal{B}(0,n)$ and $\partial\mathcal{B}(n) = \partial\mathcal{B}(0,n)$. Let $\mathcal{B}_k(x) = \mathcal{B}(x,2^k)$ and $\partial\mathcal{B}_k(x) = \partial\mathcal{B}(x,2^k)$. We also write $\mathcal{B}_k = \mathcal{B}_k(0)$ and $\partial\mathcal{B}_k = \partial\mathcal{B}_k(0)$.

A sequence of points $\gamma = [\gamma(0), \gamma(1), \cdots, \gamma(l)] \subset \mathbb{Z}^d$ is called path if $|\gamma(j) - \gamma(j-1)| = 1$ for each $j = 1, 2, \cdots, l$. We let $\text{len}\gamma = l$ be the length of the path, $\Lambda(n)$ be the set of paths satisfying that

$$\gamma(0) = 0, \gamma(j) \in \mathcal{B}(n) \text{ for all } j = 0, 1, \cdots, \text{len}\gamma - 1$$
$$\gamma(\text{len}\gamma) \in \partial\mathcal{B}(n).$$

Let

$$\Gamma(n) = \{\overline{\gamma} = (\gamma^1, \gamma^2) \in \Lambda(n)^2 : \gamma^1(i) \neq \gamma^2(j) \text{ for all } (i,j) \neq (0,0)\},$$

and $\Gamma(\infty) = \bigcap_{n=1}^{\infty} \Gamma(n)$. We write $\Gamma_k = \Gamma(2^k)$.

Let S^1, S^2 be the independent simple random walks in $\mathbb{Z}^d$ started at the origin. Let

$$\tau^i(n) = \inf\{k \geq 0 : S^i(k) \in \partial\mathcal{B}(n)\},$$

and $\tau^i_k = \tau^i(2^k)$.

Theorem 1.1. *Let $d = 2$ or 3. For each $L \in \mathbb{N}$ and $\overline{\gamma} \in \Gamma(L)$, the limit*

$$\lim_{N\to\infty} P\Big((S^1[0,\tau^1(L)], S^2[0,\tau^2(L)]) = \overline{\gamma} \;\Big|\; (S^1[0,\tau^1(N)], S^2[0,\tau^2(N)]) \in \Gamma(N)\Big) =: P^\sharp(\overline{\gamma}) \tag{1.4}$$

exists. Furthermore, there exist $\delta > 0$ and $c < \infty$ depending only on the dimension such that the following holds for all L and $\overline{\gamma} \in \Gamma(L)$.

$$\Big| P\Big((S^1[0,\tau^1(L)], S^2[0,\tau^2(L)]) = \overline{\gamma} \;\Big|\; (S^1[0,\tau^1(N)], S^2[0,\tau^2(N)]) \in \Gamma(N) \Big) - P^\sharp(\overline{\gamma}) \Big| \le c e^{-\delta\sqrt{\log N}} \tag{1.5}$$

for $d = 2$,

$$\Big| P\Big((S^1[0,\tau^1(L)], S^2[0,\tau^2(L)]) = \overline{\gamma} \;\Big|\; (S^1[0,\tau^1(N)], S^2[0,\tau^2(N)]) \in \Gamma(N) \Big) - P^\sharp(\overline{\gamma}) \Big| \le c N^{-\delta} \tag{1.6}$$

for $d = 3$,

and $P^\sharp$ extends uniquely to a probability measure on $\Gamma(\infty)$.

The paper is organized as follows. Section 2 gives some preliminary propositions about Brownian motions and simple random walks. In particular, we state the Skorohod embedding which is crucial in this paper. Key estimates are given in Section 3 by using this approximation. We give the proof of Theorem 1.1 in Section 4.

2 Known Results

In this section, we give a list of definition of the objects and known results commonly used throughout this paper.

2.1 Intersection Exponent

In this subsection, we review the intersection exponent for Brownian motion and simple random walk. Let $d = 2$ or 3. Let B^1, B^2 be independent Brownian motions in $\mathbb{R}^d$. We start by stating the estimate from [5]. Let

$$T^i(n) = \inf\{t \ge 0 : |B^i(t)| = n\},$$

and write $P^{x,y} = P^{x,y}_{1,2}$ to denote probabilities assuming $B^1(0) = x, B^2(0) = y$. Then we have the following proposition.

Proposition 2.1. *([5], Corollary 3.13.) There exist $\xi = \xi_d$, $c < \infty$ and an increasing function $f : (0,2] \to (0,\infty)$ such that if $|x| = |y| = 1$, then for all $n \ge 1$*

$$f(|x-y|)n^{-\xi} \le P^{x,y}(B^1[0,T^1(n)] \cap B^2[0,T^2(n)] = \emptyset) \le c n^{-\xi}. \tag{2.1}$$

Note that ξ is called the intersection exponent for Brownian motion. Next we state the analogues for simple random walks. Let S^1, S^2 be independent simple random walks in $\mathbb{Z}^d$. Again we write $P^{x,y} = P^{x,y}_{1,2}$ to denote probabilities assuming $S^1(0) = x, S^2(0) = y$. Let

$$\tau^i(n) = \inf\{k \ge 0 : |S^i(k)| \ge n\}.$$

Then the following proposition was proved in [4].

Proposition 2.2. *([4], Theorem 1.3, Corollary 4.6.) Let ξ be the exponent in Proposition 2.1. Then there exist constants c_1, c_2 such that the following holds.*

$$c_1 n^{-\xi} \le P^{0,0}(S^1[0,\tau^1(n)] \cap S^2(0,\tau^2(n)] = \emptyset) \le c_2 n^{-\xi}, \tag{2.2}$$

$$\sup_{|x|,|y|\le m} P^{x,y}(S^1[0,\tau^1(n)] \cap S^2(0,\tau^2(n)] = \emptyset) \le c_2 \big(\frac{n}{m}\big)^{-\xi}, \tag{2.3}$$

for all $m \le n$.

Remark 2.3. *In [6], it was proved that*

$$\xi_2 = \frac{5}{4}. \tag{2.4}$$

The value of ξ_3 is not known. Rigorous estimate ([1], [5]) show that $\frac{1}{2} < \xi_3 < 1$. Simulations suggests that ξ_3 is around 0.57 (see Section 7 in [8]).

2.2 Skorohod Embedding

In this subsection, we state the strong approximation of Brownian motion by simple random walk derived from the Skorohod embedding (see [4] for details).

Proposition 2.4. *([4], Lemma 3.1, Lemma 3.2.) There exist a probability space $(\Omega, \mathcal{F}, P)$ containing a d-dimensional standard Brownian motion B and d-dimensional simple random walk S such that the following holds. For every $\epsilon > 0$ there exist $\delta > 0$ and $a < \infty$ such that*

$$P\big(\sup_{0\le t\le n} |B(t) - S(td)| \ge n^{\frac{1}{4}+\epsilon}\big) \le a\exp(-n^{\delta}). \tag{2.5}$$

Moreover, if we set

$$T(n) = \inf\{t : |B(t)| = n\}, \quad \tau(n) = \inf\{j : |S(j)| \ge n\}$$

then for every $\epsilon > 0$ there exist $\delta > 0$ and $a < \infty$ such that

$$P\big(\sup_{0\le t\le T(n)} |B(t) - S(td)| \ge n^{\frac{1}{2}+\epsilon}\big) \le a\exp(-n^{\delta}). \tag{2.6}$$

We will be using the strong Markov property at time $T(n)$. However, one slight complication that arises is the fact that $\{B(t), S(td) : t \le T(n)\}$ might contain a little information about $B(t)$ beyond time $T(n)$. To overcome this problem, we need the following proposition.

Proposition 2.5. *([4], Lemma 3.3.) There exist $\delta > 0$ and $a < \infty$ such that the following holds. For each n, there is an event $\Psi(n)$ with*

$$P(\Psi(n)) \ge 1 - a\exp(-n^{\delta})$$

such that on the event $\Psi(n)$,

$$\{B(t) : t \le \max\{T(n), \tau(n)\}\} \cup \{S(td) : t \le \max\{T(n), \tau(n)\}\}$$

and

$$\{B(t) : t \ge T(2n)\}$$

are conditionally independent given $B(T(2n))$.

2.3 Beurling Estimate

For $d = 2$, thanks to the Beurling estimate stated in Proposition 2.6 below, we can say intuitively that every path is 'hittable' for both simple random walk and Brownian motion. Now we state it.

Let B be the Brownian motion in $\mathbb{R}^2$ and S be the simple random walk in $\mathbb{Z}^2$. Then the following are well-known (see [7] for the continuous case and [2] for discrete case).

Proposition 2.6. *(i) ([7], Theorem 3.76) There exists a constant $K < \infty$ such that for any $R \ge 1$, any $x \in \mathbb{R}^2$ with $|x| \le R$, any $A \subset \mathbb{R}^2$ with $[0, R] \subset \{|z| : z \in A\}$,*

$$P^x(T(R) < T_A) \le K\big(\frac{|x|}{R}\big)^{\frac{1}{2}}, \tag{2.7}$$

where $T(R) = \inf\{t \geq 0 : |B(t)| \geq R\}$ *and* $T_A = \inf\{t \geq 0 : B(t) \in A\}$.
(ii) ([2], Theorem 2.5.2.) There exists a constant $K < \infty$ *such that for any* $n \geq 1$*, any* $x \in \mathbb{Z}^2$ *with* $|x| \leq n$*, any connected set* $A \subset \mathbb{Z}^2$ *containing the origin and such that* $\sup\{|z| : z \in A\} \geq n$*,*

$$P^x(\tau(n) < \tau_A) \leq K\big(\frac{|x|}{n}\big)^{\frac{1}{2}}, \tag{2.8}$$

where $\tau(n) = \inf\{j \geq 0 : |S(j)| \geq n\}$ *and* $\tau_A = \inf\{j \geq 0 : S(j) \in A\}$.

For $d = 3$, there is no useful analogue of Proposition 2.6. So we need some more work. Let B be the Brownian motion in $\mathbb{R}^3$. For each $\epsilon > 0$, $b < \infty$ and a curve γ in $\mathbb{R}^3$, define

$$f(\gamma) = f(\gamma, \epsilon, b) = \sup P^z(B[0, T(2n)] \cap \gamma = \emptyset),$$

where the supremum is over all z with $|z| \leq n$ such that

$$\text{dist}(z, \gamma) \leq bn^{1-\epsilon},$$

and $T(n)$ be the first hitting time of B to the boundary of disk centered at the origin with radius n. Note that P^z denotes the probability with $B(0) = z$ and $f(\gamma)$ is a function of γ. Let B' be the another Brownian motion in $\mathbb{R}^3$ and denote the first hitting time of it to the boundary of disk centered at the origin with radius n by $T'(n)$. Let

$$Z_n = Z_n(\epsilon, b) = f(B'[0, T'(2n)]).$$

Note that Z_n is a function of $B'[0, T'(2n)]$. In other words,

$$Z_n = \sup P^z(B[0, T(2n)] \cap B'[0, T'(2n)] = \emptyset \mid B'[0, T'(2n)]),$$

where the supremum is over all z with $|z| \leq n$ such that

$$\text{dist}(z, B'[0, T'(2n)]) \leq bn^{1-\epsilon},$$

The following proposition says that Brownian path is a 'hittable set' with high probability.

Proposition 2.7. *([4], Lemma 2.4.) For every* $M < \infty, \epsilon > 0, b < \infty$*, there exist* $\delta > 0$ *and* $a < \infty$ *such that for* $|x| \leq n$*,*

$$P'^x(Z_n \geq n^{-\delta}) \leq an^{-M}, \tag{2.9}$$

where P'^x *denotes probability with* $B'(0) = x$*.*

Finally, we state an analogue of this proposition for simple random walks. Let S, S' be two independent simple random walks in $\mathbb{Z}^3$. For each $\epsilon > 0$ and $b < \infty$, let

$$Z_n^\sharp = Z_n(\epsilon, b)^\sharp = \sup P^z(S[0, \tau(2n)] \cap S'[0, \tau'(2n)] = \emptyset \mid S'[0, \tau'(2n)]),$$

where the supremum is over all z with $|z| \leq n$ and

$$\text{dist}(z, S'[0, \tau'(2n)]) \leq bn^{1-\epsilon},$$

and $\tau(n)$ (resp. $\tau'(n)$) be the first hitting time of S (resp. S') to $\partial\mathcal{B}(n)$. Again note that P^z denotes the probability with $S(0) = z$ and $Z_n^\sharp$ is a function of $S'[0, \tau'(2n)]$. Then we have the following.

Proposition 2.8. *([4], Lemma 2.6.) For every* $M < \infty, \epsilon > 0, b < \infty$*, there exist* $\delta > 0$ *and* $a < \infty$ *such that for* $|x| \leq n$*,*

$$P'^x(Z_n^\sharp \geq n^{-\delta}) \leq an^{-M}, \tag{2.10}$$

where P'^x *denotes probability with* $S'(0) = x$*.*

Throughout this paper, we use Proposition 2.8 for $M = 6$.

2.4 Nonintersecting Brownian motions

In this subsection, we state convergence theorems for Brownian motion in $\mathbb{R}^2$ and $\mathbb{R}^3$ obtained in [3] and [8], respectively. Let $d = 2$ or 3, and B^1, B^2 be independent Brownian motions in $\mathbb{R}^d$. Let $\mathcal{D} = \{z \in \mathbb{R}^d : |z| \le 1\}$ and $\partial\mathcal{D} = \{z \in \mathbb{R}^d : |z| = 1\}$. For $K_1, K_2 \subset \mathcal{D}$ and $w = (w_1, w_2) \in \partial\mathcal{D}^2$ with $w_j \in K_j \cap \partial\mathcal{D}$, define

$$A_n(K_1, K_2) = \{B^1[0, T^1(e^n)] \cap B^2[0, T^2(e^n)] = \emptyset, B^1[0, T^1(e^n)] \cap K_2 = \emptyset, B^2[0, T^2(e^n)] \cap K_1 = \emptyset\},$$

where $T^i(R) = \inf\{t \ge 0 : |B^i(t)| \ge R\}$. Let

$$Q_n(K, w) = e^{n\xi} P^{w_1, w_2}(A_n(K_1, K_2)).$$

Here $\xi = \xi_d$ is the intersection exponent defined as in Section 2.1. In [3] and [8], it was shown the following convergence theorems for $d = 2$ and $d = 3$, respectively.

Proposition 2.9. *([3], Theorem 1.2 and [8], Proposition 4.8.) Let $d = 2$ or 3. For each $K_1, K_2 \subset \mathcal{D}$ and $w = (w_1, w_2) \in \partial\mathcal{D}^2$ with $w_j \in K_j \cap \partial\mathcal{D}$, the limit*

$$\lim_{n\to\infty} Q_n(K, w) =: Q(K, w) \tag{2.11}$$

exists. Moreover there exist $c < \infty$ and $\beta > 0$ depending only on the dimension such that the following holds.

$$|Q(K, w) - Q_n(K, w)| \le c e^{-\beta\sqrt{n}} Q(K, w) \qquad \text{for } d = 2, \tag{2.12}$$

$$|Q(K, w) - Q_n(K, w)| \le c e^{-\beta n} Q(K, w) \qquad \text{for } d = 3. \tag{2.13}$$

As mentioned, our main result Theorem 1.1 (or Theorem 4.1 below) is a random walk version of this proposition. Notice that the rate of convergence in Theorem 1.1 is same as that of Proposition 2.9.

3 Approximation of non-intersection probabilities

3.1 Preliminary

Fix $L \in \mathbb{N}$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma_L$. We write $w^i = \gamma^i(\text{len}\gamma^i)$ for the end point of γ^i. Assume $10L < m < n$. Let S^1, S^2 be two independent simple random walks in $\mathbb{Z}^d$ starting at w^1, w^2 respectively. Let $A_m(\overline{\gamma})$ denote the event

$$A_m(\overline{\gamma}) = \left\{ \begin{array}{l} S^1[0, \tau_m^1] \cap \gamma^2 = \emptyset, \\ S^2[0, \tau_m^2] \cap \gamma^1 = \emptyset, \\ S^1[0, \tau_m^1] \cap S^2[0, \tau_m^2] = \emptyset \end{array} \right\}. \tag{3.1}$$

The goal of this section is to prove the following proposition.

Proposition 3.1. *Let $d = 2, 3$. For all $L \in \mathbb{N}$ and $\overline{\gamma} = (\gamma^1, \gamma^2) \in \Gamma_L$, there exist $c < \infty$ and $\delta > 0$ such that for all $n > m > 10L$,*

$$|2^{(m-L)\xi} P(A_m(\overline{\gamma})) - 2^{(n-L)\xi} P(A_n(\overline{\gamma}))| \le c 2^{-\delta m^{\frac{d}{2}-\frac{1}{2}}}. \tag{3.2}$$

3.2 Several Lemmas

For $\frac{m}{3} \le j \le \frac{m}{2}$, let

$$D_j = \min\{\text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]), \text{dist}(S^2(\tau_j^2), S^1[0, \tau_j^1])\} \tag{3.3}$$

The next lemma shows the probability that a pair of random walk paths come close to each other, but still stay apart for a long time afterward, is small.

Lemma 3.2. *There exist $c < \infty$ and $\delta > 0$ such that for all $N \geq m$,*

$$P(A_N(\overline{\gamma}), D_j \leq 2^{0.99j}) \leq c2^{-(N-L)\xi}2^{-\delta j}, \tag{3.4}$$

for each $\frac{m}{3} \leq j \leq \frac{m}{2}$.

Proof. It is enough to show that

$$P(A_N(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}) \leq c2^{-(N-L)\xi}2^{-\delta j}. \tag{3.5}$$

By the strong Markov property,

$$P(A_N(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}) \leq c2^{-(N-j-1)\xi}P(A_{j+1}(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}).$$

Applying Proposition 2.8 with $\epsilon = 0.01$, $b = 1$, $S = S^1$ and $S' = S^2$, we see that there exist $\delta > 0$ and $c < \infty$ such that

$$\begin{aligned}
&P(A_{j+1}(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}) \\
&\leq P(Z^\sharp_{2^j}(0.01, 1) \geq 2^{-\delta j}) + P(A_{j+1}(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}, Z^\sharp_{2^j}(0.01, 1) \leq 2^{-\delta j}) \\
&\leq c2^{-6j} + P(A_{j+1}(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}, Z^\sharp_{2^j}(0.01, 1) \leq 2^{-\delta j}).
\end{aligned}$$

By the strong Markov property,

$$P(A_{j+1}(\overline{\gamma}), \text{dist}(S^1(\tau_j^1), S^2[0, \tau_j^2]) \leq 2^{0.99j}, Z^\sharp_{2^j}(0.01, 1) \leq 2^{-\delta j}) \leq 2^{-\delta j}P(A_j(\overline{\gamma})).$$

Since $P(A_j(\overline{\gamma})) \leq c2^{-(j-L)\xi}$, the lemma is finished.

□

Let

$$F_m = \{D_{\frac{m}{3}} \geq 2^{\frac{0.99m}{3}}\}. \tag{3.6}$$

By Lemma 3.2, there exists $\delta > 0$ such that

$$P(A_N(\overline{\gamma}), F_m^c) \leq c2^{-(N-L)\xi}2^{-\delta m}, \tag{3.7}$$

for every $N \geq m$.

For each $i = 1, 2$, define

$$\sigma^i = \sigma^i_m = \inf\{k \geq \tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}) : S^i(k) \in \partial\mathcal{B}(S^i(\tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})), 2^{\frac{m}{4}})\}.$$

The next lemma shows that conditioned on $A_N(\overline{\gamma})$, if the random walk stays at a point x near the boundary of a ball, then it hits the boundary near by x with high probability.

Lemma 3.3. *There exist $\delta > 0$ and $c < \infty$ such that for each $N \geq m$,*

$$P(A_N(\overline{\gamma}), \sigma^i < \tau^i_{\frac{m}{3}}) \leq c2^{-(N-L)\xi}2^{-\delta m}. \tag{3.8}$$

Proof. By the strong Markov property,

$$P(A_N(\overline{\gamma}), \sigma^i < \tau^i_{\frac{m}{3}}) \leq c2^{-(N-\frac{m}{3})\xi}P(A_{\frac{m}{3}}(\overline{\gamma}), \sigma^i < \tau^i_{\frac{m}{3}}).$$

Since $\sigma^i > \tau^i_{\frac{m}{3}-1}$, we see that

$$P(A_{\frac{m}{3}}(\overline{\gamma}), \sigma^i < \tau^i_{\frac{m}{3}}) \leq E_{3-i}\big(E_i\big(\mathbf{1}\{A_{\frac{m}{3}-1}(\overline{\gamma})\}P_i^{S^i(\tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}))}(\sigma^i < \tau^i_{\frac{m}{3}})\big)\big),$$

where P_i^x denotes the probability measure of S^i with $S^i(0) = x$. It is easy to see that there exist $\delta > 0$ and $c < \infty$ such that

$$P_i^{S^i(\tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}))}(\sigma^i < \tau^i_{\frac{m}{3}}) \leq c2^{-\delta m},$$

and the lemma is proved.

□

Let $G_m = \{S^i[\tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}), \tau^i_{\frac{m}{3}}] \subset \mathcal{B}(S^i(\tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})), 2^{\frac{m}{4}}),$ for $i = 1, 2\}$ From Lemma 3.3, we have

$$P(A_N(\overline{\gamma}), G_m^c) \le c2^{-(N-L)\xi}2^{-\delta m}. \tag{3.9}$$

Let S be the another simple random walk whose probability measure is denoted by P^z when we assume $S(0) = z$. Then let

$$Z_m^i = \sup P^z(S[0, \tau_{\frac{m}{3}}] \cap S^i[0, \tau_{\frac{m}{3}}] = \emptyset), \tag{3.10}$$

where the supremum is over all z with

$$\text{dist}(z, S^i[0, \tau^i(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})]) \le 2^{\frac{11m}{60}}.$$

Note that Z_m^i is a function of $S^i[0, \tau_{\frac{m}{3}}]$. By Proposition 2.8, we see that there exist $\delta > 0$ and $c < \infty$ such that

$$P_i(Z_m^i \ge 2^{-\delta m}) \le c2^{-6m}. \tag{3.11}$$

Therefore, if we set $H_m^i = \{Z_m^i < 2^{-\delta m}\}$ and $H_m = H_m^1 \cap H_m^2$, we have

$$P(A_N(\overline{\gamma}), H_m^c) \le c2^{-(N-L)\xi}2^{-\delta m}. \tag{3.12}$$

3.3 Coupling

Using the strong Markov property, we see that

$$P(A_N(\overline{\gamma}), F_m, G_m, H_m) = E\big(\mathbf{1}\{A_{\frac{m}{3}}(\overline{\gamma}), F_m, G_m, H_m\} P_{1,2}^{S^1(\tau^1_{\frac{m}{3}}), S^2(\tau^2_{\frac{m}{3}})}(R_{\frac{m}{3},N})\big), \tag{3.13}$$

where we let $R_{\frac{m}{3},N}$ be the event

$$R_{\frac{m}{3},N} = \left\{ \begin{array}{l} \overline{S}^1[0, \tau_N^1] \cap (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) = \emptyset \\ \overline{S}^2[0, \tau_N^2] \cap (S^1[0, \tau^1_{\frac{m}{3}}] \cup \gamma^1) = \emptyset \\ \overline{S}^1[0, \tau_N^1] \cap \overline{S}^2[0, \tau_N^2] = \emptyset \end{array} \right\}. \tag{3.14}$$

Here $\overline{S}^1$ and $\overline{S}^2$ are independent simple random walks starting at $S^1(\tau^1_{\frac{m}{3}})$ and $S^2(\tau^2_{\frac{m}{3}})$, respectively, and we use same notation $\tau^i(l), \tau_k^i$ for the hitting time of $\overline{S}^i$. More precisely, let

$$\tau^i(l) = \inf\{j \ge 0 : \overline{S}^i \in \partial\mathcal{B}(l)\}$$

and $\tau_k^i = \tau^i(2^k)$. Throughout this section we will let $(B^1, \overline{S}^1)$ and $(B^2, \overline{S}^2)$ be two independent Brownian motion - random walk pairs coupled as in Section 2.2. Assume $B^i(0) = \overline{S}^i(0) = S^i(\tau^i_{\frac{m}{3}}) =: w^i_{m/3}$. Let

$$T^i(l) = \inf\{t \ge 0 : |B^i| = l\}, \tag{3.15}$$

and $T_k^i = T^i(2^k)$. From now on, we assume the event $A_{\frac{m}{3}}(\overline{\gamma}) \cap F_m \cap G_m \cap H_m$ holds and compare the probability that two Brownian motions do not intersect each other with the probability that simple random walks do not intersect. For this purpose, let

$$\text{PATH}_f^i = \text{PATH}_{f,m}^i = \{z \in \mathbb{R}^d : \text{dist}(z, S^i[0, \tau^i_{\frac{m}{3}}] \cup \gamma^i) \le 2^{\frac{11m}{60}}\}. \tag{3.16}$$

be a fattened path of $S^i[0, \tau^i_{\frac{m}{3}}] \cup \gamma^i$. Set

$$\beta = \inf\left\{ \frac{m}{3} \le k \le N : \begin{array}{c} \overline{S}^1[\tau_k^1, \tau_N^1] \cap (\overline{S}^2[0, \tau_k^2] \cup S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) = \emptyset \\ \overline{S}^2[\tau_k^2, \tau_N^2] \cap (\overline{S}^1[0, \tau_k^1] \cup S^1[0, \tau^1_{\frac{m}{3}}] \cup \gamma^1) = \emptyset \\ \overline{S}^1[\tau_k^1, \tau_N^1] \cap \overline{S}^2[\tau_k^2, \tau_N^2] = \emptyset \end{array} \right\}, \tag{3.17}$$

where $\beta = \infty$ if no such k exists. Note that $\beta = \infty$ implies $\overline{S}^1(\tau_N^1) = \overline{S}^2(\tau_N^2)$ and

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1(\tau_N^1) = \overline{S}^2(\tau_N^2)) \le c2^{-N}.$$

Here $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}$ denotes the probability measure of $(\overline{S}^1, \overline{S}^2)$ with $\overline{S}^i(0) = w^i_{m/3}$. By definition of β, we see that $\beta = \frac{m}{3}$ implies that $R_{\frac{m}{3},N}$ holds. Therefore, if we let $J_{m,N}$ be the event

$$J_{m,N} = \{B^1[0,T_N^1] \cap \mathrm{PATH}_f^2 = \emptyset, B^2[0,T_N^2] \cap \mathrm{PATH}_f^1 = \emptyset, B^1[0,T_N^1] \cap B^2[0,T_N^2] = \emptyset\}, \quad (3.18)$$

then

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}) \le P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},N}) + \sum_{k=\frac{m}{3}+1}^{N} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k) + c2^{-N}. \quad (3.19)$$

Since

$$\min\{\mathrm{dist}(w^1_{m/3}, \mathrm{PATH}_f^2), \mathrm{dist}(w^2_{m/3}, \mathrm{PATH}_f^1)\} \ge 2^{\frac{0.99m}{3}-1}$$

on the event F_m, we see that $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}) > 0$. In this section, we will estimate $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$ for $\frac{m}{3} < k \le N$ assuming that F_m, G_m and H_m hold.

Before starting estimates of $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$, we give an intuitive idea here. If $\beta = k$ and $\frac{m}{3} < k \le N$, then we have

$$\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap (\overline{S}^2[0,\tau_k^2] \cup S^2[0,\tau_{\frac{m}{3}}^2] \cup \gamma^2) \ne \emptyset$$

or

$$\overline{S}^2[\tau_{k-1}^2, \tau_k^2] \cap (\overline{S}^1[0,\tau_k^1] \cup S^1[0,\tau_{\frac{m}{3}}^1] \cup \gamma^1) \ne \emptyset.$$

Assume $\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap (\overline{S}^2[0,\tau_k^2] \cup S^2[0,\tau_{\frac{m}{3}}^2] \cup \gamma^2) \ne \emptyset$. If $\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap \overline{S}^2[0,\tau_k^2] \ne \emptyset$, then roughly speaking, $B^1[T_{k-1}^1, T_k^1]$ intersects $B^2[0,T_k^2]$ with high probability because of the strong approximation in Proposition 2.4 and the fact Brownian path is a 'hittable' set. Hence by the strong Markov property, $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$ is much smaller than $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N})$. A slight difficulty arises when $k = N$ and $\overline{S}^1[\tau_{N-1}^1, \tau_N^1]$ intersects $\overline{S}^2[0,\tau_N^2]$ around $\overline{S}^2(\tau_N^2)$. In this case, we cannot conclude that $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$ is much smaller than $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N})$ by using the strong approximation as above. However, the probability that $\overline{S}^1[\tau_{N-1}^1, \tau_N^1]$ intersects $\overline{S}^2[0,\tau_N^2]$ around $\overline{S}^2(\tau_N^2)$ is small, so we can overcome this difficulty (see subsection 3.3.3 for details). Next, assume $\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap S^2[0,\tau_{\frac{m}{3}}^2] \ne \emptyset$ and $d = 2$. Then by the strong approximation, $B^1[T_{k-1}^1, T_k^1]$ intersects $S^2[0,\tau_{\frac{m}{3}}^2] \cup B^2[0,T_N^2]$ with high probability. Hence by using the strong Markov property, we conclude $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$ is much smaller than $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N})$. For $d = 3$, we need more careful observation because Brownian motion does not intersect simple random walk path with probability one. To overcome this problem we first consider the case that $\frac{21m}{60} < k$. In this case, the probability that $\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap \mathcal{B}(2^{\frac{m}{3}})$ is small, so by the strong Markov property, we can say that $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \beta = k)$ is much smaller than $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N})$ (see subsection 3.3.2). For $\frac{m}{3} < k \le \frac{21m}{60}$, by the strong approximation, one can show that $B^1[T_{k-1}^1, T_k^1]$ intersects $\mathrm{PATH}_f^2 \cup B^2[0,T_N^2]$ with high probability under the assumption that F_m, G_m and H_m hold (see subsection 3.3.1). Hence we have the same conclusion. The same idea works for the case that $\overline{S}^1[\tau_{k-1}^1, \tau_k^1] \cap \gamma^2 \ne \emptyset$.

3.3.1 Bounds for $\frac{m}{3} < k \le \frac{21m}{60}$

Let $\frac{m}{3} < k \le \frac{21m}{60}$. It is easy to see that $\beta = k$ implies that

$$\overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap (\overline{S}^2[0, \tau^2_k] \cup S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset$$

or

$$\overline{S}^2[\tau^2_{k-1}, \tau^2_k] \cap (\overline{S}^1[0, \tau^1_k] \cup S^1[0, \tau^1_{\frac{m}{3}}] \cup \gamma^1) \neq \emptyset.$$

We assume that the first event holds. (Similar arguments work for the second one.)

Let

$$Q^i = \{ \sup_{0 \le s \le T^i_{k+1}} |B^i(s) - \overline{S}^i(ds)| \ge 2^{\frac{31k}{60}} \},$$

and $Q = Q^1 \cup Q^2$. Let $\Psi^1(2^{k+1}), \Psi^2(2^{k+1})$ be the events given in Proposition 2.5 for $(B^1, \overline{S}^1)$ and $(B^2, \overline{S}^2)$, respectively, and let $\Psi = \Psi^1(2^{k+1}) \cap \Psi^2(2^{k+1})$.

Lemma 3.4. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P^{w^1_{m/3}, w^2_{m/3}}_{1,2}(J_{m,N}, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap \overline{S}^2[0, \tau^2_k] \neq \emptyset) \le c 2^{-(N - \frac{m}{3})\xi} 2^{-\delta k}. \tag{3.20}$$

Proof. There exist $c < \infty$ and $\delta > 0$ such that

$$P(J_{m,N}, \Psi^c) \le c 2^{-(N-k-2)\xi} \exp(-2^{\delta k}). \tag{3.21}$$

By Proposition 2.5, Q and $\{B^1(t) : t \ge T^1_{k+2}\} \cup \{B^2(t) : t \ge T^2_{k+2}\}$ are conditionally independent given $B^1(T^1_{k+2}), B^1(T^1_{k+2})$ on the event Ψ. Hence by Proposition 2.4,

$$\begin{aligned} P(J_{m,N}, \Psi, Q) &\le P(B^1[T^1_{k+2}, T^1_N] \cap B^2[T^2_{k+2}, T^2_N] = \emptyset, \Psi, Q) \\ &\le c 2^{-(N-k-2)\xi} \exp(-2^{\delta k}). \end{aligned} \tag{3.22}$$

Now we give an upper bound of

$$P(J_{m,N}, \Psi, Q^c, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap \overline{S}^2[0, \tau^2_k] \neq \emptyset). \tag{3.23}$$

By the strong Markov property, this probability is bounded above by

$$c 2^{-(N-k-2)\xi} P(J_{m,k+1}, \Psi, Q^c, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap \overline{S}^2[0, \tau^2_k] \neq \emptyset).$$

Assume Q^c holds. Then it is easy to see that

$$dT^i(2^{k-1} - 2^{\frac{31k}{60}}) \le \tau^i_{k-1} < \tau^i_k \le dT^i(2^k + 2^{\frac{31k}{60}}).$$

Hence on the event $Q^c \cap \{\overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap \overline{S}^2[0, \tau^2_k] \neq \emptyset\}$, we see that there exist s, t with

$$\begin{aligned} dT^1(2^{k-1} - 2^{\frac{31k}{60}}) \le s \le dT^1(2^k + 2^{\frac{31k}{60}}), \\ 0 \le t \le dT^2(2^k + 2^{\frac{31k}{60}}) \end{aligned}$$

such that $\overline{S}^1(s) = \overline{S}^2(t)$. For such s and t, we have

$$|B^1(\frac{s}{d}) - B^2(\frac{t}{d})| \le 2^{\frac{31k}{60}+1}.$$

Namely, the following event holds,

$$D_k := \{\mathrm{dist}\big(B^1[T^1(2^{k-1} - 2^{\frac{31k}{60}}), T^1(2^k + 2^{\frac{31k}{60}})], B^2[0, T^2(2^k + 2^{\frac{31k}{60}})]\big) \le 2^{\frac{31k}{60}+1}\}. \tag{3.24}$$

Let

$$Z_k = \sup P^z(B[0,T_{k+1}] \cap B^2[0,T_{k+1}] = \emptyset),$$

where the supremum is over all z with $z \in \mathcal{B}(2^k + 2^{\frac{31k}{60}})$ and

$$\text{dist}(z, B^2[0, T^2(2^k + 2^{\frac{31k}{60}})) \le 2^{\frac{31k}{60}+1}.$$

We let H_k be the event $\{Z_k \le 2^{-\delta k}\}$. By Proposition 2.7, there exists $\delta > 0$ such that

$$P(H_k) \le 2^{-6k}.$$

Therefore, we have only to estimate

$$P(J_{m,k+1}, \Psi, D_k, H_k^c).$$

On the event $J_{m,k+1} \cap D_k \cap H_k^c$, $B^1[T^1(2^{k-1} - 2^{\frac{31k}{60}}), T^1_{k+1}]$ does not intersect $B^2[0, T^2_{k+1}]$ nevertheless B^1 comes close to $B^2[0, T^2_{k+1}]$ which is a hittable set. By the strong Markov property,

$$P(J_{m,k+1}, \Psi, D_k, H_k^c) \le c 2^{-\delta k} 2^{-(k-\frac{m}{3})\xi},$$

and this finishes the proof.

□

Lemma 3.5. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset) \le c 2^{-(N-\frac{m}{3})\xi} \exp(-2^{\delta k}). \quad (3.25)$$

Proof. Recall Ψ and Q are the events given before the statement of Lemma 3.4. By (3.21) and (3.22), it suffices to estimate

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,N}, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset, \Psi, Q^c).$$

By the strong Markov property, this probability is bounded above by

$$c 2^{-(N-k)\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}\big(J_{m,k+1}, \Psi, Q^c, \overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset\big).$$

On the event $Q^c \cap \{\overline{S}^1[\tau^1_{k-1}, \tau^1_k] \cap (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset\}$, it is easy to see that there exists t with

$$T^1(2^{k-1} - 2^{\frac{31k}{60}}) \le t \le T^1(2^k + 2^{\frac{31k}{60}})$$

such that

$$\text{dist}\big(B^1(t), (S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2)\big) \le 2^{\frac{31k}{60}}.$$

Since $k \le \frac{21m}{60}$, we have $\frac{31k}{60} \le \frac{651m}{3600} < \frac{11m}{60}$. Therefore,

$$B^1[0, T^1_{k+1}] \cap \text{PATH}^2_f \neq \emptyset,$$

and the lemma is finished.

□

3.3.2 Bounds for $\frac{21m}{60} < k \le N-3$

From now we assume that $\frac{21m}{60} < k \le N-3$. The similar argument in the proof of Lemma 3.4 gives the following lemma, so we omit the proof.

Lemma 3.6. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}, \overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap \overline{S}^2[0,\tau^2_k] \neq \emptyset) \le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.26}$$

Let Ψ and Q be the events defined before the statement of Lemma 3.4. By (3.21) and (3.22), in order to prove

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}, \beta = k) \le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k},$$

for $\frac{21m}{60} < k \le N-3$, it is enough to show the following lemma.

Lemma 3.7. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}, \overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap (S^2[0,\tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset, \Psi, Q^c) \le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.27}$$

Proof. By the strong Markov property, the left hand side of (3.27) is bounded above by

$$c2^{-(N-k)\xi}P_{1,2}^{w^1_{m/3},w^2_{m/3}}\big(J_{m,k+1}, \Psi, Q^c, \overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap (S^2[0,\tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset\big).$$

Assume $d = 3$ and $\frac{21m}{60} < k \le \frac{20m}{31}$ so that $\frac{31k}{60} \le \frac{m}{3}$. If $\overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap (S^2[0,\tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset$, then $\overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap \mathcal{B}(2^{\frac{m}{3}}) \neq \emptyset$. On the other hand, on the event Q^c, we have

$$3T^1(2^{k-1} - 2^{\frac{31k}{60}}) \le \tau^1_{k-1} \le \tau^1_k \le 3T^1(2^k + 2^{\frac{31k}{60}}).$$

Since $\frac{31k}{60} \le \frac{m}{3}$, we have

$$B^1[T^1(2^{k-1} - 2^{\frac{31k}{60}}), T^1(2^k + 2^{\frac{31k}{60}})] \cap \mathcal{B}(2^{\frac{m}{3}+1}) \neq \emptyset.$$

For $k > \frac{21m}{60}$, a standard estimate shows that

$$P_1(B^1[T^1(2^{k-1} - 2^{\frac{31k}{60}}), T^1(2^k + 2^{\frac{31k}{60}})] \cap \mathcal{B}(2^{\frac{m}{3}+1}) \neq \emptyset) \le c2^{-(k-\frac{m}{3})}.$$

Using the strong Markov property at $T^1(2^{k-1}-2^{\frac{31k}{60}})$ first, and then estimating $P(J_{m,k-2})$, we have

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}\big(J_{m,k+1}, \Psi, Q^c, \overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap (S^2[0,\tau^2_{\frac{m}{3}}] \cup \gamma^2) \neq \emptyset\big) \le c2^{-(k-\frac{m}{3})}2^{-(k-\frac{m}{3})\xi}.$$

Therefore, the proof for $d = 3$ and $\frac{21m}{60} < k \le \frac{20m}{31}$ is finished.

Next we assume $d = 3$ and $\frac{20m}{31} < k \le N-3$. In this case, if $\overline{S}^1[\tau^1_{k-1},\tau^1_k] \cap \mathcal{B}(2^{\frac{m}{3}}) \neq \emptyset$ and Q^c hold, then

$$B^1[T^1(2^{k-1} - 2^{\frac{31k}{60}}), T^1(2^k + 2^{\frac{31k}{60}})] \cap \mathcal{B}(2^{\frac{31k}{60}+1}) \neq \emptyset. \tag{3.28}$$

Since this event occur with probability at most $c2^{-\frac{k}{3}}$, the lemma is proved for $d = 3$.

Assume $d = 2$. In this case, the probability of the event (3.28) is bounded below by $1/k$, so we need to change the proof. Assume $\frac{21m}{60} < k \le \frac{20m}{31}$. (For the otherwise, the proof is almost same as this case. So we only consider this case.) Let

$$\eta = \inf\{t \ge T^1(2^{k-1} - 2^{\frac{31k}{60}}) : B^1(t) \in \mathcal{B}(2^{\frac{m}{3}+1})\}.$$

We already showed that if $\overline{S}^1[\tau^1_{k-1},\tau^1_k]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset$ and Q^c hold, then $\eta\le T^1(2^k+2^{\frac{31k}{60}})$. By the Proposition 2.6, we see that

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3},w^2_{m/3}}\big(J_{m,k+1},\eta\le T^1(2^k+2^{\frac{31k}{60}})\big)\\
&\le E_2^{w^2_{m/3}}\big(E_1^{w^1_{m/3}}\big(\mathbf{1}\{J_{m,k-2},\eta\le T^1(2^k+2^{\frac{31k}{60}}),B^1[\eta,T^1_{k+1}]\cap B^2[0,T^2_{k+1}]=\emptyset\}\big)\big)\\
&\le P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,k-2})c2^{-\frac{k}{5}}\\
&\le c2^{-(k-\frac{m}{3})\xi}2^{-\frac{k}{5}},
\end{aligned}$$

and the lemma is proved for all cases.

□

3.3.3 Bounds for $N-2\le k\le N$

Finally, we give estimates for $N-2\le k\le N$. Since a proof is similar, we only consider for $k=N$. By definition of β in (3.17), we see that $\beta=N$ implies the event

$$\bigcup_{i=1,2}\{\overline{S}^i[\tau^i_{N-1},\tau^i_N]\cap(\overline{S}^{3-i}[0,\tau^{3-i}_N]\cup S^{3-i}[0,\tau^{3-i}_{\frac{m}{3}}]\cup\gamma^{3-i})\neq\emptyset\}. \tag{3.29}$$

We will only give bounds on the probability of the event for $i=1$ in (3.29). First we show the following lemma.

Lemma 3.8. *There exist $\delta>0$ and $c<\infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset)\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \tag{3.30}$$

Proof. Recall Q is the event defined before the statement of Lemma 3.4 ($k=N$ in this case). By Proposition 2.4,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(Q)\le c\exp(-(2^{\delta N})). \tag{3.31}$$

Assume Q^c holds. Then $dT^1(2^{N-1}-2^{\frac{31N}{60}})\le\tau^1_{N-1}$. Therefore, if $\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset$, we have

$$B^1[T^1_{N-2},\infty)\cap\mathcal{B}(2^{\frac{31N}{60}+1})\neq\emptyset. \tag{3.32}$$

For $d=3$, the probability of the event (3.32) is bounded above by $c2^{-\frac{N}{3}}$. Therefore, by the strong Markov property,

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset)\\
&\le P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N-2},Q^c,B^1[T^1_{N-2},\infty)\cap\mathcal{B}(2^{\frac{31N}{60}+1})\neq\emptyset)+c\exp(-(2^{\delta N}))\\
&\le c2^{-\frac{N}{3}}2^{-(N-\frac{m}{3})\xi},
\end{aligned}$$

for $d=3$.

Next we consider the two dimensional case. Assume $\overline{S}^1[\tau^1(2^N-2^{\frac{31N}{60}}),\tau^1_N]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset$ and Q^c holds. This implies that $\overline{S}^1[\tau^1(2^N-2^{\frac{31N}{60}}),\tau^1_N]\cap\mathcal{B}(2^{\frac{31N}{60}})\neq\emptyset$. On the event Q^c, we have

$$2T^1(2^N-2^{\frac{31N}{60}+1})\le\tau^1(2^N-2^{\frac{31N}{60}})\le\tau^1_N\le 2T^1(2^N+2^{\frac{31N}{60}}).$$

Therefore,

$$B^1[T^1(2^N-2^{\frac{31N}{60}+1}),T^1(2^N+2^{\frac{31N}{60}})]\cap\mathcal{B}(2^{\frac{31N}{60}+1})\neq\emptyset. \tag{3.33}$$

Using Proposition 2.6, the probability of the event (3.33) is bounded above by $c2^{-\frac{N}{3}}$. Hence by the strong Markov property,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1(2^N-2^{\frac{31N}{60}}),\tau^1_N]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset)\leq c2^{-\frac{N}{3}}2^{-(N-\frac{m}{3})\xi}.$$

Assume $\overline{S}^1[\tau^1_{N-1},\tau^1(2^N-2^{\frac{31N}{60}})]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset$ and Q^c holds. This implies that

$$B^1[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1_N]\cap\mathcal{B}(2^{\frac{31N}{60}+1})\neq\emptyset. \tag{3.34}$$

So let

$$\rho=\inf\{t\geq T^1(2^{N-1}-2^{\frac{31N}{60}}):B^1(t)\in\mathcal{B}(2^{\frac{31N}{60}+1})\}.$$

Again by using Proposition 2.6,

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1(2^N-2^{\frac{31N}{60}})]\cap(S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\neq\emptyset,Q^c)\\
&\leq P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N-2},\rho\in[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1_N],B^1[\rho,T^1_N]\cap B^2[0,T^2_N]=\emptyset)\\
&\leq P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N-2})2^{-\frac{N}{3}}\\
&\leq c2^{-(N-\frac{m}{3})\xi}2^{-\frac{N}{3}},
\end{aligned}$$

and the lemma is proved.

□

To estimate the probability of (3.29), we have only to show the following lemma.

Lemma 3.9. *There exist $\delta>0$ and $c<\infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\overline{S}^2[0,\tau^2_N]\neq\emptyset)\leq c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \tag{3.35}$$

Before we start to prove this lemma, we need to prepare several lemmas.

Lemma 3.10. *There exist $\delta>0$ and $c<\infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\overline{S}^2[\tau^2(2^N-2^{\frac{2N}{3}}),\tau^2_N]\neq\emptyset)\leq c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \tag{3.36}$$

Proof. Let Q be the event defined in the proof of Lemma 3.8. Let

$$\sigma=\inf\{k\geq\tau^2(2^N-2^{\frac{2N}{3}}):\overline{S}^2_k\in\partial\mathcal{B}(\overline{S}^2(\tau^2(2^N-2^{\frac{2N}{3}})),2^{\frac{3N}{4}})\}.$$

If Q^c holds and $\sigma<\tau^2_N$, then

$$\overline{\sigma}:=\inf\{t\geq T^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}}):B^2(t)\in\partial\mathcal{B}(B^2(T^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}}),2^{\frac{3N}{4}}-2^{\frac{31N}{60}})\}\leq T^2(2^N+2^{\frac{31N}{60}}). \tag{3.37}$$

It is easy to see that the probability of (3.37) is bounded above by $c2^{-\delta N}$ for some $c<\infty$ and $\delta>0$. Hence by the strong Markov property,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\overline{S}^2[\tau^2(2^N-2^{\frac{2N}{3}}),\tau^2_N]\neq\emptyset,\sigma<\tau^2_N)\leq c2^{-\delta N}2^{-(N-\frac{m}{3})\xi}.$$

Now assume $\sigma<\tau^2_N$. Then $\overline{S}^2[\tau^2(2^N-2^{\frac{2N}{3}}),\tau^2_N]\subset\mathcal{B}(\overline{S}^2(\tau^2(2^N-2^{\frac{2N}{3}})),2^{\frac{3N}{4}})$. Therefore $\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\overline{S}^2[\tau^2(2^N-2^{\frac{2N}{3}}),\tau^2_N]\neq\emptyset$ implies that

$$\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\mathcal{B}(\overline{S}^2(\tau^2(2^N-2^{\frac{2N}{3}})),2^{\frac{3N}{4}})\neq\emptyset. \tag{3.38}$$

If Q^c and (3.38) hold, we see that

$$B^1[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1(2^N+2^{\frac{31N}{60}})]\cap\mathcal{B}(B^2(T^2(2^N-2^{\frac{2N}{3}})),2^{\frac{3N}{4}+1})\neq\emptyset. \tag{3.39}$$

For any $x\in\partial\mathcal{B}(2^N-2^{\frac{2N}{3}})$, we have

$$P_1(B^1[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1(2^N+2^{\frac{31N}{60}})]\cap\mathcal{B}(x,2^{\frac{3N}{4}+1})\neq\emptyset)\le c2^{-\frac{N}{4}}.$$

By the strong Markov property,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1_N]\cap\overline{S}^2[\tau^2(2^N-2^{\frac{2N}{3}}),\tau^2_N]\neq\emptyset,\sigma\ge\tau^2_N)\le c2^{-\delta N}2^{-(N-\frac{m}{3})\xi},$$

and hence prove the lemma.

□

Remark 3.11. *Similar arguments in the proof of Lemma 3.10 give that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1(2^N-2^{\frac{2N}{3}}),\tau^1_N]\cap\overline{S}^2[0,\tau^2_N]\neq\emptyset)\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \tag{3.40}$$

By Lemma 3.10 and Remark 3.11, we have only to show the following lemma to prove Lemma 3.9.

Lemma 3.12. *There exist $\delta>0$ and $c<\infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1(2^N-2^{\frac{2N}{3}})]\cap\overline{S}^2[0,\tau^2(2^N-2^{\frac{2N}{3}})]\neq\emptyset)\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \tag{3.41}$$

Proof. Let Q be the event defined in the proof of Lemma 3.8. If

$$\overline{S}^1[\tau^1_{N-1},\tau^1(2^N-2^{\frac{2N}{3}})]\cap\overline{S}^2[0,\tau^2(2^N-2^{\frac{2N}{3}})]\neq\emptyset$$

and Q^c holds, then we have

$$\mathrm{dist}(B^1[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})],B^2[0,T^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})])\le 2^{\frac{31N}{60}+1}. \tag{3.42}$$

Let

$$Z=\sup P^z(B[0,T_N]\cap B^2[0,T^2_N]=\emptyset),$$

where the supremum is over all z with $z\in\mathcal{B}(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})$ and

$$\mathrm{dist}(z,B^2[0,T^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})])\le 2^{\frac{31N}{60}+1}.$$

Then by Proposition 2.7,

$$P_2^{w^2_{m/3}}(Z\ge 2^{-\delta N})\le c2^{-6N},$$

for some $\delta>0$ and $c<\infty$. Therefore,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\overline{S}^1[\tau^1_{N-1},\tau^1(2^N-2^{\frac{2N}{3}})]\cap\overline{S}^2[0,\tau^2(2^N-2^{\frac{2N}{3}})]\neq\emptyset,Q^c)$$

is bounded above by

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N},\mathrm{dist}(B^1[T^1(2^{N-1}-2^{\frac{31N}{60}}),T^1(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})],B^2[0,T^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})])$$
$$\le 2^{\frac{31N}{60}+1},Z\le 2^{-\delta N})+c2^{-6N}.$$

Using the strong Markov property for B^1, we see that this probability is bounded above by

$$2^{-\delta N}P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N-2}),$$

and hence the proof is finished.

□

3.3.4 Conclusion Lower Bound

Combining estimates obtained in subsections 3.3.1, 3.3.2 and 3.3.3 with (3.19), we have the following proposition.

Proposition 3.13. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}) \le P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}) + c2^{-(N-\frac{m}{3})\xi}2^{-\delta m}, \tag{3.43}$$

on the event $F_m \cap G_m \cap H_m$.

3.3.5 Upper bound

From this subsection, we will give an upper bound of $P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N})$ by using $P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N})$ on the event $F_m \cap G_m \cap H_m$. For this purpose, define

$$\beta^\sharp = \inf\left\{ \frac{m}{3} \le k \le N : \begin{array}{c} B^1[T^1_k, T^1_N] \cap (B^2[0,T^2_k] \cup \mathrm{PATH}^2_f) = \emptyset \\ B^2[T^2_k, T^2_N] \cap (B^1[0,T^1_k] \cup \mathrm{PATH}^1_f) = \emptyset \\ B^1[T^1_k, T^1_N] \cap B^2[T^2_k, T^2_N] = \emptyset \end{array} \right\}. \tag{3.44}$$

Note that $\beta^\sharp \le N$ almost surely and $\beta^\sharp = \frac{m}{3}$ implies $J_{m,N}$ holds. Therefore,

$$\begin{aligned} P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}) &= P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}, \frac{m}{3} \le \beta^\sharp \le N) \\ &= P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}, J_{m,N}) + \sum_{k=\frac{m}{3}+1}^{N} P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}, \beta^\sharp = k) \\ &\le P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}) + \sum_{k=\frac{m}{3}+1}^{N} P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}, \beta^\sharp = k). \end{aligned} \tag{3.45}$$

We will give bounds for the second term in the right hand side of (3.45). For $\frac{m}{3}+1 \le k \le N$, $\beta^\sharp = k$ implies that

$$\bigcup_{i=1,2} \{B^1[T^1_k, T^1_N] \cap B^2[T^2_k, T^2_N] = \emptyset\} \cap \{B^i[T^i_{k-1}, T^i_k] \cap (B^{3-i}[0, T^{3-i}_k] \cup \mathrm{PATH}^{3-i}_f) \ne \emptyset\}. \tag{3.46}$$

We only consider the case $i = 1$ in (3.46). The idea to estimate $P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}, \beta^\sharp = k)$ is essentially same as that for $P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N}, \beta = k)$. Roughly speaking, if two Brownian motions intersect, then with high probability, corresponding two simple random walks also intersect.

3.3.6 Bounds for $\frac{21m}{60} \le k \le N-3$

Lemma 3.14. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_k, T^1_N] \cap B^2[T^2_k, T^2_N] = \emptyset, B^1[T^1_{k-1}, T^1_k] \cap B^2[0, T^2_k] \ne \emptyset, R_{\frac{m}{3},N}) \le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.47}$$

Proof. Since the idea is quite similar as in the proof of Lemma 3.4, we will just sketch the proof. By the strong Markov property, the probability in the left hand side of (3.47) can be bounded above by

$$c2^{-(N-k)\xi} P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_{k-1}, T^1_k] \cap B^2[0, T^2_k] \ne \emptyset, R_{\frac{m}{3},k+1}).$$

By Proposition 2.4, if $B^1[T^1_{k-1},T^1_k]\cap B^2[0,T^2_k]\neq\emptyset$, then $\overline{S}^1$ comes close to $\overline{S}^2[0,\tau^2_k]$ during $[\tau^1_{k-1},\tau^1_k]$ with probability greater than $1-c\exp(-2^{\delta k})$, for some $\delta>0$ and $c<\infty$. By Proposition 2.8, once $\overline{S}^1$ comes close to $\overline{S}^2[0,\tau^2_k]$ during $[\tau^1_{k-1},\tau^1_k]$, then $\overline{S}^1$ intersects $\overline{S}^2[0,\tau^2_{k+1}]$ until τ^1_{k+1} with probability greater than $1-2^{-\delta k}$. Hence by using the strong Markov property, the lemma can be proved.

□

Lemma 3.15. *There exist $\delta>0$ and $c<\infty$ such that*

$$P^{w^1_{m/3},w^2_{m/3}}_{1,2}(B^1[T^1_k,T^1_N]\cap B^2[T^2_k,T^2_N]=\emptyset, B^1[T^1_{k-1},T^1_k]\cap \mathit{PATH}^2_f\neq\emptyset, R_{\frac{m}{3},N})\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.48}$$

Proof. Similar idea used in the proof of Lemma 3.7 works here. So we just state the idea of the proof.

First let $d=3$. The probability that $B^1[T^1_k,T^1_N]$ does not intersect $B^2[T^2_k,T^2_N]$ is bounded above by $c2^{-(N-k)\xi}$. Assume $B^1[T^1_{k-1},T^1_k]\cap\mathrm{PATH}^2_f\neq\emptyset$, then B^1 enters in $\mathcal{B}(2^{\frac{m}{3}})$ during $[T^1_{k-1},T^1_k]$. The probability that $B^1[T^1_{k-1},T^1_k]\cap\mathcal{B}(2^{\frac{m}{3}})\neq\emptyset$ is at most $c2^{-(k-\frac{m}{3})}$. Finally, using $P^{w^1_{m/3},w^2_{m/3}}_{1,2}(R_{\frac{m}{3},k-2})\le c2^{-(k-\frac{m}{3})\xi}$ and the strong Markov property, the lemma is finished for $d=3$.

Next let $d=2$. In this case, if B^1 enters in $\mathcal{B}(2^{\frac{m}{3}})$ during $[T^1_{k-1},T^1_k]$, then $\overline{S}^1$ enters $\mathcal{B}(2^{(\frac{m}{3}\vee\frac{31N}{60})+1})$ during $[\tau^1_{k-1},\tau^1_k]$ with probability greater than $1-\exp(-2^{\delta k})$. Once $\overline{S}^1[\tau^1_{k-1},\tau^1_k]\cap\mathcal{B}(2^{(\frac{m}{3}\vee\frac{31N}{60})+1})\neq\emptyset$, the probability that $\overline{S}^1$ intersects $\overline{S}^2[0,\tau^2_k]$ until τ^1_k is greater than $1-2^{-\delta k}$. Therefore, by using the strong Markov property, we finish the proof of the lemma for $d=2$.

□

3.3.7 Bounds for $\frac{m}{3}+1\le k\le\frac{21m}{60}$

We can prove the following lemma by using the same idea of Lemma 3.14. So we omit its proof.

Lemma 3.16. *There exist $\delta>0$ and $c<\infty$ such that*

$$P^{w^1_{m/3},w^2_{m/3}}_{1,2}(B^1[T^1_k,T^1_N]\cap B^2[T^2_k,T^2_N]=\emptyset, B^1[T^1_{k-1},T^1_k]\cap B^2[0,T^2_k]\neq\emptyset, R_{\frac{m}{3},N})\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.49}$$

For $\frac{m}{3}+1\le k\le\frac{21m}{60}$, we have only to show the following lemma.

Lemma 3.17. *There exist $\delta>0$ and $c<\infty$ such that*

$$P^{w^1_{m/3},w^2_{m/3}}_{1,2}(B^1[T^1_k,T^1_N]\cap B^2[T^2_k,T^2_N]=\emptyset, B^1[T^1_{k-1},T^1_k]\cap \mathit{PATH}^2_f\neq\emptyset, R_{\frac{m}{3},N})\le c2^{-(N-\frac{m}{3})\xi}2^{-\delta k}. \tag{3.50}$$

Proof. We will give a full proof of this lemma. Recall the definition of PATH^2_f in (3.16). Let

$$2\mathrm{PATH}^2_f=\{z\in\mathbb{R}^d:\mathrm{dist}(z,S^2[0,\tau^2_{\frac{m}{3}}]\cup\gamma^2)\le 2^{\frac{11m}{60}+1}\},$$

and

$$Q^i=\{\sup_{0\le t\le T^i_{k+1}}|\overline{S}^i(dt)-B^i(t)|\ge 2^{\frac{31k}{60}}\}.$$

We write $Q=Q^1\cup Q^2$. By Proposition 2.4, we see that

$$P^{w^1_{m/3},w^2_{m/3}}_{1,2}(Q)\le c\exp(-2^{\delta k}),$$

for some $\delta > 0$ and $c < \infty$. Let $\Psi^1(2^{k+1})$, $\Psi^2(2^{k+1})$ be the event in Proposition 2.5 for $(B^1, \overline{S}^1)$, $(B^2, \overline{S}^2)$, respectively. Let $\Psi = \Psi^1(2^{k+1}) \cap \Psi^2(2^{k+1})$. By Proposition 2.5,

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\Psi^c) \le c \exp(-2^{\delta k}),$$

for some $\delta > 0$ and $c < \infty$. Recall that on the event Ψ,

$$\bigcup_{i=1,2} \{B^i(t) : t \le T^i_{k+1} \vee \tau^i_{k+1}\} \cup \{\overline{S}^i(dt) : t \le T^i_{k+1} \vee \tau^i_{k+1}\}$$

and

$$\bigcup_{i=1,2} \{B^i(t) : t \ge T^i_{k+2}\}$$

are conditionally independent given $B^1(T^1_{k+2})$ and $B^2(T^2_{k+2})$. Therefore,

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(B^1[T^1_k, T^1_N] \cap B^2[T^2_k, T^2_N] = \emptyset, B^1[T^1_{k-1}, T^1_k] \cap \mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},N}, Q^c, \Psi) \\
&\le c 2^{-(N-k)\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(B^1[T^1_{k-1}, T^1_k] \cap \mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},k+1}, Q^c, \Psi).
\end{aligned}$$

From now we will estimate for $P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(B^1[T^1_{k-1}, T^1_k] \cap \mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},k+1}, Q^c)$.

First, let $d = 2$. If Q^c holds, then it is easy to see that

$$\frac{\tau^1(2^{k-1} - 2^{\frac{31k}{60}})}{2} \le T^1_{k-1} \le T^1_k \le \frac{\tau^1(2^k + 2^{\frac{31k}{60}})}{2}. \tag{3.51}$$

Therefore, on the event $Q^c \cap \{B^1[T^1_{k-1}, T^1_k] \cap \mathrm{PATH}^2_f \neq \emptyset\}$, we have

$$\mathrm{dist}(\overline{S}^1(2t), S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \le 2^{\frac{11m}{60}} + 2^{\frac{31k}{60}} \le 2^{\frac{11m}{60}+1},$$

for some $t \in [\frac{\tau^1(2^{k-1} - 2^{\frac{31k}{60}})}{2}, \frac{\tau^1(2^k + 2^{\frac{31k}{60}})}{2}]$. Here we use $k \le \frac{21m}{60}$ in the last inequality. Hence,

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(B^1[T^1_{k-1}, T^1_k] \cap \mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},k+1}, Q^c) \\
&\le P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[\tau^1(2^{k-1} - 2^{\frac{31k}{60}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap 2\mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},k+1}).
\end{aligned} \tag{3.52}$$

Let

$$\sigma = \inf\{j \ge \tau^1(2^{k-1} - 2^{\frac{31k}{60}}) : \overline{S}^1(j) \in 2\mathrm{PATH}^2_f\}.$$

Then the right hand side in (3.52) is bounded above by

$$\begin{aligned}
E_2^{w^2_{m/3}}\big(E_1^{w^1_{m/3}}\big(&\mathbf{1}\{\overline{S}^1[0, \tau^1_{k-2}] \cap \overline{S}^2_{k-2} = \emptyset, \sigma \le \tau^1(2^k + 2^{\frac{31k}{60}})\} \\
&\times P^{\overline{S}^1(\sigma)}(\overline{S}^1[0, \tau^1_{k+1}] \cap (\overline{S}^2[0, \tau^2_{k+1}] \cup S[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) = \emptyset)\big)\big).
\end{aligned} \tag{3.53}$$

Since $\overline{S}^2[0, \tau^2_{k+1}] \cup S[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2$ is a path from the origin to $\partial\mathcal{B}(2^{k+1})$,

$$\overline{S}^1(\sigma) \in \mathcal{B}(2^{\frac{m}{3}} + 2^{\frac{11m}{60}+1})$$

and

$$\mathrm{dist}(\overline{S}^1(\sigma), S^2[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) \le 2^{\frac{11m}{60}+1},$$

by using Proposition 2.6, we see that

$$P^{\overline{S}^1(\sigma)}(\overline{S}^1[0, \tau^1_{k+1}] \cap (\overline{S}^2[0, \tau^2_{k+1}] \cup S[0, \tau^2_{\frac{m}{3}}] \cup \gamma^2) = \emptyset) \le c 2^{-\delta k},$$

for some $\delta > 0$ and $c < \infty$. Hence (3.53) is bounded above by

$$c2^{-\delta k} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[0, \tau^1_{k-2}] \cap \overline{S}^2_{k-2} = \emptyset) \le c2^{-\delta k} 2^{-(k-\frac{m}{3})\xi},$$

and the proof for $d = 2$ is finished.

Next we estimate for $d = 3$. Recall the events F_m, G_m and H_m in (3.13). By (3.52), we need to estimate

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[\tau^1(2^{k-1} - 2^{\frac{31k}{60}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap 2\text{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},k+1})$$

on the event $F_m \cap G_m \cap H_m$. For this end, we decompose 2PATH^2_f into three parts U_1, U_2 and U_3 as follows.

$$\begin{aligned}
U_1 &= \{z \in \mathbb{R}^d : \text{dist}(z, \gamma^2) \le 2^{\frac{11m}{60}+1}\} \\
U_2 &= \{z \in \mathbb{R}^d : \text{dist}(z, S^2[0, \tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})]) \le 2^{\frac{11m}{60}+1}\} \\
U_3 &= \{z \in \mathbb{R}^d : \text{dist}(z, S^2[\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}), \tau^2_{\frac{m}{3}}]) \le 2^{\frac{11m}{60}+1}\}
\end{aligned}$$

Since $\gamma^2 \in \mathcal{B}(2^L)$ and $L \le \frac{m}{10}$, it is easy to see that

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[\tau^1(2^{k-1} - 2^{\frac{31k}{60}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap U_1 \neq \emptyset, R_{\frac{m}{3},k+1}) \le c2^{-\delta k} 2^{-(k-\frac{m}{3})\xi},$$

for some $\delta > 0$ and $c < \infty$. Since $S^2[\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}), \tau^2_{\frac{m}{3}}] \subset \mathcal{B}(S^2(\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})), 2^{\frac{m}{4}})$ on the event G_m, we see that $U_3 \subset \mathcal{B}(S^2(\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})), 2^{\frac{m}{4}+1})$. Therefore $\overline{S}^1[\tau^1((2^{k-1} - 2^{\frac{31k}{60}}) \vee 2^{\frac{m}{3}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap U_3 \neq \emptyset$ implies that

$$\overline{S}^1[\tau^1((2^{k-1} - 2^{\frac{31k}{60}}) \vee 2^{\frac{m}{3}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap \mathcal{B}(S^2(\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}})), 2^{\frac{m}{4}+1}). \tag{3.54}$$

However,

$$|\overline{S}^1(\tau^1((2^{k-1} - 2^{\frac{31k}{60}}) \vee 2^{\frac{m}{3}}) - S^2(\tau^2(2^{\frac{m}{3}} - 2^{\frac{2m}{9}}))| \ge 2^{\frac{0.99m}{3}},$$

on the event F_m. So the probability of (3.54) is bounded above by $c2^{-\frac{m}{24}}$ for some $c < \infty$. Using the strong Markov property,

$$P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[\tau^1((2^{k-1} - 2^{\frac{31k}{60}}) \vee 2^{\frac{m}{3}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap U_3 \neq \emptyset, R_{\frac{m}{3},k+1}) \le c2^{-\delta k} 2^{-(k-\frac{m}{3})\xi},$$

for some $\delta > 0$ and $c < \infty$. Finally we consider U_2. Let

$$\sigma^\sharp = \inf\{j \ge \tau^1(2^{k-1} - 2^{\frac{31k}{60}}) : \overline{S}^1(j) \in U_2\}.$$

Then by the strong Markov property,

$$\begin{aligned}
&P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(\overline{S}^1[\tau^1(2^{k-1} - 2^{\frac{31k}{60}}), \tau^1(2^k + 2^{\frac{31k}{60}})] \cap U_2 \neq \emptyset, R_{\frac{m}{3},k+1}) \\
&\le E_2^{w^2_{m/3}}\big(E_1^{w^1_{m/3}}\big(\mathbf{1}\{R_{\frac{m}{3},k-2}, \sigma^\sharp \le \tau^1(2^k + 2^{\frac{31k}{60}})\} \\
&\times P_1^{\overline{S}^1(\sigma^\sharp)}(\overline{S}^1[0, \tau^1_{k+1}] \cap S^2[0, \tau^2_{\frac{m}{3}}] = \emptyset)\big)\big).
\end{aligned} \tag{3.55}$$

Note that $P_1^{\overline{S}^1(\sigma^\sharp)}(\overline{S}^1[0, \tau^1_{k+1}] \cap S^2[0, \tau^2_{\frac{m}{3}}] = \emptyset) \le Z^2_m$. Hence on the event H_m, the right hand side of (3.55) can be bounded above by $c2^{-\delta k} 2^{-(k-\frac{m}{3})\xi}$ for some $\delta > 0$ and $c < \infty$, and the lemma is proved.

□

3.3.8 Bounds for $k = N-2, N-1$, and $k = N$

Again, we will only consider the case that $k = N$, as in Section 3.3.3. Other cases can be proved similarly.

Lemma 3.18. *There exist $\delta > 0$ and $c < \infty$ such that*

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_{N-1},T^1_N] \cap (B^2[0,T^2_N] \cup \mathit{PATH}^2_f) \neq \emptyset, R_{\frac{m}{3},N}) \leq c2^{-(N-\frac{m}{3})\xi}2^{-\delta N}. \quad (3.56)$$

Proof. We will sketch the proof. First we consider the following probability,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_{N-1},T^1_N] \cap \mathrm{PATH}^2_f \neq \emptyset, R_{\frac{m}{3},N}). \quad (3.57)$$

The probability that B^1 enters $\mathcal{B}_{\frac{m}{3}}$ during $[T^1_{N-1},T^1_N]$ is bounded above by $c2^{-(N-\frac{m}{3})}$ for $d = 3$. Therefore by using the strong Markov property, we see that (3.57) can be bounded above by $c2^{-(N-\frac{m}{3})}2^{-(N-\frac{m}{3})\xi}$. Since $N \geq m$, we have $2^{-(N-\frac{m}{3})}2^{-(N-\frac{m}{3})\xi} \leq 2^{-\frac{N}{2}}2^{-(N-\frac{m}{3})\xi}$.

For $d = 2$, we use Proposition 2.6 as follows. Assume B^1 enters $\mathcal{B}_{\frac{m}{3}}$ during $[T^1_{N-1},T^1_N]$. Then by a similar argument given in the proof of Lemma 3.8, $\overline{S}^1$ also enters $\mathcal{B}(2^{\frac{2N}{3}})$ during $[\tau^1_{N-1},\tau^1_N]$ with probability greater than $1-2^{-\frac{N}{6}}$. After $\overline{S}^1$ enters $\mathcal{B}(2^{\frac{2N}{3}})$, it follows from Proposition 2.6 that the probability that $\overline{S}^1$ does not intersect $\overline{S}^2[0,\tau^2_N]$ until it reaches $\partial\mathcal{B}(2^N)$ is bounded above by $c2^{-\frac{N}{6}}$. Combining these estimate, we see that (3.57) can be bounded above by $c2^{-\frac{N}{6}}2^{-(N-\frac{m}{3})\xi}$ for $d = 2$.

Therefore, in order to show (3.56), we need to estimate the following probability,

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_{N-1},T^1_N] \cap B^2[0,T^2_N] \neq \emptyset, R_{\frac{m}{3},N}). \quad (3.58)$$

By similar arguments as in Lemma 3.10 and Remark 3.11, we have

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1_{N-1},T^1_N] \cap B^2[T^2(2^N-2^{\frac{2N}{3}}),T^2_N] \neq \emptyset, R_{\frac{m}{3},N}) \leq c2^{-\delta N}2^{-(N-\frac{m}{3})\xi}$$
$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(B^1[T^1(2^N-2^{\frac{2N}{3}}),T^1_N] \cap B^2[0,T^2_N] \neq \emptyset, R_{\frac{m}{3},N}) \leq c2^{-\delta N}2^{-(N-\frac{m}{3})\xi},$$

for some $\delta > 0$ and $c < \infty$. So, assume $B^1[T^1_{N-1},T^1(2^N-2^{\frac{2N}{3}})] \cap B^2[0,T^2(2^N-2^{\frac{2N}{3}})] \neq \emptyset$. Then by Proposition 2.4,

$$\mathrm{dist}\big(\overline{S}^1[\tau^1(2^{N-1}-2^{\frac{31N}{60}}),\tau^1(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})],\overline{S}^2[0,\tau^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})]\big) \leq 2^{\frac{31N}{60}}, \quad (3.59)$$

with probability greater than $1-\exp(-2^{\delta N})$. Let

$$\overline{U} = \Big\{\mathrm{dist}\big(\overline{S}^1[\tau^1(2^{N-1}-2^{\frac{31N}{60}}),\tau^1(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})],\overline{S}^2[0,\tau^2(2^N-2^{\frac{2N}{3}}+2^{\frac{31N}{60}})]\big) \leq 2^{\frac{31N}{60}}\Big\},$$

then by modifying the proof of Lemma 3.10, we see that

$$P_{1,2}^{w^1_{m/3},w^2_{m/3}}(\overline{U} \cap R_{\frac{m}{3},N}) \leq c2^{-\delta N}2^{-(N-\frac{m}{3})\xi},$$

which gives the proof of the lemma.

□

3.3.9 Conclusion Upper Bound

Combining estimates obtained in subsections 3.3.6, 3.3.7 and 3.3.8 with (3.45) and Proposition 3.13, we have the following.

Proposition 3.19. *There exist $\delta > 0$ and $c < \infty$ such that*

$$|P_{1,2}^{w^1_{m/3},w^2_{m/3}}(R_{\frac{m}{3},N}) - P_{1,2}^{w^1_{m/3},w^2_{m/3}}(J_{m,N})| \leq c2^{-\delta m}2^{-(N-\frac{m}{3})\xi}, \quad (3.60)$$

on the event $F_m \cap G_m \cap H_m$.

4 Proof of Main Theorem

4.1 Cauchy sequence

Fix $L \in \mathbb{N}$ and $\overline{\gamma} \in \Gamma_L$. For $m \geq 10L$, define

$$Q(m, \overline{\gamma}) = 2^{(m-L)\xi} P(A_m(\overline{\gamma})). \tag{4.1}$$

We will show the following Theorem.

Theorem 4.1. *There exist $\delta > 0$ and $c < \infty$ depending only on the dimension such that for all $n \geq m \geq 10L$, we have*

$$|Q(m, \overline{\gamma}) - Q(n, \overline{\gamma})| \leq c2^{-\delta\sqrt{m}}, \quad \text{for } d = 2, \tag{4.2}$$

$$|Q(m, \overline{\gamma}) - Q(n, \overline{\gamma})| \leq c2^{-\delta m}, \quad \text{for } d = 3. \tag{4.3}$$

Proof. Fix $L \in \mathbb{N}$, $\overline{\gamma} \in \Gamma_L$ and $n \geq m \geq 10L$. Then by (3.7), (3.9) and (3.12), we have

$$\begin{aligned}
&|Q(m, \overline{\gamma}) - Q(n, \overline{\gamma})| \\
&\leq |2^{(m-L)\xi} P(A_m(\overline{\gamma}) \cap F_m \cap G_m \cap H_m) - 2^{(n-L)\xi} P(A_n(\overline{\gamma}) \cap F_m \cap G_m \cap H_m)| + c2^{-\delta m},
\end{aligned}$$

for some $\delta > 0$ and $c < \infty$. By the strong Markov property,

$$\begin{aligned}
&|2^{(m-L)\xi} P(A_m(\overline{\gamma}) \cap F_m \cap G_m \cap H_m) - 2^{(n-L)\xi} P(A_n(\overline{\gamma}) \cap F_m \cap G_m \cap H_m)| \\
&= |2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{A_{\frac{m}{3}}(\overline{\gamma}) \cap F_m \cap G_m \cap H_m\} 2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},m})\big) \\
&- 2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{A_{\frac{m}{3}}(\overline{\gamma}) \cap F_m \cap G_m \cap H_m\} 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},n})\big)|
\end{aligned} \tag{4.4}$$

By Proposition 3.19, we have

$$\begin{aligned}
|P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},m})\big) - P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m})| &\leq c2^{-\delta m} 2^{-\frac{2m}{3}\xi} \\
|P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},n}) - P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})| &\leq c2^{-\delta m} 2^{-(n-\frac{m}{3})\xi},
\end{aligned}$$

on the event $F_m \cap G_m \cap H_m$. Therefore, the right hand side of (4.4) is bounded above by

$$\begin{aligned}
&2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{V^m\} 2^{\frac{2m}{3}\xi} |P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},m}) - P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m})|\big) \\
&+ 2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{V^m\} |2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m}) - 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})|\big) \\
&+ 2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{V^m\} 2^{(n-\frac{m}{3})\xi} |P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(R_{\frac{m}{3},n}) - P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})|\big) \\
&\leq 2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{V^m\} |2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m}) - 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})|\big) \\
&+ c2^{-\delta m} 2^{(\frac{m}{3}-L)\xi} P(V^m),
\end{aligned}$$

where $V^m = A_{\frac{m}{3}}(\overline{\gamma}) \cap F_m \cap G_m \cap H_m$. By Proposition 2.9,

$$|2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m}) - 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})| \leq c2^{-\delta\sqrt{m}},$$

for $d = 2$ and

$$|2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m}) - 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})| \leq c2^{-\delta m},$$

for $d = 3$ on the event V^m. Hence

$$\begin{aligned}
&2^{(\frac{m}{3}-L)\xi} E\big(\mathbf{1}\{V^m\} |2^{\frac{2m}{3}\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,m}) - 2^{(n-\frac{m}{3})\xi} P_{1,2}^{w^1_{m/3}, w^2_{m/3}}(J_{m,n})|\big) \\
&\leq c2^{-\delta m^{\frac{d}{2}-\frac{1}{2}}} 2^{(\frac{m}{3}-L)\xi} P(V^m).
\end{aligned}$$

Finally, by the strong Markov property,

$$P(V^m) \le P(A_{L+1}(\overline{\gamma}))c2^{-(\frac{m}{3}-L)\xi},$$

and the proof is finished.

□

From Theorem 4.1, we get the following corollary immediately.

Corollary 4.2. *There exist $\delta > 0$ and $c < \infty$ such that the following holds. For each $L \in \mathbb{N}$ and $\overline{\gamma} \in \Gamma_L$, there exists $Q(\overline{\gamma}) \in (0,1)$ such that*

$$\lim_{m\to\infty} Q(m,\overline{\gamma}) = Q(\overline{\gamma}) \tag{4.5}$$

$$|Q(m,\overline{\gamma}) - Q(\overline{\gamma})| \le c2^{-\delta m^{\frac{d}{2}-\frac{1}{2}}}. \tag{4.6}$$

Especially, there exists a $\alpha \in (0,1)$ such that

$$\lim_{n\to\infty} P\big((S^1[0,\tau_n^1], S^2[0,\tau_n^2]) \in \Gamma_n\big)2^{n\xi} = \alpha \tag{4.7}$$

$$|P\big((S^1[0,\tau_n^1], S^2[0,\tau_n^2]) \in \Gamma_n\big)2^{n\xi} - \alpha| \le c2^{-\delta n^{\frac{d}{2}-\frac{1}{2}}}. \tag{4.8}$$

Corollary 4.3. *There exist $\delta > 0$ and $c < \infty$ such that the following holds. For each $L \in \mathbb{N}$ and $\overline{\gamma} \in \Gamma_L$, the limit*

$$\lim_{N\to\infty} P((S^1[0,\tau_L^1], S^2[0,\tau_L^2]) = \overline{\gamma} \mid (S^1[0,\tau_N^1], S^2[0,\tau_N^2]) \in \Gamma_N) \tag{4.9}$$

exists. If we denote the limit by $P^\sharp(\overline{\gamma})$, then

$$|P((S^1[0,\tau_L^1], S^2[0,\tau_L^2]) = \overline{\gamma} \mid (S^1[0,\tau_N^1], S^2[0,\tau_N^2]) \in \Gamma_N) - P^\sharp(\overline{\gamma})| \le c2^{-\delta N^{\frac{d}{2}-\frac{1}{2}}}. \tag{4.10}$$

Proof. Fix $L \in \mathbb{N}$ and $\overline{\gamma} \in \Gamma_L$. Let

$$p(\overline{\gamma}) = P((S^1[0,\tau_L^1], S^2[0,\tau_L^2]) = \overline{\gamma}).$$

Then

$$\begin{aligned} &P((S^1[0,\tau_L^1], S^2[0,\tau_L^2]) = \overline{\gamma} \mid (S^1[0,\tau_N^1], S^2[0,\tau_N^2]) \in \Gamma_N) \\ &= \frac{p(\overline{\gamma})P(A_N(\overline{\gamma}))}{P((S^1[0,\tau_N^1], S^2[0,\tau_N^2]) \in \Gamma_N)} \end{aligned}$$

By Corollary 4.2, if we let $P^\sharp(\overline{\gamma})$ be

$$\frac{p(\overline{\gamma})2^{L\xi}Q(\overline{\gamma})}{\alpha},$$

the proof is finished. □

Remark 4.4. *In order to simplify the notations, all results above were stated for the first hitting time of $\partial\mathcal{B}(2^N)$ instead of $\partial\mathcal{B}(N)$. However there is no essential difference between them and similar arguments also work for the latter case. Since it is easy to extend above results to the hitting time of $\partial\mathcal{B}(N)$, we leave the details to the reader.*

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