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Multitype Contact Process on \mathbb{Z} : Extinction and Interface

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Abstract

We consider a two-type contact process on \mathbb{Z} in which both types have equal finite range and supercritical infection rate. We show that a given type becomes extinct with probability 1 if and only if, in the initial configuration, it is confined to a finite interval $[-L, L]$ and the other type occupies infinitely many sites both in $(-\infty, L)$ and (L, ∞) . Additionally, we show that if both types are present in finite number in the initial configuration, then there is a positive probability that they are both present for all times. Finally, it is shown that, starting from the configuration in which all sites in $(-\infty, 0]$ are occupied by type 1 particles and all sites in $(0, \infty)$ are occupied by type 2 particles, the process ρ_t defined by the size of the interface area between the two types at time t is tight.

Key words: Interacting Particle Systems, Interfaces, Multitype Contact Process.

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1 Introduction

The contact process on \mathbb{Z} is the spin system with generator

$$\Omega f(\zeta) = \sum_x (f(\zeta^x) - f(\zeta)) c(x, \zeta); \quad \zeta \in \{0, 1\}^{\mathbb{Z}}$$

where

$$\begin{cases} \zeta^x(y) = \zeta(y) & \text{if } x \neq y; \\ \zeta^x(x) = 1 - \zeta(x); \end{cases} \quad c(x, \zeta) = \begin{cases} 1 & \text{if } \zeta(x) = 1; \\ \lambda \sum_y \zeta(y) \cdot p(y - x) & \text{if } \zeta(x) = 0; \end{cases}$$

for $\lambda > 0$ and $p(\cdot)$ a probability kernel. We take p to be symmetric and to have finite range $R = \max\{x : p(x) > 0\}$.

The contact process is usually taken as a model for the spread of an infection; configuration $\zeta \in \{0, 1\}^{\mathbb{Z}}$ is the state in which an infection is present at $x \in \mathbb{Z}$ if and only if $\zeta(x) = 1$. With this in mind, the dynamics may be interpreted as follows: each infected site waits an exponential time of parameter 1, after which it heals, and additionally each infected site waits an exponential time of parameter λ , after which it chooses, according to the kernel p , some other site to which the infection is transmitted if not already present.

We refer the reader to [13] for a complete account of the contact process. Here we mention only the most fundamental fact. Let $\bar{\zeta}$ and $\mathbf{0}$ be the configurations identically equal to 1 and 0, respectively, $S(t)$ the semi-group associated to Ω , \mathbb{P}_λ the probability measure under which the process has rate λ and ζ_t^0 the configuration at time t , started from the configuration where only the origin is infected. There exists λ_c , depending on p , such that

- if $\lambda \leq \lambda_c$, then $\mathbb{P}_\lambda(\zeta_t^0 \neq \mathbf{0} \forall t) = 0$ and $\delta_{\bar{\zeta}} S(t) \rightarrow \delta_{\mathbf{0}}$;
- if $\lambda > \lambda_c$, then $\mathbb{P}_\lambda(\zeta_t^0 \neq \mathbf{0} \forall t) > 0$ and $\delta_{\bar{\zeta}} S(t)$ converges, as $t \rightarrow \infty$, to some non-trivial invariant measure.

Again, see [13] for the proof. Throughout this paper, we fix $\lambda > \lambda_c$.

The multitype contact process was introduced in [15] as a modification of the above system. Here we consider a two-type contact process, defined as the particle system $(\xi_t)_{t \geq 0}$ with state space $\{0, 1, 2\}^{\mathbb{Z}}$ and generator

$$\Lambda f(\xi) = \sum_{x: \xi(x) \neq 0} (f(\xi^{x,0}) - f(\xi)) + \sum_{x: \xi(x) = 0} [(f(\xi^{x,1}) - f(\xi)) c_1(x, \xi) + (f(\xi^{x,2}) - f(\xi)) c_2(x, \xi)]; \quad \xi \in \{0, 1, 2\}^{\mathbb{Z}},$$

where

$$\begin{cases} \xi^{x,i}(y) = \xi(y) & \text{if } x \neq y; \\ \xi^{x,i}(x) = i, \\ i = 0, 1, 2; \end{cases} \quad c_i(x, \xi) = \lambda \sum_y \mathbb{1}_{\{\xi(y)=i\}} \cdot p(y - x), \quad i = 1, 2.$$

($\mathbb{1}$ denotes indicator function).

This is thought of as a model for competition of two biological species. Each site in \mathbb{Z} corresponds to a region of space, which can be either empty or occupied by an individual of one of the two species. Occupied regions become empty at rate 1, meaning natural death of the occupant, and

empty regions become occupied at a rate that depends on the number of individuals of each species living in neighboring sites, and this means a new birth. The important point is that occupancy is strong in the sense that, if a site has an individual of, say, type 1, the only way it will later contain an individual of type 2 is if the current individual dies and a new birth occurs originated from a type 2 individual.

Let us point out some properties of the above dynamics. First, it is symmetric for the two species: both die and give birth following the same rules and restrictions. Second, if only one of the two species is present in the initial configuration, then the process evolves exactly like in the one-type contact process. Third, if we only distinguish occupied sites from non-occupied ones, thus ignoring which of the two types is present at each site, again we see the evolution of the one-type contact process. We also point out that both the contact and the multitype contact processes can be constructed with families of Poisson processes which are interpreted as *transmissions* and *healings*. These families are called graphical constructions, or Harris constructions. Although we will provide formal definitions in the next section, we will implicitly adopt the terminology of Harris constructions in the rest of this Introduction.

The first question we address is: for which initial configurations does a given type (say, type 1) become extinct with probability one? By extinction we mean: for some time t_0 (and hence all $t \geq t_0$), $\xi_{t_0}(x) \neq 1$ for all x . We prove

Theorem 1.1. *Assume at least one site is occupied by a 1 in ξ_0 . The 1's become extinct with probability one if and only if there exists $L > 0$ such that*

- (A) $\xi_0(x) \neq 1 \forall x \notin [-L, L]$ and
- (B) $\#\{x \in (-\infty, -L] : \xi_0(x) = 2\} = \#\{x \in [L, \infty) : \xi_0(x) = 2\} = \infty$.

(# denotes cardinality). This result is a generalization of Theorem 1.1. in [1], which is the exact same statement in the nearest neighbour context (*i.e.*, $p(1) = p(-1) = 1/2$). Although there are some points in common between our proof and the one in that work, our general approach is completely different. Additionally, their methods do not readily apply in our setting, as we now briefly explain. Let $\mathcal{L} = \{\xi \in \{0, 1, 2\}^{\mathbb{Z}} : \xi(x) = 1, \xi(y) = 2 \implies x < y\}$. When the range $R = 1$, $\xi_0 \in \mathcal{L}$ implies $\xi_t \in \mathcal{L}$ for all $t \geq 0$. In [1], the proof of both directions of the equivalence of Theorem 1.1 rely on this fact (see for example Corollary 3.1 in that paper), which does not hold for $R > 1$.

If both types are present in finite number in the initial configuration, there is obviously positive probability that one of them is present for all times, because the process is supercritical. The following theorem says that there is positive probability that they are *both* present for all times.

Theorem 1.2. *Assume that*

$$0 < \#\{x : \xi_0(x) = 1\}, \#\{x : \xi_0(x) = 2\} < \infty.$$

Then, with positive probability, for all $t \geq 0$ there exist $x, y \in \mathbb{Z}$ such that $\xi_t(x) = 1, \xi_t(y) = 2$.

Now assume that $R > 1$. Define the “heaviside” configuration as $\xi^h = \mathbb{1}_{(-\infty, 0]} + 2 \cdot \mathbb{1}_{(0, \infty)}$ and denote by ξ_t the two-type contact process with initial condition $\xi_0 = \xi^h$. Define

$$r_t = \sup\{x : \xi_t(x) = 1\}, \quad l_t = \inf\{x : \xi_t(x) = 2\}, \quad \rho_t = r_t - l_t.$$

We have $\rho_0 = -1$, and at a given time t both events $\{\rho_t > 0\}$ and $\{\rho_t < 0\}$ have positive probability. If $\rho_t > 0$, we call the interval $[l_t, r_t]$ the *interface area*. The question we want to ask is: if t is large, is it reasonable to expect a large interface? We answer this question negatively.

Theorem 1.3. *The law of $(\rho_t)_{t \geq 0}$ is tight; that is, for any $\epsilon > 0$, there exists $L > 0$ such that $\mathbb{P}(|\rho_t| > L) < \epsilon$ for every $t \geq 0$.*

There are several works concerning interface tightness in one-dimensional particle systems, the first of which is [5], where interface tightness is established for the voter model. Others are [3], [4], [17] and [2].

In [2], it is shown that interface tightness also occurs on another variant of the contact process, namely the grass-bushes-trees model considered in [8], with both species having same infection rate and non-nearest neighbor interaction. The difference between the grass-bushes-trees model and the multitype contact process considered here is that, in the former, one of the two species, say the 1's, is privileged in the sense that it is allowed to invade sites occupied by the 2's. For this reason, from the point of view of the 1's, the presence of the 2's is irrelevant. It is thus possible to restrict attention to the evolution of the 1's, and it is shown that they form barriers that prevent entrance from outside; with this at hand, interface tightness is guaranteed regardless of the evolution of the 2's. Here, however, we do not have this advantage, since we cannot study the evolution of any of the species while ignoring the other.

Our results depend on a careful examination of the temporal dual process; that is, rather than moving forward in time and following the descendency of individuals, we move backwards in time and trace ancestries. The dual of the multitype contact process was first studied by Neuhauser in [15] and may be briefly described as follows. Each site $x \in \mathbb{Z}$ at (primal) time s has a random (and possibly empty) *ancestor sequence*, which is a list of sites $y \in \mathbb{Z}$ such that the presence of an infection in $(y, 0)$ would imply the presence of an infection in (x, s) (in other words, such that there is an infection path connecting $(y, 0)$ and (x, s)). The ancestors on the list are ranked in decreasing order; the idea is that if the first ancestor is not occupied in ξ_0 , then we look at the second, and so on, until we find the first on the list that is occupied in ξ_0 , and take its type as the one passed to x . We denote this sequence $(\eta_{1,s}^x, \eta_{2,s}^x, \dots)$. By moving in time in the opposite direction as that of the original process and using the graphical representation of the contact process for “negative” primal times, we can define the *ancestry process of x* , $((\eta_{1,t}^x, \eta_{2,t}^x, \dots))_{t \geq 0}$. The process given by the first element of the sequence, $(\eta_{1,t}^x)_{t \geq 0}$, is called the *first ancestor process*. We point out three key properties of the ancestry process:

- **First ancestor processes have embedded random walks.** In [15] it is proven that, on the event that a site x has a nonempty ancestry at all times $t \geq 0$, we can define an increasing sequence of random *renewal times* $(\tau_n^x)_{n \geq 0}$ with the property that the space-time increments $(\eta_{1, \tau_{n+1}^x}^x - \eta_{1, \tau_n^x}^x, \tau_{n+1}^x - \tau_n^x)$ are independent and identically distributed. This fact enormously simplifies the study of the first ancestor process, which is not markovian and at first seems very complicated.
- **Ancestries coalesce.** If we are to use the dual process to obtain information about the joint distribution of the states at sites x and y at a given time, we must study the joint behavior of two ancestry processes, specially of two first ancestor processes. The intuitive picture is that this behavior resembles that of two random walks that are independent until they meet, at

which time they coalesce. We give a new approach to formalizing this notion, one that we believe provides a clear understanding of the picture and allows for detailed results.

In order to follow two first ancestor processes simultaneously, we define *joint renewals* $(\tau_n^{x,y})_{n \geq 0}$ and argue that the law of the processes after a joint renewal only depends on their initial difference at the instant of the renewal. Thus, the discrete-time process defined by the difference between the two processes at the instants of renewals is a Markov chain on \mathbb{Z} . For this chain, zero is an absorbing state and corresponds to coalescence of first ancestors. We also show that, far from the origin, the transition probabilities of the chain become close to a symmetric measure on \mathbb{Z} , and from this fact we are able to show that the tail of the distribution of the hitting time of 0 for the chain looks like the one associated to a simple random walk on \mathbb{Z} . From this construction and estimate we also bound the expected distance between ancestors at a given time.

- **Ancestries become sparse with time.** Consider the system of coalescing random walks in which each site of \mathbb{Z} starts with one particle at time 0. The density of occupied sites at time t , which is equal to the probability of the origin being occupied, tends to 0 as $t \rightarrow \infty$. We prove a similar result for our ancestry sequences. Fix a *truncation level* N and, at dual time t , mark the N first ancestors of each site at dual time 0 (this gives the set $\{\eta_{n,t}^x : 1 \leq n \leq N, x \in \mathbb{Z} : \text{the ancestry of } x \text{ reaches time } t\}$). We show that the density of this random set tends to 0 as $t \rightarrow \infty$, and estimate the speed of this convergence depending on N .

From this last fact, we can immediately prove Theorem 1.1 under the stronger hypothesis that all sites outside $[-L, L]$ are occupied by 2's in ξ_0 . To obtain the general case, we then use a structure called a *descendancy barrier*, whose existence was established in [2]. Theorem 1.2 is obtained quite easily from Theorem 1.1. The proof of Theorem 1.3 is more intricate, and follows the main steps of [5], which studies the voter model.

We believe that our results and general approach may prove useful in other questions concerning the multitype contact process, in particular those that relate to almost sure properties of the trajectories $t \mapsto \xi_t$ of the process, as opposed to properties of its limit measures.

2 Ancestry process

We will start describing the familiar construction of the one-type contact process from its graphical representation. We will then show how the same representation can be used to construct the multitype contact process, present the definition of the ancestry process together with some facts from [15], and finally prove a simple lemma.

Suppose given a collection of independent Poisson processes on $[0, \infty)$:

$$(D^x)_{x \in \mathbb{Z}} \text{ with rate } 1, \quad (N^{(x,y)})_{x,y \in \mathbb{Z}} \text{ with rate } \lambda \cdot p(y-x).$$

A *Harris construction* H is a realization of all such processes. H can thus be understood as a point measure on $(\mathbb{Z} \cup \mathbb{Z}^2) \times [0, \infty)$. Sometimes we abuse notation and denote the collection of processes itself by H . Given $(x, t) \in \mathbb{Z} \times [0, \infty)$, let $\theta(x, t)(H)$ be the Harris construction obtained by shifting H so that (x, t) becomes the space-time origin. By translation invariance of the space-time construction, $\theta(x, t)(H)$ and H have the same distribution. We will also write $H_{[0,t]}$ to denote the restriction of H to $\mathbb{Z} \times [0, t]$, and refer to such restrictions as finite-time Harris constructions.

Given a Harris construction H and $(x, s), (y, t) \in \mathbb{Z} \times [0, \infty)$ with $s < t$, we write $(x, s) \leftrightarrow (y, t)$ (in H) if there exists a piecewise constant, right-continuous function $\gamma : [s, t] \rightarrow \mathbb{Z}$ such that

- $\gamma(s) = x, \gamma(t) = y$;
- $\gamma(r) \neq \gamma(r-)$ if and only if $r \in N^{(\gamma(r-), \gamma(r))}$;
- $\nexists s \leq r \leq t$ with $r \in D^{\gamma(r)}$.

One such function γ is called a *path* determined by H . The points in the processes $\{D^x\}$ are usually called *death marks*, and the points in $\{N^{(x,y)}\}$ are called *arrows*. Thus, a path can be thought of as a line going up from (x, s) to (y, t) following the arrows and not crossing any death marks.

Given $A \subset \mathbb{Z}, (x, t) \in \mathbb{Z} \times [0, \infty)$ and a Harris construction H , put

$$[\zeta_t^A(x)](H) = \mathbb{1}_{\{\text{For some } y \in A, (y, 0) \leftrightarrow (x, t) \text{ in } H\}}.$$

Under the law of H , (ζ_t^A) has the distribution of the contact process with parameter λ , kernel p and initial state $\mathbb{1}_A$; see [6] for details. From now on, we omit dependency on the Harris construction and write (for instance) ζ_t instead of $\zeta_t(H)$.

Before going into the multitype contact process, we list some properties of the one-type contact process that will be very useful. Fix $(x, s) \in \mathbb{Z} \times [0, \infty)$ and $t > s$. Define the time of death and maximal distance traveled until time t for an infection that starts at (x, s) ,

$$T^{(x,s)} = \inf\{s' > s : \nexists y : (x, s) \leftrightarrow (y, s')\},$$

$$M_t^{(x,s)} = \sup\{|y - x| : (x, s) \leftrightarrow (y, s') \text{ for some } s' \in [s, t]\}$$

(these only depend on H and are thus well-defined regardless of $\xi_s(x)$). When $s = 0$, we omit it and write T^x, M_t^x . If $A \subset \mathbb{Z}$, we also define $T^A = \inf\{t \geq 0 : \nexists x \in A, y \in \mathbb{Z} : (x, 0) \leftrightarrow (y, t)\}$. We start by observing that $M_t^{(x,s)}$ is stochastically dominated by a multiple of a Poisson random variable, so there exist $\kappa, c, C > 0$ such that

$$\mathbb{P}(M_t^x > \kappa t) \leq C e^{-ct} \quad \forall x \in \mathbb{Z}, t \geq 0. \quad (2.1)$$

Remark 2.1. *Throughout this paper, letters that denote universal constants whose particular values are irrelevant, such as c, C, κ in the above, will have different values in different contexts and will sometimes change from line to line in equations.*

Next, since we are taking $\lambda > \lambda_c$, we have $\mathbb{P}(T^x = \infty) = \mathbb{P}(T^0 = \infty) > 0$ for all x , and

$$\mathbb{P}(T^x = T^y = \infty) \geq \mathbb{P}(T^0 = \infty)^2 > 0, \quad \forall x, y \in \mathbb{Z}. \quad (2.2)$$

This follows from the self-duality of the contact process and the fact that its upper invariant measure has positive correlations; see [12]. Our last property is that there exist $c, C > 0$ such that, for any $A \subset \mathbb{Z}$ and $t > 0$,

$$\mathbb{P}(t < T^A < \infty) \leq C e^{-ct}. \quad (2.3)$$

For the case $R = 1$, this is Theorem 2.30 in [13]. The proof uses a comparison with oriented percolation and can be easily adapted to the case $R > 1$.

We now turn to our treatment of the multitype contact process and its dual, the ancestry process, through graphical constructions. We will proceed in three steps.

Step 1: Graphical construction of the multitype contact process. The construction is done as for the one-type contact process, with the difference that we must ignore the arrows whose endpoints are

already occupied. This was first done in [15]; there, an algorithmic procedure is provided to find the state of each site at a given time. Here we provide an approach that is formally different but amounts to the same. Fix $(x, t) \in \mathbb{Z} \times [0, \infty)$, a Harris construction H and $\xi_0 \in \{0, 1, 2\}^{\mathbb{Z}}$. Let Γ_1^x be the set of paths γ that connect points of $\mathbb{Z} \times \{0\}$ to (x, t) in H . Assume that $\#\Gamma_1^x < \infty$; this happens with probability one if H is sampled from the processes described above. For the moment, also assume that $\Gamma_1^x \neq \emptyset$. Given $\gamma, \gamma' \in \Gamma_1^x$, let us write $\gamma < \gamma'$ if there exists $\bar{s} \in (0, t)$ such that $\gamma(s) = \gamma'(s) \forall s \in [\bar{s}, t]$ and $\gamma(\bar{s}) \neq \gamma(\bar{s}-), \gamma'(\bar{s}) = \gamma'(\bar{s}-)$. From the fact that these paths are all piecewise constant, have finitely many jumps and the same endpoint, we deduce that $<$ is a total order on Γ_1^x . We can then find γ_1^* , the maximal path in Γ_1^x . Now define $\Gamma_2^x = \{\gamma \in \Gamma_1^x : \gamma(0) \neq \gamma_1^*(0)\}$ and γ_2^* as the maximal path in Γ_2^x . Then define $\Gamma_3^x = \{\gamma \in \Gamma_2^x : \gamma(0) \neq \gamma_2^*(0)\}$, and so on, until $\Gamma_N^x = \emptyset$. For $1 \leq n < N$, denote $\xi_{n,t}^x = \gamma_n^*(0)$, and for $n \geq N$ put $\xi_{n,t}^x = \Delta$. We claim that

$$\forall n < N, \forall s \text{ such that } \gamma_n^*(s-) \neq \gamma_n^*(s), \text{ we have } \gamma_n^*(s) \notin \zeta_{s-}^{\{\xi_{n+1,t}^x, \dots, \xi_{N-1,t}^x\}} \quad (2.4)$$

(Here ζ continues to denote the one-type contact process defined from H). In words, if γ_n^* makes a jump that lands on a space-time point (x, s) , then for some positive ϵ the set $\{x\} \times [s - \epsilon, s)$ cannot be reached by paths coming from $\xi_{n+1,t}^x, \dots, \xi_{N-1,t}^x$, and so the jump is not obstructed. If this were not the case, we could obtain $m < n, s \in [0, t]$ and γ with $\gamma(0) = \xi_{m,t}^x$ and $\gamma_m^*(s-) \neq \gamma_m^*(s) = \gamma(s) = \gamma(s-)$. But we could then construct a path γ' coinciding with γ on $[0, s]$ and with γ_m^* on $(s, t]$, and γ' would contradict the maximality that defined γ_m^* .

If $\xi_0(\xi_{n,t}^x) = 0$ for all $n < N$, put $\xi_t(x) = 0$. Otherwise, if $k = \min\{n : \xi_0(\xi_{n,t}^x) \neq 0\}$, put $\xi_t(x) = \xi_0(\xi_{k,t}^x)$. In this second case, using (2.4), we see that there is a path connecting $(\xi_{k,t}^x, 0)$ to (x, t) which is not obstructed by any of the paths connecting $\{y \neq \xi_{k,t}^x : \xi_0(y) \neq 0\} \times \{0\}$ to (x, t) . Finally, if $\Gamma_1^x = \emptyset$, put $\xi_{n,t}^x = \Delta$ for every n and set $\xi_t(x) = 0$. We can proceed similarly for every $x \in \mathbb{Z}$ and get a configuration $(\xi_t(x))_{x \in \mathbb{Z}}$. It now follows that $(\xi_t(x))_{x \in \mathbb{Z}}$ has the distribution of the multitype contact process at time t with initial state ξ_0 . Additionally, by applying this construction to every $t > 0$, we get a trajectory $(\xi_t)_{t \geq 0}$ of the process that is right continuous with left limits. We will sometimes write $\xi_t(H)$ to make the dependence on the Harris construction explicit.

Step 2: Ancestry process. This will be our main object of investigation. Fix $x \in \mathbb{Z}$, $t \in [0, \infty)$ and a Harris construction H . Let Ψ_1^x be the set of paths ψ connecting $(x, 0)$ to $\mathbb{Z} \times \{t\}$ in H . Assume for the moment that $\#\Psi_1^x > 0$. Given $\psi, \psi' \in \Psi_1^x$, write $\psi < \psi'$ if there exists $\bar{s} \in (0, t)$ such that $\psi(s) = \psi'(s) \forall s \in [0, \bar{s})$ and $\psi(\bar{s}) \neq \psi(\bar{s}-), \psi'(\bar{s}) = \psi'(\bar{s}-)$. As before, we can check that $<$ is a total order on Ψ_1^x . Let ψ_1^* be the maximal path, define $\Psi_2^x = \{\psi \in \Psi_1^x : \psi(t) \neq \psi_1^*(t)\}$, let ψ_2^* be the maximal path in Ψ_2^x , and so on, until $\Psi_N^x = \emptyset$. For $1 \leq n < N$, we then define $\eta_{n,t}^x = \psi_n^*(t)$. For $n \geq N$, we put $\eta_{n,t}^x = \Delta$. Finally, if $\Psi_1^x = \emptyset$, we put $\eta_{n,t}^x = \Delta$ for all n .

Applying this construction for every $t > 0$, we get a $(\mathbb{Z} \cup \{\Delta\})^\infty$ -valued process $t \mapsto (\eta_{1,t}^x, \eta_{2,t}^x, \dots)$. We call it the *ancestry process* of x . $\eta_{n,t}^x$ is called the *nth ancestor of x at time t* . Of course, we may also jointly take, for every $x \in \mathbb{Z}$, the processes $(\eta_{1,t}^x, \eta_{2,t}^x, \dots)_{t \geq 0}$. Finally, we will write $\eta_{n,t}^x(H)$ when we need to make the dependence on the Harris construction explicit.

Given $x \in \mathbb{Z}$ and $0 \leq s \leq t$, we define $\eta_{n,t}^{(x,s)} = \eta_{n,t-s}^x(\theta(0,s)(H))$ (that is, the n th ancestor in the graph that grows from (x, s) up to time t). Also, when $n = 1$, we omit it, writing $\eta_t^x, \eta_t^{(x,s)}$ instead of $\eta_{1,t}^x, \eta_{1,t}^{(x,s)}$. Finally, we write $\eta_{*,t}^{(x,s)} = \{\eta_{n,t}^{(x,s)} \in \mathbb{Z} : n \geq 1\}$ (notice that we are excluding the Δ state), and similarly for $\eta_{*,t}^x$. The set $\eta_{*,t}^x$ has the same distribution as $\zeta_t^{\{x\}}$, the set of infected sites of a one-type contact process started from the configuration where only x is infected. For this reason,

if we ignore the Δ state, the process $t \mapsto (\eta_{1,t}^x, \eta_{2,t}^x, \dots)$ can be seen as a one-type contact process started from $\mathbb{1}_{\{x\}}$ and such that at each time, infected sites have numerical ranks.

Step 3: Joint construction of the multitype contact process and the ancestry process. We now explain the duality relation that connects the two processes described above. Fix $t > 0$ and let $H = ((D^x), (N^{(x,y)}))$ be a Harris construction on $\mathbb{Z} \times [0, t]$ (this means that the Poisson processes that constitute H are only defined on $[0, t]$). Let $\mathcal{A}_t(H)$ be the Harris construction on $[0, t]$ obtained from H by inverting the direction of time and of the arrows; formally, $\mathcal{A}_t(H) = ((\hat{D}^x), (\hat{N}^{(x,y)}))$, where $\hat{D}^x(s) = D^x(t-s)$, $\hat{N}^{(x,y)}(s) = N^{(y,x)}(t-s)$, $0 \leq s \leq t$, $x, y \in \mathbb{Z}$. Two immediate facts are that the laws of $H_{[0,t]}$ and $\mathcal{A}_t(H)$ are equal and that $(x, 0) \leftrightarrow (y, t)$ in H if and only if $(y, 0) \leftrightarrow (x, t)$ in $\mathcal{A}_t(H)$.

Given $\xi_0 \in \{0, 1, 2\}^{\mathbb{Z}}$, we now take the pair

$$(\xi_s(H))_{0 \leq s \leq t}, (\eta_{n,s}^x(\mathcal{A}_t(H)))_{n \in \mathbb{N}, 0 \leq s \leq t, x \in \mathbb{Z}}.$$

We thus have a coupling of the contact process started at ξ_0 and the ancestry process of every site, both up to time t . We should think of time for the multitype contact process as running on the opposite direction as time for the ancestry processes. This is illustrated on Figure 1.

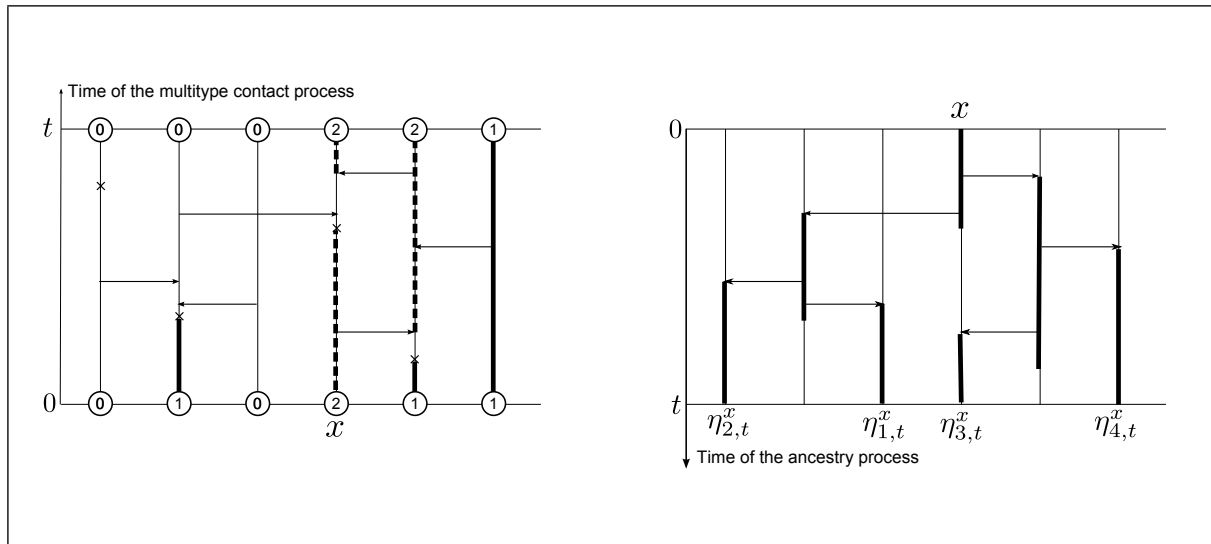


Figure 1: On the left, we have the multitype contact process starting from $\xi_0 \in \{0, 1, 2\}^{\mathbb{Z}}$ and following a Harris construction H . Time is increasing from bottom to top, thick lines represent the evolution of the 1's and dashed lines represent the evolution of the 2's. On the right, we have the ancestry process of $x \in \mathbb{Z}$ following $\mathcal{A}_t(H)$. Time is increasing from top to bottom. The facts that $\xi_0(\eta_{1,t}^x) = \xi_0(\eta_{2,t}^x) = 0$ and $\xi_0(\eta_{3,t}^x) = 2$ imply that $\xi_t(x) = 2$.

We now write our fundamental duality relation: for each $x \in \mathbb{Z}$,

$$\xi_t(x) = \begin{cases} 0, & \text{if for each } n, \text{ either } \xi_0(\eta_{n,t}^x) = 0 \text{ or } \eta_{n,t}^x = \Delta; \\ \xi_0(\eta_{n^*(x),t}^x), & \text{otherwise,} \end{cases} \quad (2.5)$$

where $n^*(x) = \inf\{n : \xi_0(\eta_{n,t}^x) \neq 0\}$. This can be seen at work in Figure 1. Its formal justification is straightforward and relies on the following. Given $x \in \mathbb{Z}$, define a bijection $\mathcal{J}_t : \Gamma_1^x \rightarrow \Psi_1^x$ by

$(\mathcal{J}_t \gamma)(s) = \gamma((t-s)^+)$; $\mathcal{J}_t \gamma$ is thus γ ran backwards and repaired so that we get a right continuous path. We then have $\gamma < \gamma' \Leftrightarrow \mathcal{J}_t \gamma' \mathcal{J}_t \gamma$, so all maximality properties can be translated from one space of paths to the other.

The obvious utility of (2.5) is that it allows us to relate properties of the distributions of the multitype contact process and of the ancestry processes at a fixed time. Let us state two such relations. First,

$$\mathbb{P}(\forall x, \xi_t(x) \neq 1) = \mathbb{P} \left(\begin{array}{l} \forall x, \text{ either } \eta_{*,t}^x = \emptyset, \text{ or} \\ \xi_0 \text{ is equal to 0 on } \eta_{*,t}^x, \text{ or} \\ \xi_0(\eta_{n^*(x),t}^x) = 2 \end{array} \right), \quad (2.6)$$

where $n^*(x)$ was defined after (2.5). This will be useful for our proof of Theorem 1.1. Second, taking $\xi_0 = \xi^h$ (the heaviside configuration defined in the Introduction) and ρ_t as in the Introduction,

$$\mathbb{P}(|\rho_t| > L) = \mathbb{P}(|\sup\{x : \eta_{1,t}^x \leq 0\} - \inf\{x : \eta_{1,t}^x > 0\}| > L). \quad (2.7)$$

This will be useful in our proof of Theorem 1.3. We have now concluded Step 3.

Even though the relations of Step 3 will be extremely useful, our standard point of view will be the one of Step 2. This means that, from now on, unless explicitly stated otherwise, we will have an infinite-time Harris construction H used to jointly define, for every $x \in \mathbb{Z}$, the ancestry processes $((\eta_{n,t}^x)_{n \in \mathbb{N}})_{t \geq 0}$. Whenever we mention a function of the Harris construction, such as $T^{(x,s)}$ or $M_t^{(x,s)}$, we mean to apply it to the Harris construction used to define the ancestry process.

The following is an easy consequence of the definition of the ancestry process with the ordering of paths ' defined above.

Lemma 2.2. (i.) Let $0 < s < t$, assume that $\eta_s^x \neq \Delta$ and $T^{(\eta_s^x, s)} > t$. Then, $\eta_t^x = \eta_t^{(\eta_s^x, s)}$. In particular, if $T^{(\eta_s^x, s)} = \infty$, then for all $s' > s$ we have $\eta_{s'}^x = \eta_{s'}^{(\eta_s^x, s)}$.

(ii.) Let $0 \leq s < t, z_1, \dots, z_N \in \mathbb{Z}$ and assume

$$\begin{aligned} \eta_{i,s}^x \neq \Delta, \eta_{*,t}^{(\eta_{i,s}^x, s)} = \emptyset, \quad 1 \leq i < n \\ \eta_{n,s}^x \neq \Delta, (\eta_{1,t}^{(\eta_{n,s}^x, s)}, \dots, \eta_{N,t}^{(\eta_{n,s}^x, s)}) = (z_1, \dots, z_N) \end{aligned}$$

(that is, the first $n-1$ ancestors of x at time s do not reach time t , but the n -th one does, with ancestors z_1, \dots, z_N at time t). Then, we have $(\eta_{1,t}^x, \dots, \eta_{N,t}^x) = (z_1, \dots, z_N)$.

Given $x \in \mathbb{Z}$, on $\{T^x = \infty\}$, define

$$\tau_1^x = \inf\{t \geq 1 : T^{(\eta_t^x, t)} = \infty\},$$

the first time after 1 at which the first ancestor of x lives forever. It is useful to think of τ_1^x as the result of a sequence of attempts, as we now explain and illustrate on Figure 2. Define $\sigma_1^x \equiv 1$ and, for $n \geq 1$,

$$\sigma_{n+1}^x = \begin{cases} +\infty & \text{if } \sigma_n^x = +\infty; \\ T^{(\eta_{\sigma_n^x}^x, \sigma_n^x)} & \text{otherwise.} \end{cases} \quad (2.8)$$

σ_n^x is thought of as the time of the n -th attempt to find a first ancestor of x that lives forever. If $(\eta_{\sigma_1^x}^x, \sigma_1^x) = (\eta_1^x, 1)$ lives forever (that is, if $T^{(\eta_1^x, 1)} = \infty$), we have $\tau_1^x = \sigma_1^x$ and say that the first

attempt succeeds. Otherwise, we must wait until $\sigma_2^x = T^{(\eta_1^x, 1)}$ to start the second attempt. This is because for any $t \in (\sigma_1^x, \sigma_2^x)$, we have $\eta_t^{(\eta_{\sigma_1^x}^x, \sigma_1^x)} = \eta_t^x$ as a consequence of Lemma 2.2(i.), so in particular $(\eta_{\sigma_1^x}^x, \sigma_1^x) \leftrightarrow (\eta_t^x, t)$ and then $T^{(\eta_t^x, t)} \leq T^{(\eta_{\sigma_1^x}^x, \sigma_1^x)} < \infty$. Next, if $(\eta_{\sigma_2^x}^x, \sigma_2^x)$ lives forever, then the second attempt succeeds and we have $\tau_1^x = \sigma_2^x$; otherwise we must wait for $(\eta_{\sigma_2^x}^x, \sigma_2^x)$ to die, and so on.

On $\{T^x = \infty\}$, also define $\tau_0^x \equiv 0$ and, for $n \geq 1$,

$$\tau_{n+1}^x = \tau_n^x + \tau_1^0 \circ \theta(\eta_{\tau_n^x}^x, \tau_n^x).$$

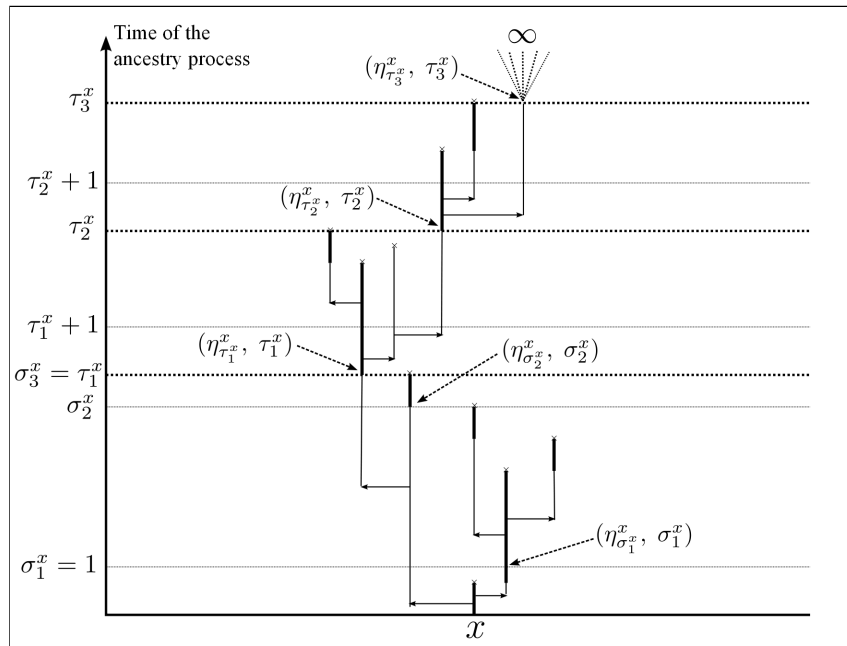


Figure 2: Renewal times. We detail the attempts σ_i^x to find the first renewal, τ_1^x . The first ancestor process is given by the thick line. The top of the figure means that $(\eta_{\tau_3^x}^x, \tau_3^x)$ lives forever.

For the sake of readability, we will sometimes write $\tilde{\mathbb{P}}^x(\cdot)$ and $\tilde{\mathbb{E}}^x(\cdot)$ instead of $\mathbb{P}(\cdot|T^x = \infty)$ and $\mathbb{E}(\cdot|T^x = \infty)$. In Proposition 1 in [15], it is shown that under $\tilde{\mathbb{P}}^x$, the times τ_n^x work as renewal times for the process η_t^x , that is, the (Time length, Trajectory) pairs

$$(\tau_{n+1}^x - \tau_n^x, t \in [0, \tau_{n+1}^x - \tau_n^x]) \mapsto \eta_{\tau_n^x + t}^x - \eta_{\tau_n^x}^x$$

are independent and identically distributed. This follows from an idea of Kuczek ([9]) which has become an important tool in the particle systems literature. In our current setting, it can be explained as follows. The probability $\tilde{\mathbb{P}}^x$ is the original probability for the process conditioned on the event $\{(x, 0) \text{ lives forever}\}$. But $(x, 0)$ being connected to $(\eta_{\tau_1^x}^x, \tau_1^x)$ and $(\eta_{\tau_1^x}^x, \tau_1^x)$ living forever imply that $(x, 0)$ lives forever, the event of the former conditioning. This and the fact that, under \mathbb{P} , restrictions of H to disjoint time intervals are independent yield that, under $\tilde{\mathbb{P}}^x$, the shifted Harris construction $\theta(\eta_{\tau_1^x}^x, \tau_1^x)(H)$ has same law as H . The argument is then repeated for all $\tau_n^x, n \geq 1$.

We now list the properties of the renewal times that we will need.

Proposition 2.3. (i.) $\tilde{\mathbb{P}}^0(\tau_n^0 < \infty) = 1 \forall n$.

(ii.) For $n \geq 0$, let

$$H_n = H_{[0, \tau_n^0(H)]}, \quad H_{n+} = \theta(\eta_{\tau_n^0}^0, \tau_n^0)(H).$$

Given an event A on finite-time Harris constructions and an event B on Harris constructions, we have

$$\tilde{\mathbb{P}}^0(H_n \in A, H_{n+} \in B) = \tilde{\mathbb{P}}^0(H_n \in A) \cdot \tilde{\mathbb{P}}^0(H \in B).$$

(iii.) Under $\tilde{\mathbb{P}}^x$, the \mathbb{Z} -valued process $(\eta_{\tau_n^x}^x)_{n \geq 0}$ is a symmetric random walk starting at x and with transitions

$$P(z, w) = \tilde{\mathbb{P}}^0(\eta_{\tau_1^0}^0 = w - z).$$

(iv.) There exist $c, C > 0$ such that

$$\tilde{\mathbb{P}}^0(\tau_1^0 \vee M_{\tau_1}^0 > r) \leq Ce^{-cr}.$$

Except for part (ii.), the above proposition is contained in Proposition (1), page 474, of [15] ((i.) and (iv.) are explicitly on the statement of the proposition and (iii.) is a direct consequence of (i.)). Part (ii.) is an adaption of Lemma 7 in [14] to our context; since its proof also uses ideas similar to the ones of Proposition (1) in [15], we omit it.

To conclude this section, we prove some simple properties of the first ancestor process.

Remark 2.4. Every time we write events involving a random variable η that may take the value Δ , such as $\{\eta \leq 0\}$, we mean $\{\eta \neq \Delta, \eta \leq 0\}$. This applies to part (iii) of the following lemma. Also, by convention we put $\mathbb{E}(f(\eta)) = \mathbb{E}(f(\eta); \eta \neq \Delta)$ for every function $f : \mathbb{Z} \rightarrow \mathbb{R}$.

Lemma 2.5. (i.) There exist $c, C > 0$ such that, for all $0 \leq a < b$,

$$\tilde{\mathbb{P}}^0(\exists n : \tau_n^0 \in [a, b]) \leq Ce^{-c(b-a)}.$$

(ii.) There exists $C > 0$ such that, for all $0 \leq s < t$,

$$\tilde{\mathbb{E}}^0((\eta_t^0)^2 - (\eta_s^0)^2) \leq C + C(t - s).$$

(iii.) There exist $c, C > 0$ such that for all $l \geq 0$,

$$\mathbb{P}(|\eta_t^0| > l) \leq Ce^{-cl^2/t} + Ce^{-cl}.$$

Proof. Define on $\{T^0 = \infty\}$, for $t \geq 0$,

$$\tau_{t-} = \sup\{\tau_n^0 \leq t : n \in \mathbb{N}\}, \quad \tau_{t+} = \inf\{\tau_n^0 \geq t : n \in \mathbb{N}\}, \quad \psi_t = M_{\tau_{t+}}^{(\eta_{\tau_{t-}}, \tau_{t-})} \vee (\tau_{t+} - \tau_{t-}).$$

Using Proposition 2.3(ii.) and (iv.),

$$\begin{aligned}
\tilde{\mathbb{P}}^0(\psi_t > x) &= \sum_{k=0}^{\infty} \tilde{\mathbb{P}}^0(\tau_k^0 < t, \tau_{k+1}^0 \geq t, \psi_t > x) \\
&= \sum_{k=0}^{\infty} \int_0^t \tilde{\mathbb{P}}^0\left(\tau_1^0 \geq t-s, M_{\tau_1^0}^0 \vee \tau_1^0 > x\right) \tilde{\mathbb{P}}^0(\tau_k^0 \in ds) \\
&\leq \sum_{k=0}^{\infty} \sum_{i=1}^{\lceil t \rceil} \int_{i-1}^i \left[\tilde{\mathbb{P}}^0(\tau_1^0 \geq t-s) \wedge \tilde{\mathbb{P}}^0\left(M_{\tau_1^0}^0 \vee \tau_1^0 > x\right) \right] \tilde{\mathbb{P}}^0(\tau_k^0 \in ds) \\
&\leq \sum_{i=1}^{\lceil t \rceil} \left[C e^{-c(t-i)} \wedge C e^{-cx} \right] \sum_{k=0}^{\infty} \tilde{\mathbb{P}}^0(\tau_k^0 \in [i-1, i]) \\
&= \sum_{i=1}^{\lceil t \rceil} \left[C e^{-c(t-i)} \wedge C e^{-cx} \right] \tilde{\mathbb{E}}^0(\#\{n : \tau_n^0 \in [i-1, i]\}). \tag{2.9}
\end{aligned}$$

Observe that the above expectation is less than 1, because there is at most one renewal in each unit interval. (2.9) is thus less than

$$C \sum_{i=1}^{\infty} [e^{-ci} \wedge e^{-cx}] \leq C \lceil x \rceil e^{-cx} + C \sum_{i=\lceil x \rceil+1}^{\infty} e^{-ci} \leq C e^{-cx};$$

since this does not depend on t , we get

$$\tilde{\mathbb{P}}(\psi_t > x) \leq C e^{-cx} \tag{2.10}$$

for some $c, C > 0$ and all $t \geq 0$. Let us now prove the two statements of the lemma.

(i.) For $0 \leq a < b$,

$$\tilde{\mathbb{P}}^0(\#\{n : \tau_n^0 \in [a, b]\}) \leq \tilde{\mathbb{P}}^0(\tau_{a+} - \tau_{a-} > b-a) \leq \tilde{\mathbb{P}}^0(\psi_a > b-a) \leq C e^{-c(b-a)}.$$

(ii.) The definition of ψ_t and (2.10) imply

$$|\eta_t^0 - \eta_{\tau_{t-}^0}^0|, |\eta_{\tau_{t+}^0}^0 - \eta_t^0|, |\eta_{\tau_{t+}^0}^0 - \eta_{\tau_{t-}^0}^0| \leq 2\psi_t; \tag{2.11}$$

$$\bar{C} := \sup_{t \geq 0} \tilde{\mathbb{E}}^0((\psi_t)^2) < \infty. \tag{2.12}$$

Next, for $t > 0$, since $\tau_{\lceil t \rceil}^0 \geq \lceil t \rceil \geq t$, we have $\tau_{t+} \in \{\tau_0^0, \tau_1^0, \dots, \tau_{\lceil t \rceil}^0\}$, so

$$\tilde{\mathbb{E}}^0\left((\eta_{\tau_{t+}^0}^0)^2\right) \leq \tilde{\mathbb{E}}^0\left(\max_{1 \leq i \leq \lceil t \rceil} (\eta_{\tau_i^0}^0)^2\right)$$

By the reflection principle (see [7], page 285), the expectation on the right-hand side is less than $2 \tilde{\mathbb{E}}^0((\eta_{\tau_{\lceil t \rceil}^0}^0)^2) = 2 \lceil t \rceil \tilde{\mathbb{E}}^0((\eta_{\tau_1^0}^0)^2)$, so we have

$$\tilde{\mathbb{E}}^0((\eta_{\tau_{t+}^0}^0)^2) \leq C \cdot \lceil t \rceil \leq C \cdot (t+1). \tag{2.13}$$

With (2.11), (2.12) and (2.13) at hand, we are ready to estimate

$$\begin{aligned} & \tilde{\mathbb{E}}^0 \left((\eta_t^0)^2 - (\eta_s^0)^2 \right) \\ &= \tilde{\mathbb{E}}^0 \left((\eta_t^0)^2 - (\eta_{\tau_{t+}}^0)^2 \right) + \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{s+}}^0)^2 - (\eta_s^0)^2 \right) + \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{t+}}^0)^2 - (\eta_{\tau_{s+}}^0)^2 \right). \end{aligned} \quad (2.14)$$

Let us treat each of the three terms separately.

$$\begin{aligned} & \tilde{\mathbb{E}}^0 \left((\eta_t^0)^2 - (\eta_{\tau_{t+}}^0)^2 \right) \\ &= \tilde{\mathbb{E}}^0 \left((\eta_t^0 - \eta_{\tau_{t-}}^0 + \eta_{\tau_{t-}}^0)^2 + (\eta_{\tau_{t+}}^0 - \eta_{\tau_{t-}}^0 + \eta_{\tau_{t-}}^0)^2 \right) \\ &= \tilde{\mathbb{E}}^0 \left((\eta_t^0 - \eta_{\tau_{t-}}^0)^2 + 2\eta_{\tau_{t-}}^0 (\eta_t^0 - \eta_{\tau_{t-}}^0) - (\eta_{\tau_{t+}}^0 - \eta_{\tau_{t-}}^0)^2 - 2\eta_{\tau_{t-}}^0 (\eta_{\tau_{t+}}^0 - \eta_{\tau_{t-}}^0) \right) \\ &= \tilde{\mathbb{E}}^0 \left((\eta_t^0 - \eta_{\tau_{t-}}^0)^2 \right) - \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{t+}}^0 - \eta_{\tau_{t-}}^0)^2 \right) \end{aligned} \quad (2.15)$$

since, by the independence of increments between different pairs of renewals and symmetry, $\tilde{\mathbb{E}}^0(\eta_{\tau_{t-}}^0 (\eta_t^0 - \eta_{\tau_{t-}}^0)) = \tilde{\mathbb{E}}^0(\eta_{\tau_{t-}}^0 (\eta_{\tau_{t+}}^0 - \eta_{\tau_{t-}}^0)) = 0$. Using (2.11), (2.15) can be bounded by $2 \tilde{\mathbb{E}}^0((2\psi_t)^2)$, then by (2.12) we get

$$\tilde{\mathbb{E}}^0 \left((\eta_t^0)^2 - (\eta_{\tau_{t+}}^0)^2 \right) \leq 2\bar{C}. \quad (2.16)$$

Similarly,

$$\tilde{\mathbb{E}}^0 \left((\eta_{\tau_{s+}}^0)^2 - (\eta_s^0)^2 \right) \leq 2\bar{C}. \quad (2.17)$$

Finally,

$$\begin{aligned} \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{t+}}^0)^2 - (\eta_{\tau_{s+}}^0)^2 \right) &= \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{t+}}^0 - \eta_{\tau_{s+}}^0)^2 + 2\eta_{\tau_{s+}}^0 (\eta_{\tau_{t+}}^0 - \eta_{\tau_{s+}}^0) \right) \\ &= \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{t+}}^0 - \eta_{\tau_{s+}}^0)^2 \right) = \int_s^t \tilde{\mathbb{E}}^0 \left((\eta_{\tau_{(t-r)+}}^0)^2 \right) \tilde{\mathbb{P}}^0(\tau_{s+} \in dr). \end{aligned}$$

By (2.13), this is less than

$$C \int_s^t (t-r+1) \tilde{\mathbb{P}}^0(\tau_{s+} \in dr) \leq C(t-s+1).$$

Putting this, (2.16) and (2.17) back in (2.14) completes the proof.

(iii.) For $l \geq 0$,

$$\mathbb{P}(|\eta_t^0| > l) = \mathbb{P}(t < T^0 < \infty, |\eta_t^0| > l) + \mathbb{P}(T^0 = \infty) \tilde{\mathbb{P}}^0(|\eta_t^0| > l).$$

The first term is less than

$$\mathbb{P}(T^0 < \infty, M_{T^0}^0 > l) \leq \mathbb{P}(l/\kappa < T^0 < \infty) + \mathbb{P}(M_{l/\kappa}^0 > l),$$

where κ is as in (2.1). Now use (2.1) and (2.3) to get that this last sum is less than Ce^{-cl} . Next, we have

$$\tilde{\mathbb{P}}^0(|\eta_t^0| > l) \leq \tilde{\mathbb{P}}^0 \left(\max_{1 \leq i \leq \lfloor t \rfloor} |\eta_{\tau_i^0}^0| > l/2 \right) + \tilde{\mathbb{P}}^0(\psi_t > l/2),$$

because there are at most $\lfloor t \rfloor$ renewals until time t . By (2.10), $\tilde{\mathbb{P}}^0(\psi_t > l/2) \leq Ce^{-cl}$. By Proposition 2.1.2 in [10], $\tilde{\mathbb{P}}^0 \left(\max_{1 \leq i \leq \lfloor t \rfloor} |\eta_{\tau_i^0}^0| > l/2 \right) \leq Ce^{-cl^2/\lfloor t \rfloor} \leq Ce^{-cl^2/t}$. This completes the proof. \blacksquare

3 Pairs and sets of ancestries

In this section, we study the joint behavior of ancestral paths. For pairs of ancestries, we define joint renewal points that have properties similar to the ones just discussed for single renewals, and then use these properties to study the speed of coalescence of first ancestors. For sets of ancestries, we show that, given $N > 0$, the overall density of sites of \mathbb{Z} occupied by ancestors of rank smaller than or equal to N at time t tends to 0 as $t \rightarrow \infty$.

Let us define our sequence of *joint renewal times*. Fix $x, y \in \mathbb{Z}$ and on $\{T^x = T^y = \infty\}$ define

$$\tau_1^{x,y} = \inf\{t \geq 1 : T(\eta_t^x, t) = T(\eta_t^y, t) = \infty\},$$

the first time after 1 at which both the first ancestor of x and the one of y live forever. In parallel with (2.8), define $\sigma_1^{x,y} \equiv 1$ and, for $n \geq 1$,

$$\sigma_{n+1}^{x,y} = \begin{cases} +\infty & \text{if } \sigma_n^{x,y} = +\infty \\ T\left(\eta_{\sigma_n^{x,y}}^x, \sigma_n^{x,y}\right) \wedge T\left(\eta_{\sigma_n^{x,y}}^y, \sigma_n^{x,y}\right) & \text{otherwise.} \end{cases}$$

The sequence of attempts in this case works as follows. We start asking if both $(\eta_{\sigma_1^{x,y}}^x, \sigma_1^{x,y})$ and $(\eta_{\sigma_1^{x,y}}^y, \sigma_1^{x,y})$ live forever. If so, we set $\tau_1^{x,y} = \sigma_1^{x,y}$. Otherwise, we wait until one of them dies out; this happens at time $\sigma_2^{x,y} = T\left(\eta_{\sigma_1^{x,y}}^x, \sigma_1^{x,y}\right) \wedge T\left(\eta_{\sigma_1^{x,y}}^y, \sigma_1^{x,y}\right)$. We then look at $(\eta_{\sigma_2^{x,y}}^x, \sigma_2^{x,y})$ and $(\eta_{\sigma_2^{x,y}}^y, \sigma_2^{x,y})$, and so on.

Also define $\tau_0^{x,y} \equiv 0$ and, for $n \geq 1$, on $\{T^x = T^y = \infty, \eta_{\tau_n^{x,y}}^x = z, \eta_{\tau_n^{x,y}}^y = w\}$, define

$$\tau_{n+1}^{x,y} = \tau_n^{x,y} + \tau_1^{z,w} \circ \theta(0, \tau_n^{x,y}).$$

For $x, y \in \mathbb{Z}$, we write $\tilde{\mathbb{P}}^{x,y}(\cdot) = \mathbb{P}(\cdot | T^x = T^y = \infty)$ and $\tilde{\mathbb{E}}^{x,y}(\cdot) = \mathbb{E}(\cdot | T^x = T^y = \infty)$. Note that $\tilde{\mathbb{P}}^{x,x} = \tilde{\mathbb{P}}^x$, $\tilde{\mathbb{E}}^{x,x} = \tilde{\mathbb{E}}^x$ and $\tau_n^{x,x} = \tau_n^x$ for any x and n . We have the following analog of Lemma 2.3:

Proposition 3.1. (i.) $\tilde{\mathbb{P}}^{x,y}(\tau_n^{x,y} < \infty) = 1 \forall n, x, y$.

(ii.) For $n \geq 0$, let

$$H_n = H_{[0, \tau_n^{x,y}(H)]}, \quad H_{n+} = \theta(0, \tau_n^{x,y})(H).$$

Given an event A on finite-time Harris constructions, an event B on Harris constructions and $z, w \in \mathbb{Z}$, we have

$$\tilde{\mathbb{P}}^{x,y}(H_n \in A, \eta_{\tau_n^{x,y}}^x = z, \eta_{\tau_n^{x,y}}^y = w, H \in B) = \tilde{\mathbb{P}}^{x,y}(H_n \in A, \eta_{\tau_n^{x,y}}^x = z, \eta_{\tau_n^{x,y}}^y = w) \cdot \tilde{\mathbb{P}}^{z,w}(H \in B).$$

(iii.) Under $\tilde{\mathbb{P}}^{x,y}$, the \mathbb{Z}^2 -valued process $(\eta_{\tau_n^{x,y}}^x, \eta_{\tau_n^{x,y}}^y)_{n \geq 0}$ is a Markov chain starting at (x, y) and with transitions

$$P((a, b), (c, d)) = \tilde{\mathbb{P}}^{a,b}(\eta_{\tau_1^{a,b}}^a = c, \eta_{\tau_1^{a,b}}^b = d).$$

In particular, if $\{T^x = T^y = \infty\}$ and $\eta_{\tau_m^{x,y}}^x = \eta_{\tau_m^{x,y}}^y$, then $\eta_{\tau_n^{x,y}}^x = \eta_{\tau_n^{x,y}}^y$ for all $n \geq m$.

(iv.) There exist $c, C > 0$ such that, for any x, y ,

$$\tilde{\mathbb{P}}^{x,y}(\max(\tau_1^{x,y}, M_{\tau_1^{x,y}}^x, M_{\tau_1^{x,y}}^y) > r) \leq C e^{-cr}.$$

We omit the proof since it is an almost exact repetition of the one of Lemma 2.3; the only difference is that, when looking for renewals, we must inspect two points instead of one.

We now study the behavior of the discrete time Markov chain mentioned in part (iii.) of the above proposition. Our first objective is to show that the time it takes for two ancestries to coalesce has a tail that is similar to that of the time it takes for two independent simple random walks on \mathbb{Z} to meet. This fact will be extended to continuous time in Lemma 3.3; in Section 5, we will establish other similarities between pairs of ancestries and pairs of coalescing random walks.

Lemma 3.2. (i.) For $z \in \mathbb{Z}$, let π_z denote the probability on \mathbb{Z} given by

$$\pi_z(w) = \tilde{\mathbb{P}}^{0,z} \left(\eta_{\tau_1^0,z}^z - \eta_{\tau_1^0,z}^0 = z + w \right), \quad w \in \mathbb{Z}.$$

There exist a symmetric probability π on \mathbb{Z} and $c, C > 0$ such that

$$\|\pi_z - \pi\|_{TV} \leq C e^{-c|z|} \quad \forall z \in \mathbb{Z},$$

where $\|\cdot\|_{TV}$ denotes total variation distance.

(ii.) There exists $C > 0$ such that, for all $x, y \in \mathbb{Z}$ and $n \in \mathbb{N}$,

$$\tilde{\mathbb{P}}^{x,y} \left(\eta_{\tau_n^x,y}^x \neq \eta_{\tau_n^x,y}^y \right) \leq \frac{C|x-y|}{\sqrt{n}}.$$

Proof. (i.) Fix $z \in \mathbb{Z}$. For simplicity of notation, we will go through the proof in the case $z > 0$; however, it will be clear how to treat the case $z < 0$. Let us take two random Harris constructions H^1 and H^2 defined on a common space with probability measure \mathbb{P} , under which H^1 and H^2 are independent and both have the original, unconditioned distribution obtained from the construction with Poisson processes. Define H^3 as a superposition of H^1 and H^2 , as follows. We include in H^3 :

- from H^1 , all death marks in sites that belong to $(-\infty, \lfloor z/2 \rfloor]$ and all arrows whose starting points belong to $(-\infty, \lfloor z/2 \rfloor]$;
- from H^2 , all death marks in sites that belong to $(\lfloor z/2 \rfloor, \infty)$ and all arrows whose starting points belong to $(\lfloor z/2 \rfloor, \infty)$.

Then, H^3 has same law as H^1 and H^2 . We will write all processes and times defined so far as functions of these Harris constructions: for $i \in \{1, 2, 3\}$, we may take

$$\begin{aligned} & \eta_{n,t}^{(x,s)}(H^i) \text{ for } x \in \mathbb{Z}, n \in \mathbb{N}, s < t; \\ & M_t^{(x,s)}(H^i) \text{ and } T^{(x,s)}(H^i) \text{ for } x \in \mathbb{Z}, s < t; \\ & \tau_n^{(x,y)}(H^i) \text{ on } \{T^x(H^i) = T^y(H^i) = \infty\}, \text{ for } x, y \in \mathbb{Z}, n \in \mathbb{N}, \end{aligned}$$

as defined before and nothing new is involved. On the event $\{T^0(H^1) = T^z(H^2) = \infty\}$, define

$$\tilde{\tau}^{0,z} = \inf\{t \geq 1 : T^{(\eta_t^0(H^1),t)}(H^1) = T^{(\eta_t^z(H^2),t)}(H^2) = \infty\}.$$

Our definition of $\tilde{\tau}^{0,z}$ is similar to the one of first joint renewal time of two first ancestor processes. However, for $\tilde{\tau}^{0,z}$, we follow a different Harris construction for each ancestor process. We can also think of $\tilde{\tau}^{0,z}$ as the result of a “sequence of attempts”, and define corresponding stopping times similar to the ones illustrated on Figure 2. The same proof that establishes Proposition 3.1(iv.) can be repeated here to show that there exist $c, C > 0$ such that

$$\mathbb{P} \left(T^0(H^1) = T^z(H^2) = \infty, \tilde{\tau}^{0,z} \vee M_{\tilde{\tau}^{0,z}}^0(H^1) \vee M_{\tilde{\tau}^{0,z}}^z(H^2) > r \right) \leq C e^{-cr}. \quad (3.1)$$

Now define

$$\begin{aligned} X^{0,z} &= \begin{cases} \eta_{\tau_1^{0,z}(H^3)}^z(H^3) - \eta_{\tau_1^{0,z}(H^3)}^0(H^3) & \text{if } T^0(H^3) = T^z(H^3) = \infty \\ \Delta & \text{otherwise;} \end{cases} \\ Y^{0,z} &= \begin{cases} \eta_{\tilde{\tau}^{0,z}}^z(H^2) - \eta_{\tilde{\tau}^{0,z}}^0(H^1) & \text{if } T^0(H^1) = T^z(H^2) = \infty, \\ \Delta & \text{otherwise.} \end{cases} \end{aligned}$$

Note that

$$\pi_z(\cdot) = \mathbb{P}(X^{0,z} = z + \cdot \mid X^{0,z} \neq \Delta), \quad (3.2)$$

where π_z is defined in the statement of the lemma. Also define

$$\pi(\cdot) = \mathbb{P}(Y^{0,z} = z + \cdot \mid Y^{0,z} \neq \Delta). \quad (3.3)$$

By the definition of $Y^{0,z}$ from independent Harris constructions, π is symmetric and does not depend on z . To conclude the proof, we have two tasks. First, to show that $X^{0,z} = Y^{0,z}$ with high probability when z is large. Second, to show that this implies that, when z is large, $\|\pi_z - \pi\|_{TV}$ is small.

Let κ be as in (2.1) and define $t^* = z/3\kappa$. Consider the events

$$\begin{aligned} \mathcal{L}_1 &= \{M_{t^*}^0(H^1) \vee M_{t^*}^z(H^2) < z/2\}, \\ \mathcal{L}_2 &= \{T^0(H^1) \wedge T^z(H^2) < t^*\}, \\ \mathcal{L}_3 &= \left\{ \begin{array}{l} T^0(H^1) = T^z(H^2) = T^0(H^3) = T^z(H^3) = \infty, \\ \tau_1^{0,z}(H^3) < t^*, \tilde{\tau}^{0,z} < t^* \end{array} \right\}. \end{aligned}$$

On \mathcal{L}_1 , we have $\{(x, t) : 0 \leq t \leq t^*, (0, 0) \leftrightarrow (x, t) \text{ in } H^1\} \subset (-\infty, \lfloor z/2 \rfloor] \times [0, t^*]$. Since the restriction of H^1 to $(-\infty, \lfloor z/2 \rfloor] \times [0, t^*]$ coincides with the restriction of H^3 to the same set and similar considerations apply to H^2 and the set $(\lfloor z/2 \rfloor, \infty) \times [0, t^*]$, we get that, on \mathcal{L}_1 ,

$$\eta_{n,t}^0(H^1) = \eta_{n,t}^0(H^3), \quad \eta_{n,t}^z(H^2) = \eta_{n,t}^z(H^3), \quad \forall n \in \mathbb{N}, 0 \leq t \leq t^*. \quad (3.4)$$

We now claim that, if the event $\mathcal{L} := (\mathcal{L}_1 \cap \mathcal{L}_2) \cup (\mathcal{L}_1 \cap \mathcal{L}_3)$ occurs, then $X^{0,z} = Y^{0,z}$. To see this, assume first that $\mathcal{L}_1 \cap \mathcal{L}_2$ occurs. Then, by the definition of \mathcal{L}_2 , we either have $T^0(H^1) < t^*$ or $T^z(H^2) < t^*$. In any case we have $Y^{0,z} = \Delta$ and, also using (3.4), we either get $T^0(H^3) < t^*$ or $T^z(H^3) < t^*$, so $X^{0,z} = \Delta$. Now assume $\mathcal{L}_1 \cap \mathcal{L}_3$ occurs. Define

$$\begin{aligned} t_1 &= \tilde{\tau}^{0,z}, & a_1 &= \eta_{t_1}^0(H^1), & b_1 &= \eta_{t_1}^z(H^2); \\ t_2 &= \tau_1^{0,z}(H^3), & a_2 &= \eta_{t_2}^0(H^3), & b_2 &= \eta_{t_2}^z(H^3). \end{aligned}$$

By (3.4) and the fact that $t_1, t_2 \leq t^*$, if we show that $t_1 = t_2$, we get $a_1 = a_2$ and $b_1 = b_2$, hence $Y^{0,z} = b_1 - a_1 = b_2 - a_2 = X^{0,z}$. Assume $t_1 \leq t_2$. Again by (3.4) and the fact that $t_1 \leq t^*$, we have

$$\eta_{t_1}^0(H^3) = \eta_{t_1}^0(H^1) = a_1, \quad \eta_{t_1}^z(H^3) = \eta_{t_1}^z(H^2) = b_1. \quad (3.5)$$

The definition of t_1 implies that $T^{(a_1, t_1)}(H^1) = T^{(b_1, t_1)}(H^2) = \infty$, and then we obviously have $(a_1, t_1) \leftrightarrow \mathbb{Z} \times \{t^*\}$ in H^1 and $(b_1, t_1) \leftrightarrow \mathbb{Z} \times \{t^*\}$ in H^2 ; since we are assuming \mathcal{L}_1 occurs, the paths that make these connections are also available in H^3 . This gives

$$T^{(a_1, t_1)}(H^3) > t^*, \quad T^{(b_1, t_1)}(H^3) > t^*. \quad (3.6)$$

We now use (3.5), (3.6) and the definition of a_2, b_2 in Lemma 2.2(i.) to conclude that

$$\eta_{t_2}^{(a_1, t_1)}(H^3) = a_2, \quad \eta_{t_2}^{(b_1, t_1)}(H^3) = b_2. \quad (3.7)$$

By the definition of t_2 , we also have $T^{(a_2, t_2)}(H^3) = T^{(b_2, t_2)}(H^3) = \infty$; together with (3.7) this yields

$$T^{(a_1, t_1)}(H^3) = T^{(b_1, t_1)}(H^3) = \infty. \quad (3.8)$$

Since $t_2 = \tau_1^{0, z}(H^3)$ is defined as the infimum over all times $t \geq 1$ that satisfy $T^{(\eta_t^0, t)}(H^3) = T^{(\eta_t^z, t)}(H^3) = \infty$, we see from (3.5) and (3.8) that $t_2 \leq t_1$, so $t_2 = t_1$. A similar set of arguments show that $t_2 \leq t_1$ implies $t_1 = t_2$. This completes the proof of the claim.

Now note that the event \mathcal{L}^c is contained in the union of:

$$\begin{aligned} & \{M_{t^*}^0(H^1) > z/2\}, \{M_{t^*}^z(H^2) > z/2\}, \\ & \{t^* < T^0(H^1) < \infty\}, \{t^* < T^0(H^3) < \infty\}, \{t^* < T^z(H^2) < \infty\}, \{t^* < T^z(H^3) < \infty\}, \\ & \{T^0(H^1) = T^z(H^2) = \infty, \tilde{\tau}^{0, z} > t^*\}, \{T^0(H^3) = T^z(H^3) = \infty, \tau_1^{0, z}(H_3) > t^*\}. \end{aligned}$$

Using our choice of t^* , (2.1), (2.3), Proposition 3.1(iii.) and (3.1), the probability of any of these events decreases exponentially with z .

We thus have $\mathbb{P}(X^{0, z} \neq Y^{0, z}) \leq Ce^{-cz}$, so

$$\sum_{w \in \mathbb{Z} \cup \{\Delta\}} |\mathbb{P}(X^{0, z} = w) - \mathbb{P}(Y^{0, z} = w)| \leq Ce^{-cz}. \quad (3.9)$$

Then,

$$\begin{aligned} \|\pi_z - \pi\|_{TV} &= \frac{1}{2} \sum_{w \in \mathbb{Z}} |\pi_z(w) - \pi(w)| = \frac{1}{2} \sum_{w \in \mathbb{Z}} \left| \frac{\mathbb{P}(X^{0, z} = w)}{\mathbb{P}(X^{0, z} \neq \Delta)} - \frac{\mathbb{P}(Y^{0, z} = w)}{\mathbb{P}(Y^{0, z} \neq \Delta)} \right| \\ &\leq \frac{1}{2\mathbb{P}(X^{0, z} \neq \Delta)} \sum_{w \in \mathbb{Z}} |\mathbb{P}(X^{0, z} = w) - \mathbb{P}(Y^{0, z} = w)| \\ &\quad + \frac{1}{2} \left| \frac{1}{\mathbb{P}(Y^{0, z} \neq \Delta)} - \frac{1}{\mathbb{P}(X^{0, z} \neq \Delta)} \right| \sum_{w \in \mathbb{Z}} \mathbb{P}(Y^{0, z} = w) \\ &\leq \frac{1}{\mathbb{P}(X^{0, z} \neq \Delta)} Ce^{-cz} + \frac{|\mathbb{P}(X^{0, z} \neq \Delta) - \mathbb{P}(Y^{0, z} \neq \Delta)|}{\mathbb{P}(X^{0, z} \neq \Delta) \cdot \mathbb{P}(Y^{0, z} \neq \Delta)} \\ &\leq \frac{1}{\mathbb{P}(X^{0, z} \neq \Delta)} Ce^{-cz} + \frac{1}{\mathbb{P}(X^{0, z} \neq \Delta)} \frac{1}{\mathbb{P}(Y^{0, z} \neq \Delta)} Ce^{-cz}, \end{aligned}$$

where we have applied (3.2) and (3.3) in the second equality and (3.9) in the last two inequalities. We have $\mathbb{P}(X^{0, z} \neq \Delta) = \mathbb{P}(T^0(H^3) = T^z(H^3) = \infty)$ and $\mathbb{P}(Y^{0, z} \neq \Delta) = \mathbb{P}(T^0(H^1) = T^z(H^2) = \infty)$, and these probabilities are bounded away from zero uniformly in z by (2.2). We thus get $\|\pi_z - \pi\|_{TV} \leq Ce^{-cz}$. Adapting the proof to the case $z < 0$, we get $\|\pi_z - \pi\|_{TV} \leq Ce^{-c|z|}$ for any $z \in \mathbb{Z}$.

(ii.) For $n \geq 0$, $x, y \in \mathbb{Z}$, define

$$X_n^{x, y} = \begin{cases} \eta_{\tau_n^{x, y}}^y - \eta_{\tau_n^{x, y}}^x, & \text{if } T^x = T^y = \infty \\ \Delta, & \text{otherwise.} \end{cases}$$

Using Proposition 3.1(iii.) and translation invariance, we see that under $\tilde{\mathbb{P}}^{x,y}$, $X_n^{x,y}$ is a Markov chain that starts at $y - x$ and has transitions

$$\tilde{\mathbb{P}}^{x,y}(X_{n+1}^{x,y} = z + w \mid X_n^{x,y} = z) = \tilde{\mathbb{P}}^{0,z}(X_1^{0,z} = z + w) = \pi_z(w).$$

In particular, 0 is an absorbing state. (ii.) follows from Theorem 6.1 in Section 6. Here, let us ensure that the four conditions in the beginning of that section are satisfied by π_z and π . Conditions (6.1) and (6.4) are already established. Condition (6.2) is straightforward to check and (6.3) follows from (3.1) and Proposition 3.1(iv.). \blacksquare

We now want to define a random time $J^{x,y}$ that will work as a “first renewal after coalescence” for the first ancestors of x and y , a time after which the two processes evolve together with the law of a single first ancestor process. Some care should be taken, however, to treat the cases in which the ancestries of x or of y die out. With this in mind, we put

$$J^{x,y} = \begin{cases} \inf\{\tau_n^{x,y} : \eta_{\tau_n^{x,y}}^x = \eta_{\tau_n^{x,y}}^y\} & \text{on } \{T^x = T^y = \infty\}; \\ \inf\{\tau_n^x : \tau_n^x > T^y\} & \text{on } \{T^x = \infty, T^y < \infty\}; \\ \inf\{\tau_n^y : \tau_n^y > T^x\} & \text{on } \{T^y = \infty, T^x < \infty\}; \\ 0 & \text{on } \{T^x < \infty \text{ and } T^y < \infty\}. \end{cases}$$

This definition is symmetric: $J^{x,y} = J^{y,x}$.

Lemma 3.3. (i.) *There exists $C > 0$ such that, for any $x, y \in \mathbb{Z}$ and $t \geq 0$,*

$$\mathbb{P}(J^{x,y} > t) \leq \frac{C|x - y|}{\sqrt{t}}.$$

(ii.) *Conditioned to $\{T^x = \infty\}$, the process $t \mapsto (\eta_{1,t}^0, \eta_{2,t}^0, \dots) \circ \theta(\eta_{J^{x,y}}^x, J^{x,y})$ is independent of $J^{x,y}$. Additionally, the law of $t \mapsto (\eta_{1,t}^0, \eta_{2,t}^0, \dots) \circ \theta(\eta_{J^{x,y}}^x, J^{x,y})$ conditioned to $\{T^x = \infty\}$ is equal to the law of $t \mapsto (\eta_{1,t}^0, \eta_{2,t}^0, \dots)$ conditioned to $\{T^0 = \infty\}$.*

Proof. (i.) By Proposition 3.1(iv.), there exists $\gamma > 0$ such that $\beta := \sup_{z,w} \tilde{\mathbb{E}}^{z,w}(e^{\gamma \tau_1^{z,w}}) < \infty$. Using Chebyshev's inequality, for every $x, y \in \mathbb{Z}$ and $t \geq 0$ we have

$$\tilde{\mathbb{P}}^{x,y}(\tau_n^{x,y} > t) \leq e^{-\gamma t} \tilde{\mathbb{E}}^{x,y}(e^{\gamma \tau_n^{x,y}}) = e^{-\gamma t} \sum_{z,w \in \mathbb{Z}} \tilde{\mathbb{E}}^{x,y} \left(e^{\gamma \tau_n^{x,y}} \cdot \mathbb{1}\{\eta_{\tau_{n-1}^{x,y}}^x = z, \eta_{\tau_{n-1}^{x,y}}^y = w\} \right).$$

By Proposition 3.1(ii.), this is equal to

$$e^{-\gamma t} \sum_{z,w \in \mathbb{Z}} \tilde{\mathbb{E}}^{x,y} \left(e^{\gamma \tau_{n-1}^{x,y}} \cdot \mathbb{1}\{\eta_{\tau_{n-1}^{x,y}}^x = z, \eta_{\tau_{n-1}^{x,y}}^y = w\} \right) \cdot \tilde{\mathbb{E}}^{z,w} \left(e^{\gamma \tau_1^{z,w}} \right) \leq \beta e^{-\gamma t} \tilde{\mathbb{E}}^{x,y} \left(e^{\gamma \tau_{n-1}^{x,y}} \right).$$

Iterating, we get $\tilde{\mathbb{P}}^{x,y}(\tau_n^{x,y} > t) \leq e^{-\gamma t} \beta^n$, so, putting $n^* = n^*(t) = \lfloor \frac{\gamma}{2 \log \beta} t \rfloor$, we have $\tilde{\mathbb{P}}^{x,y}(\tau_{n^*}^{x,y} > t) \leq e^{-\gamma t/2}$. This together with Lemma 3.2 gives

$$\tilde{\mathbb{P}}^{x,y}(J^{x,y} > t) \leq \tilde{\mathbb{P}}^{x,y} \left(\eta_{\tau_{n^*}^{x,y}}^x \neq \eta_{\tau_{n^*}^{x,y}}^y \right) + \tilde{\mathbb{P}}^{x,y}(\tau_{n^*}^{x,y} > t) \leq \frac{C|x - y|}{\sqrt{t}} \quad (3.10)$$

for some $C > 0$.

Note that if $T^x = \infty, T^y < t/2$ and there exists some n such that $\tau_n^x \in [t/2, t]$, then $J^{x,y} \leq t$, and similarly exchanging the roles of x and y . Using (2.3), Lemma 2.5(i.) and (3.10), we thus have

$$\begin{aligned} & \mathbb{P}(J^{x,y} > t) \\ & \leq \mathbb{P}(T^x = \infty, T^y < \infty, J^{x,y} > t) + \mathbb{P}(T^x < \infty, T^y = \infty, J^{x,y} > t) + \mathbb{P}(T^x = T^y = \infty, J^{x,y} > t) \\ & \leq \mathbb{P}(t/2 < T^y < \infty) + \mathbb{P}(T^x = \infty, \exists n : \tau_n^x \in [t/2, t]) \\ & \quad + \mathbb{P}(t/2 < T^x < \infty) + \mathbb{P}(T^y = \infty, \exists n : \tau_n^y \in [t/2, t]) \\ & \quad + \mathbb{P}(T^x = T^y = \infty) \cdot \tilde{\mathbb{P}}^{x,y}(J^{x,y} > t) \leq \frac{C|x-y|}{\sqrt{t}}. \end{aligned}$$

(ii.) Let A be a borelian of $[0, \infty)$ and B be an event in the σ -field of Harris constructions. Using Proposition 2.3 (ii.),

$$\begin{aligned} & \tilde{\mathbb{P}}^x(J^{x,y} \in A, T^y < \infty, \theta(\eta_{J^{x,y}, J^{x,y}}^x)(H) \in B) = \\ & \sum_{n=1}^{\infty} \tilde{\mathbb{P}}^x(\tau_{n-1}^x < T^y \leq \tau_n^x, \tau_n^x \in A, \theta(\eta_{\tau_n^x, \tau_n^x}^x)(H) \in B) = \\ & \tilde{\mathbb{P}}^0(H \in B) \cdot \sum_{n=1}^{\infty} \tilde{\mathbb{P}}^x(\tau_{n-1}^x < T^y \leq \tau_n^x, \tau_n^x \in A) = \tilde{\mathbb{P}}^0(H \in B) \cdot \tilde{\mathbb{P}}^x(J^{x,y} \in A, T^y < \infty) \end{aligned}$$

Using Proposition 3.1 (ii) and the fact that $\tilde{\mathbb{P}}^{z,z} = \tilde{\mathbb{P}}^z$ for any z ,

$$\begin{aligned} & \tilde{\mathbb{P}}^x(J^{x,y} \in A, T^y = \infty, \theta(\eta_{J^{x,y}, J^{x,y}}^x)(H) \in B) = \\ & \frac{\mathbb{P}(T^x = T^y = \infty)}{\mathbb{P}(T^x = \infty)} \sum_{n=1}^{\infty} \sum_{z \in \mathbb{Z}} \tilde{\mathbb{P}}^{x,y} \left(\begin{array}{l} \eta_{\tau_{n-1}^x}^x \neq \eta_{\tau_{n-1}^y}^y, \eta_{\tau_n^x}^x = \eta_{\tau_n^y}^y = z, \\ \tau_n^{x,y} \in A, \theta(z, \tau_n^{x,y})(H) \in B \end{array} \right) = \\ & \frac{\mathbb{P}(T^x = T^y = \infty)}{\mathbb{P}(T^x = \infty)} \tilde{\mathbb{P}}^0(H \in B) \cdot \tilde{\mathbb{P}}^{x,y}(J^{x,y} \in A) = \tilde{\mathbb{P}}^0(H \in B) \cdot \tilde{\mathbb{P}}^x(J^{x,y} \in A, T^y = \infty). \end{aligned}$$

Putting things together we get

$$\tilde{\mathbb{P}}^x(J^{x,y} \in A, \theta(\eta_{J^{x,y}, J^{x,y}}^x)(H) \in B) = \tilde{\mathbb{P}}^0(H \in B) \cdot \tilde{\mathbb{P}}^x(J^{x,y} \in A).$$

The claim is a direct consequence of this equality. ■

Lemma 3.4. *There exist $c, C > 0$ such that, for any $x, y \in \mathbb{Z}, N \geq 1$ and $t \geq 0$,*

$$\mathbb{P}(T^x, T^y > t, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) \neq (\eta_{1,t}^y, \dots, \eta_{N,t}^y)) \leq Ce^{CN-ct} + \frac{C|x-y|}{\sqrt{t}}.$$

Proof. There exists $\delta > 0$ such that, given a finite set $A \subset \mathbb{Z}$, we have

$$\mathbb{P}(T^A < \infty) > \delta^{|A|}. \tag{3.11}$$

We can for instance take δ as the probability of a particle dying out before having any children, an observe that this occurs independently for different sites. Define

$$\sigma_N = \sup\{s \geq 0 : 0 < \#\eta_{*,s}^0 \leq N\}; \quad \tilde{\sigma}_N = \inf\{s > t : 0 < \#\eta_{*,s}^0 \leq N\},$$

with the convention $\inf \emptyset = \infty$. Then, $\tilde{\sigma}_N$ is a stopping time and $\{\sigma_N > t\} = \{\tilde{\sigma}_N < \infty\}$. Using this, (3.11) and the Strong Markov Property, we have

$$\begin{aligned} \mathbb{P}(t < T^0 < \infty) &\geq \sum_{A \subset \mathbb{Z}, 0 < \#A \leq N} \mathbb{P}(\tilde{\sigma}_N < \infty, \eta_{*, \tilde{\sigma}_N}^0 = A) \cdot \mathbb{P}(T^A < \infty) \\ &\geq \delta^N \sum_{A \subset \mathbb{Z}, 0 < \#A \leq N} \mathbb{P}(\tilde{\sigma}_N < \infty, \eta_{*, \tilde{\sigma}_N}^0 = A) = \delta^N \mathbb{P}(\tilde{\sigma}_N < \infty) = \delta^N \mathbb{P}(\sigma_N > t). \end{aligned}$$

We then have $\mathbb{P}(\sigma_N > t) \leq \delta^{-N} \mathbb{P}(t < T^0 < \infty)$; also using (2.3), we obtain

$$\mathbb{P}(\sigma_N > t) \leq C_1 e^{C_2 N - c_1 t}. \quad (3.12)$$

for some $C_1, c_1, C_2 > 0$.

Let $x, y \in \mathbb{Z}$; assume that $T^x = T^y = \infty$ and $J^{x,y} + \sigma_N \circ \theta(\eta_{J^{x,y}}^x, J^{x,y}) \leq t$. This means that first, (η^x) and (η^y) have the first joint renewal at some space-time point $(\eta_{J^{x,y}}^x, J^{x,y}) = (\eta_{J^{x,y}}^y, J^{x,y})$ with $J^{x,y} \leq t$, and second, that the ancestry process of $(\eta_{J^{x,y}}^x, J^{x,y})$ never has less than N elements after time t . We must then have $z_1, \dots, z_N \in \mathbb{Z}$ such that

$$\eta_{n, t - J^{x,y}}^0 \circ \theta(\eta_{J^{x,y}}^x, J^{x,y}) = \eta_{n, t - J^{x,y}}^0 \circ \theta(\eta_{J^{x,y}}^y, J^{x,y}) = z_n, \quad 1 \leq n \leq N.$$

Lemma 2.2 then implies that $\eta_{n,t}^x = \eta_{n,t}^y = z_n$, $1 \leq n \leq N$, and we have thus shown that

$$\begin{aligned} &\{T^x = T^y = \infty, J^{x,y} + \sigma_N \circ \theta(\eta_{J^{x,y}}^x, J^{x,y}) \leq t\} \\ &\subset \{T^x = T^y = \infty, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) = (\eta_{1,t}^y, \dots, \eta_{N,t}^y)\}. \end{aligned}$$

Then,

$$\begin{aligned} &\mathbb{P}(T^x = T^y = \infty, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) \neq (\eta_{1,t}^y, \dots, \eta_{N,t}^y)) \\ &\leq \mathbb{P}(T^x = T^y = \infty, J^{x,y} + \sigma_N \circ \theta(\eta_{1, J^{x,y}}^x, J^{x,y}) > t) \\ &\leq \mathbb{P}(J^{x,y} > t/2) + \mathbb{P}(\sigma_N \circ \theta(\eta_{1, J^{x,y}}^x, J^{x,y}) > t/2 \mid T^x = \infty) \\ &\leq \frac{C|x-y|}{\sqrt{t}} + C e^{CN-ct}, \end{aligned}$$

where in the last inequality we used Lemma 3.3(i.) in the first term and Lemma 3.3(ii.) and (3.12) in the second.

Finally, we have

$$\begin{aligned} &\mathbb{P}(T^x, T^y > t, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) \neq (\eta_{1,t}^y, \dots, \eta_{N,t}^y)) \\ &\leq \mathbb{P}(t < T^x < \infty) + \mathbb{P}(t < T^y < \infty) + \mathbb{P}(T^x = T^y = \infty, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) \neq (\eta_{1,t}^y, \dots, \eta_{N,t}^y)) \\ &\leq 2C e^{-ct} + C e^{CN-ct} + \frac{C|x-y|}{\sqrt{t}} \leq C e^{CN-ct} + \frac{C|x-y|}{\sqrt{t}}. \end{aligned}$$

■

Proposition 3.5. *There exist $C, \gamma > 0$ such that, for any $N \geq 1$ and $t \geq 0$,*

$$\mathbb{P}(0 \in \{\eta_{n,t}^x : x \in \mathbb{Z}, 1 \leq n \leq N\}) \leq C \frac{N}{t^\gamma}.$$

Proof. Fix a real $t \geq 0$ and a positive integer l with $l > N$. Define $\Gamma = \{0, \dots, l-1\}$ and

$$\Lambda = \bigcap_{\{x,y\} \subset \Gamma} \left(\{T^x \leq t\} \cup \{T^y \leq t\} \cup \{(\eta_{1,t}^x, \dots, \eta_{N,t}^x) = (\eta_{1,t}^y, \dots, \eta_{N,t}^y)\} \right),$$

that is, for all sites in Γ that have non-empty ancestry at time t , the first N terms of the ancestor sequence at time t must coincide. We can use Lemma 3.4 to bound the probability of Λ^c :

$$\mathbb{P}(\Lambda^c) \leq \sum_{\{x,y\} \subset \Gamma} \mathbb{P}(T^x, T^y > t, (\eta_{1,t}^x, \dots, \eta_{N,t}^x) \neq (\eta_{1,t}^y, \dots, \eta_{N,t}^y)) \leq \frac{Cl^3}{\sqrt{t}} + Cl^2 e^{CN-ct} \quad (3.13)$$

since there are less than l^2 choices for $\{x, y\}$ and for any of them, $|x - y| \leq l$.

Let $\eta_{n,t}^\Gamma = \{\eta_{n,t}^x \in \mathbb{Z} : x \in \Gamma\}$ and $\eta_{N-,t}^\Gamma = \{\eta_{n,t}^x \in \mathbb{Z} : x \in \Gamma, 1 \leq n \leq N\}$. We have

$$\sum_{r=0}^{l-1} \sum_{n=1}^N \mathbb{P}(\Lambda, \{\eta_{n,t}^\Gamma \subset (r + l\mathbb{Z})\}) \leq \sum_{n=1}^N \mathbb{P}(\Lambda) \leq N.$$

A consequence of the above inequality is that there exists $r^* \in \{0, \dots, l-1\}$ such that

$$\sum_{n=1}^N \mathbb{P}(\Lambda, \{\eta_{n,t}^\Gamma \subset (r^* + l\mathbb{Z})\}) \leq \frac{N}{l}. \quad (3.14)$$

Finally, for $z \in \mathbb{Z}$ let $\Gamma_z = -r^* + lz + \Gamma$. The idea is that 0 seen from Γ_0 is the same as r^* seen from Γ . Let $\Lambda_z, \eta_{n,t}^{\Gamma_z}$ and $\eta_{N-,t}^{\Gamma_z}$ be defined from Γ_z as $\Lambda, \eta_{n,t}^\Gamma$ and $\eta_{N-,t}^\Gamma$ are defined from Γ . We can now proceed to our upper bound:

$$\begin{aligned} \mathbb{P}(0 \in \{\eta_{n,t}^x : x \in \mathbb{Z}, 1 \leq n \leq N\}) &\leq \sum_{z \in \mathbb{Z}} \mathbb{P}(0 \in \eta_{N-,t}^{\Gamma_z}) \\ &= \sum_{z \in \mathbb{Z}} \mathbb{P}(\{0 \in \eta_{N-,t}^{\Gamma_z}\}, \Lambda_z) + \sum_{z \in \mathbb{Z}} \mathbb{P}(\{0 \in \eta_{N-,t}^{\Gamma_z}\}, \Lambda_z^c) \\ &\leq \sum_{z \in \mathbb{Z}} \sum_{n=1}^N \mathbb{P}(\Lambda_z, \{\eta_{n,t}^{\Gamma_z} = \{0\}\}) + \sum_{z \in \mathbb{Z}} \sum_{x \in \Gamma_z} \sum_{n=1}^N \mathbb{P}(\Lambda_z^c, \{\eta_{n,t}^x = 0\}) \\ &= \sum_{n=1}^N \sum_{z \in \mathbb{Z}} \mathbb{P}(\Lambda, \{\eta_{n,t}^\Gamma = \{r^* + lz\}\}) + \sum_{x \in \Gamma} \sum_{n=1}^N \sum_{z \in \mathbb{Z}} \mathbb{P}(\Lambda^c, \{\eta_{n,t}^x = r^* + lz\}). \end{aligned} \quad (3.15)$$

Noting that

$$\cup_{z \in \mathbb{Z}} (\Lambda \cap \{\eta_{n,t}^\Gamma = \{r^* + lz\}\}) = \Lambda \cap \{\eta_{n,t}^\Gamma \in (r^* + l\mathbb{Z})\}, \cup_{z \in \mathbb{Z}} \{\eta_{n,t}^x = r^* + lz\} = \{\eta_{n,t}^x \in r^* + l\mathbb{Z}\}$$

and the unions are disjoint, (3.15) is equal to

$$\begin{aligned} &\sum_{n=1}^N \mathbb{P}(\Lambda, \{\eta_{n,t}^\Gamma \in (r^* + l\mathbb{Z})\}) + \sum_{x \in \Gamma} \sum_{n=1}^N \mathbb{P}(\Lambda^c; \{\eta_{n,t}^x \in r^* + l\mathbb{Z}\}) \\ &\leq \sum_{n=1}^N \mathbb{P}(\Lambda, \{\eta_{n,t}^\Gamma \in (r^* + l\mathbb{Z})\}) + \sum_{x \in \Gamma} \sum_{n=1}^N \mathbb{P}(\Lambda^c) \leq \frac{N}{l} + \frac{CNl^4}{\sqrt{t}} + CNl^3 e^{CN-ct}. \end{aligned}$$

We now put $l = t^{\frac{1}{9}}$; we have thus obtained

$$\begin{aligned} \mathbb{P}(0 \in \{\eta_{n,t}^x : x \in \mathbb{Z}, 1 \leq n \leq N\}) &\leq \left(\frac{N}{t^{1/9}} + \frac{CNt^{4/9}}{t^{1/2}} + CNt^{1/3}e^{CN-ct} \right) \wedge 1 \\ &\leq \frac{N}{t^{1/9}} + \frac{CN}{t^{1/18}} + (CNt^{1/3}e^{CN-ct}) \wedge 1. \end{aligned}$$

The first two terms are already in the form we want, and it is straightforward to show that, for some $C' > 0$, we have $(CNt^{1/3}e^{CN-ct}) \wedge 1 \leq C'N/t$ for all N, t . ■

4 Extinction, Survival and Coexistence

In this section we prove Theorems 1.1 and 1.2. Our three key ingredients will be a result about extinction under a stronger hypothesis (Lemma 4.1), an estimate for the edge speed of one of the types when obstructed by the other (Lemma 4.2) and the formation of “descendancy barriers” for the contact process on \mathbb{Z} (Lemma 4.3).

We recall our notation from the Introduction: the letters ξ and η will be used for the multitype contact process and the ancestry process, respectively. Throughout this section, in contrast with the rest of the paper, Harris constructions and statements related to them, such as “ $(x, s) \leftrightarrow (y, t)$ ”, refer to the construction for ξ rather than the one for η .

Lemma 4.1. *For the process (ξ_t) with initial state ξ_0 such that there exists $A > 0$ such that $\xi_0(x) = 2$ for all x with $|x| \geq A$, the 1’s almost surely die out, i.e. almost surely there exists t such that $\xi_t(x) \neq 1 \forall x$.*

Proof. Using (2.6), for any $t_0 > 0$ we have

$$\begin{aligned} \mathbb{P}(\exists t : \forall x, \xi_t(x) \neq 1) &\geq \mathbb{P}(\forall x, \xi_{t_0}(x) \neq 1) = \mathbb{P} \left(\begin{array}{l} \forall x, \text{ either } \eta_{*,t_0}^x = \emptyset, \text{ or} \\ \xi_0 \text{ is equal to 0 on } \eta_{*,t_0}^x, \text{ or} \\ \xi_0(\eta_{n^*(x),t_0}^x) = 2 \end{array} \right) \\ &\geq \mathbb{P} \left(\begin{array}{l} \forall x, \text{ either } \eta_{*,t_0}^x = \emptyset \text{ or} \\ \eta_{1,t_0}^x \in (-A, A)^c \end{array} \right) \geq 1 - \sum_{i=-A}^A \mathbb{P}(\exists x : \eta_{1,t_0}^x = i). \end{aligned}$$

By (3.5) each of the probabilities in the last sum converges to 0 as $t_0 \rightarrow \infty$. ■

Lemma 4.2. *Fix $\beta > 0$. For any $\epsilon > 0$, there exists $K > 0$ such that, if $\xi_0 = \xi^H = \mathbb{1}_{(-\infty, 0]} + 2 \cdot \mathbb{1}_{(0, \infty)}$, then*

$$\mathbb{P}(\sup\{x : \xi_t(x) = 1\} \leq K + \beta t \forall t) > 1 - \epsilon.$$

Proof. For $K > 0$, consider the events

$$\begin{aligned} A_n &= \{\xi_n(x) = 1 \text{ for some } x \geq K/2 + \beta n/2\}, \\ B_n &= \{(x, n) \leftrightarrow (y, t) \text{ for some } x < K/2 + \beta n/2, y \geq K + \beta n, t \in [n, n+1]\}, \end{aligned}$$

$n \in \{0, 1, 2, \dots\}$. Now, using Lemma 2.5 (iii.),

$$\begin{aligned}
\mathbb{P}(\cup_{n=0}^{\infty} A_n) &\leq \sum_{n=0}^{\infty} \sum_{x=K/2+\beta n/2}^{\infty} \mathbb{P}(\eta_{1,n}^x \leq 0) \leq \sum_{n=0}^{\infty} \sum_{x=K/2+\beta n/2}^{\infty} \mathbb{P}(|\eta_{1,n}^0| \geq x) \\
&\leq \sum_{n=0}^{\infty} \sum_{x=K/2+\beta n/2}^{\infty} (Ce^{-cx^2/n} + Ce^{-cx}) \\
&\leq C \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} e^{-\frac{c}{n}(\frac{K}{2}+\frac{\beta n}{2}+i)^2} + C \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} e^{-c(\frac{K}{2}+\frac{\beta n}{2}+i)} \\
&\leq C \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} e^{-c(\frac{\beta}{2}+\frac{K}{2n}+\frac{i}{n})(\frac{K}{2}+\frac{\beta n}{2}+i)} + C \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} e^{-c(\frac{K}{2}+\frac{\beta n}{2}+i)} \\
&\leq C \sum_{n=0}^{\infty} \sum_{i=0}^{\infty} e^{-c\frac{\beta}{2}(\frac{K}{2}+\frac{\beta n}{2}+i)} + Ce^{-\frac{cK}{2}} \left(\sum_{n=0}^{\infty} e^{-\frac{c\beta n}{2}} \right) \left(\sum_{i=0}^{\infty} e^{-ci} \right) \xrightarrow{K \rightarrow \infty} 0.
\end{aligned}$$

Next, event B_n requires the existence of a path that advances a distance of at least $K/2 + \beta n/2$ in a unit time interval; by a comparison with a multiple of a Poisson random variable as in (2.1), this occurs with probability smaller than $Ce^{-c(K/2+\beta n/2)}$ for some $c, C > 0$, so $\mathbb{P}(\cup_n B_n) \leq \sum_n \mathbb{P}(B_n) \xrightarrow{K \rightarrow \infty} 0$ as well. This gives $\mathbb{P}(\cap_n (A_n^c \cap B_n^c)) \rightarrow 1$ as $K \rightarrow \infty$, and to conclude the proof note that on $\cap_n (A_n^c \cap B_n^c)$, the set $\{(x, t) : \xi_t(x) = 1\}$ is contained in $\{(x, t) : x < K + \beta t\}$. ■

For $\rho > 0$, define $V(\rho) = \{(x, t) \in \mathbb{Z} \times [0, \infty) : -\rho t \leq x \leq \rho t\}$. We say that site 0 forms a ρ -descendancy barrier (according to the Harris construction H) if

- (D1) for any $x, y \in \mathbb{Z}$ and $t \geq 0$ with $(x, 0) \leftrightarrow (y, t)$ and $(y, t) \in V(\rho)$, we have $(0, 0) \leftrightarrow (y, t)$;
- (D2) for any $x, y \in \mathbb{Z}$ with opposite signs and $t \geq 0$ such that $(x, 0) \leftrightarrow (y, t)$, we have $(0, 0) \leftrightarrow (y, t)$.

Say that $x \in \mathbb{Z}$ forms a ρ -descendancy barrier if the origin forms a ρ -descendancy barrier according to $\theta(x, 0)(H)$.

Lemma 4.3. *For any $\epsilon > 0$, there exists $\beta, K > 0$ such that*

$$\mathbb{P}(\exists x \in [0, K] : x \text{ forms a } \beta\text{-descendancy barrier}) > 1 - \epsilon.$$

The proof is in [2]; see Proposition 2.7 and the definition of the event \mathcal{A}_2 in page 10 of that paper. Before the proof of Theorem 1.1, we state two more lemmas. Their proofs are straightforward and we omit them. For the following, as is usual, we abbreviate $\{x : \xi_t(x) = i\}$ as $\{\xi_t = i\}$.

Lemma 4.4. *Let $(\xi'_t), (\xi''_t)$ be two realizations of the multitype contact process built with the same Harris construction and such that*

$$\{\xi'_0 = 1\} \supset \{\xi''_0 = 1\}, \quad \{\xi'_0 = 2\} \subset \{\xi''_0 = 2\}.$$

Then,

$$\{\xi'_t = 1\} \supset \{\xi''_t = 1\}, \quad \{\xi'_t = 2\} \subset \{\xi''_t = 2\} \quad \forall t \geq 0.$$

This can be verified by looking at the generator of the multitype contact process. It is an attractiveness property: defining the partial order $\xi' \prec \xi'' \iff \{\xi' = 1\} \supset \{\xi'' = 1\}, \{\xi' = 2\} \subset \{\xi'' = 2\}$, the above lemma says that the set $\{(\xi', \xi'') : \xi' \prec \xi''\}$ is invariant under the coupled dynamics we are considering.

Lemma 4.5. *Assume at least one site is occupied by a 1 in ξ_0 and let $K \in \mathbb{N}$.*

(i.) *If conditions (A) and (B) of Theorem 1.1 are satisfied, then almost surely there exist (random) $L', a_1, a_2 \in \mathbb{Z}$ with $L' > 0, a_1 < -L', a_2 > L'$ such that*

$$\begin{aligned} (A') \quad & \xi_1(x) \neq 1 \quad \forall x \notin [-L', L']; \\ (B') \quad & \#\{x \in (-\infty, L'] : \xi_1(x) = 2\} = \#\{x \in [L', \infty) : \xi_1(x) = 2\} = \infty; \\ (C') \quad & \xi_1(x) = 2 \quad \forall x \in [a_1 - K, a_1] \cup [a_2, a_2 + K]. \end{aligned}$$

(ii.) *Assume condition (A) of Theorem 1.1 is satisfied but condition (B) is not (say, with finitely many 2's in $[0, \infty)$) and, for a given $a \in \mathbb{Z}$, we have $\xi_0(a) = 1$. Then, with positive probability,*

$$\begin{aligned} (A'') \quad & \xi_1(x) = 1 \quad \forall x \in [a, a + K]; \\ (B'') \quad & \xi_1(x) \neq 2 \quad \forall x > a + K. \end{aligned}$$

(iii.) *Assume the hypotheses of Theorem 1.2 are satisfied and, for given $b, c \in \mathbb{Z}$ with $b < c$, we have $\xi_0(b) = i, \xi_0(c) = j$, where $\{i, j\} = \{1, 2\}$. Then, with positive probability,*

$$\begin{aligned} (A''') \quad & \xi_1(x) \neq j \quad \forall x \in (-\infty, c - K); \\ (B''') \quad & \xi_1(x) = i \quad \forall x \in [c - K, c); \\ (C''') \quad & \xi_1(x) = j \quad \forall x \in [c, c + K); \\ (D''') \quad & \xi_1(x) \neq i \quad \forall x \in [c + K, \infty). \end{aligned}$$

Proof of Theorem 1.1. We first prove that, if conditions (A) and (B) in the statement of the theorem are satisfied, then the 1's almost surely become extinct. Fix $\epsilon > 0$. As in Lemma 4.3, choose β, K_1 corresponding to ϵ , then as in Lemma 4.2, choose K_2 corresponding to ϵ and β . Let $K = K_1 + K_2 + 2R$. Using Lemma 4.5(i.) with this value of K and relabeling time so as to start looking at the process at time 1, we may assume that there exist a_1, a_2, L' such that (A'), (B') and (C') are satisfied by ξ_0 (rather than by ξ_1).

Let $(\xi_t^1), (\xi_t^2), (\xi_t^{12})$ and (ξ_t^{21}) be realizations of the multitype contact process all built using the same Harris construction as the original process (ξ_t) and having initial configurations

$$\begin{aligned} \xi_0^1 &= \mathbb{1}_{(a_1, a_2)} + 2 \cdot \mathbb{1}_{[a_1 - K, a_1]} + 2 \cdot \mathbb{1}_{[a_2, a_2 + K]}; \\ \xi_0^2 &= \mathbb{1}_{(a_1, a_2)} + 2 \cdot \mathbb{1}_{(a_1, a_2)^c}; \\ \xi_0^{12} &= \mathbb{1}_{(-\infty, a_2)} + 2 \cdot \mathbb{1}_{[a_2, \infty)}; \\ \xi_0^{21} &= 2 \cdot \mathbb{1}_{(-\infty, a_1]} + \mathbb{1}_{(a_1, \infty)}. \end{aligned}$$

By a series of comparisons and uses of the previous lemmas, we will show that in ξ^1 , the 1's become extinct with high probability. An application of Lemma 4.4 to the pair ξ^1, ξ then implies that in ξ , the 1's become extinct with high probability.

Define the events

$$\mathcal{G}_1 = \{\forall t, \inf\{\xi_t^{21} = 1\} > a_1 - K_2 - \beta t\}, \quad \mathcal{G}_2 = \{\forall t, \sup\{\xi_t^{12} = 1\} < a_2 + K_2 + \beta t\}.$$

By the choice of K_2 , we have $\mathbb{P}(\mathcal{G}_1), \mathbb{P}(\mathcal{G}_2) > 1 - \epsilon$. Defining $W = \{(x, t) : a_1 - K_2 - \beta t < x < a_2 + K_2 + \beta t\}$ and applying Lemma 4.4 to the pairs ξ^{12}, ξ^2 and ξ^{21}, ξ^2 , we get that

$$\text{on } \mathcal{G}_1 \cap \mathcal{G}_2, \quad \{(x, t) : \xi_t^2(x) = 1\} \subset W. \quad (4.1)$$

Also define

$$\mathcal{G}_3 = \{\exists b_1 \in [a_1 - K, a_1 - K + K_1] : b_1 \text{ forms a } \beta\text{-descendancy barrier}\};$$

$$\mathcal{G}_4 = \{\exists b_2 \in [a_2 + K - K_1, a_2 + K] : b_2 \text{ forms a } \beta\text{-descendancy barrier}\}.$$

The choice of K_1 and β gives $\mathbb{P}(\mathcal{G}_3), \mathbb{P}(\mathcal{G}_4) > 1 - \epsilon$. Put $W_+ = \{(x, t) : a_1 - K_2 - 2R - \beta t < x < a_2 + K_2 + 2R + \beta t\}$; a consequence of the definition of descendancy barriers is that

$$\text{on } \mathcal{G}_3 \cap \mathcal{G}_4, \quad \forall (x, t) \in W_+, \quad \xi_t^1(x) = 0 \Rightarrow \xi_t^2(x) = 0. \quad (4.2)$$

Indeed, assume that $\mathcal{G}_3 \cap \mathcal{G}_4$ occurs, $(x, t) \in W_+$ and $\xi_t^2(x) \neq 0$. Then, there exist $y \in \mathbb{Z}$ with $\xi_0^2(y) \neq 0$ and a path $\gamma : [0, t] \rightarrow \mathbb{Z}$ determined by the graphical construction and connecting $(y, 0)$ to (x, t) , as explained in the beginning of Section 2. We can also choose b_1, b_2 as in the definition of $\mathcal{G}_3, \mathcal{G}_4$. Using the facts that $(x, t) \in W_+, b_1 \in [a_1 - K, a_1 - K + K_1], b_2 \in [a_2 + K - K_1, a_2 + K]$, we see that at least one of the following five cases hold:

- $y \in [a_1 - K, a_2 + K]$. Then, $\xi_0^1(y) \neq 0$, so $\xi_t^1(y) \neq 0$.
- $(x, t) \in V(b_1)$. Then, by (D1) in the definition of descendancy barrier we must have $(b_1, 0) \leftrightarrow (x, t)$; since $\xi_0^1(b_1) \neq 0$ also holds, we get $\xi_t^1(x) \neq 0$.
- $(x, t) \in V(b_2)$. We get $\xi_t^1(x) \neq 0$ as in the previous case.
- $y - b_1, x - b_1$ have opposite signs. Then, by (D2) we have $(b_1, 0) \leftrightarrow (x, t)$, so $\xi_t^1(x) \neq 0$.
- $y - b_2, x - b_2$ have opposite signs. We get $\xi_t^1(x) \neq 0$ as in the previous case.

We now claim that, on $\cap_{i=1}^4 \mathcal{G}_i, \{(x, t) : \xi_t^1(x) = 1\} = \{(x, t) : \xi_t^2(x) = 1\}$. This claim, together with Lemma 4.1, will imply that with probability larger than $1 - 4\epsilon$, the 1's die out in ξ^1 , and we will be done. To prove the claim, we start observing that $\{(x, t) : \xi_t^1(x) = 1\} \supset \{(x, t) : \xi_t^2(x) = 1\}$ always holds by Lemma 4.4. To establish the opposite inclusion in the occurrence of the good events, suppose to the contrary that for some $t, \{\xi_t^1 = 1\} \not\subset \{\xi_t^2 = 1\}$. Then we can find (x^*, t^*) such that $\xi_{t^*}^1(x^*) = 1, \xi_{t^*}^2(x^*) \neq 1$ and $\{\xi_t^1 = 1\} = \{\xi_t^2 = 1\} \forall t \in [0, t^*]$. We must then have $\xi_{t^*}^1(x^*) = 0$, since $\xi_{t^*}^1(x^*) = 2$ would be incompatible with $\xi_{t^*}^1(x^*) = 1$ and $\xi_{t^*}^1(x^*) = 1$ would imply, by the choice of $t^*, \xi_{t^*}^2(x^*) = 1$ and then $\xi_{t^*}^2(x^*) = 1$, a contradiction. Now, since $\xi_{t^*}^1(x^*) = 0$ and $\xi_{t^*}^1(x^*) = 1$ there must exist y^* with $|y^* - x^*| \leq R$ such that $\xi_{t^*}^1(y^*) = \xi_{t^*}^1(y^*) = 1$ and there exists an arrow from (y^*, t^*) to (x^*, t^*) . But then, again by the choice of $t^*, \xi_{t^*}^1(y^*) = 1$ implies $\xi_{t^*}^2(y^*) = 1$, so $\xi_{t^*}^2(y^*) = 1$. Using (4.1), we can then conclude that $(y^*, t^*) \in W$, so (x^*, t^*) is in the interior of W_+ . This, (4.2) and $\xi_{t^*}^1(x^*) = 0$ imply that $\xi_{t^*}^2(x^*) = 0$, so $\xi_{t^*}^2(x^*) = 1$, another contradiction. This completes the proof.

To prove the converse, we start noting that the case where there are infinitely many 1's in ξ_0 is trivial because then, at any $t \geq 0$ there almost surely exists some $x \in \mathbb{Z}$ such that $\xi_0(x) = 1$ and no death mark is present on $\{x\} \times [0, t]$, so the 1's are almost surely always present. We must thus

show that, if condition (A) of the theorem is satisfied but condition (B) is not, then the 1's have positive probability of surviving. We will first treat the case of $\xi_0 = \xi_0^{21K} := 2 \cdot \mathbb{1}_{(-\infty, 0)} + \mathbb{1}_{[0, K]}$; let us show that

$$\lim_{K \rightarrow \infty} \mathbb{P}(\forall t, \{\xi_t^{21K} = 1\} \neq \emptyset) = 1. \quad (4.3)$$

Fix $\epsilon > 0$ and choose β, K_1 and K_2 as before. We will need another constant K_3 whose choice will depend on the following. Let $\alpha > 0$ be the edge speed for our contact process (i.e., the almost sure limit as $t \rightarrow \infty$ of $\frac{1}{t} \sup\{y : \exists x \in (-\infty, 0] : (x, 0) \leftrightarrow (y, t)\}$). See Theorem 2.19 in [12] for the existence of the limit; the proof is easily seen to apply to our non-nearest neighbor context). Given $\alpha' \in (0, \alpha)$, we have

$$\lim_{K' \rightarrow \infty} \mathbb{P}(\forall t, \exists x \in [0, K'], y > \alpha' t : (x, 0) \leftrightarrow (y, t)) = 1. \quad (4.4)$$

This is a consequence of the definition of α and the fact that $\lim_{K' \rightarrow \infty} \mathbb{P}(\forall t, \exists x \in [0, K'], y \in \mathbb{Z} : (x, 0) \leftrightarrow (y, t)) = 1$; we omit the details. We may assume that the β we have chosen is strictly smaller than α : it is readily verified that if $x \in \mathbb{Z}$ forms a β -descendancy barrier, then it forms a β' -descendancy barrier for any $\beta' < \beta$; hence, we may decrease β if required. We choose K_3 such that, putting $K' = K_3$ and $\alpha' = \beta$, the probability in (4.4) is larger than $1 - \epsilon$. Set $K = K_1 + K_2 + K_3 + 2R$. Recycling some of the notation from before, define (ξ_t^{21}) with the same Harris construction as that of (ξ_t) , with

$$\xi_0^{21} = 2 \cdot \mathbb{1}_{(-\infty, 0)} + \mathbb{1}_{[0, \infty)}$$

and the events

$$\begin{aligned} \mathcal{G}_1 &= \{\forall t, \sup\{\xi_t^{21} = 2\} < K_2 + \beta t\}; \\ \mathcal{G}_2 &= \{\exists x \in (K_2 + 2R, K_2 + 2R + K_1] : x \text{ forms a } \beta\text{-descendancy barrier}\}; \\ \mathcal{G}_3 &= \{\forall t, \exists x \in (K_2 + 2R + K_1, K], y > K_2 + 2R + K_1 + \beta t : (x, 0) \leftrightarrow (y, t)\}. \end{aligned}$$

By the choices of K_1, K_2 and K_3 , we get $\mathbb{P}(\cap_{i=1}^3 \mathcal{G}_i) > 1 - 3\epsilon$. We can argue as before to the effect that, on $\mathcal{G}_1 \cap \mathcal{G}_2$, $\{\xi_t^{21} = 2\} = \{\xi_t = 2\}$ holds for all t , so $\sup\{\xi_t = 2\} < K_2 + \beta t$ for all t . Additionally, on \mathcal{G}_3 , for every t there exists $y > K + \beta t$ such that $\xi_t(y) \neq 0$, so it must be the case that $\xi_t(y) = 1$. This shows that for all t , $\{\xi_t = 1\} \neq \emptyset$. The proof of (4.3) is now complete.

Now assume that $\xi_0 \in \{0, 1, 2\}^{\mathbb{Z}}$ is a general configuration containing at least one 1 and satisfying (A) but not (B) in the statement of the theorem. Reflecting ξ_0 about 0 if necessary, we may assume that there are finitely many 2's in $[0, \infty)$ (such a reflection does not alter the property we want to prove, since the law of the Harris construction is invariant by it). Choose $K > 0$ such that $\mathbb{P}(\forall t, \{\xi_t^{21K} = 1\} \neq \emptyset) > 0$; this is possible by (4.3). Next, choose $a \in \mathbb{Z}$ such that $\xi_0(a) = 1$. Then, with positive probability, ξ_1 satisfies (A'') and (B'') in Lemma 4.5 corresponding to the chosen a and K . On this event, we can use Lemma 4.4 with $\xi_1, \xi_1^{a, 21K}$, where $\xi_1^{a, 21K} = 2 \cdot \mathbb{1}_{(-\infty, a)} + \mathbb{1}_{[a, a+K]}$ to conclude that the 1's survive with positive probability. Together with the Markov property, this gives

$$\mathbb{P}(\forall t, \{\xi_t = 1\} \neq \emptyset) \geq \mathbb{P}(\xi_1 \text{ satisfies (A''), (B'')}) \cdot \mathbb{P}(\forall t, \{\xi_1^{a, 21K} = 1\} \neq \emptyset) > 0.$$

■

Proof of Theorem 1.2. Choose K such that the probability in (4.3) is larger than $2/3$. Using the Markov property and Lemma 4.5(iii.) as in the end of the above proof, it suffices to prove the

statement for $\xi_0 = i \cdot \mathbb{1}_{[-K-1, -1]} + j \cdot \mathbb{1}_{[0, K]}$, with $\{i, j\} = \{1, 2\}$. Let $\xi_0^{ijk} = i \cdot \mathbb{1}_{(-\infty, 0)} + j \cdot \mathbb{1}_{[0, K]}$, $\xi_0^{Kij} = i \cdot \mathbb{1}_{[-K-1, -1]} + j \cdot \mathbb{1}_{[0, \infty)}$; again we will consider the processes $(\xi_t^{ijk}), (\xi_t^{Kij}), (\xi_t)$ defined with the same Harris construction. By the choice of K , the event $\{\forall t, \{\xi_t^{ijk} = j\} \neq \emptyset\} \cap \{\forall t, \{\xi_t^{Kij} = i\} \neq \emptyset\}$ has positive probability. Comparing the two pairs ξ^{ijk}, ξ and ξ^{Kij}, ξ via Lemma 4.4, we see that $\{\forall t, \{\xi_t^{ijk} = j\} \neq \emptyset\} \cap \{\forall t, \{\xi_t^{Kij} = i\} \neq \emptyset\} \subset \{\forall t, \{\xi_t = j\} \neq \emptyset\} \cap \{\forall t, \{\xi_t = i\} \neq \emptyset\}$, completing the proof. \blacksquare

5 Interface tightness

We now carry out the proof outlined at the end of the Introduction. It is instructive to restate Theorem 1.3 in its dualized form, which follows from (2.7):

Theorem 1.3, dual version For any $\epsilon > 0$, there exists $L > 0$ such that

$$\mathbb{P}(|\sup\{x : \eta_t^x \leq 0\} - \inf\{x : \eta_t^x > 0\}| > L) < \epsilon \text{ for every } t \geq 0.$$

We start with two Lemmas concerning the expectation of the distance between two first ancestors. Lemma 5.1 shows a resemblance to the case of two random walks that evolve independently until they meet, at which time they coalesce. Lemma 5.2 is a generalization that allows us to integrate over the event of death of a preassigned set of sites.

Lemma 5.1. *There exists $C > 0$ such that, for all $x < y \in \mathbb{Z}$ and $t \geq 0$,*

$$(i.) \mathbb{E}(|\eta_t^y - \eta_t^x|) \leq C(y - x);$$

$$(ii.) \mathbb{E}((\eta_t^y - \eta_t^x)^-) \leq C, \text{ where } z^- = -(z \wedge 0).$$

Proof. By translation invariance, it suffices to treat $x = 0 < y$. It also suffices to prove (i.) and (ii.) for t sufficiently large (not depending on x, y), because

$$\begin{aligned} \mathbb{E}(|\eta_t^y - \eta_t^0|) &\leq y + \mathbb{E}(|\eta_t^0|) + \mathbb{E}(|\eta_t^y - y|) \leq y + \mathbb{E}(M_t^0) + \mathbb{E}(M_t^y) = y + 2\mathbb{E}(M_t^0); \\ \mathbb{E}((\eta_t^y - \eta_t^0)^-) &\leq \mathbb{E}((\eta_t^0)^+) + \mathbb{E}((\eta_t^y - y)^-) \leq \mathbb{E}(M_t^0) + \mathbb{E}(M_t^y) = 2\mathbb{E}(M_t^0), \end{aligned}$$

and these expectations grow polynomially in t , by comparisons with Poisson random variables. Finally,

$$\begin{aligned} \mathbb{E}(|\eta_t^0 - \eta_t^y|) &= \sum_{z, w} |z - w| \mathbb{P}(\eta_t^0 = z, \eta_t^y = w) \\ &= \sum_{z, w} |z - w| \mathbb{P}(\eta_t^0 = z, \eta_t^y = w, T^{(z, t)} = T^{(w, t)} = \infty) \mathbb{P}(T^{(z, t)} = T^{(w, t)} = \infty)^{-1} \\ &\leq C \mathbb{E}(|\eta_t^0 - \eta_t^y|, T^0 = T^y = \infty); \end{aligned}$$

the second equality follows from independence of $H_{[0, t]}$ and $\theta(0, t)(H)$, and the inequality follows from (2.2). We can treat $\mathbb{E}((\eta_t^y - \eta_t^0)^-)$ similarly, so it suffices to prove (i.) and (ii.) on the event $\{T^0 = T^y = \infty\}$.

(i.) We have

$$\begin{aligned} \mathbb{E}(|\eta_t^y - \eta_t^0|; T^0 = T^y = \infty) &\leq y + \mathbb{E}(|\eta_t^0|; T^0 = T^y = \infty, J^{0, y} > t) + \mathbb{E}(|\eta_t^y - y|; T^0 = T^y = \infty, J^{0, y} > t) \\ &= y + 2\mathbb{E}(|\eta_t^0|; T^0 = T^y = \infty, J^{0, y} > t) \end{aligned} \tag{5.1}$$

by symmetry. By Cauchy-Schwarz, this last expectation is less than

$$\left(\mathbb{E}((\eta_t^0)^2; T^0 = T^y = \infty, J^{0,y} > t) \cdot \mathbb{P}(T^0 = T^y = \infty, J^{0,y} > t) \right)^{\frac{1}{2}}. \quad (5.2)$$

Let us estimate the expectation.

$$\begin{aligned} & \mathbb{E}((\eta_t^0)^2; T^0 = T^y = \infty, J^{0,y} > t) \\ & < \frac{1}{\mathbb{P}(T^0 = \infty)} \cdot \mathbb{E}((\eta_t^0)^2; T^0 = T^y = \infty, J^{0,y} > t) \\ & \leq \tilde{\mathbb{E}}^0((\eta_t^0)^2; J^{0,y} > t) \\ & = \tilde{\mathbb{E}}^0((\eta_t^0)^2) - \tilde{\mathbb{E}}^0((\eta_t^0)^2; J^{0,y} \leq t) \\ & = \tilde{\mathbb{E}}^0((\eta_t^0)^2) - \tilde{\mathbb{E}}^0((\eta_t^0 - \eta_{J^{0,y}}^0)^2 + (\eta_{J^{0,y}}^0)^2 + 2\eta_{J^{0,y}}^0(\eta_t^0 - \eta_{J^{0,y}}^0); J^{0,y} \leq t). \end{aligned} \quad (5.3)$$

By Lemma 3.3(ii.), we have

$$\tilde{\mathbb{E}}^0((\eta_t^0 - \eta_{J^{0,y}}^0)^2; J^{0,y} \leq t) = \int_0^t \tilde{\mathbb{E}}^0((\eta_{t-s}^0)^2) \cdot \tilde{\mathbb{P}}^0(J^{0,y} \in ds), \quad (5.4)$$

$$\tilde{\mathbb{E}}^0(\eta_{J^{0,y}}^0(\eta_t^0 - \eta_{J^{0,y}}^0); J^{0,y} \leq t) = 0. \quad (5.5)$$

Using (5.4) and (5.5) and ignoring the term $(\eta_{J^{0,y}}^0)^2$, the expression in (5.3) is less than

$$\begin{aligned} & \tilde{\mathbb{E}}^0((\eta_t^0)^2) - \int_0^t \tilde{\mathbb{E}}^0((\eta_{t-s}^0)^2) \cdot \tilde{\mathbb{P}}^0(J^{0,y} \in ds) \\ & \leq \tilde{\mathbb{E}}^0((\eta_t^0)^2) \cdot \tilde{\mathbb{P}}^0(J^{0,y} > t) + \int_0^t \tilde{\mathbb{E}}^0((\eta_t^0)^2 - (\eta_{t-s}^0)^2) \cdot \tilde{\mathbb{P}}^0(J^{0,y} \in ds) \\ & \leq (C_1 t + C_2) \frac{C y}{\sqrt{t}} + \int_0^t (C_1 s + C_2) \tilde{\mathbb{P}}^0(J^{0,y} \in ds) \end{aligned}$$

by Lemma 2.5(ii.) and Lemma 3.3(i.). Now we can continue as in Lemma 1 in [5]: the above is less than

$$C y \sqrt{t} + \frac{C y}{\sqrt{t}} + C \int_0^t \tilde{\mathbb{P}}^0(J^{0,y} > u) du + C \leq C y \sqrt{t} + C \int_0^t \frac{y}{\sqrt{u}} du \leq C y \sqrt{t}$$

when $t \geq 1$. This and another application of Lemma 3.3(i.) show that (5.2) is less than $\sqrt{C y \sqrt{t} \cdot \frac{C y}{\sqrt{t}}} \leq C y$; going back to (5.1), we get

$$\mathbb{E}(|\eta_t^0 - \eta_t^y|; T^0 = T^y = \infty) \leq C y.$$

(ii.) To treat the expectation on the event $\{T^0 = T^y = \infty\}$, we will separately consider two cases, depending on whether or not the ancestor processes of 0 and y had a joint renewal in inverted order before time t . To this end, define

$$\tau^* = \inf \left\{ \tau_n : \eta_{\tau_n}^y < \eta_{\tau_n}^0 \right\}$$

(we set $\tau^* = \infty$ if the set is empty). Now,

$$\begin{aligned} & \mathbb{E}((\eta_t^y - \eta_t^0)^-; T^0 = T^y = \infty, \tau^* \leq t) \\ & \leq \sum_{z < w} \int_0^t \tilde{\mathbb{E}}^{z,w}(|\eta_{t-s}^w - \eta_{t-s}^z|) \cdot \mathbb{P}(T^0 = T^y = \infty, \eta_{\tau^*}^y = z, \eta_{\tau^*}^0 = w, \tau^* \in ds) \end{aligned} \quad (5.6)$$

For each z, w , we have $\tilde{\mathbb{E}}^{z,w}(|\eta_{t-s}^w - \eta_{t-s}^z|) \leq \mathbb{P}(T^z = T^w = \infty)^{-1} \cdot C|w - z| \leq C|w - z|$ by part (i.) and (2.2). Then, (5.6) is less than

$$\begin{aligned} & C \sum_{z < w} \int_0^t (w - z) \mathbb{P}(T^0 = T^y = \infty, \eta_{\tau^*}^y = z, \eta_{\tau^*}^0 = w; \tau^* \in ds) \\ & \leq C \sum_{z < w} (w - z) \mathbb{P}(T^0 = T^y = \infty, \eta_{\tau^*}^y = z, \eta_{\tau^*}^0 = w, \tau^* < \infty) \\ & = C \mathbb{E}((\eta_{\tau^*}^y - \eta_{\tau^*}^0)^-; T^0 = T^y = \infty, \tau^* < \infty), \end{aligned} \quad (5.7)$$

which is bounded by Lemma 6.6.

Finally, as in Lemma 2.5, define on the event $\{T^0 = T^y = \infty\}$ the random variables $\tau_{t-}^{0,y}, \tau_{t+}^{0,y}$ and

$$\phi_t = M_{\tau_{t+}^{0,y}} \left(\eta_{\tau_{t-}^{0,y}}^0, \tau_{t-}^{0,y} \right) \vee M_{\tau_{t+}^{0,y}} \left(\eta_{\tau_{t-}^{0,y}}^y, \tau_{t-}^{0,y} \right).$$

We then have

$$\left| \eta_t^0 - \eta_{\tau_{t-}^{0,y}}^0 \right|, \left| \eta_t^y - \eta_{\tau_{t-}^{0,y}}^y \right| \leq \phi_t$$

on $\{T^0 = T^y = \infty\}$. Since on $\{T^0 = T^y = \infty, \tau^* > t\}$, $\eta_{\tau_{t-}^{0,y}}^0 \leq \eta_{\tau_{t-}^{0,y}}^y$ also holds, we have

$$\mathbb{E}((\eta_t^y - \eta_t^0)^-; T^0 = T^y = \infty, \tau^* > t) \leq \mathbb{E}(2\phi_t; T^0 = T^y = \infty, \tau^* > t). \quad (5.8)$$

As in the proof of Lemma 2.5, we can then show that $\mathbb{E}(\phi_t; T^0 = T^y = \infty)$ is bounded uniformly in y and t . Putting together (5.7) and (5.8), we get the result. \blacksquare

Lemma 5.2. *There exist $c, C > 0$ such that, for all $x < y \in \mathbb{Z}, t \geq 0$ and finite $A \subset \mathbb{Z}$,*

$$(i.) \mathbb{E}(|\eta_t^y - \eta_t^x|; T^A < t) \leq C(y - x)e^{-c|A|};$$

$$(ii.) \mathbb{E}((\eta_t^y - \eta_t^x)^-; T^A < t) \leq Ce^{-c|A|}.$$

Proof. Since both estimates are treated similarly, we will only show part (ii.):

$$\begin{aligned} & \mathbb{E}((\eta_t^y - \eta_t^x)^-; T^A < t) \\ & = \sum_{k=1}^{\infty} \mathbb{E}((\eta_t^y - \eta_t^x)^-; T^A < t, M_{T^A}^x \vee M_{T^A}^y = k) \\ & \leq \sum_{k=1}^{\infty} \sum_{i=-k}^k \sum_{j=-k}^k \mathbb{E}((\eta_{t-T^A}^{(y+j, T^A)} - \eta_{t-T^A}^{(x+i, T^A)})^-; T^A < t, M_{T^A}^x \vee M_{T^A}^y = k) \\ & \leq \sum_{k=1}^{\infty} \int_0^t \left(\sum_{i=-k}^k \sum_{j=-k}^k \mathbb{E}((\eta_{t-s}^{y+j} - \eta_{t-s}^{x+i})^-) \right) \mathbb{P}(T^A \in ds, M_{T^A}^x \vee M_{T^A}^y = k) \end{aligned}$$

If $x + i < y + j$, then $\mathbb{E} \left((\eta_{t-s}^{y+j} - \eta_{t-s}^{x+i})^- \right) \leq C$ by Lemma 5.1(ii.). If $x + i > y + j$, since we also have $(x + i) - (y + j) < 2k$, we get $\mathbb{E} \left(\left| \eta_{t-s}^{y+j} - \eta_{t-s}^{x+i} \right| \right) \leq 2Ck$ by Lemma 5.1(i.). Hence, in all cases the expectation is less than Ck , and the above sum is less than

$$\begin{aligned} C \sum_{k=1}^{\infty} k^3 \mathbb{P}(T^A < t, M_{T^A}^x \vee M_{T^A}^y = k) &\leq C \mathbb{E} \left((M_{T^A}^x \vee M_{T^A}^y)^3; T^A < \infty \right) \\ &\leq C \mathbb{E} \left((M_{T^A}^x)^3; T^A < \infty \right) + C \mathbb{E} \left((M_{T^A}^y)^3; T^A < \infty \right). \end{aligned} \quad (5.9)$$

Now, by Cauchy-Schwarz,

$$\mathbb{E} \left((M_{T^A}^x)^3; T^A < \infty \right) \leq \left(\mathbb{E} \left((M_{T^A}^x)^6; T^A < \infty \right) \cdot \mathbb{P}(T^A < \infty) \right)^{1/2}. \quad (5.10)$$

The probability in the right-hand side decreases exponentially with $|A|$ (see Section 11b in [6]). Using the bound

$$\mathbb{P}(M_{T^A}^x > l, T^A < \infty) \leq \mathbb{P} \left(\frac{l}{\sigma} < T^A < \infty \right) + \mathbb{P} \left(M_{l/\sigma}^0 > l \right)$$

with large σ and using (2.3) again, we see that the expectation on the right-hand side of (5.10) is uniformly bounded in x and A . This and the same bound applied to $\mathbb{E} \left((M_{T^A}^y)^3; T^A < \infty \right)$ in (5.9) complete the proof. \blacksquare

Let $z \in \mathbb{Z}, z > 0$ and $t \geq 0$. The following lemma shows that the expected number of $x \in \mathbb{Z}$ such that $\eta_t^x > 0 \geq \eta_t^{x+z}$ is bounded uniformly in z and t . It also illustrates the usefulness of Lemma 5.2.

Lemma 5.3. *There exist $c, C > 0$ such that, for any integer $z \geq 1$, real $t \geq 0$ and finite $A \subset \mathbb{Z}$,*

$$(i.) \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x > 0 \geq \eta_t^{x+z}, T^{x+A} < t) \leq C e^{-c|A|};$$

$$(ii.) \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x \leq 0 < \eta_t^{x+z}, T^{x+A} < t) \leq C|z|e^{-c|A|}.$$

Proof. We start proceeding like in Lemma 4 in [5], noticing that, by translation invariance,

$$\mathbb{P}(\eta_t^x > 0 \geq \eta_t^{x+z}, T^{x+A} < t) = \mathbb{P}(\eta_t^0 > -x \geq \eta_t^z, T^A < t),$$

$$\mathbb{P}(\eta_t^x \leq 0 < \eta_t^{x+z}, T^{x+A} < t) = \mathbb{P}(\eta_t^0 \leq -x < \eta_t^z, T^A < t)$$

and summing over x to obtain

$$\sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x > 0 \geq \eta_t^{x+z}, T^{x+A} < t) = \mathbb{E} \left((\eta_t^z - \eta_t^0)^-; T^A < t \right),$$

$$\sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x \leq 0 < \eta_t^{x+z}, T^{x+A} < t) = \mathbb{E} \left(|\eta_t^0 - \eta_t^z|; T^A < t \right);$$

see Lemma 4 in [5] for more details. Also recall our conventions about the Δ state in Remark 2.4. Now, it suffices to apply Lemma 5.2. \blacksquare

Fix $0 < s < t$. Given integers $x < y$, we say that x and y make an $[s, t]$ -inversion if $\eta_t^{(x,s)} > 0 \geq \eta_t^{(y,s)}$. Given $A \subset \mathbb{Z}$, we say that A contains an $[s, t]$ -inversion if there is some pair $x, y \in A$ such that x and y make an $[s, t]$ -inversion. Of course,

$$\begin{aligned} \exists x, y \in A, x < y : \eta_t^{(x,s)} > 0 \geq \eta_t^{(y,s)} &\Leftrightarrow \\ \exists x, y \in A, x < y : \eta_t^{(x,s)} > 0 \geq \eta_t^{(y,s)}, T^{(z,s)} < t \forall z \in (x, y) \cap A. \end{aligned}$$

or, in other words,

Lemma 5.4. *$A \subset \mathbb{Z}$ contains an $[s, t]$ -inversion if and only if there exist $x, y \in A, x < y$ such that x and y make an $[s, t]$ -inversion and $T^{(z,s)} < t$ for every $z \in (x, y) \cap A$.*

Again fix $0 < s < t$. For $x \in \mathbb{Z}$ such that $\eta_{*,t}^x \neq \emptyset$, let n be the smallest integer such that $(\eta_{n,s}^x, s)$ survives up to time t (as in the statement of Lemma 2.2(ii.)). Define $R_{s,t}^x = \eta_{n,s}^x$. Then define $R_{s,t} = \{R_{s,t}^x : x \in \mathbb{Z}, \eta_{*,t}^x \neq \emptyset\}$. This will be understood as a set of “relevant” sites. The reason is that, by Lemma 2.2 we have $\eta_t^x = \eta_t^{(R_{s,t}^x, s)}$ for all x such that $\eta_{*,t}^x \neq \emptyset$, hence $\eta_t^{(y,s)}$ is relevant to determine the values of $\{\eta_t^x : x \in \mathbb{Z}\}$ if and only if $y \in R_{s,t}$.

Finally define

$$\mathcal{G}_{s,t} = \{R_{s,t} \text{ contains no } [s, t]\text{-inversions}\}. \quad (5.11)$$

The following is a direct consequence of the equality $\eta_t^a = \eta_t^{(R_{s,t}^a, s)}$.

Lemma 5.5. *Let $a, b \in \mathbb{Z}, a < b$. On $\mathcal{G}_{s,t} \cap \{T^a > t\} \cap \{T^b > t\}$, a and b make a $[0, t]$ -inversion if and only if $R_{s,t}^a > R_{s,t}^b$ and $\eta_t^{(R_{s,t}^a, s)} > 0 \geq \eta_t^{(R_{s,t}^b, s)}$.*

Proposition 5.6. $\lim_{s \rightarrow \infty} \inf_{t \geq s} \mathbb{P}(\mathcal{G}_{s,t}) = 1$.

Proof. We fix $s < t$ and an integer N to be chosen later. We will write \mathcal{G}, R instead of $\mathcal{G}_{s,t}, R_{s,t}$, and in general omit the dependence on s, t, N .

Fix d with $1 > d \geq \mathbb{P}(0 \in \{\eta_{n,s}^x : x \in \mathbb{Z}, 1 \leq n \leq N\})$ and let X be a random variable with uniform distribution on $\{0, \dots, \lceil 1/d \rceil - 1\}$ and independent of the Harris construction. Define

$$\hat{R} = \{\eta_{n,s}^x : x \in \mathbb{Z}, 1 \leq n \leq N\} \cup (X + \lceil 1/d \rceil \mathbb{Z}).$$

\hat{R} is a random subset of \mathbb{Z} ; its law is invariant with respect to shifts in \mathbb{Z} and $\mathbb{P}(0 \in \hat{R}) \leq 2d$. Additionally, it only depends on the Harris construction on times in $[0, s]$, and of course on X . Put $\mathcal{S} = \{x \in \mathbb{Z} : \eta_{*,t}^{(x,s)} \neq \emptyset\}$. Note that by the definition of R , we have $R \subset \mathcal{S}$; also, by our conventions, when we say for example $\eta_t^{(x,s)} \geq 0$, we are implying that $x \in \mathcal{S}$.

We will proceed in three steps.

Step 1. We start establishing an upper bound for the probability of \hat{R} containing an $[s, t]$ -inversion. Using Lemma 5.4, we have

$$\left\{ \begin{array}{l} \hat{R} \text{ contains an} \\ [s, t]\text{-inversion} \end{array} \right\} \subset \left\{ \begin{array}{l} \exists x, y \in \hat{R}, x < y : (x, y) \cap \hat{R} \cap \mathcal{S} = \emptyset, \\ x, y \text{ make an } [s, t]\text{-inversion} \end{array} \right\},$$

so that

$$\begin{aligned}
\mathbb{P} \left(\hat{R} \text{ contains an } [s, t]\text{-inversion} \right) &\leq \sum_{x < y} \mathbb{P}([x, y] \cap \hat{R} \cap \mathcal{S} = \{x, y\}, \eta_t^{(x,s)} > 0 \geq \eta_t^{(y,s)}) \\
&\leq \sum_{x \in \mathbb{Z}, z \geq 1} \sum_{A \subset (0, z)} \mathbb{P}([x, x+z] \cap \hat{R} = \{x, x+z\} \cup (x+A), x+A \subset \mathcal{S}^c, \eta_t^{(x,s)} > 0 \geq \eta_t^{(x+z,s)}) \\
&= \sum_{z, A} \mathbb{P}([0, z] \cap \hat{R} = \{0, z\} \cup A) \cdot \sum_{x \in \mathbb{Z}} \mathbb{P}(T^{x+A} < t-s, \eta_{t-s}^x > 0 \geq \eta_{t-s}^{x+z}),
\end{aligned}$$

where we have written $y = x + z$, used the facts that $\{[x, x+z] \cap \hat{R} = \{x, x+z\} \cup (x+A)\}$ and $\{x+A \subset \mathcal{S}^c, \eta_t^{(x,s)} > 0 \geq \eta_t^{(x+z,s)}\}$ are independent and that $\mathbb{P}([x, x+z] \cap \hat{R} = \{x, x+z\} \cup (x+A)) = \mathbb{P}([0, z] \cap \hat{R} = \{0, z\} \cup A)$ by translation invariance. Applying Lemma 5.3 to the inner sum, we get that the above is less than

$$\begin{aligned}
&C \sum_{z, A} e^{-c(\#A)} \mathbb{P}([0, z] \cap \hat{R} = \{0, z\} \cup A) \\
&\leq C \sum_{k \geq 0} e^{-ck} \sum_{z \geq k+1} \sum_{A \subset (0, z): \#A=k} \mathbb{P}([0, z] \cap \hat{R} = \{0, z\} \cup A) \\
&\leq C \sum_{k \geq 0} e^{-ck} \mathbb{P}(0 \in \hat{R}) \leq Cd.
\end{aligned}$$

We thus have

$$\mathbb{P}(\hat{R} \text{ contains an } [s, t]\text{-inversion}) \leq Cd. \quad (5.12)$$

Step 2. Our next goal is to bound the probability of

$$\{R \cup \hat{R} \text{ contains an } [s, t]\text{-inversion}, \hat{R} \text{ contains no } [s, t]\text{-inversion}\} \quad (5.13)$$

We claim that this event is contained in

$$\left\{ \begin{array}{l} \exists x < y, x \in R - \hat{R}, (x, y) \cap \hat{R} \cap \mathcal{S} = \emptyset, \\ x, y \text{ make an } [s, t]\text{-inversion} \end{array} \right\} \cup \left\{ \begin{array}{l} \exists x < y, y \in R - \hat{R}, (x, y) \cap \hat{R} \cap \mathcal{S} = \emptyset, \\ x, y \text{ make an } [s, t]\text{-inversion} \end{array} \right\}. \quad (5.14)$$

Indeed, assume that (5.13) occurs. Then, by Lemma 5.4, there exist $x < y$ in $R \cup \hat{R}$ such that $(x, y) \cap \hat{R} \cap \mathcal{S} \subset (x, y) \cap (R \cup \hat{R}) \cap \mathcal{S} = \emptyset$ and x and y make an $[s, t]$ -inversion. We cannot have $x, y \in \hat{R}$, because this would imply that \hat{R} contains an $[s, t]$ -inversion, contradicting the definition of the event in (5.13). Hence, at least one of x, y is in $R - \hat{R}$, so at least one of the events in (5.14) occurs.

The probability of the first event in (5.14) is less than

$$\begin{aligned}
& \sum_{x < y} \mathbb{P}(x \in R - \hat{R}, (x, y) \cap \hat{R} \cap \mathcal{S} = \emptyset, \eta_t^{(x,s)} > 0 \geq \eta_t^{(y,s)}) \\
& \leq \sum_{x \in \mathbb{Z}, z \geq 1} \sum_{A \subset (0, z)} \sum_{a \in \mathbb{Z}, m > N} \sum_{(a_1, \dots, a_{m-1}) \in \mathbb{Z}^{m-1}} \\
& \quad \mathbb{P} \left(\begin{array}{l} \eta_{m,s}^{x+a} = x, \eta_{i,s}^{x+a} = x + a_i \ \forall i < m, \\ (x, x+z) \cap \hat{R} = x + A, \\ x + a_i \notin \mathcal{S} \ \forall i < m, x + A \subset \mathcal{S}^c, \eta_t^{(x,s)} > 0 \geq \eta_t^{(x+z,s)} \end{array} \right) \\
& \leq \sum_{z, A, a, m, (a_i)} \mathbb{P}(\eta_{m,s}^a = 0, \eta_{i,s}^a = a_i \ \forall i < m, (0, z) \cap \hat{R} = A) \cdot \\
& \quad \sum_{x \in \mathbb{Z}} \mathbb{P}(T^{x+a_i} < t - s \ \forall i < m, T^{x+A} < t - s, \eta_{t-s}^x > 0 \geq \eta_{t-s}^{x+z}) \\
& \leq C \sum_{z, A, a, m, (a_i)} e^{-c((\#A) \vee m)} \mathbb{P}(\eta_{m,s}^a = 0, \eta_{i,s}^a = a_i \ \forall i < m, (0, z) \cap \hat{R} = A) \\
& \leq C \sum_{z, A, a, m} e^{-c((\#A) \vee m)} \mathbb{P}(\eta_{m,s}^a = 0, (0, z) \cap \hat{R} = A) \\
& = C \sum_{k \geq 0} \sum_{m > N} e^{-c(k \vee m)} \sum_{a \in \mathbb{Z}} \sum_{z \geq k+1} \sum_{A \subset (0, z): \#A=k} \mathbb{P}(0 = \eta_{m,s}^a, (0, z) \cap \hat{R} = A) \\
& = C \sum_{k \geq 0} \sum_{m > N} e^{-c(k \vee m)} \sum_{a \in \mathbb{Z}} \sum_{z \geq k+1} \mathbb{P}(0 = \eta_{m,s}^a, \#((0, z) \cap \hat{R}) = k). \tag{5.15}
\end{aligned}$$

Now note that, since $X + [1/d]\mathbb{Z} \subset \hat{R}$, there are no intervals of length larger than $[1/d]$ that do not intersect \hat{R} . Hence, when $z > \frac{k+2}{d}$, we have $\#((0, z) \cap \hat{R}) > k$, hence $\mathbb{P}(\#((0, z) \cap \hat{R}) = k) = 0$. When $z \leq \frac{k+2}{d}$, we use the bound $\mathbb{P}(0 = \eta_{m,s}^a, \#((0, z) \cap \hat{R}) = k) \leq \mathbb{P}(0 = \eta_{m,s}^a)$. So the expression in (5.15) is less than

$$C \sum_{k \geq 0} \sum_{m > N} \frac{k+2}{d} e^{-c(k \vee m)} \sum_{a \in \mathbb{Z}} \mathbb{P}(0 = \eta_{m,s}^a). \tag{5.16}$$

The inner sum is less than

$$\sum_{a \in \mathbb{Z}} \mathbb{P}(0 \in \eta_{*,s}^a) = \mathbb{E} \#\{a \in \mathbb{Z} : 0 \in \eta_{*,s}^a\}.$$

By a routine comparison with a Poisson random variable, the latter is less than Cs for some $C > 0$. Hence the expression in (5.16) is less than

$$\frac{Cs}{d} \sum_{k \geq 0} \sum_{m \geq N} (k+2) e^{-c(k \vee m)} \leq \frac{Cs}{d} \sum_{k \geq 0} (k+2) e^{-(c/2)k} \sum_{m \geq N} e^{-(c/2)m} \leq C \frac{s}{d} e^{-cN}$$

for some $c, C > 0$.

By symmetry, the same bound applies to the second event in (5.14), so we have:

$$\mathbb{P}(R \cup \hat{R} \text{ contains an } [s, t]\text{-inversion, } \hat{R} \text{ contains no } [s, t]\text{-inversion}) \leq C \frac{s}{d} e^{-cN}. \tag{5.17}$$

Step 3. From (5.12) and (5.17), we have

$$\begin{aligned} \mathbb{P}(\mathcal{G}^c) &\leq \mathbb{P}\left(\begin{array}{c} R \cup \hat{R} \text{ contains an} \\ [s, t]\text{-inversion} \end{array} \right) \\ &\leq \mathbb{P}\left(\begin{array}{c} \hat{R} \text{ contains an} \\ [s, t]\text{-inversion} \end{array} \right) + \mathbb{P}\left(\begin{array}{c} R \cup \hat{R} \text{ contains an } [s, t]\text{-inversion,} \\ \hat{R} \text{ contains no } [s, t]\text{-inversion} \end{array} \right) \leq \bar{C}d + \bar{C}\frac{s}{d}e^{-\bar{c}N} \end{aligned}$$

for some $\bar{c} > 0, 0 < \bar{C} < \infty$. Since these bounds are uniform in t ,

$$\sup_{t \geq s} \mathbb{P}(\mathcal{G}_{s,t}) \leq \bar{C}d + \bar{C}\frac{s}{d}e^{-\bar{c}N}.$$

Take C, γ as in Proposition 3.5 and set $N = \lceil s^{\gamma/2} \rceil$. Recall from the beginning of this proof that d must satisfy

$$\mathbb{P}(0 \in \{\eta_{n,s}^x : x \in \mathbb{Z}, 1 \leq n \leq N\}) \leq d < 1.$$

By Proposition 3.5, we may put $d = C\frac{N}{s^\gamma}$ (provided that s is large enough so that this is less than 1).

We then conclude that $\sup_{t \geq s} \mathbb{P}(\mathcal{G}_{s,t}^c) \leq \bar{C}C\frac{\lceil s^{\gamma/2} \rceil}{s^\gamma} + \bar{C}\frac{s^{1+\gamma}}{C\lceil s^{\gamma/2} \rceil}e^{-\bar{c}\lceil s^{\gamma/2} \rceil} \rightarrow 0$ as $s \rightarrow \infty$. \blacksquare

Following the terminology in [5], define $B_t = \#\{(x, y) : x < y, \eta_t^x > 0 \geq \eta_t^y\}$. Our next-to-last result before the proof of Theorem 1.3 will be

Proposition 5.7. *The process $(B_t)_{t \geq 0}$ is tight.*

Proof. Let $\epsilon > 0$. By Proposition 5.6, there exists s such that $\mathbb{P}(\mathcal{G}_{s,t}^c) < \epsilon/2$ for any $t > s$. Fix $t > s$. Using Lemma 5.5, we have

$$\begin{aligned} \mathbb{E}(B_t; \mathcal{G}_{s,t}) &= \sum_{a < b} \mathbb{P}(\eta_t^a > 0 \geq \eta_t^b, \mathcal{G}_{s,t}) \\ &\leq \sum_{a < b} \sum_{x < y} \mathbb{P}(R_{s,t}^a = y, R_{s,t}^b = x, \eta_t^{(x,s)} \leq 0 < \eta_t^{(y,s)}) \\ &\leq \sum_{a < b} \sum_{x < y} \mathbb{P}(y \in \eta_{*,s}^a, x \in \eta_{*,s}^b, \eta_t^{(x,s)} \leq 0 < \eta_t^{(y,s)}) \\ &\leq \sum_{z \geq 1} \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^{(x,s)} \leq 0 < \eta_t^{(x+z,s)}) \sum_{a < b} \mathbb{P}(x+z \in \eta_{*,s}^a, x \in \eta_{*,s}^b). \quad (5.18) \end{aligned}$$

The inner sum is equal to $\sum_{a < b} \mathbb{P}(z \in \eta_{*,s}^a, 0 \in \eta_{*,s}^b)$ by translation invariance. By (2.1), there exist c (that depends on s) such that

$$\mathbb{P}(z \in \eta_{*,s}^a) \wedge \mathbb{P}(0 \in \eta_{*,s}^b) \leq \mathbb{P}(M_s^0 > |a-z|) \wedge \mathbb{P}(M_s^0 > |b|) \leq e^{-c(|a-z| \vee |b|)},$$

so

$$\sum_{a < b} \mathbb{P}(z \in \eta_{*,s}^a, 0 \in \eta_{*,s}^b) \leq \sum_{a < b} (e^{-c(|a-z| \vee |b|)}).$$

We write $a = (i - j)/2$, $b = (i + j)/2$ with $i, j \in \mathbb{Z}$, $j \geq 0$ and use the bound $e^{-c|z-a|}$ if $i \leq z/2$ and $e^{-c|b|}$ if $i > z/2$; the above is less than

$$\sum_{i \leq z/2} \sum_{j \geq 0} e^{-c(z-(i-j)/2)} + \sum_{i > z/2} \sum_{j \geq 0} e^{-c((i+j)/2)} \leq C \left(\sum_{i \leq z/2} e^{-c(z-i/2)} + \sum_{i > z/2} e^{-ci/2} \right) \leq C e^{-cz}.$$

Using this and Lemma 5.3, we see that the expression in (5.18) is less than

$$C \sum_{z \geq 1} e^{-cz} \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^{(x,s)} \leq 0 < \eta_t^{(x+z,s)}) \leq C \sum_{z \geq 1} z e^{-cz} < \infty.$$

Recall that the choice of t in (s, ∞) was arbitrary, so the above derivation shows that we can choose L large enough that $\frac{\mathbb{E}(B_t; \mathcal{G}_{s,t})}{L} < \frac{\epsilon}{2}$ for all $t > s$, and thus

$$\mathbb{P}(B_t > L) \leq \mathbb{P}(\mathcal{G}_{s,t}^c) + \mathbb{P}(B_t > L, \mathcal{G}_{s,t}) \leq \frac{\epsilon}{2} + \frac{\mathbb{E}(B_t; \mathcal{G}_{s,t})}{L} < \epsilon,$$

Noticing that the trajectories of (B_t) are right continuous with left limits, we can increase L if necessary so that this inequality also holds for $t \leq s$, completing the proof. \blacksquare

Proof of Theorem 1.3. We separately show that $(\rho_t \wedge 0)$ and $(\rho_t \vee 0)$ are tight. We start with the first. Given $L > 0$, for the event $\{\rho_t > L\}$ to occur, there necessarily exist two sites x, y such that $y - x > L$ and $\eta_t^y \leq 0 < \eta_t^x$. If $N < L$ and $\{B_t < N\}$ also occurs, then we cannot have more than N sites $z \in (x, y)$ such that $\eta_{*,t}^z \neq \emptyset$, because every such site makes a $[0, t]$ -inversion either with x or with y and thus increases B_t by one. So we have, for all $t \geq 0$,

$$\begin{aligned} \mathbb{P}(B_t < N, \rho_t > L) &\leq \sum_{x < y, y-x > L} \mathbb{P}(\eta_t^x > 0 \geq \eta_t^y, \hat{T}^{(x,y) \setminus A} < t \text{ for some } A \subset (x, y), \#A < N) \\ &\leq \sum_{z > L} \sum_{A \subset (0, z); \#A < N} \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x > 0 \geq \eta_t^{x+z}, \hat{T}^{(x, x+z) \setminus (x+A)} < \infty). \end{aligned}$$

Using Lemma 5.3 on the innermost sum and counting the possible choices of A , the above is less than

$$C \sum_{z > L} \left[\binom{z}{0} + \dots + \binom{z}{N-1} \right] e^{-c(z-N)},$$

which tends to 0 as $L \rightarrow \infty$. So, given $\epsilon > 0$, choose $N > 0$ such that $\mathbb{P}(B_t \geq N) < \epsilon/2 \forall t$, then choose L such that $\mathbb{P}(B_t < N, \rho_t > L) < \epsilon/2 \forall t$, so that $\mathbb{P}(\rho_t > L) \leq \mathbb{P}(B_t \geq N) + \mathbb{P}(B_t < N, \rho_t > L) < \epsilon \forall t$, and we are done.

Now we treat $(\rho_t \vee 0)$. This is easier: given $L > 0$, for $\{\rho_t < -L\}$ to occur we must have $x < y$ such that $\eta_t^x < 0 \leq \eta_t^y$ and $\eta_{*,t}^w = \emptyset \forall w \in (x, y)$. Then, for any t ,

$$\begin{aligned} \mathbb{P}(\rho_t < -L) &\leq \sum_{x < y, y-x > L} \mathbb{P}(\eta_t^x \leq 0 < \eta_t^y, \hat{T}^{(x,y)} < t) \\ &\leq \sum_{z > L} \sum_{x \in \mathbb{Z}} \mathbb{P}(\eta_t^x \leq 0 < \eta_t^{x+z}, \hat{T}^{(x, x+z)} < t) \leq C \sum_{z \geq L} z e^{-cz}, \end{aligned}$$

which tends to zero as $L \rightarrow \infty$. \blacksquare

6 Estimate for a perturbed random walk

In what follows, π and $(\pi_z)_{z \in \mathbb{Z}}$ are probability distributions on \mathbb{Z} . We assume:

$$\pi \text{ is symmetric (i.e. } \pi(-x) = \pi(x) \forall x); \quad (6.1)$$

$$\pi(x), \pi_z(x) > 0 \text{ for all } x \in \mathbb{Z}, z \in \mathbb{Z} - \{0\}; \quad (6.2)$$

$$\text{There exist } f, F > 0 \text{ such that } \pi(x), \pi_z(x) < Fe^{-f|x|} \text{ for all } x \in \mathbb{Z}, z \in \mathbb{Z}; \quad (6.3)$$

$$\text{There exist } g, G > 0 \text{ such that } \|\pi_z - \pi\|_{TV} < Ge^{-g|z|} \text{ for all } z \in \mathbb{Z}. \quad (6.4)$$

Given $x \in \mathbb{Z}$, let \mathbb{P}_x be a probability under which a process (X_n) is a Markov chain with transitions $P(z, w) = \pi_z(w - z)$ and $\mathbb{P}_x(X_0 = x) = 1$. Define $H_0 = \inf\{n \geq 0 : X_n = 0\}$.

Theorem 6.1. *There exists $C > 0$ such that, for $x \in \mathbb{Z}$,*

$$\mathbb{P}_x(H_0 > N) < \frac{C|x|}{\sqrt{N}}.$$

The proof of Theorem 6.1 will be carried out in a series of results. Fix $L > 0$ such that $Ge^{-gL} < 1$ and let $I = [-L, L]$. Put $\epsilon_z = Ge^{-g|z|}$ for $z \in I^c$ and $\epsilon_z = 1$ for $z \in I$. A consequence of (6.4) is that, for all $z \in \mathbb{Z}$, there exist probabilities g_z, b_z^1, b_z^2 on \mathbb{Z} such that

$$\pi_z = \epsilon_z b_z^1 + (1 - \epsilon_z)g_z; \quad (6.5)$$

$$\pi = \epsilon_z b_z^2 + (1 - \epsilon_z)g_z. \quad (6.6)$$

(Of course, if $z \in I$ we must have $b_z^1 = \pi_z, b_z^2 = \pi$).

We will construct the process (X_n) coupled with other processes of interest. Let (X_n, Z_n) be a Markov chain on $\mathbb{Z} \times \{0, 1\}$ with transitions

$$Q((x, i), (y, j)) = \begin{cases} \epsilon_x \cdot b_x^1(y - x) & \text{if } j = 1; \\ (1 - \epsilon_x) \cdot g_x(y - x) & \text{if } j = 0. \end{cases} \quad (6.7)$$

We write \mathbb{P}_x to represent any probability for this chain with $X_0 = x$, regardless of the law of Z_0 . This abuse of notation is justified by the fact that Z_0 has no influence on the distribution of the other variables of the chain, nor on the random variables to be defined below. Let (\mathcal{F}_n) be the natural filtration of the chain, and $T = \inf\{n \geq 1 : Z_n = 1\}$.

Let $(\Psi_z)_{z \in \mathbb{Z}}$ be random variables defined on the same probability space as the chain above, independent of the chain and with laws $\Psi_z \stackrel{d}{=} b_z^2$. Additionally, let $(\Phi_n)_{n \geq 0}$ be a random walk with increment law π , initial state 0, also defined on the same space as the previous variables and independent of them. For $n \geq 0$, define

$$Y_n = \begin{cases} X_n, & \text{if } n < T; \\ X_{T-1} + \Psi_{X_{T-1}} + \Phi_{n-T}, & \text{if } n \geq T. \end{cases} \quad (6.8)$$

We can use (6.5) and (6.6) to check that under \mathbb{P}_x , (X_n) is a Markov chain with transitions $P(z, w) = \pi_z(w - z)$ and initial state x , and (Y_n) is a random walk with increment distribution π and initial state x . They satisfy $X_n = Y_n$ on $\{T > n\}$.

We now define some more stopping times. Let $H_0^Y = \inf\{n \geq 0 : Y_n = 0\}$, $H_I = \inf\{n \geq 0 : X_n \in I\}$, $\tau_0 \equiv 0$, $\tau_1 = H_I \wedge T$ and $\tau_{k+1} = \tau_k + \tau_1 \circ \theta_{\tau_k}$ for $k \geq 1$, where θ_t denotes the shift operation $\theta_t((X_n, Z_n)_{n \geq 0}) = (X_{t+n}, Z_{t+n})_{n \geq 0}$. Note that $\{\tau_1 = n\}$ occurs if and only if $X_0, \dots, X_{n-1} \notin I, Z_1, \dots, Z_{n-1} = 0$ and either $X_n \in I$ or $Z_n = 1$ (or both). Also, $\tau_k \leq H_I$ for all k and, if $\tau_k = H_I$ and $m > k$, then $\tau_m = H_I$. Finally, we have $\tau_1 \leq H_0^Y$, because if $Y_n = 0$ for some n , then either $X_n = 0$, in which case $\tau_1 \leq H_I \leq H_0 \leq n$, or $X_n \neq 0$, in which case $\tau_1 \leq T < n$.

We will need the following standard facts about random walk on \mathbb{Z} :

Lemma 6.2. (i.) $\mathbb{P}_x(H_0^Y > N) \leq \frac{C|x|}{\sqrt{N}}$ for some $C > 0$ and all $x \in \mathbb{Z}$;
(ii.) $\mathbb{E}_x(\#\{n < H_0^Y : Y_n = y\}) \leq C|y|$ for some $C > 0$ and all $x, y \in \mathbb{Z}$.

Proof. (i) is in [16]: see **P4** in Section 32 and Section 29. For (ii), we have $\mathbb{E}_x(\#\{n < H_0^Y : Y_n = y\}) \leq \mathbb{E}_y(\#\{n < H_0^Y : Y_n = y\}) = \mathbb{P}_y(H_0^Y < H_{y+}^Y)^{-1}$, where $H_{y+}^Y = \inf\{n \geq 1 : Y_n = y\}$, so it suffices to show that $\mathbb{P}_y(H_0^Y < H_{y+}^Y) > c/y$ for some $c > 0$ and all $y \in \mathbb{Z}$. This can be done using Thomson's Principle for electric networks (see for example [11], Theorem 9.10 and Section 21.2 for the infinite network case): if $y > 0$, take the unit flow $\theta(\vec{z}\vec{w}) = 1$ if $z \in \{1, \dots, y\}$, $w = z - 1$ and $\theta(\vec{z}\vec{w}) = 0$ otherwise, and similarly if $y < 0$. ■

Lemma 6.3. (i.) $\mathbb{P}_x(\tau_1 = \infty) = 0 \forall x \in \mathbb{Z}$.
(ii.) There exist $c, C > 0$ such that $\mathbb{P}_x(|X_{\tau_1}| > r, \tau_1 < H_I) \leq Ce^{-cr}$ for all $x \in \mathbb{Z}, r \geq 0$.
(iii.) $A := \sup_{x \in \mathbb{Z}} \mathbb{E}_x(|X_{\tau_1}|) < \infty$.

Proof. For (i.), since $\tau_1 \leq H_0^Y$, $\mathbb{P}_x(\tau_1 > N) \leq \mathbb{P}_x(H_0^Y > N) \xrightarrow{N \rightarrow \infty} 0$ by Lemma 6.2(i.). For (ii.), note that if $x \in I$, then $\tau_1 = H_I = 0$ and the stated inequality is trivial. If $x \notin I$,

$$\begin{aligned} \mathbb{P}_x(|X_{\tau_1}| > r, \tau_1 < H_I) &= \mathbb{P}_x(|X_{\tau_1}| > r, \tau_1 < \infty, \tau_1 < H_I) \\ &= \sum_{n=0}^{\infty} \sum_{z \in I^c} \mathbb{P}_x(X_0, \dots, X_n \in I^c, X_n = z, Z_1, \dots, Z_n = 0, Z_{n+1} = 1, X_{n+1} \in I^c, |X_{n+1}| > r) \\ &= \sum_{n=0}^{\infty} \sum_{z \in I^c} \mathbb{P}_x(Y_0, \dots, Y_n \in I^c, Y_n = z, Z_1, \dots, Z_n = 0, Z_{n+1} = 1, X_{n+1} \in I^c, |X_{n+1}| > r), \end{aligned}$$

simply because the events are the same. This is further equal to

$$\begin{aligned} &\sum_{n=0}^{\infty} \sum_{z \in I^c} \mathbb{P}_x(Y_0, \dots, Y_n \in I^c, Y_n = z, Z_1, \dots, Z_n = 0) \cdot \mathbb{P}_z(Z_1 = 1, X_1 \in I^c, |X_1| > r) \\ &\leq \sum_{n=0}^{\infty} \sum_{z \in I^c} \mathbb{P}_x(Y_0, \dots, Y_n \in I^c, Y_n = z) \cdot \epsilon_z \cdot b_z^1\{w : |w| \geq |r - z|\}, \end{aligned}$$

using (6.7). Since $\epsilon_z b_z^1(E) \leq \pi_z(E)$ for any event E , we have $\epsilon_z b_z^1(E) \leq \epsilon_z \wedge \pi_z(E)$. The above is less than

$$\begin{aligned} &\sum_{z \in I^c} (\epsilon_z \wedge \pi_z\{w : |w| \geq |r - z|\}) \mathbb{E}_x(\#\{n < H_0^Y : Y_n = z\}) \\ &\leq C \sum_{z \in \mathbb{Z}} |z| ((Ge^{-g|z|}) \wedge (Fe^{-f|r-z|})) \leq Ce^{-cr}; \end{aligned}$$

we have used Lemma 6.2(ii.) to bound the expectation. This finishes the proof of (ii.). Finally, we have

$$\forall x, \mathbb{E}_x(|X_{\tau_1}|) = \mathbb{E}_x(|X_{\tau_1}|; \tau_1 < H_I) + \mathbb{E}_x(|X_{\tau_1}|; \tau_1 = H_I);$$

the first term can be bounded uniformly in x by summing the two sides of the inequality in (ii.) and the second term is less than L , so (iii.) is proved. ■

Corollary 6.4. *Increasing L if necessary,*

(i.) $\sigma := \inf_{x \in \mathbb{Z}} \mathbb{P}_x(\tau_1 = H_I < \infty) > 0$;

(ii.) For all x , $\mathbb{P}_x(\tau_k < H_I) \leq (1 - \sigma)^k$;

(iii.) For all x , $\mathbb{P}_x(H_I = \infty) = 0$.

Proof. For (i.), note that

$$\begin{aligned} \mathbb{P}_x(\tau_1 = H_I < \infty) &= 1 - \mathbb{P}_x(\tau_1 = H_I = \infty) - \mathbb{P}_x(\tau_1 < \infty, \tau_1 < H_I) \\ &= 1 - 0 - \mathbb{P}_x(\tau_1 < \infty, |X_{\tau_1}| > L) \geq 1 - Ce^{-cL}, \end{aligned}$$

which can be made positive by increasing L . Now, if $k \geq 1$,

$$\mathbb{P}_x(\tau_k < H_I) = \mathbb{E}_x(\mathbb{1}_{\{\tau_{k-1} < H_I\}} \mathbb{P}_{X_{\tau_{k-1}}}(\tau_1 < H_I)) \leq (1 - \sigma) \mathbb{P}_x(\tau_{k-1} < H_I)$$

by (i.), and continuing we get (ii.) Finally, note that

$$\mathbb{P}_x(H_I = \infty) \leq \mathbb{P}_x(H_I = \infty, \tau_k < \infty \forall k) + \sum_{k=1}^{\infty} \mathbb{P}_x(H_I = \infty, \tau_k = \infty).$$

The first term is zero by (ii.) and, using Lemma 6.3(i.),

$$\mathbb{P}_x(\tau_k = \infty) = \sum_{i=0}^{k-1} \mathbb{P}_x(\tau_i < \infty, \tau_{i+1} = \infty) = \sum_{i=0}^{k-1} \mathbb{E}_x(\mathbb{1}_{\{\tau_i < \infty\}} \mathbb{P}_{X_{\tau_i}}(\tau_1 = \infty)) = 0,$$

so (iii.) follows. ■

Lemma 6.5. *There exists $C > 0$ such that, for all $x \in \mathbb{Z}$,*

$$\mathbb{P}_x(H_I > N) \leq \frac{C|x|}{\sqrt{N}}.$$

Proof. If $x \in I$, the left-hand side is zero. Assume that $x \notin I$.

$$\begin{aligned} \mathbb{P}_x(H_I > N) &= \mathbb{P}_x(H_I = \infty) + \sum_{k=0}^{\infty} \mathbb{P}_x(H_I > N, \tau_k < \tau_{k+1} = H_I < \infty) \\ &= \sum_{k=0}^{\infty} \mathbb{P}_x \left(\sum_{i=1}^{k+1} (\tau_i - \tau_{i-1}) > N, \tau_k < \tau_{k+1} = H_I < \infty \right) \\ &\leq \sum_{k=0}^{\infty} \sum_{i=1}^{k+1} \mathbb{P}_x \left(\tau_i - \tau_{i-1} > \frac{N}{k+1}, \tau_k < H_I \right) \end{aligned}$$

We will show that, for $k \geq 0$ and $1 \leq i \leq k + 1$,

$$\mathbb{P}_x(\tau_i - \tau_{i-1} > l, \tau_k < H_I) \leq \frac{C|x|}{\sqrt{l}}(1 - \sigma)^{k-2} \quad (6.9)$$

for some $C > 0$. So the above sum is less than

$$\sum_{k=0}^{\infty} \sum_{i=1}^{k+1} C|x| \sqrt{\frac{k+1}{t}} (1 - \sigma)^{k-2} \leq \frac{C'|x|}{\sqrt{t}}$$

as required. To get (6.9), note that, if $i \leq k$, by Corollary 6.4(ii.),

$$\mathbb{P}_x(\tau_i - \tau_{i-1} > l, \tau_k < H_I) = \mathbb{E}_x(\mathbb{1}_{\{\tau_i - \tau_{i-1} > l\}} \mathbb{P}_{X_{\tau_i}}(\tau_{k-i} < H_I)) \leq (1 - \sigma)^{k-i} \mathbb{P}_x(\tau_i - \tau_{i-1} > l),$$

so for any $i \in \{1, \dots, k + 1\}$,

$$\mathbb{P}_x(\tau_i - \tau_{i-1} > l, \tau_k < H_I) \leq (1 - \sigma)^{k-i} \mathbb{P}_x(\tau_i - \tau_{i-1} > l). \quad (6.10)$$

(the bound for $i = k + 1$ is trivial, since $(1 - \sigma)^{-1} > 1$). Now, using the fact that $\tau_i - \tau_{i-1}$ can only be positive when $\tau_{i-1} < H_I$, the inequality $\tau_1 \leq H_0^Y$ and Lemma 6.2(i.),

$$\begin{aligned} \mathbb{P}_x(\tau_i - \tau_{i-1} > l) &= \mathbb{E}_x(\mathbb{1}_{\{\tau_{i-1} < H_I\}} \mathbb{P}_{X_{\tau_{i-1}}}(\tau_1 > l)) \leq \mathbb{E}_x(\mathbb{1}_{\{\tau_{i-1} < H_I\}} \mathbb{P}_{X_{\tau_{i-1}}}(H_0^Y > l)) \\ &\leq (C/\sqrt{l}) \mathbb{E}_x(\mathbb{1}_{\{\tau_{i-1} < H_I\}} |X_{\tau_{i-1}}|). \end{aligned}$$

If $i = 1$, the above expectation is equal to $|x| \mathbb{1}_{\{x \in I^c\}}$; if $i > 1$ it is equal to

$$\begin{aligned} \mathbb{E}_x(\mathbb{1}_{\{\tau_{i-2} < H_I\}} \mathbb{E}_x(|X_{\tau_{i-1}}| \cdot \mathbb{1}_{\{\tau_{i-1} < H_I\}} | \mathcal{F}_{\tau_{i-2}})) &= \mathbb{E}_x(\mathbb{1}_{\{\tau_{i-2} < H_I\}} \mathbb{E}_{X_{\tau_{i-2}}}(|X_{\tau_1}| \cdot \mathbb{1}_{\{\tau_1 < H_I\}})) \\ &\leq A \mathbb{P}_x(\tau_{i-2} < H_I) \leq A(1 - \sigma)^{i-2} \end{aligned}$$

by Lemma 6.3(iii.) and Corollary 6.4(ii.). So, for any $i \in \{1, \dots, k + 1\}$,

$$\mathbb{P}_x(\tau_i - \tau_{i-1} > l) \leq AC \frac{|x|}{\sqrt{l}} (1 - \sigma)^{i-2}. \quad (6.11)$$

Putting together (6.10) and (6.11), we get (6.9). ■

From here to the proof of Theorem 6.1, it is a matter of reapplying the ideas that established Corollary 6.4 and Lemma 6.5, so we simply sketch the main steps.

Define $T' = \inf\{n \geq 0 : \{X_0, \dots, X_n\} \cap I \neq \emptyset, \{X_0, \dots, X_n\} \cap I^c \neq \emptyset\}$, $\lambda_0 = 0$, $\lambda_1 = T' \wedge H_0$, $\lambda_{k+1} = \lambda_k + \lambda_1 \circ \theta_{\lambda_k}$ for $k \geq 1$. From (6.2), we get

$$\delta := \inf_{x \in I} P(x, 0) = \inf_{x \in I} \pi_x(-x) > 0. \quad (6.12)$$

Two consequences are

$$\sup_{x \in I} \mathbb{P}_x(\lambda_1 > N) \leq (1 - \delta)^N \quad (6.13)$$

and

$$\inf_{x \in I} \mathbb{P}_x(\lambda_1 = H_0 < \infty) \geq \delta. \quad (6.14)$$

Now, (6.13) and Corollary 6.4(iii.) together imply

$$\forall x \in \mathbb{Z}, \mathbb{P}_x(\lambda_1 = \infty) = 0. \quad (6.15)$$

Also, (6.14) gives

$$\forall x \in \mathbb{Z}, \mathbb{P}_x(\lambda_k < H_0) \leq (1 - \delta)^{\lfloor k/2 \rfloor}; \quad (6.16)$$

this is justified by the fact that, if $\lambda_k < H_0$, then at least $\lfloor k/2 \rfloor$ times X_n must have left I without touching the origin. As in the proof of Corollary 6.4 (iii.), (6.15) and (6.16) are used to establish

$$\forall x, \mathbb{P}_x(H_0 = \infty) = 0. \quad (6.17)$$

The last ingredient is an analog of Lemma 6.3 (iii.),

$$B := \sup_{x \in I} \mathbb{E}_x(|X_{\lambda_1}|) = \sup_{x \in I} \mathbb{E}_x(|X_{\lambda_1}|; \lambda_1 < H_0) < \infty, \quad (6.18)$$

which follows from (6.3) and the fact that I is finite.

We can now write

$$\mathbb{P}_x(H_0 > N) = \mathbb{P}_x(H_0 = \infty) + \sum_{k=0}^{\infty} \mathbb{P}_x(H_0 > N, \lambda_k < \lambda_{k+1} = H_0 < \infty)$$

and then, as in the preceding proof, use (6.15), Lemma 6.5, (6.13), (6.16), and (6.18) to show that the above sum is less than $\frac{C|x|}{\sqrt{N}}$ for some $C > 0$.

To conclude, we mention the following result, for use in the proof of Lemma 5.1. We omit its proof since it is simply a repetition of the above arguments.

Lemma 6.6. *Let $H_{(-\infty, 0)} = \inf\{n : X_n < 0\}$. Then,*

$$\sup_{x > 0} \mathbb{E}_x(|X_{H_{(-\infty, 0)}}|; H_{(-\infty, 0)} < H_0) \leq \sup_{x > 0} \mathbb{E}_x |X_{H_{(-\infty, 0)}}| < \infty.$$

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