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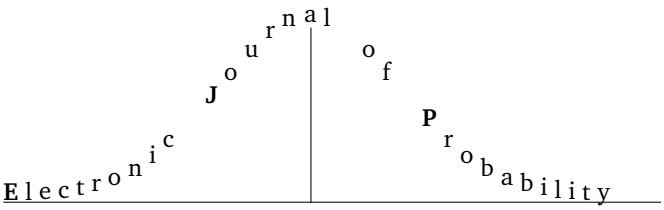

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Expected Lengths of Minimum Spanning Trees for Non-identical Edge Distributions

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Abstract

An exact general formula for the expected length of the minimal spanning tree (MST) of a connected (possibly with loops and multiple edges) graph whose edges are assigned lengths according to independent (not necessarily identical) distributed random variables is developed in terms of the multivariate Tutte polynomial (alias Potts model). Our work was inspired by Steele's formula based on two-variable Tutte polynomial under the model of uniformly identically distributed edge lengths. Applications to wheel graphs and cylinder graphs are given under two types of edge distributions.

Key words: Expected Length, Minimum Spanning Tree, The Tutte Polynomial, The Multivariate Tutte Polynomial, Random Graph, Wheel Graph, Cylinder Graph.

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1 Introduction

For a finite and connected graph $G = (V, E)$ with vertex set V and edge set E, we denote the total length of its minimum spanning tree (MST) as

$$L_{MST}(G) = \sum_{e \in E(MST(G))} \xi_e, \tag{1}$$

where ξ_e is the length of the edge $e \in E$. We are interested in studying the expected length of the MST of G, denoted as $\mathbb{E}L_{MST}(G)$, for nonnegative random variables ξ_e with independent (not necessarily identical) distribution F_e. For the special case that $F_e = F$ for every edge e, we use notation $\mathbb{E}L^F_{MST}(G)$. In particular, if F is the uniform distribution on the interval $(0, 1)$, i.e., $U(0, 1)$, or exponential distribution with rate 1, i.e., exp(1), then the expected length of the MST of the graph G is denoted as $\mathbb{E}L^u_{MST}(G)$ or $\mathbb{E}L^e_{MST}(G)$ respectively.

Frieze [Fri85] first studied $\mathbb{E}L^F_{MST}(G)$ and showed that for a complete graph K_n on n vertices,

$$\lim_{n\to\infty} \mathbb{E}L^F_{MST}(K_n) = \zeta(3)/F'(0), \quad where\ \zeta(3) = \sum_{k=1}^{\infty} k^{-3} = 1.202....$$

This implies that

$$\lim_{n\to\infty} \mathbb{E}L^e_{MST}(K_n) = \lim_{n\to\infty} \mathbb{E}L^u_{MST}(K_n) = \zeta(3).$$

Later, this result was extended and strengthened in many different ways, refer to [Ste87; FM89; Jan95]. Moreover, Steele [Ste02] started the investigation on exact formulae for the expected lengths of MSTs and discovered the following nice formula

$$\mathbb{E}L^u_{MST}(G) = \int_0^1 \frac{(1-t)}{t} \frac{T_x(G; 1/t, 1/(1-t))}{T(G; 1/t, 1/(1-t))}\, \mathrm{d}t, \tag{2}$$

where $T(G; x, y)$ is the standard Tutte polynomial of G and $T_x(G; x, y)$ is the partial derivative of $T(G; x, y)$ with respect to x. This can be easily extended, see [LZ09], to the case of general independent and identical distributed (i.i.d.) edge lengths as

$$\mathbb{E}L^F_{MST}(G) = \int_0^{\infty} \frac{1 - F(t)}{F(t)} \frac{T_x(G; x, y)}{T(G; x, y)}\, \mathrm{d}t, \tag{3}$$

where $x = 1/F(t)$, $y = 1/(1 - F(t))$. For discussions about the standard Tutte polynomial and the multivariate Tutte polynomial, see Section 2.

In this paper, we provide an exact formula for the expected lengths of MSTs for any finite, connected graph G, in which the edge length distributions are not necessarily identical. This generalizes formulae (2) and (3) considerably.

The standard notations $G\backslash\{e, e \in A\}$ and $G/\{e, e \in A\}$ are used for graphs obtained by deleting or contracting the edges in $A \subseteq E$ from G respectively. Following the ideas in [Ste02] and [LZ09], a relationship between the length of MST and the number of components in the graph is first obtained. Then by relating the expected number of components to the multivariate Tutte polynomial, we obtain the main theorem of this paper as the following:

Theorem 1 (General Formula). *Let $G = (V, E)$ be a finite, connected graph, in which each edge $e \in E$ has a positive random length ξ_e with independent distribution $F_e(t) = P(\xi_e \leq t)$. If we set $E'(t) = \{e : 0 < F_e(t) < 1\}$ and $G'(t) = G/\{e : F_e(t) = 1\}$, then*

$$\mathbb{E}L_{MST}(G) = \int_0^\infty \left(\prod_{e \in E'(t)} \frac{1}{1 + \nu_e(t)} Z_q(G'(t); 1, \mathbf{v}(t)) - 1 \right) \mathrm{d}t, \tag{4}$$

where $\nu_e(t) = F_e(t)/(1 - F_e(t))$, $\mathbf{v}(t) = \{\nu_e(t), e \in E'(t)\}$, and $Z_q(G'(t); 1, \mathbf{v}(t))$ is the partial derivative of the multivariate Tutte polynomial $Z(G'(t); q, \mathbf{v}(t))$ with respect to q evaluated at $q = 1$.

Note that if the edge length distributions are i.i.d., then the above formula is reduced to (3) by relations between the multivariate Tutte polynomial and the standard Tutte polynomial. See Section 2 for more details.

The seemingly complicated general formula applied to specific graphs shows a more explicit form. In this paper, the applications of this generalized formula are illustrated in two families of graphs: wheel graphs and cylinder graphs. These graphs are assumed to have two different types of edges following two different types of edge length distributions. In particular, we are interested in $U(0, 1)$ and exp(1) edge length distributions. By applying our generalized formula, we compare $\mathbb{E}L_{MST}(G)$ between graphs with switched edge types. In addition, we show that Theorem 1 also provides a new angle to work on the problem of the expected lengths of MSTs of the complete graph. For more applications of Theorem 1, refer to [Zha08].

A wheel graph, see Figure 1 for an example, is often used in network to illustrate the simple topology. To distinguish the two different type of edges, we use dark and thick lines to draw rims, and lighter and thinner lines to draw spokes. The letters r and s beside the edges indicate the edge type as rim and a spoke respectively.

Definition 1 (Wheel Graphs with two types of edges). *The wheel graph W_n is defined as the joint $K_1 + C_n$, where K_1 is the (trivial) complete graph on 1 node (which is known as the hub) and C_n is the cycle graph of n vertices. The edges of C_n are called* rims *and the edges of a wheel which include the hub are called* spokes.

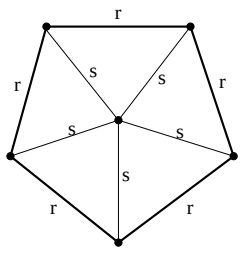


Figure 1: Wheel Graph W_5 with Two Types of Edges

For a wheel graph W_n, we denote the edge lengths on rims and spokes as ξ_r and ξ_s respectively. Taking into account of different edge types, the multivariate Tutte polynomial can be calculated explicitly following similar ideas in [LZ09]. Then by applying Theorem 1, we obtain the following:

Theorem 2. *For a wheel graph W_n with the nonnegative lengths of rims and spokes following distributions $F_r(t) = P(\xi_r \le t)$ and $F_s(t) = P(\xi_s \le t)$ respectively, if we assume $b = \min\{t : (F_r(t)-1)(F_s(t)-1) = 0\}$, then*

$$\lim_{n\to\infty} \frac{1}{n}\mathbb{E}L_{MST}(W_n) = \int_0^b \frac{(1-F_r(t))^2(1-F_s(t))}{1-F_r(t)+F_r(t)F_s(t)}\,\mathrm{d}t.$$

Note that b in the above theorem is allowed to be infinity, see discussions in Section 4.1.2. Theorem 2 shows that $\mathbb{E}L_{MST}(W_n)$ converges to a constant with a scaling of the number of vertices. Comparing the expected length of the wheel graph with that of the complete graph, the cubic graph [Pen98] and the almost regular graph studied in [BFM98; FRT00], we see that the scaling of convergence is related to the graph density. Recall that for a simple graph $G = (V, E)$, the graph density is defined as

$$D(G) = \frac{2|E|}{|V|(|V|-1)}.$$

One may check that all the graphs mentioned above satisfy the following identity:

$$\lim_{n\to\infty} D(G)\mathbb{E}L^u_{MST}(G) = C_G,$$

for some constant C_G. We conjecture that this identity holds for all simple graphs.

If we specify the two types of edge length distribution in the wheel graph as $U(0,1)$ and $\exp(1)$, then the asymptotic values of $\mathbb{E}L_{MST}(W_n)/n$ follow immediately after Theorem 2. In addition, we compare $\mathbb{E}L^{ue}_{MST}(W_n)/n$ and $\mathbb{E}L^{eu}_{MST}(W_n)/n$, the values of the expected length of the MST for the wheel graph with switched edge length distributions. See Section 4.1 for more details.

As another example, we apply Theorem 1 to the cylinder graph $P_n \times C_k$, which was studied by Hutson and Lewis [HL07] under i.i.d uniform distribution. Since it is natural to divide the edges of the cylinder graph into two types, we study $\mathbb{E}L_{MST}(P_n \times C_k)$ under general non-identical edge distribution.

Definition 2 (Cylinder Graphs with two types of edges)**.** *A cylinder graph of length n is defined as $P_n \times C_k$, a Cartesian product of a path P_n and a cycle C_k. The edges on the paths P_n and the cycles C_k are called type 1 edges and type 2 edges respectively.*

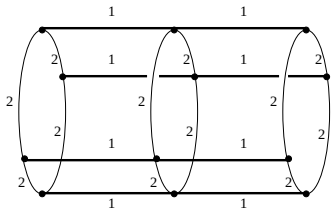


Figure 2: The Cylinder $P_2 \times C_4$ with Two types of Edges

Following similar ideas in [HL07], but with more complicated derivation and identifying various quantities, we obtain a representation of $Z(P_n \times C_k)$ in terms of $Z(P_0 \times C_k)$ and a transfer matrix $A(q, \mathbf{v})$. See more details in Section 4.2. Applying Theorem 1, we obtain

Theorem 3. *For a cylinder graph $P_n \times C_k$ with $k \geq 2$, let $\lambda(q, \mathbf{v})$ be the Perron-Frobenius eigenvalue of the transfer matrix $A(q, \mathbf{v})$, defined in (25).*

(a). If the edges on the paths and cycles have lengths following distributions exp(1) *and $U(0,1)$ respectively, then*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} L^{eu}_{MST}(P_n \times C_k) = \frac{e^{-k}}{k} + \int_0^1 \frac{\lambda_q(1, \mathbf{v}(t))}{\lambda(1, \mathbf{v}(t))}\, \mathrm{d}t,$$

where $\mathbf{v}(t) = \{v_1(t), v_2(t)\}$, $v_1(t) = e^t - 1$ and $v_2(t) = t/(1-t)$.

(b). If the edges on the paths and cycles have lengths following distributions $U(0,1)$ and exp(1) *respectively, then*

$$\lim_{n\to\infty} \frac{1}{n} \mathbb{E} L^{ue}_{MST}(P_n \times C_k) = \int_0^1 \frac{\lambda_q(1, \mathbf{u}(t))}{\lambda(1, \mathbf{u}(t))}\, \mathrm{d}t,$$

where $\mathbf{u}(t) = \{u_1(t), u_2(t)\}$, $u_1(t) = t/(1-t)$ and $u_2(t) = e^t - 1$.

Note that the two integrals in cases (a) and (b) are different. From numerical results in Table 3, we conjecture that the integral in (a) is bigger than the one in (b). Thus, if normalized by the length of the cylinder graph, we conjecture that the asymptotic value of $\mathbb{E} L^{eu}_{MST}(P_n \times C_k)$ is always bigger than $\mathbb{E} L^{ue}_{MST}(P_n \times C_k)$ by a quantity larger than e^{-k}/k.

Our proposed general formula also applies to non-simple graphs, i.e. graphs with loops (edges joining a vertex to itself) or multiple edges (two or more edges connecting the same pair of vertices). This feature together with the non-i.i.d. edge length assumption enables us to study the random minimum spanning tree problem in much more complicated situations. In addition, this generalized formula applied to the complete graph serves as a general formula for the expected length of any simple connected graph. The remaining of the paper is organized as follows: We discuss the main properties of the multivariate Tutte polynomial related to our results and compare it with the standard Tutte polynomial in Section 2. These are also used in the proof of Theorem 1 in Section 3, where the relationship between the standard Tutte polynomial and the expected length of MST is reviewed and analyzed. In Section 4, several applications of Theorem 1 are given.

2 The Multivariate Tutte Polynomial

The multivariate Tutte polynomial has been known to physicists for many years in various forms, but it is relative new to mathematicians. The first comprehensive survey was given by Sokal [Sok05]. Since the multivariate Tutte polynomial contains all the edge lengths as variables, it is generally considered to be more flexible to use than its two-variate version (standard Tutte polynomial). For example, it allows simpler forms of deletion-contraction identity and parallel-reduction identity. In this section, we first review basic properties of the multivariate Tutte polynomial and compare it with the standard Tutte polynomial. Then the relationship of the general formula (4) with formulae (2) and (3) is explored, Finally, we discuss how the application of this formula to the complete graph generalizes the expected lengths of MSTs of all finite simple graphs.

Definition 3 (The Standard Tutte polynomial)**.** *The Tutte polynomial of a graph G is a two-variable polynomial defined as*

$$T(G; x, y) = \sum_{A \subseteq E} (x-1)^{r(E)-r(A)} (y-1)^{|A|-r(A)}, \tag{5}$$

where $k(A)$ denotes the number of connected components in the subgraph (V,A), and $r(A)$ is the rank function defined as $r(A) = |V| - k(A)$.

The standard Tutte polynomial is a special case of the multivariate Tutte polynomial.

Definition 4 (The Multivariate Tutte Polynomial)**.** *For a finite graph $G = (V,E)$ (not necessarily simple or connected), its multivariate Tutte polynomial is defined as*

$$Z(G;\ q,\ \mathbf{v}) = \sum_{A \subseteq E} q^{k(A)} \prod_{e \in A} v_e, \tag{6}$$

where q and $\mathbf{v} = \{v_e,\ e \in E\}$ are variables.

If we set the edge weights v_e to the same value v, then a two-variable polynomial $Z(G;q,v)$ is obtained, which is essentially equivalent to the standard Tutte polynomial $T(G;x,y)$ in the following way:

$$T(G;x,y) = (x-1)^{-k(E)}(y-1)^{-|V|}Z(G;(x-1)(y-1),y-1). \tag{7}$$

From Definition 4, we can see that by setting $q = 1$, the multivariate Tutte polynomial is much simplified as

$$Z(G;1,\mathbf{v}) = \sum_{A \subseteq E} \prod_{e \in A} v_e = \prod_{e \in E} (1 + v_e). \tag{8}$$

Actually as formula (7) shows, $q = 1$ in $Z(G;q,\mathbf{v})$ corresponds to $(x-1)(y-1) = 1$ in $T(G;x,y)$. Thus, by applying (7) and setting $v_e = v = y - 1$ in (8), one easily obtains the well-known simplied version of the standard Tutte polynomial on the hyperbola $H_1 = \{(x-1)(y-1) = 1\}$ as

$$T(G;x,y) = x^{|E|}(x-1)^{|V|-k(E)-|E|}. \tag{9}$$

While it may not be that easy to observe that the standard Tutte polynomial simplifies on H_1, the simplification of $Z(G;q,\mathbf{v})$ at $q = 1$ is almost trivial to see.

In the practice of computing the multivariate Tutte polynomial for specific graphs, it is usually hard to apply Definition 4 directly. The following identities, which can be verified by Definition 4, for the multivariate Tutte polynomial are often used to simplify the computation. Moreover, these identities are usually in simpler forms than their two-variate versions. For a detailed discussion of the multivariate Tutte polynomial, we refer to [Sok05].

Union Graph Identity

1. If graphs G_1 and G_2 are disjoint, then

$$Z(G_1 \cup G_2;q,\mathbf{v}) = Z(G_1;q,\mathbf{v})Z(G_2;q,\mathbf{v}).$$

2. If graphs G_1 and G_2 share one common vertex but no common edges, then

$$Z(G_1 \cup G_2;q,\mathbf{v}) = \frac{Z(G_1;q,\mathbf{v})Z(G_2;q,\mathbf{v})}{q}. \tag{10}$$

Duality Identity

For a connected planar graph $G=(V,E)$, let $G^*=(V^*,E^*)$ be its dual graph, then

$$Z(G;q,\mathbf{v})=q^{|V|-|E|-1}\left(\prod_{e\in E}v_e\right)Z\left(G^*;q,q/\mathbf{v}\right). \tag{11}$$

Deletion-Contraction Identity

For any $e\in E$,

$$Z(G;q,\mathbf{v})=Z(G\backslash e;q,\mathbf{v}_{\neq\{e\}})+v_eZ(G/e;q,\mathbf{v}_{\neq\{e\}}), \tag{12}$$

where $\mathbf{v}_{\neq\{e\}}=\{v_f,f\in E\backslash e\}$. In particular, if e is a loop or bridge, then this identity is simplified as

$$Z(G;q,\mathbf{v})=\begin{cases}(1+v_e)Z(G/e;q,\mathbf{v}), & \text{if } e \text{ is a loop}\\ (q+v_e)Z(G/e;q,\mathbf{v}), & \text{if } e \text{ is a bridge}\end{cases}.$$

Note that a bridge in a connected graph is defined as an edge whose removal disconnects the graph.

Parallel-Reduction Identity

Another useful feature of the multivariate Tutte polynomial is the simple parallel-reduction identity. That is, we can replace m parallel edges $e_1,\ldots,e_m$, which join the same pair of vertices x,y, by a single edge e with weight

$$v_e=\prod_{i=1}^{m}(1+v_{e_i})-1, \tag{13}$$

without changing the value of the multivariate polynomial Z.

From Theorem 1, we see that if all the edge lengths follow an identical distribution F, then $v_e(t)=v(t)=F(t)/(1-F(t))$ for any $e\in E$ and $Z(G;q,v(t))$ becomes a two-variate polynomial. For $y=v(t)+1$ and $x=q/v(t)+1$,

$$Z(G;q,v(t))=(x-1)(y-1)^{|V|}T(G;x,y),$$

and

$$Z_q(G;q,v)=(y-1)^{|V|-1}T(G;x,y)\left((x-1)\frac{T_x(G;x,y)}{T(G;x,y)}+1\right).$$

Specifically, for $q=(x-1)(y-1)=1$, $T(G;x,y)=x^{|E|}(x-1)^{|V|-|E|-1}$. Therefore,

$$Z_q(G;1,v)=y^{|E|}\left((x-1)\frac{T_x(G;x,y)}{T(G;x,y)}+1\right),$$

where $(x-1)(y-1)=1$. This shows that for i.i.d. edge lengths, the general formula (4) is reduced to formula (3). In addition, if the identical edge length distribution is specified to be $U(0,1)$, then by setting $x=1/t$ and $y=1/(1-t)$, one obtains Steele's formula (2).

In fact, when applied to a complete graph, the general formula (4) generalizes the expected lengths of MSTs for all simple connected graphs $G=(V,E)$. In order to see this, one can supplement edges

with large lengths to G. To be more precise, for any two vertices $i,\ j \in V$, if the edge $ij \notin E$, then set $\xi_{ij} = M$ and $E = E \cup ij$. Eventually, a complete graph K_n is formed. If the supplemented edge length M is chosen to be a large number, say $M = \max\{\xi_e, e \in E\} + 1$, then there is no chance for the supplemented edges to be selected in the MST. Therefore, $\mathbb{E}L_{MST}(G)$ is the same as the expected length of MST for the complete graph K_n, on which the edge weight ξ'_e satisfies the following condition:

$$\xi'_e = \begin{cases} \xi_e, \text{ if } e \in E, \\ M, \text{ if } e \notin E. \end{cases}$$

Therefore, it is useful to exam the multivariate Tutte polynomial of a complete graph, more details are discussed in Section 4.3.

3 Proof of the Main Theorem

It is well known that the length of MST of a graph with random edge lengths can be represented in terms of the number of components, refer to [AB92; Ste02; Jan95] for edge lengths distributed uniformly on the interval $(0, 1)$ and [Gam05; LZ09] for simple, finite, connected graphs with general i.i.d. nonnegative edge distributions. The essence of the idea is summarized in the following lemma. For the completeness, we include its proof. Note that this lemma is not restricted to simple graph or identical edge distributions. It holds for graphs with non-random edge lengths as well.

Lemma 1. *For any finite, connected graph $G = (V, E)$ (not necessarily simple) with independent random edge lengths, we have*

$$L_{MST}(G) = \int_0^\infty (k(t) - 1)\,\mathrm{d}t,$$

where $k(t)$ is the number of components in the random graph $G(t) = (V, E(t))$ where the edge set $E(t)$ is defined to consist of all edges in G with length no more than t.

Proof: Given a finite, connected graph $G = (V, E)$, consider a continuous time random graph process $G(t) = (V, E(t))$ with the edge set $E(t) = \{e : \xi_e \le t\}$. Let $N(t)$ be the number of MST edges selected up to time t, *i.e.,*

$$N(t) = \sum_{e \in E(t)} \mathbb{I}(e \in E(MST(G))) = \sum_{e \in E(MST(G))} \mathbb{I}(\xi_e \le t).$$

Then $k(t) = |V| - N(t)$, since the selection of each MST edge in the random graph process decreases the number of components by 1.

Hence a nice representation for the length of MST is obtained as the following:

$$\begin{aligned} L_{MST}(G) &= \sum_{e \in E(MST(G))} \xi_e = \sum_{e \in E(MST(G))} \int_0^\infty \mathbb{I}(t < \xi_e)\,\mathrm{d}t \\ &= \int_0^\infty \sum_{e \in E(MST(G))} (1 - \mathbb{I}(\xi_e \le t))\,\mathrm{d}t \\ &= \int_0^\infty (|V| - 1 - N(t))\,\mathrm{d}t = \int_0^\infty (k(t) - 1)\,\mathrm{d}t \end{aligned}$$

Note that the integration limit is only up to $\max_{e\in E(G)}\xi_e$, since $k(t)=1$ for $t>\max_{e\in E(G)}\xi_e$, if G is connected. This finishes the proof of Lemma 1. One can check that the above argument does not require G to be a simple graph, since neither a loop nor a multiple edge can be selected in the MST of G. □

Proof of Theorem 1: Inspired by Steele's work [Ste02], we relate $k(t)$ to the multivariate Tutte polynomial of the graph $G(t)$. In this way, we not only allow the graph to have multiple edges and loops, but also allow the edge weights to follow non-identical distributions. Assume $\xi_e \sim F_e(t)$, then the moment generating function of $k(t)$ is

$$\phi(s)=\mathbb{E}\exp(sk(t))=\sum_{A\subseteq E}\left(\prod_{e\in A}F_e(t)\right)\left(\prod_{e\in E\setminus A}(1-F_e(t))\right)\ e^{sk(A)}.$$

Since the edge lengths may follow different distributions, it is possible that for some $t>0$, there exists an edge e such that $F_e(t)=1$. Let $E'(t)=\{e\in E: F_e(t)<1\}$, then $\phi(s)$ is the same as the above formula with the edge set E replaced by $E'(t)$.

Let $G'(t)$ be the graph obtained from $G(t)$ by contracting each pair of endpoints of the edge in $E\setminus E'(t)$ into a single vertex (loops may be formed). Then we can rewrite $\phi(s)$ in terms of the multivariate Tutte polynomial as the following:

$$\begin{aligned}\phi(s) &= \prod_{e\in E'(t)}(1-F_e(t))\sum_{A\subseteq E'(t)}e^{sk(A)}\left(\prod_{e\in A}\frac{F_e(t)}{1-F_e(t)}\right)\\ &= \prod_{e\in E'(t)}\frac{1}{1+v_e(t)}\ Z(G'(t);\ e^s,\ \mathbf{v}(t)),\end{aligned}$$

where $\mathbf{v}(t)=\{v_e(t),\ e\in E'(t)\}$ and $v_e(t)=F_e(t)/(1-F_e(t))$.

Therefore, if we denote $Z_q(G;q,\mathbf{v})$ as the partial derivative of the multivariate Tutte polynomial with respect to q, then

$$\mathbb{E}k(t)=\phi'(0)=\prod_{e\in E'(t)}\frac{1}{1+v_e(t)}\ Z_q(G'(t);\ 1,\ \mathbf{v}(t)).$$

This finishes the proof of Theorem 1. □

4 Applications

Theorem 1 gives a most generalized version of the exact formula of $\mathbb{E}L_{MST}(G)$ in the sense that it not only allows the edge distributions to be non-identical, but also allows the graphs to be non-simple. In this section, we apply this general theorem to two specific families of graphs: wheel graphs and cylinder graphs. For both families of graphs, the edges are divided into two groups by the type of edge length distribution. We first derive the multivariate Tutte polynomial explicitly, then study the exact values and asymptotic values of the expected lengths of MSTs. At the end of this section, we apply Theorem 1 to complete graphs. Since Frieze [Fri85] first studied $\mathbb{E}L^F_{MST}(K_n)$ more than twenty years ago, many research has been done for the expected lengths of MSTs for the complete graph, for example, [Jan95; Gam05]. However, We show that our approach provides a new insight to the problem.

4.1 The Wheel Graph

The wheel graph, defined in Section 1, is an important class of planar graphs both in theory and in applications. It has nice properties such as self-duality, see [ST50; Tut50]. In [LZ09], the wheel graph was used as an example to study the expected length of MST for graphs with general i.i.d. edge lengths. In this paper, we consider the wheel graphs with rims and spokes following edge distributions F_r and F_s respectively. The graph structure of the wheel graph enables us to compute its multivariate Tutte polynomial with two types of edges explicitly. By applying the general formula (4), we obtain a representation of $\mathbb{E}L_{MST}(W_n)$. In Section 4.1.2, we substitute $U(0,1)$ and $\exp(1)$ distributions as special cases of F_r and F_s and compute the exact values and asymptotical values of the expected lengths of MSTs for the wheel graph in these cases.

4.1.1 The Multivariate Tutte Polynomial of Wheel Graphs

For a wheel graph W_n, we denote the edge lengths on rims and spokes as v_r, v_s respectively.

Theorem 4 (The Multivariate Tutte Poly for W_n)**.** *For a wheel graph W_n, the multivariate Tutte Polynomial is*

$$Z(W_n; q, \mathbf{v}) = q(q-2)v_r^n + q(\alpha^n + \beta^n), \tag{14}$$

where $\mathbf{v} = \{v_r, v_s\}$*, and*
$\alpha, \beta = \frac{1}{2}\left(2v_r + v_r v_s + v_s + q \pm \sqrt{v_r^2 v_s^2 + 2v_r v_s^2 + v_s^2 + 4v_r v_s - 2q v_r v_s + 2q v_s + q^2}\right)$.

Proof: The theorem is proved by considering the recursive relations among the multivariate Tutte polynomials of W_n and its subgraphs X_n , Y_n , Z_n. This is similar to the idea in deriving the standard Tutte polynomial for wheel graphs in [LZ09], except that there are now two types of edges in each graph, see Figure 3. For short, we denote $Z(G)$ as the multivariate Tutte polynomial $Z(G; q, \mathbf{v})$ of

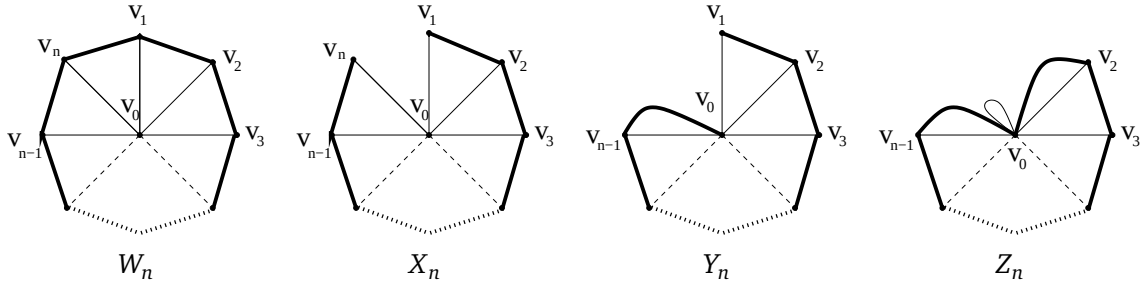


Figure 3: W_n,X_n , Y_n, and Z_n with Two Types of Edges

graph G. From the deletion-contraction identity (12), we obtain the following recursive relations:

$$\begin{aligned}
Z(W_{n+1}) &= Z(X_{n+1}) + v_r v_s(1+v_s)Z(Y_n) + v_r^2 v_s(1+v_s)Z(Z_n) + v_r Z(W_n),\\
Z(X_{n+1}) &= (q + v_r + v_s)Z(X_n) + v_r v_s(1+v_s)Z(Y_n),\\
Z(Y_{n+1}) &= Z(X_n) + v_r(1+v_s)Z(Y_n),\\
Z(Z_{n+1}) &= (1+v_s)Z(Y_n) + v_r(1+v_s)Z(Z_n),
\end{aligned}$$

with initial conditions:

$$\begin{aligned} Z(W_2) &= Z(X_2) + v_r(1+v_r)((q+v_s)^2 + (q-1)v_s^2), \\ Z(X_2) &= (q+v_s)^2(q+v_r) + (q-1)v_r v_s^2, \\ Z(Y_2) &= (q+v_s)(q+v_r) + (q-1)v_r v_s, \\ Z(Z_2) &= (1+v_r)(1+v_s)q. \end{aligned}$$

Then generating functions are formed as

$$F(t) = \sum_{n\geq 2} Z(X_n)t^n, \;\; G(t) = \sum_{n\geq 2} Z(Y_n)t^n, \;\; P(t) = \sum_{n\geq 2} Z(Z_n)t^n, \;\; Q(t) = \sum_{n\geq 2} Z(W_n)t^n.$$

By solving these generating functions, we obtain

$$Q(t) = -q(q+qv_r+v_s+v_r v_s)t - q^2 + \frac{q(q-2)}{1-tv_r} + \frac{q}{1-\alpha t} + \frac{q}{1-\beta t},$$

where

$$\alpha, \beta = \frac{1}{2}\left(2v_r + v_r v_s + v_s + q \pm \sqrt{v_r^2 v_s^2 + 2v_r v_s^2 + v_s^2 + 4v_r v_s - 2qv_r v_s + 2qv_s + q^2}\right).$$

Since the multivariate Tutte polynomial of W_n is the coefficient of the term t^n in the series expansion of $Q(t)$,

$$Z(W_n; q, \mathbf{v}) = q(q-2)v_r^n + q(\alpha^n + \beta^n),$$

which finishes the proof of Theorem 4. □

4.1.2 The Expected Lengths of MSTs of Wheel Graphs

For a wheel graph W_n, we assume the edge lengths on rims and spokes ξ_r and ξ_s follow distributions F_r and F_s respectively. In addition, let $b_r = \min\{t : F_r(t) = 1\}$ and $b_s = \min\{t : F_s(t) = 1\}$. Note that it is possible for both b_r and b_s to be infinity. If this is the case, we use the convention that $\mathbb{I}(b_r < b_s) = 0$. The exact values of the expected lengths of MSTs of the wheel graph are then given in the following proposition.

Proposition 1. *For a wheel graph W_n, we have*

$$\begin{aligned} &\mathbb{E}L_{MST}(W_n) \\ = &\int_0^{\min(b_r,b_s)} (F_r(t))^n(1-F_s(t))^n + n(1-F_r(t))(1-F_s(t))\,\mathrm{d}t \\ &+ \int_0^{\min(b_r,b_s)} \frac{nF_r(t)F_s(t)(1-F_r(t))(1-F_s(t))}{1-F_r(t)+F_r(t)F_s(t)}((F_r(t))^{n-1}(1-F_s(t))^{n-1} - 1)\,\mathrm{d}t \\ &+ \mathbb{I}(b_r < b_s)\int_{b_r}^{b_s} (1-F_s(t))^n\,\mathrm{d}t. \end{aligned}$$

From Proposition 1, exact and asymptotic values of $\mathbb{E}L_{MST}(W_n)$ can be computed explicitly when F_r and F_s are specialized to specific distributions. In these cases, we use special notations for the expected lengths of MSTs as the following: $\mathbb{E}L^{ue}_{MST}(W_n)$, if $F_r \sim U(0,1)$ and $F_s \sim \exp(1)$; and $\mathbb{E}L^{eu}_{MST}(W_n)$, if $F_r \sim \exp(1)$ and $F_s \sim U(0,1)$. It is easy to see that for both of these cases, the values of expected length of MST should be between $\mathbb{E}L^{u}_{MST}(W_n)$ and $\mathbb{E}L^{e}_{MST}(W_n)$. From Proposition 1, we have

Corollary 1. *For wheel graphs W_n,*

$$\mathbb{E}L^{ue}_{MST}(W_n) = \frac{e^{-n}}{n} + ne^{-1} + \int_0^1 t^n e^{-nt} + n\frac{t(1-t)e^{-t}(1-e^{-t})}{1-te^{-t}}((te^{-t})^{n-1}-1)\,\mathrm{d}t, \tag{15}$$

and

$$\begin{aligned}\mathbb{E}L^{eu}_{MST}(W_n) &= n\int_0^1 \frac{t(1-t)e^{-t}(1-e^{-t})}{e^{-t}+t-te^{-t}}((1-t)^{n-1}(1-e^{-t})^{n-1}-1)\,\mathrm{d}t \\ &\quad + ne^{-1} + \int_0^1 (1-t)^n(1-e^{-t})^n\,\mathrm{d}t.\end{aligned} \tag{16}$$

Asymptotically,

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}L^{ue}_{MST}(W_n) = \int_0^1 \frac{(1-t)^2}{e^t-t}\,\mathrm{d}t \approx 0.31637,$$

and

$$\lim_{n\to\infty}\frac{1}{n}\mathbb{E}L^{eu}_{MST}(W_n) = \int_0^1 \frac{(1-t)e^{-t}}{1-t+te^t}\,\mathrm{d}t \approx 0.32490.$$

Comparing this with the corollary in [LZ09], we see that both $\mathbb{E}L^{ue}_{MST}(W_n)$ and $\mathbb{E}L^{eu}_{MST}(W_n)$ go to infinity with a rate between the rate of $\mathbb{E}L^{u}_{MST}(W_n)$ and $\mathbb{E}L^{e}_{MST}(W_n)$ as expected. In addition, Corollary 1 shows that for large n,

$$\mathbb{E}L^{eu}_{MST}(W_n) > \mathbb{E}L^{ue}_{MST}(W_n).$$

Somewhat surprising, one can show that this inequality holds for all $n \geq 4$, by comparing formula (15) with (16), proof is not given here. Intuitively, this may be true because the wheel graph has more spanning trees containing more rims than spokes. Since the random variables distributed $U(0,1)$ are more likely to take small values, more edges with $U(0,1)$ distributed lengths available will give a shorter minimum spanning tree. However, a rigorous proof without using Proposition 1 can be difficult.

In the following proof, we let E_r and E_s denote the set of rims and spokes respectively.
Proof of Proposition 1: This proposition is a direct application of Theorem 1. The key is to find the multivariate Tutte polynomial of graph $W'_n(t) = (V, E'(t))$ for each t. Since $W'_n(t)$ is obtained by contracting all the edges e such that $F_e(t) = 1$, it depends on the size of $b_r = \min\{t : F_r(t) = 1\}$ and $b_s = \min\{t : F_s(t) = 1\}$. In the case that $b_r \neq b_s$, we have to examine the value of $Z_q(W'_n(t); 1, \mathbf{v}(t))$ in the following intervals of t.

1. If $b_r < b_s$, then for every $t \in (b_r, b_s)$, all the rims have to be contracted and the edge set $E'(t)$ contains only spokes. That is, $W'_n(t) = W_n/\{e \in E_r\}$, which is a graph with n parallel edges (spokes) joining the same pair of vertices. Then in the multivariate Tutte polynomial of $W'_n(t)$, we have $\mathbf{v}(t) = \{v_s(t)\}$. By formula (13) and Definition 4,

$$Z(W'_n(t); q, \mathbf{v}(t)) = q(q + (1 + v_s(t))^n - 1),$$

Taking derivative with respect to q and evaluating at $q = 1$,

$$Z_q(W'_n(t); 1, \mathbf{v}(t)) = (1 + v_s(t))^n + 1. \tag{17}$$

2. If $b_r > b_s$, then for any $t \in (b_s, b_r)$, all the spokes have to be contracted and the edge set $E'(t)$ contains only rims. That is, $W'_n(t) = W_n/\{e \in E_s\}$, which is a graph with one vertex and n loops (rims). Then in the multivariate Tutte polynomial of $W'_n(t)$, we have $\mathbf{v}(t) = \{v_r(t)\}$. By formula (10),

$$Z(W'_n(t); q, \mathbf{v}(t)) = q(1 + v_r(t))^n,$$

and

$$Z_q(W'_n(t); 1, \mathbf{v}(t)) = (1 + v_r(t))^n. \tag{18}$$

3. For $t < b_r$ and $t < b_s$, the edge set $E'(t) = E_r \cup E_s$ contains all the rims and spokes of the wheel graph. That is $W'_n(t) = W_n(t)$. Then the multivariate Tutte polynomial of $W'_n(t)$ is given in formula (14). By taking derivative on both sides of (14) with respect to q,

$$Z_q(W'_n(t); q, \mathbf{v}(t)) = 2(q-1)v_r^n(t) + \alpha^n + \beta^n + nq(\alpha^{n-1}\alpha_q + \beta^{n-1}\beta_q).$$

In Theorem 1, we are only interested in $Z_q(W_n; q, \mathbf{v})$ at $q = 1$. This makes the computation much simpler, since the quadratic term in the expressions of α and β in (14) is reduced to $v_r v_s + v_s + 1$. Easy computation gives

$$\alpha\big|_{q=1} = (1 + v_r)(1 + v_s), \qquad \beta\big|_{q=1} = v_r,$$

and

$$\alpha_q\big|_{q=1} = \frac{1 + v_s}{1 + v_s + v_r v_s}, \qquad \beta_q\big|_{q=1} = \frac{v_r v_s}{1 + v_s + v_r v_s}.$$

Thus,

$$\begin{aligned} Z_q(W'_n(t); 1, \mathbf{v}(t)) \;=\; & (1 + v_r(t))^n (1 + v_s(t))^n + (v_r(t))^n \\ & + n\left(\frac{(1 + v_r(t))^{n-1}(1 + v_s(t))^n}{1 + v_s(t) + v_r(t)v_s(t)} + \frac{(v_r(t))^n)v_s(t)}{1 + v_s(t) + v_r(t)v_s(t)}\right). \end{aligned} \tag{19}$$

4. For $t > b_r$ and $t > b_s$, all the rims and spokes are contracted and the edge set $E'(t)$ is empty. $W'_n(t)$ is the trivial graph with one node, which has multivariate Tutte polynomial equaling to q. Thus, for $t > \max\{b_r, b_s\}$, the integral in formula (4) varnishes.

By applying Theorem 1, we obtain

$$\begin{aligned}
& \mathbb{E}L_{MST}(W_n) \\
= \; & \mathbb{I}(b_r < b_s) \int_{b_r}^{b_s} \left((1+v_s(t))^{-n} Z_q(W_n'(t); 1, \mathbf{v}(t)) - 1\right) \mathrm{d}t \\
& + \mathbb{I}(b_r > b_s) \int_{b_s}^{b_r} \left((1+v_r(t))^{-n} Z_q(W_n'(t); 1, \mathbf{v}(t)) - 1\right) \mathrm{d}t \\
& + \int_0^{\min(b_r, b_s)} \left((1+v_r(t))^{-n}(1+v_s(t))^{-n} Z_q(W_n'(t); 1, \mathbf{v}(t)) - 1\right) \mathrm{d}t, \qquad (20)
\end{aligned}$$

where $v_r(t) = F_r(t)/(1-F_r(t))$ and $v_s(t) = F_s(t)/(1-F_s(t))$.

In the case that both b_r and b_s are infinite, $Z_q(W_n'(t); 1, \mathbf{v}(t))$ can be calculated exactly the same as in the case 3 above. The first two integrals in formula (20) are zero.

Finally, by substituting the multivariate Tutte polynomials of $W_n'(t)$ in equations (17)-(19) for $Z_q(W_n'(t); 1, \mathbf{v}(t))$ in (20), Proposition 1 is proved. □

Theorem 2 and Corollary 1 follow immediately from Proposition 1, proofs are omitted.

4.2 The Cylinder Graph

The cylinder graph $P_n \times C_k$, defined in Section 1, was studied in [HL07], where Hutson and Lewis studied $\mathbb{E}L^u_{MST}(P_n \times C_k)$ by computing the standard Tutte polynomial explicitly. We are interested in the cylinder graph because this is another example, besides the wheel graph, for which it is natural to divide the edges into two types, as illustrated in Figure 2. We first compute the multivariate Tutte polynomial for cylinder graphs with two types of edges. Then by applying Theorem 1, we study the expected length of the MST of the cylinder graph with different distributions on different types of edges.

4.2.1 The Multivariate Tutte Polynomial of Cylinder Graphs

As a common strategy in computing the multivariate Tutte polynomial of any graph, the duality identity (11) and the deletion-contraction operations (12) are conducted repeatedly to derive the the multivariate Tutte polynomial of a cylinder graph. Using the simplest cylinder graph $P_1 \times C_2$ as an example, we first illustrate the basic process in conducting the deletion-contraction operations on cylinder graphs with two types of edges. Figure 4 illustrates how the two different types of edges affect the operation. Note that we use wide and dark lines to draw type 1 edges (paths), but lighter and thinner lines to draw type 2 edges (cycles). For each edge that is being operated, we put a number 1 or 2 besides it to indicate the edge type. Each arrow represents one of the Tutte polynomial operations in (12), and the label above the arrow is the coefficient of the operation. In this system of graphs, the multivariate Tutte polynomial of any graph equals to the sum of the multivariate Tutte polynomial of the next directed graph times its coefficient.

Note that for simple notation, we denote G_n as $P_n \times C_2$, and H_n as the graph obtained by contracting all edges in top layers of the cycles in $P_n \times C_2$. In addition, we let $\mathscr{F}_{n,2} = \{G_n, H_n\}$ denote the family

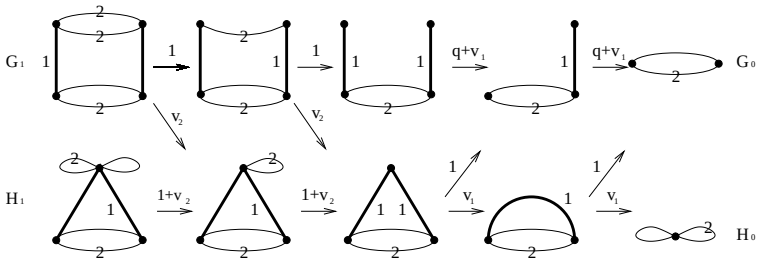


Figure 4: An Example of Operations on Cylinder $P_1 \times C_2$

of such graphs. From Figure 4, one can see clearly that

$$\begin{bmatrix} Z(G_1;q,\mathbf{v}) \\ Z(H_1;q,\mathbf{v}) \end{bmatrix} = \begin{bmatrix} 1 & v_2 \\ 0 & 1+v_2 \end{bmatrix}^2 \begin{bmatrix} q+v_1 & 0 \\ 1 & v_1 \end{bmatrix}^2 \begin{bmatrix} Z(G_0;q,\mathbf{v}) \\ Z(H_0;q,\mathbf{v}) \end{bmatrix}. \tag{21}$$

This shows that there is a recursive relation between the multivariate Tutte polynomials of the family of graphs $\{\mathscr{F}_{n,2};\ n \geq 1\}$. For cylinder graphs $P_n \times C_k$ with larger k, a similar but more complicated recursive relation exists in a family of graphs $\mathscr{F}_{n,k}$, which includes more graphs and will be defined precisely shortly. Similar to $\mathscr{F}_{n,2}$, as shown in Figure 4, the graphs in $\mathscr{F}_{n,k}$ are different from each other only at level n. In fact, these subgraphs on the level n, a special kind of *cap graphs* (defined in Appendix 5), are found to have a one-to-one relationship with noncrossing partitions of the vertices of C_k in [HL07].

A noncrossing partition Γ of the vertices $V = \{0,..,k-1\}$ of C_k, is a partition in which the convex hulls of different blocks are disjoint from each other, i.e., they do not "cross" each other. It was first introduced in [Kre72] and shown to be counted by Catalan numbers. Now the family $\mathscr{F}_{n,k}$ can be defined as

$$\mathscr{F}_{n,k} = \{\Gamma(n,k) : \Gamma \text{ is a noncrossing partition of } C_k\}, \tag{22}$$

where $\Gamma(n,k)$ is obtained by conflating vertices of subgraph at level n of the cylinder graph $P_n \times C_k$ according to the set of blocks in Γ. Thus a loop is created if two vertices are in the same set of blocks in Γ. For example, given partition $\Gamma = \{\{0,1\}\}$, $\Gamma(1,2)$ is the same as H_1 in Figure 4. Actually, the graph $\Gamma(n,k)$ is a special kind of *capped cylinder graph*, which is defined in Appendix 5.

In addition, let $\mathscr{G}$ be an ordering of the noncrossing partitions of C_k and define

$$Z(n;q,\mathbf{v}) = [Z(\Gamma(n,k);q,\mathbf{v})]_{\mathscr{G}}, \tag{23}$$

which is the column vector of the multivariate Tutte polynomial of the graphs in $\mathscr{F}_{n,k}$ relative to the ordering of $\mathscr{G}$. Then similar to equation (21), a general recursive relation can be obtained as the following:

Theorem 5. *For each $n \geq 1$, and the edge lengths $\mathbf{v} = \{v_1,\ v_2\}$,*

$$Z(n;q,\mathbf{v}) = A(q,\mathbf{v})Z(n-1;q,\mathbf{v}) = A^n(q,\mathbf{v})Z(0;q,\mathbf{v}), \tag{24}$$

where the transfer matrix

$$A(q,\mathbf{v}) = q^{1-k}v_1^k(M(v_2)\Theta)^k B\Delta(M(q/v_1)\Theta)^k B\Delta. \tag{25}$$

Note that Θ, Δ, M, and B are matrices defined for noncrossing partitions, which will be defined precisely in formulae (39)-(43) in Appendix 5. The proof of this theorem goes along the same line with the proof of Theorem 4.2 in [HL07], but is more complicated since path and cycle edges in the cylinder graph are considered to be of two types with different length distributions. The details of the proof require precise identification of various quantities, see Appendix 5. To demonstrate the use of Theorem 5, we show two examples of evaluating the multivariate Tutte polynomial of the cylinder graph in the appendix.

4.2.2 Exact Values of the Expected Lengths of MSTs of Cylinder Graphs

In this section, we study the expected lengths of MSTs for cylinder graphs with different edge length distributions on different types of edges, thereby extend the results in [HL07]. By applying the generalized formula (4) and the multivariate Tutte polynomial (24) in the previous section, both exact values and asymptotic behaviors of the expected length of MST for the cylinder graphs are investigated. While the exact value property is the focus of this section, the asymptotic behaviors are discussed in the next.

For the simplicity of notation, we let $G_{n,k} = P_n \times C_k$. In the following, the edge length distributions on type 1 (path) and type 2 (cycle) edges are denoted as F_1 and F_2 respectively. Similar to the notations used in Section 4.1.2, for specialized distributions F_1 and F_2, we denote the expected lengths of MSTs of cylinder graphs as $\mathbb{E}L^{eu}_{MST}(G_{n,k})$, if $F_1 \sim \exp(1)$ and $F_2 \sim U(0,1)$, and as $\mathbb{E}L^{ue}_{MST}(G_{n,k})$, if $F_1 \sim U(0,1)$ and $F_2 \sim \exp(1)$.

Proposition 2. *For the cylinder graph $P_n \times C_k$,*

$$\mathbb{E}L^{eu}_{MST}(G_{n,k}) = n\frac{e^{-k}}{k} - 1 + \int_0^1 (1-t)^{(n+1)k} e^{-nkt} Z_q(G_{nk}; 1, \mathbf{v}(t))\, \mathrm{d}t,$$

where $\mathbf{v}(t) = \{v_1(t), v_2(t)\}$, $v_1(t) = e^t - 1$ and $v_2(t) = t/(1-t)$.

$$\begin{aligned}\mathbb{E}L^{ue}_{MST}(G_{n,k}) &= \int_0^1 e^{-(n+1)kt}(1-t)^{nk} Z_q(G_{nk}; 1, \mathbf{u}(t))\, \mathrm{d}t \\ &\quad -1 + k\frac{e^{-(n+1)}}{n+1} + \frac{1}{n+1}\sum_{i=1}^{k} \frac{(1-e^{-(n+1)})^i - 1}{i}, \end{aligned} \tag{26}$$

where $\mathbf{u}(t) = \{u_1(t), u_2(t)\}$, $u_1(t) = t/(1-t)$, and $u_2(t) = e^t - 1$. $Z_q(G_{nk}; 1, \mathbf{u}(t))$ is the derivative of the multivariate Tutte polynomial of the cylinder graph $P_n \times C_k$ with respect to q.

Proof:
Case 1: If $F_1 \sim \exp(1)$ and $F_2 \sim U(0,1)$, then by the general formula (4) we have

$$\begin{aligned}\mathbb{E}L^{eu}_{MST}(G_{n,k}) &= \int_0^1 \left((1+v_2(t))^{-(n+1)k}(1+v_1(t))^{-nk} Z_q(G_{nk}; 1, \mathbf{v}(t)) - 1\right) \mathrm{d}t \\ &\quad + \int_1^\infty \left((1+v_1(t))^{-nk} Z_q(G'_{nk}; 1, \mathbf{v}(t)) - 1\right) \mathrm{d}t, \end{aligned} \tag{27}$$

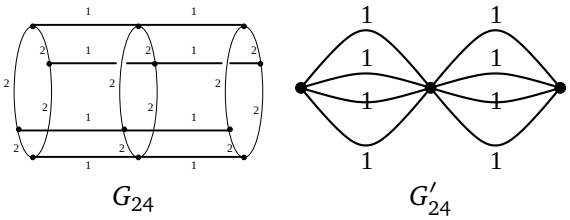


Figure 5: An Example of $P_2 \times C_4$ with Contracted Cycles

where $\mathbf{v}(t) = \{v_1(t), v_2(t)\}$, $v_1(t) = e^t - 1$, $v_2(t) = t/(1-t)$ and the graph G'_{nk} is obtained by contracting the edges on all the cycles in $P_n \times C_k$. An example of G'_{24} is pictured in Figure 5.

By formulae (10) and (13), it is not hard to check that

$$Z(G'_{nk}; q, \mathbf{v}) = q(q + (1+v_1)^k - 1)^n,$$

and

$$Z_q(G'_{nk}; 1, \mathbf{v}) = (1+v_1)^{nk} + n(1+v_1)^{k(n-1)}.$$

Hence for $v_1(t) = e^t - 1$, the second integral in formula (27) of $\mathbb{E}L^{ue}_{MST}(G_{n,k})$ can be computed easily as

$$\int_1^\infty \left(e^{-nkt}(e^{nkt} + ne^{(n-1)kt}) - 1\right) \, \mathrm{dt} = n\frac{e^{-k}}{k}.$$

Hence the first part of Proposition 2 is proved.

Case 2: If $F_1 \sim U(0,1)$ and $F_2 \sim \exp(1)$, we have

$$\begin{aligned} \mathbb{E}L^{ue}_{MST}(G_{n,k}) &= \int_0^1 \left((1+u_2(t))^{-(n+1)k}(1+u_1(t))^{-nk} Z_q(G_{nk}; 1, \mathbf{u}(t)) - 1\right) \, \mathrm{d}t \\ &\quad + \int_1^\infty \left((1+u_2(t))^{-(n+1)k} Z_q(G'_{nk}; 1, \mathbf{u}(t)) - 1\right) \, \mathrm{d}t, \end{aligned} \tag{28}$$

where $\mathbf{u}(t) = \{u_1(t), u_2(t)\}$, $u_1(t) = t/(1-t)$, $u_2(t) = e^t - 1$. Note that we use $\mathbf{u}(t)$ to denote the edge weight vector in this case, in order to differentiate it with the one in case 1. The graph G'_{nk} is obtained by contracting the edges on all the paths in $P_n \times C_k$. More precisely, G'_{nk} becomes $(C_k)_{n+1}$ with type 2 edges, that is a k-cycle with $n+1$ parallel edges on each side. An example of G'_{24} is pictured in Figure 6.

While the standard Tutte polynomial of G'_{nk} for any n may be complicated to compute directly, the multivariate Tutte polynomial is much easier to obtain. This is largely due to the parallel-reduction identity (13) of the multivariate Tutte polynomial. The contracted graph G'_{nk} may be considered as a cycle C_k with weight $u = (1+u_2)^{n+1} - 1$ for every edge. In addition, since $u_2(t) = e^t - 1$, we have $u(t) = e^{(n+1)t} - 1$.

One can check that the multivariate Tutte polynomial of C_k with $u = u_e$ for every edge e is

$$Z(C_k; q, \mathbf{u}) = (q+u)^k + (q-1)u^k.$$

Therefore,

$$Z(G'_{nk}; q, \mathbf{u}(t)) = (q + e^{(n+1)t} - 1)^k + (q-1)(e^{(n+1)t} - 1)^k.$$

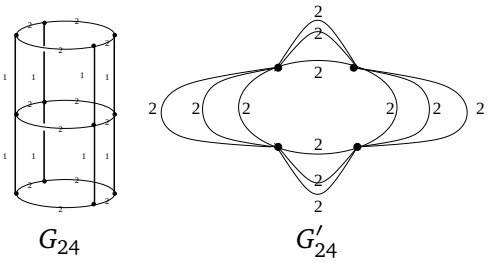


Figure 6: An Example of $P_2 \times C_4$ with Contracted Paths

Thus

$$Z_q(G'_{nk}; 1, \mathbf{u}(t)) = ke^{(k-1)(n+1)t} + (e^{(n+1)t} - 1)^k.$$

Plug these into the second integral, which we name as $I_2^{ue}(n,k)$, in formula (28) and obtain

$$\begin{aligned} I_2^{ue}(n,k) &= \int_1^\infty \left((1+u_2(t))^{-(n+1)k} Z_q(G'_{nk}; 1, \mathbf{u}(t)) - 1\right) \mathrm{d}t \\ &= \int_1^\infty ke^{-(n+1)t} + (1-e^{-(n+1)t})^k - 1 \, \mathrm{d}t \\ &= k\frac{e^{-(n+1)}}{n+1} + \int_{1-e^{-(n+1)}}^1 \frac{t^k - 1}{(n+1)(1-t)} \, \mathrm{d}t, \end{aligned}$$

by a change of variable.

By expanding the term $t^k - 1$, we obtain

$$I_2^{ue}(n,k) = k\frac{e^{-(n+1)}}{n+1} + \frac{1}{n+1}\sum_{i=1}^{k} \frac{(1-e^{-(n+1)})^i - 1}{i}. \tag{29}$$

Hence the second part of Proposition 2 is also proved. □

Note that $I_2^{ue}(n,k) \to 0$, as n goes to infinity. We will use this property in calculating the asymptotic value of $\mathbb{E}L_{MST}^{ue}(G_{n,k})/n$ in the next section.

Now the values of $\mathbb{E}L_{MST}^{ue}(G_{n,k})$ and $\mathbb{E}L_{MST}^{eu}(G_{n,k})$ can be computed for any cylinder graph as long as its multivariate Tutte polynomial is known. In Tables 1 and 2, we compare the exact values of $n^{-1}\mathbb{E}L_{MST}(G_{n,k})$

for the cases of $\{F_1(t) = F_2(t) = 1 - e^{-t}\}$, $\{F_1(t) = 1 - e^{-t}$ and $F_2(t) = t\}$, $\{F_1(t) = t$ and $F_2(t) = 1 - e^{-t}\}$, and $\{F_1(t) = F_2(t) = t\}$. While the last case was checked in Hutson and Lewis [HL07], we use it to compare with other cases.

For the simplicity of notation, we denote

$$\mathbb{E}L(n) = \mathbb{E}L_{MST}(G_{n,k}). \tag{30}$$

Note that for all n in Table 1 and $n \geq 3$ in Table 2, we have

$$n^{-1}\mathbb{E}L^e(n) \geq n^{-1}\mathbb{E}L^{eu}(n) \geq n^{-1}\mathbb{E}L^{ue}(n) \geq n^{-1}\mathbb{E}L^u(n).$$

Table 1: Exact Values of $P_n \times C_2$ for Different Edge Distributions

n	$n^{-1}\mathbb{E}L^e(n)$	$n^{-1}\mathbb{E}L^{eu}(n)$	$n^{-1}\mathbb{E}L^{ue}(n)$	$n^{-1}\mathbb{E}L^u(n)$
1	1.35000	1.13134	1.06472	0.94286
2	1.11349	0.96567	0.85633	0.77958
3	1.03495	0.91046	0.78916	0.72525
4	0.99568	0.88286	0.75576	0.69809
5	0.97212	0.86630	0.73573	0.68179
6	0.95642	0.85526	0.72238	0.67093
7	0.94520	0.84737	0.71285	0.66317
8	0.93679	0.84145	0.70570	0.65735
9	0.93025	0.83685	0.70013	0.65282
10	0.92501	0.83317	0.69568	0.64920

$k = 2$

Table 2: Exact Values of $P_n \times C_3$ for Different Edge Distributions

n	$n^{-1}\mathbb{E}L^e(n)$	$n^{-1}\mathbb{E}L^{eu}(n)$	$n^{-1}\mathbb{E}L^{ue}(n)$	$n^{-1}\mathbb{E}L^u(n)$
1	2.17421	1.71031	1.82033	1.56310
2	1.64128	1.33985	1.35248	1.20091
3	1.46555	1.21652	1.20432	1.08083
4	1.37779	1.15487	1.13103	1.02083
5	1.32515	1.11787	1.08715	0.98483
6	1.29005	1.09321	1.05790	0.96084
7	1.26499	1.07559	1.03702	0.94370
8	1.24619	1.06238	1.02136	0.93084
9	1.23156	1.05210	1.00917	0.92084
10	1.21986	1.04388	0.99943	0.91284

$k = 3$

4.2.3 Asymptotic Values of the Expected Lengths of MSTs of Cylinder Graphs

Tables 1 and 2 show that the values of $\mathbb{E}L_{MST}(G_{n,k})/n$ are decreasing for all the four cases as n grows larger. Hutson and Lewis [HL07] showed that $\mathbb{E}L^u_{MST}(G_{n,k})/n$ converges to a number that can be represented by the dominant eigenvalue of their transfer matrix. For the cylinder graph with mixed edge weight distributions, we prove a similar asymptotic result in Theorem 3. Recall that Perron-Frobenius theorem says a nonnegative primitive matrix has a unique eigenvalue of maximum modulus, which is called the Perron-Frobenius eigenvalue, see Th2.1 in [Var62].

To prove Theorem 3, we first show three basic lemmas.

Lemma 2. *For $k \geq 2$, the transfer matrix $A(q, \mathbf{v})$ is primitive, i.e., there exists k such that $A^k(q, \mathbf{v}) \geq 0$.*

In particular, at $q=1$, the Perron-Frobenius eigenvalue of $A(1,\mathbf{v})$ is

$$\lambda(1,\mathbf{v}) = (1+v_1)^k(1+v_2)^k,$$

with the corresponding eigenvector as $Z(0;1,\mathbf{v})$.

Proof: To show the transfer matrix $A(q,\mathbf{v})$ given in (25) is primitive, it is enough to show it is irreducible and has all positive entries on the diagonal, see Theorem 8.5.5 in [HJ86]. Both of these two properties are easy to verify, because it is enough to show these are true for $A(q,\mathbf{v})$ at $q=v_1=v_2=1$. In this case, our transfer matrix $A(1,\mathbf{v})$ is reduced to the transfer matrix in [HL07] at $(x-1)(y-1)=1$, which was shown to be irreducible and have positive diagonal entries. In addition, since $A(q,\mathbf{v})$ is clearly nonnegative, by Perron-Frobenius theorem, it has the Perron-Frobenius eigenvalue, which we denote as $\lambda(q,\mathbf{v})$.

Since graph $\Gamma(0,k)=(V,\{E_1,E_2\})$ has edge count $|E_1|=0$, $|E_2|=k$ and graph $\Gamma(1,k)=(V',\{E_1',E_2'\})$ has $|E_1'|=k$, $|E_2'|=2k$, from the simplification formula (8) of the multivariate Tutte polynomial at $q=1$, we have

$$Z(\Gamma(0,k);1,\mathbf{v}) = (1+v_2)^k,$$

and

$$Z(\Gamma(1,k);1,\mathbf{v}) = (1+v_2)^{2k}(1+v_1)^k = (1+v_1)^k(1+v_2)^k Z(\Gamma(0,k);1,\mathbf{v}).$$

Moreover, $Z(1;1,\mathbf{v})$ is a vector of $Z(\Gamma(1,k);1,\mathbf{v})$ for all noncrossing partitions $\Gamma\in\mathscr{G}$, thus by Theorem 5,

$$Z(1;1,\mathbf{v}) = A(q,\mathbf{v})Z(0;1,\mathbf{v}) = (1+v_1)^k(1+v_2)^k Z(0;1,\mathbf{v}).$$

Therefore, $Z(0;1,\mathbf{v})$ is a positive eigenvector of the nonnegative matrix $A(1,\mathbf{v})$, with the corresponding positive eigenvalue $(1+v_1)^k(1+v_2)^k$. Thus by Theorem 8.1.30 in [HJ86], at $q=1$, the eigenvalue of maximum modulus of $A(q,\mathbf{v})$, which is the Perron-Frobenius eigenvalue, is

$$\lambda(1,\mathbf{v}) = (1+v_1)^k(1+v_2)^k.$$

This completes the proof of Lemma 2. □

Next we show that

Lemma 3. *For $\mathbf{v}=\{v_1,v_2\}$, if we let $Z_q(P_n\times C_k;1,\mathbf{v})$ be the derivative of the multivariate Tutte polynomial of the cylinder graph $P_n\times C_k$ with respect to q and evaluated at $q=1$, then*

$$\frac{Z_q(P_n\times C_k;1,\mathbf{v})}{n(1+v_2)^{(n+1)k}(1+v_1)^{nk}} \to \frac{\lambda_q(1,\mathbf{v})}{\lambda(1,\mathbf{v})},\ \textit{as } n\to\infty.$$

Proof: The proof of this lemma goes along the similar line as the proof of Theorem 7.1 in [HL07]. Adopt the notations there, we let $\xi_(q,\mathbf{v})$ be the Perron-Frobenius eigenvector. Choose a scaling of $\xi_(q,\mathbf{v})$ such that $\xi(1,\mathbf{v})=Z(0;1,\mathbf{v})$. If we let

$$r(q,\mathbf{v}) = Z(0;q,\mathbf{v}) - \xi(q,\mathbf{v}),$$

then

$$\begin{aligned} Z(n;q,\mathbf{v}) &= A^n(q,\mathbf{v})Z(0;q,\mathbf{v}) = A^n(q,\mathbf{v})(\xi(q,\mathbf{v})+r(q,\mathbf{v})) \\ &= \lambda^n(q,\mathbf{v})\xi(q,\mathbf{v}) + A^n(q,\mathbf{v})r(q,\mathbf{v}). \end{aligned} \tag{31}$$

Therefore, by taking derivative on both sides of equation (31) with respect to q,

$$\begin{aligned} Z_q(n;q,\mathbf{v}) &= n\lambda^{n-1}(q,\mathbf{v})\xi(q,\mathbf{v})\lambda_q(q,\mathbf{v}) + \lambda^n(q,\mathbf{v})\xi_q(q,\mathbf{v}) \\ &\quad + nA^{n-1}(q,\mathbf{v})r(q,\mathbf{v})A_q(q,\mathbf{v}) + A^n(q,\mathbf{v})r_q(q,\mathbf{v}). \end{aligned}$$

We are only interested in the above quantity at $q=1$, at which $\xi(1,\mathbf{v}) = Z(0;1,\mathbf{v})$ and $r(1,\mathbf{v}) = 0$. This shows that, by dividing $n\lambda^n(1,\mathbf{v})$, we obtain

$$\frac{Z_q(n;1,\mathbf{v})}{n\lambda^n(1,\mathbf{v})} = \frac{\lambda_q(1,\mathbf{v})}{\lambda(1,\mathbf{v})} Z(0;1,\mathbf{v}) + \frac{\xi_q(1,\mathbf{v})}{n} + \left(\frac{A(1,\mathbf{v})}{\lambda(1,\mathbf{v})}\right)^n \frac{r_q(1,\mathbf{v})}{n}.$$

As $n\to\infty$, obviously $\xi_q(1,\mathbf{v})/n \to 0$. Since the matrix $A(q,\mathbf{v})$ is primitive

$$\left(\frac{A(1,\mathbf{v})}{\lambda(1,\mathbf{v})}\right)^n \to D(1,\mathbf{v}), \text{ as } n\to\infty,$$

for some matrix $D(1,\mathbf{v})$, according to Theorem 8.5.1 in [HJ86]. Hence

$$\left(\frac{A(1,\mathbf{v})}{\lambda(1,\mathbf{v})}\right)^n \frac{r_q(1,\mathbf{v})}{n} \to 0, \text{ as } n\to\infty.$$

Consequently,

$$\frac{Z_q(n;1,\mathbf{v})}{n\lambda^n(1,\mathbf{v})} \to \frac{\lambda_q(1,\mathbf{v})}{\lambda(1,\mathbf{v})} Z(0;1,\mathbf{v}), \text{ as } n\to\infty.$$

Since $Z_q(n;1,\mathbf{v})$ is a vector of $Z_q(\Gamma(n,k);1,\mathbf{v})$ for all $\Gamma\in\mathscr{G}$, we have for any noncrossing partition Γ of C_k,

$$\frac{Z_q(\Gamma(n,k);1,\mathbf{v})}{n\lambda^n(1,\mathbf{v})Z(\Gamma(0,k);1,\mathbf{v})} \to \frac{\lambda_q(1,\mathbf{v})}{\lambda(1,\mathbf{v})}. \text{ as } n\to\infty.$$

If Γ is the partition consisting of k isolated vertices, then $\Gamma(n,k)$ is the cylinder graph $P_n\times C_k$ and $\Gamma(0,k)$ is the cycle C_k with only type 2 edges. Therefore, by Lemma 2 and the fact that $Z(C_k;1,\mathbf{v}) = (1+v_2)^k$, Lemma 3 is proved. □

Now by dominated convergence theorem, Theorem 3 follows easily from the next lemma, details are omitted.

Lemma 4.

$$0 < \frac{Z_q(P_n\times C_k;1,\mathbf{v})}{n(1+v_2)^{(n+1)k}(1+v_1)^{nk}} \le \frac{(n+1)k}{nk}.$$

Proof of Lemma 4: Let the edge subset $A = A_1\cup A_2$, where the set A_1, A_2 consists of type 1 and type 2 edges in A respectively. By the definition of the multivariate Tutte polynomial,

$$Z(G_{nk};q,\mathbf{v}) = \sum_{A\subseteq E} q^{k(A)} v_1^{|A_1|} v_2^{|A|-|A_1|},$$

where $0\le|A_1|\le nk$ and $0\le|A_2|\le(n+1)k$.

As such, by taking derivative on both sides of the above,

$$Z_q(G_{nk};q,\mathbf{v}) = \sum_{A\subseteq E} k(A) q^{k(A)-1} v_1^{|A_1|} v_2^{|A|-|A_1|},$$

then

$$Z_q(G_{nk};1,\mathbf{v}) = \sum_{m=0}^{|E|} \sum_{|A|=m} k(A) v_1^{|A_1|} v_2^{m-|A_1|}. \tag{32}$$

Since the number of components in a graph is bounded by the number of vertices the graph has, for the cylinder graph $P_n \times C_k$,

$$1 \le k(A) \le (n+1)k.$$

Then from equation (32) and the fact that $|E| = (2n+1)k$,

$$\begin{aligned} Z_q(P_n \times C_k;1,\mathbf{v}) &\le (n+1)k \sum_{m=0}^{(2n+1)k} \sum_{|A|=m} v_1^{|A_1|} v_2^{m-|A_1|} \\ &= (n+1)k \sum_{m=0}^{(2n+1)k} \sum_{i=0}^{nk} \left(\sum_{|A_1|=i,|A|=m} 1 \right) v_1^i v_2^{m-i} \\ &= (n+1)k \sum_{m=0}^{(2n+1)k} \sum_{i=0}^{nk} \binom{nk}{i} \binom{(n+1)k}{m-i} v_1^i v_2^{m-i} \\ &= (n+1)k(1+v_1)^{nk}(1+v_2)^{(n+1)k}, \end{aligned}$$

which completes the proof of Lemma 4. □

Theorem 3 shows that to calculate the asymptotic value of $\mathbb{E}L_{MST}(G_{nk})/n$ for a specific value of k, one only needs to find $\lambda_q(1,\mathbf{v})$. The calculation of $\lambda_q(q,\mathbf{v})$ in general is hard, since we do not have enough information for $\lambda(q,\mathbf{v})$ at $q \neq 1$. However, following the idea of using the characteristic polynomial of $A(q,\mathbf{v})$ mentioned in [HL07], one can find the following without much difficulty

$$\lambda_q(1,\mathbf{v}) = -\frac{P_q(\lambda(1,\mathbf{v});1,\mathbf{v})}{P_\lambda(\lambda(1,\mathbf{v});1,\mathbf{v})}, \tag{33}$$

where $P(\lambda;q,\mathbf{v})$ is the characteristic polynomial of $A(q,\mathbf{v})$, $P_q(\lambda(1,\mathbf{v});1,\mathbf{v})$ and $P_\lambda(\lambda(1,\mathbf{v});1,\mathbf{v})$ are obtained by taking derivative of $P(\lambda;q,\mathbf{v})$ with respect to q and λ respectively, then evaluated at $q=1$.

Finally, recall that we used $\mathbb{E}L(n)$ as a shorthand for $\mathbb{E}L_{MST}(G_{n,k})$ in (30), where $G_{n,k}$ denotes the cylinder graph $P_n \times C_k$. With the calculation of the transfer matrix for $k = 2,\ 3$ at the end of Section 4.2.1, we compute the asymptotic values of $n^{-1}\mathbb{E}L(n)$ by applying Theorem 3 and formula (33). The results are shown in Table 3.

This table shows that for $k=2$ and $k=3$, as $n \to \infty$,

$$n^{-1}\mathbb{E}L^e(n) \ge n^{-1}\mathbb{E}L^{eu}(n) \ge n^{-1}\mathbb{E}L^{ue}(n) \ge n^{-1}\mathbb{E}L^u(n).$$

One may notice that in Table 3, the difference between $n^{-1}\mathbb{E}L^{eu}(n)$ and $n^{-1}\mathbb{E}L^{ue}(n)$ is bigger than e^{-k}/k. From Theorem 3, this means that for the cases of $k = 2,\ 3$ the integral

$$\int_0^1 \frac{\lambda_q(1,\mathbf{v}(t))}{\lambda(1,\mathbf{v}(t))}\,\mathrm{d}t$$

Table 3: Asymptotic Values of the Ratio of the Expected Length of the Cylinder Graph $P_n \times C_k$ to the Length n

	$k=2$	$k=3$
$n^{-1}\mathbb{E}L^e(n)$	0.87790	1.14579
$n^{-1}\mathbb{E}L^{eu}(n)$	0.80005	0.96989
$n^{-1}\mathbb{E}L^{ue}(n)$	0.65564	0.91171
$n^{-1}\mathbb{E}L^u(n)$	0.61661	0.84085

is bigger if $\mathbf{v}(t) = \{v_1(t), v_2(t)\}$, $v_1(t) = e^t - 1$ and $v_2(t) = t/(1-t)$. We expect the same inequality holds for larger k too.

Conjecture 1. *For any $k \geq 2$, as $n \to \infty$,*

$$n^{-1}\mathbb{E}L^e(n) \geq n^{-1}\mathbb{E}L^{eu}(n) \geq n^{-1}\mathbb{E}L^{ue}(n) \geq n^{-1}\mathbb{E}L^u(n).$$

4.3 Discussions on The Complete Graph

Since the multivariate Tutte polynomial enables us to control the length of each edge, we expect a lot more applications of the general formula (4). For example, it is known that $\mathbb{E}L_{MST}(K_n)$ converges to a finite number as long as the identical edge distribution F satisfies $F'(0) \neq 0$, but diverges if $F'(0) = 0$. For a complete graph with certain portion of edge lengths following distribution $U(0,1)$ and the other edge lengths following a distribution F, with $F'(0) = 0$, it would be interesting to know for what portion and configuration of edges with distribution $F'(0) > 0$ that the expected length of MST of this complete graph still converges to a constant. In this section, we discuss the simplest problem in this category, that is, the expected lengths of MSTs of a complete graph with a portion of edges removed.

While the complete graph is the most dense simple graph, removing one edge should not change the value of expected length of MST too much for large n. Heuristically, removing two edges should not make much difference either. Then, an intriguing problem is to find the maximum number $m(K_n)$ of edges that can be removed arbitrarily from K_n without changing the asymptotic value of the expected length of the MST.

Let us first consider two extreme cases. Since the degree of each vertex is $n-1$, it is trivial to see that $m(K_n) < n-1$. Otherwise, the remaining graph may become disconnected and the length of the MST is infinity. Moreover, if $n-2$ edges incident to a common vertex, say v_1 are removed, then the vertex v_1 is incident to only one edge, which becomes a bridge and has to be included in the MST by the cut property. By properties of the MST, the remaining graph G_n satisfies

$$\mathbb{E}L_{MST}(G_n) = \mathbb{E}\xi_e + \mathbb{E}L_{MST}(K_{n-1}),$$

whose asymptotic value is obviously different from the asymptotic value of $\mathbb{E}L_{MST}(K_n)$ by the quantity $\mathbb{E}\xi_e$. Moreover, it is easy to see that for both of these cases, if the m edges are not removed from a single vertex, then the MST of the remaining graph may be much shorter. In particular, the MST in the first case at least has finite length. Therefore, we formulate the following conjecture.

Conjecture 2. *Removing arbitrary $m \leq n-1$ edges from the complete graph K_n, the expected length of the MST of the remaining graph is the largest if these m edges share a common end point.*

In fact, by examining the number of spanning trees of a complete graph with $m < n-1$ edges removed, we suspect that the smallest number of spanning trees appear in the resulting graph if these m edges share a common end point. This might be a possible approach to proving Conjecture 2.

Let $K_n^{-m} = (V, E^{-m})$ denote the graph obtained by removing m edges which share a common vertex from the complete graph K_n, with edge set E^{-m} and same vertex set as K_n. Based on Conjecture 2, to find the maximal number $m(K_n)$, it suffices to examine graph K_n^{-m}. In [LZ09], Li and Zhang showed the difference between $\mathbb{E}L^e_{MST}(K_n)$ and $\mathbb{E}L^u_{MST}(K_n)$ to be of size $\zeta(3)/n$. More precisely, the following equation is shown that for $m = 1$.

$$\mathbb{E}L^e_{MST}(K_n) - \mathbb{E}L^u_{MST}(K_n) = \int_0^1 \frac{1}{(1-t)^m} \frac{T_x(K_n; 1/t, 1/(1-t))}{T(K_n; 1/t, 1/(1-t))}\, \mathrm{d}t \sim \zeta(3)/n. \tag{34}$$

Actually, by a careful argument, one can show that the above equation holds for m up to at least $n/3$. Then we obtain the next lemma by applying Steele's formula (2) and comparing the difference of $\mathbb{E}L_{MST}(K_n^{-m})$ and $\mathbb{E}L^u_{MST}(K_n)$. The Tutte polynomial of K_n^{-m} is found to be related to $T(K_n; x, y)$ by some algebraic manipulation.

Lemma 5. *In a complete graph K_n, for both the cases of $U(0,1)$ and $\exp(1)$ edge length distributions, we can remove at least $o(n)$ edges which share a common vertex from K_n, such that the asymptotic value of the expected length of the MST of K_n^{-m} is the same as that of K_n. That is for any $m = o(n)$,*

$$\lim_{n\to\infty} \mathbb{E}L^u_{MST}(K_n^{-m}) = \lim_{n\to\infty} \mathbb{E}L^u_{MST}(K_n) = \zeta(3),$$

and

$$\lim_{n\to\infty} \mathbb{E}L^e_{MST}(K_n^{-m}) = \lim_{n\to\infty} \mathbb{E}L^e_{MST}(K_n) = \zeta(3).$$

Proof: First, we prove the case of $U(0,1)$ distribution. It is well known that the standard Tutte polynomial is simplified on $H_1 = \{(x-1)(y-1) = 1\}$ as in formula (9). Thus, if we let $N = \binom{n}{2}$ denote the number of edges in the complete graph K_n, then

$$T(K_n; 1/t, 1/(1-t)) = t^{1-n}(1-t)^{n-N-1},$$

and

$$T(K_n^{-m}; 1/t, 1/(1-t)) = (1-t)^m T(K_n; 1/t, 1/(1-t)).$$

This shows that

$$\begin{aligned} &\mathbb{E}L^u_{MST}(K_n^{-m}) - \mathbb{E}L^u_{MST}(K_n) \\ = &\int_0^1 \frac{1-t}{t} \frac{1}{T(K_n; 1/t, 1/(1-t))} \left(\frac{T_x(K_n^{-m}; 1/t, 1/(1-t))}{(1-t)^m} - T_x\left(K_n; \frac{1}{t}, \frac{1}{1-t}\right)\right) \mathrm{d}t. \end{aligned} \tag{35}$$

By Definition 3 of the Tutte polynomial,

$$T_x(K_n; 1/t, 1/(1-t)) = \sum_{A\subseteq E} (k(A)-1) t^{|A|-n+2} (1-t)^{n-|A|-2},$$

and

$$T_x(K_n^{-m};1/t,1/(1-t)) = \sum_{A\subseteq E^{-m}} (k(A)-1)t^{|A|-n+2}(1-t)^{n-|A|-2}.$$

Since K_n and K_n^{-m} share the same vertex set, then given the set A, which is a subset of both E and E^{-m}, $k(A)$ is the same in both K_n and K_n^{-m}. In addition, since $E^{-m} \subset E$, we have

$$T_x(K_n^{-m};1/t,1/(1-t)) < T_x(K_n;1/t,1/(1-t)).$$

By (36), this implies

$$\begin{aligned}\mathbb{E}L^u_{MST}(K_n^{-m}) - \mathbb{E}L^u_{MST}(K_n) &\leq \int_0^1 \frac{1-(1-t)^m}{t(1-t)^{m-1}} \frac{T_x(K_n;1/t,1/(1-t))}{T(K_n;1/t,1/(1-t))}\,\mathrm{d}t \\ &\leq \int_0^1 \frac{m}{(1-t)^{m-1}} \frac{T_x(K_n;1/t,1/(1-t)}{T(K_n;1/t,1/(1-t))}\,\mathrm{d}t. \qquad (36)\end{aligned}$$

As one can show (34) holds for any $m = o(n)$, then as $n \to \infty$,

$$\int_0^1 \frac{m}{(1-t)^{m-1}} \frac{T_x(K_n;1/t,1/(1-t)}{T(K_n;1/t,1/(1-t))}\,\mathrm{d}t \to 0.$$

In addition, it is obvious that for any $m \leq n-1$,

$$\mathbb{E}L^u_{MST}(K_n^{-m}) \geq \mathbb{E}L^u_{MST}(K_n).$$

Therefore, by squeeze theorem, for any $m = o(n)$,

$$\mathbb{E}L^u_{MST}(K_n^{-m}) - \mathbb{E}L^u_{MST}(K_n) \to 0, \text{ as } n \to \infty,$$

which completes the proof of the first part.

The proof for the case of exp(1) distribution follows similar steps as above. Details are omitted. Note that we have one less factor $(1-t)$ in the representation of $\mathbb{E}L^e_{MST}(K_n^{-m}) - \mathbb{E}L^e_{MST}(K_n)$. So the maximal number edges allowed to remove is one less than that in the uniform distribution case. The proof of Lemma 5 is thus complete. □

However, we actually believe that the result could be sharpened to ϵn for a small constant ϵ. The proof of this might require a much tighter bound than (36). For more detailed discussions, refer to [Zha08].

5 Appendix: Proof of Theorem 5

The procedure of finding a general recursive relation between the multivariate Tutte polynomials of $\mathscr{F}_{n,k}$ in the case of two different types of edges goes similar to the one obtaining the standard Tutte polynomial in [HL07], but more complicated and requires precise identification of various quantities. For the completeness of this paper, we first recall a few definitions and lemmas from [HL07] and adopt their notations.

Assume C_k has vertices $\{0,...,k-1\}$, for each i, let $v(i)$ denote the set of vertices that are in the same block with i in the noncrossing Γ. To each partition Γ of C_k, $k+1$ *cap graphs* $\Gamma^0,\ldots,\Gamma^k$ are associated. They all have the same vertex set, namely the set of blocks of Γ, but different edge set. The edge set of Γ^i includes all edges connecting j to $j+1$, for $j=k-i..k-1$. Thus a loop is created if $j+1$ and j are in the same block.

Definition 5 (Capped Cylinder Graph)**.** *The* capped cylinder graph $\Gamma(n,j)$ *is obtained by attaching the cap graph* Γ^j *to the ends of cylinders* $P_{n-1}\times C_k$*, where each vertex* $(n-1,l)$ *of* $P_{n-1}\times C_k$ *is joined to* $v(l)$ *of* Γ^j *by a type 1 edge, for each* $l\in(0,...,k-1)$*.*

Note that $\Gamma(n,k)$ is the same as the one defined in Section 4.2.1.

Definition 6 (Pinched Capped Cylinder Graph)**.** *For* $n\geq 1$*, let* $\pi\Gamma(n,j)$ *be the* pinched capped cylinder graph *obtained from first switching the edge types in* $\Gamma(n,j)$*, then contracting all the edges in* C_k *at level 0. Thereby, in* $\pi\Gamma(n,j)$ *the edges on cycles are of type 1 and the edges on paths are of type 2.*

Note that this definition is different from the one in [HL07], in that the edge types are specified explicitly here.

Lemma (Hutson-Lewis)**.** *Let* Γ *be a partition of* C_k*, and let* Φ *be the partition obtained from* Γ *by conflating the blocks of a pair of adjacent vertices of* C_k*. If* Γ *is noncrossing, then* Φ *is noncrossing.*

Lemma (Hutson-Lewis)**.** *The dual partition of a noncrossing partition is noncrossing.*

Let Γ^* denote the dual of the noncrossing partition Γ of C_k and G^* denote the dual graph of a planar graph G. It is easy to see that the cylinder graph, the capped cylinder graph and the pinched capped cylinder graph are all planar graphs. The dual of a partition and the dual of a planar graph is related by the following theorem.

Theorem (Hutson-Lewis: Duality)**.** *For* $n\geq 1$*,*

$$\Gamma(n,0)^* = \pi\Gamma^*(n,k), \tag{37}$$

and

$$\pi\Gamma(n,0)^* = \Gamma^*(n-1,k). \tag{38}$$

Now given noncrosssing partitions Γ and Φ of C_k, we define the matrices Θ, Δ, M and B used in the transfer matrix in Section 4.2.1. Define function ϕ as $\phi(x)=x+1 \bmod k$ and let $\phi(\Gamma)$ denote the noncrossing partition obtained by applying ϕ to each of the vertices of Γ. Then the rotation matrix Θ is defined as

$$\Theta_{\Gamma\Phi} = \begin{cases} 1 & \text{if } \phi(\Phi)=\Gamma, \\ 0 & \text{otherwise,} \end{cases} \tag{39}$$

and the dual matrix Δ as

$$\Delta_{\Gamma\Phi} = \begin{cases} 1 & \text{if } \Phi=\Gamma^*, \\ 0 & \text{otherwise.} \end{cases} \tag{40}$$

The multivariate Tutte matrix $M(u)$ is defined as the following:

- If $0\sim 1$, i.e., vertices 0 and 1 are in the same block in Γ, then

$$M(u)_{\Gamma\Phi} = \begin{cases} 1+u & \text{if } \Phi=\Gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{41}$$

• If $0 \nsim 1$, i.e., vertices 0 and 1 are not in the same block of vertices, let Σ be the partition obtained by conflating the two blocks containing 0 and 1 in Γ , then

$$M(u)_{\Gamma\Phi} = \begin{cases} 1 & \text{if } \Phi = \Gamma, \\ u & \text{if } \Phi = \Sigma, \\ 0 & \text{otherwise.} \end{cases} \tag{42}$$

Let $m_{\mathscr{G}} = \min\{|V(\Gamma(n,0))|, \Gamma \in \mathscr{G}\}$, then the matrix

$$B_{\Gamma\Phi} = \begin{cases} q^{|V(\Gamma(n,0))|-m_{\mathscr{G}}} & \text{if } \Phi = \Gamma, \\ 0 & \text{otherwise.} \end{cases} \tag{43}$$

For a more detailed discussion of noncrossing partitions and capped graphs, we refer to [HL07].

Proof of Theorem 5: Given an ordering $\mathscr{G}$ of the noncrossing partitions of C_k, we want to find a recursive relation between the family of $\{\Gamma(n,k),\ \Gamma \in \mathscr{G}\}$ and $\{\Gamma(n-1,k),\ \Gamma \in \mathscr{G}\}$. As for the computation of the Tutte polynomial for any graph, we first choose an edge and apply the deletion-contraction identity (12).

For a noncrossing partition Γ, if $0 \sim 1$ in Γ, then there exists a loop edge of type 2 at vertex $v(0)$ in graph Γ^k. Therefore,

$$Z(\Gamma(n,k);q,\mathbf{v}) = (1+v_2)Z(\Gamma(n,k-1);q,\mathbf{v}).$$

If $0 \nsim 1$ in Γ, then there exists an edge of type 2 between vertex $v(0)$ and $v(1)$ in graph Γ^k. Since we have not removed any edge from the path P_n, the graph remains connected even if the edge e is deleted. Therefore, the edge e is not a bridge, and

$$Z(\Gamma(n,k);q,\mathbf{v}) = Z(\Gamma(n,k-1);q,\mathbf{v}) + v_2 Z(\Sigma(n,k-1);q,\mathbf{v}).$$

By the definition of the multivariate Tutte matrix $M(u)$, we have

$$Z(n;q,\mathbf{v}) = M(v_2)[Z(\Gamma(n,k-1));q,\mathbf{v}]_{\mathscr{G}}.$$

Based on whether $1 \sim 2$ in Γ, as discussed above, we have a similar relation between $[Z(\Gamma(n,k-1);q,\mathbf{v})]_{\mathscr{G}}$ and $[Z(\Gamma(n,k-2);q,\mathbf{v})]_{\mathscr{G}}$, but with a different coefficient matrix. To make the representation easier, note that $1 \sim 2$ in Γ if and only if $0 \sim 1$ in $\phi^{-1}(\Gamma)$. Therefore,

$$[Z(\Gamma(n,k-1);q,\mathbf{v})]_{\mathscr{G}} = \Theta M(v_2)\Theta^{-1}[Z(\Gamma(n,k-2);q,\mathbf{v})]_{\mathscr{G}}.$$

In general, considering the edge between $v(j)$ and $v(j+1)$, we have

$$[Z(\Gamma(n,k-j);q,\mathbf{v})]_{\mathscr{G}} = \Theta^j M(v_2)\Theta^{-j}[Z(\Gamma(n,k-j-1);q,\mathbf{v})]_{\mathscr{G}}.$$

Hence,

$$\begin{aligned} Z(n;q,\mathbf{v}) &= M(v_1)(\Theta M(v_2)\Theta^{-1})\cdots(\Theta^{k-1}M(v_2)\Theta^{-k+1})[Z(\Gamma(n,0));q,\mathbf{v}]_{\mathscr{G}} \\ &= (M(v_2)\Theta)^k[Z(\Gamma(n,0));q,\mathbf{v}]_{\mathscr{G}}, \end{aligned} \tag{44}$$

since $\Theta^{-k+1} = \Theta$.

Now that for a given noncrossing partition Γ, $Z(\Gamma(n,k);q,\mathbf{v})$ is related to $Z(\Gamma(n,0);q,\mathbf{v})$, we are left with finding the relation between $Z(\Gamma(n,0);q,\mathbf{v})$ and $Z(\Gamma(n-1,k);q,\mathbf{v})$. This will be shown in the following steps:

$$Z(\Gamma(n,0);q,\mathbf{v}) \xrightarrow{(1)} Z(\pi\Gamma^*(n,k);q,\mathbf{v}) \xrightarrow{(2)} Z(\pi\Gamma(n,0);q,\mathbf{v}) \xrightarrow{(3)} Z(\Gamma(n-1,k);q,\mathbf{v})$$

For steps (1) and (3), the duality relation (11) for the multivariate Tutte polynomial and Hutson and Lewis's Duality Theorem are used. Step (2) follows a similar procedure as in deriving (44), but requires precise identification of edge types. In the following, we present the details of each step.

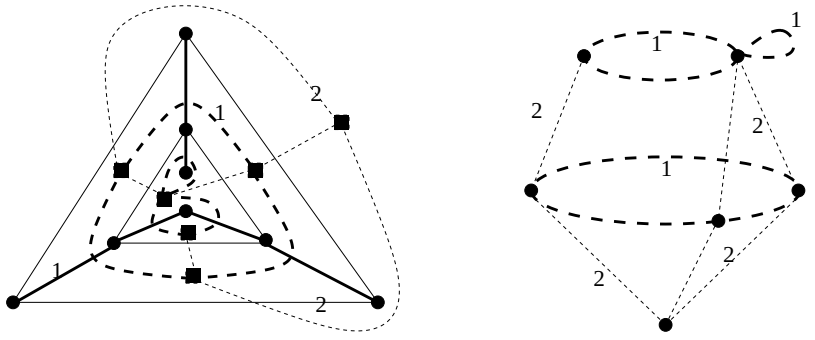
Step (1): For any noncrossing partition Γ, graph $\Gamma(n,0)$ has the same number of edges of type 1 as the number of edges of type 2 as nk. Thus,

$$|E(\Gamma(n,0))| = nk + nk = 2nk, \text{ and } |V(\Gamma(n,0))| \in [1,k] + nk.$$

Therefore, $m_{\mathscr{G}} = nk+1$ and

$$\begin{aligned}[Z(\Gamma(n,0);q,\mathbf{v})]_{\mathscr{G}} &= q^{-nk}(v_1v_2)^{nk}\left[q^{|V(\Gamma(n,0))|-nk-1}Z\left(\pi\Gamma^*(n,k);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}} \\ &= q^{-nk}(v_1v_2)^{nk}B\left[Z\left(\pi\Gamma^*(n,k);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}}. \end{aligned} \tag{45}$$

Step (2): Since the edges in the dual graph $G^* = (V^*, E^*)$ join the neighboring regions in $G = (V,E)$, the edge $e^* \in E^*$ must cross an edge $e \in E$ which separates the two regions in any drawing. We say such pairs of edges e and e^* are of the same type. Then we find that in the dual graph of $\Gamma(n,0)$, the edge types are interchanged between cycle edges and path edges. More precisely, in $\Gamma(n,0)^*$ the edges on the paths are of type 2 and the edges on the cycles are of type 1. By Hutson and Lewis's Duality Theorem and our definition of the pinched capped cylinder graph, we can see that $\Gamma(n,0)^*$ is the same as $\pi\Gamma^*(n,k)$. The different types of edges are illustrated in Figure 7. The graph and the dual are specified by the solid lines and the dashed lines respectively. As explained in Section 4.2.1, darker and wider lines are used for type 1 edges. Note that this is a graph modified from Fig 6 in [HL07].



(a) $\Gamma(3,0)$ and its dual $\Gamma(3,0)^*$ (b) $\Gamma(3,0)^* = \pi\Gamma^*(3,3)$

Figure 7: An Example of Capped Cylinder Graph and its Dual for $\Gamma = \{\{0,1\},\{2\}\}$

By noticing that the edges on the nth level of the pinched capped cylinder $\pi\Gamma(n,k)$ are of type 1, we have

$$\begin{aligned}\left[Z\left(\pi\Gamma^*(n,k);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}} &= \Delta\left[Z\left(\pi\Gamma(n,k);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}} \\ &= \Delta\left(M\left(\frac{q}{\nu_1}\right)\Theta\right)^k\left[Z\left(\pi\Gamma(n,0);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}}. \end{aligned} \tag{46}$$

Step (3): For any noncrossing partition Γ, graph $\pi\Gamma(n,0)$ has $(n-1)k$ edges of type 1 and nk edges of type 2. Thus,

$$|E(\pi\Gamma(n,0))| = (2n-1)k,$$

$$|V(\pi\Gamma(n,0))| = |V(\Gamma(n,0))| - (k-1) \in [1,k] + nk - k + 1.$$

Therefore, $m_{\mathscr{G}} = nk-k+2$ and

$$\begin{aligned}&\left[Z\left(\pi\Gamma(n,0);q,\frac{q}{\mathbf{v}}\right)\right]_{\mathscr{G}} \\ =\ & q^{nk-k+1}(\nu_1)^{-(n-1)k}(\nu_2)^{-nk}\left[q^{|V(\pi\Gamma(n,0))|-(nk-k+2)}Z(\Gamma^*(n-1,k);q,\mathbf{v})\right]_{\mathscr{G}} \\ =\ & q^{nk-k+1}(\nu_1)^{-(n-1)k}(\nu_2)^{-nk}B\Delta Z(n-1;q,\mathbf{v}). \end{aligned} \tag{47}$$

Combining formulae (44)-(47), we obtain

$$\begin{aligned} Z(n;q,\mathbf{v}) &= q^{1-k}\nu_1^k(M(\nu_2)\Theta)^k B\Delta\left(M\left(\frac{q}{\nu_1}\right)\Theta\right)^k B\Delta Z(n-1;q,\mathbf{v}) \\ &= A(q,\mathbf{v})Z(n-1;q,\mathbf{v}). \end{aligned}$$

Therefore,

$$Z(n;q,\mathbf{v}) = A^n(q,\mathbf{v})Z(0;q,\mathbf{v}),$$

which finishes the proof of Theorem 5. □

In the following, we examine examples of cylinder graph $P_n \times C_k$ for $k=2$ and $k=3$.

First, for the cylinder graphs $P_n \times C_2$, let an ordering of the partitions of the cycle C_2 be

$$\mathscr{G} = \{\{\{0\},\{1\}\},\{0,1\}\}.$$

For this particular case, a recursive relation can be found directly by applying deletion and contraction operations, as illustrated in Figure 4. Here we mainly use this as an example to demonstrate Theorem 5. Given the ordering $\mathscr{G}$, one can check that

$$\Theta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\quad \Delta = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},\quad M(u) = \begin{bmatrix} 1 & u \\ 0 & 1+u \end{bmatrix},\quad B = \begin{bmatrix} q & 0 \\ 0 & 1 \end{bmatrix}.$$

Therefore,

$$A(q,\mathbf{v}) = \begin{bmatrix} \nu_2(2+\nu_2)(2\nu_1+q)+(\nu_1+q)^2 & \nu_2(2+\nu_2)\nu_1^2 \\ (1+\nu_2)^2(2\nu_1+q) & (1+\nu_2)^2\nu_1^2 \end{bmatrix}.$$

If $\Gamma = \{\{0\},\{1\}\}$, then $\Gamma(0,2)$ becomes the cycle C_2, which is the same as graph G_0 in Figure 4, with the multivariate Tutte polynomial

$$Z(P_0 \times C_2; q, \mathbf{v}) = Z(\Gamma(0,2); q, \mathbf{v}) = q(q + 2v_2 + v_2^2).$$

If $\Gamma = \{0,1\}$, then $\Gamma(0,2)$ has one vertex and two loop edges, which is the same as graph H_0 in Figure 4, with the multivariate Tutte polynomial

$$Z(\Gamma(0,2); q, \mathbf{v}) = q(1 + v_2)^2.$$

Hence,

$$Z(0; q, \mathbf{v}) = \begin{bmatrix} q(q + 2v_2 + v_2^2) \\ q(1 + v_2)^2 \end{bmatrix}.$$

Applying Theorem 5, we could compute $Z(n; q, \mathbf{v}) = [Z(\Gamma(n,2)); q, \mathbf{v}]_{\mathscr{G}}$ iteratively by formula (24)

For $n = 1$,

$$\begin{aligned} Z(P_1 \times C_2; q, \{v_1, v_2\}) &= (v_2(2 + v_2)(2v_1 + q) + (v_1 + q)^2)q(q + 2v_2 + v_2^2) \\ &\quad + v_2(2 + v_2)v_1^2 q(1 + v_2)^2. \end{aligned} \tag{48}$$

One can compare formula (48) with the standard Tutte polynomial of the cylinder graph $P_1 \times C_2$ by letting $q = (x-1)(y-1)$ and $v_1 = v_2 = y - 1$. The relation (7) between standard and multivariate Tutte polynomial gives

$$T(P_1 \times C_2; x, y) = x^2 + 2xy + 2y^2 + 2x^2y + xy^2 + x + y + x^3 + y^3,$$

which is the same as the corresponding result in [HL07].

Following the same steps, the multivariate Tutte polynomial of $P_n \times C_2$ with two types of edges can be computed for larger n. In particular, for $n = 2$,

$$\begin{aligned} & Z(P_2 \times C_2; q, \{v_1, v_2\}) \\ = \; & v_1^2 v_2(2 + v_2)((1 + v_2)^2(2v_1 + q)q(q + 2v_2 + v_2^2) + (1 + v_2)^4 v_1^2 q) \\ & + (v_2(2 + v_2)(2v_1 + q) + (v_1 + q)^2)v_2 v_1^2(2 + v_2)q(1 + v_2)^2 \\ & + (v_2(2 + v_2)(2v_1 + q) + (v_1 + q)^2)^2 q(q + 2v_2 + v_2^2). \end{aligned}$$

□

Second, for the cylinder graphs $P_n \times C_3$, let an ordering of the partitions of the cycle C_3 be

$$\mathscr{G} = \{\{\{0\},\{1\},\{2\}\},\{\{0,1\},\{2\}\},\{\{0,2\},\{1\}\},\{\{1,2\},\{0\}\},\{\{0,1,2\}\}\}.$$

Then one can check that

$$\Theta = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad \Delta = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix},$$

and

$$M(u) = \begin{bmatrix} 1 & u & 0 & 0 & 0 \\ 0 & 1+u & & 0 & 0 \\ 0 & 0 & 1 & 0 & u \\ 0 & 0 & 0 & 1 & u \\ 0 & 0 & 0 & 0 & 1+u \end{bmatrix}, \quad B = \begin{bmatrix} q^2 & 0 & 0 & 0 & 0 \\ 0 & q & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 \\ 0 & 0 & 0 & q & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

The initial vector of $Z(0;q,\mathbf{v})$ can be computed as

$$Z(0;q,\mathbf{v}) = \begin{bmatrix} (q+v_2)^3+(q-1)v_2^3 \\ q(q+2v_2+v_2^2)(1+v_2) \\ q(q+2v_2+v_2^2)(1+v_2) \\ q(q+2v_2+v_2^2)(1+v_2) \\ q(1+v_2)^3 \end{bmatrix}.$$

Substitute the above into formula (24), one can find the multivariate Tutte polynomial of $P_n \times C_3$ with two types of edges for any n. In particular, for $n = 1$,

$$\begin{aligned} & Z(P_1 \times C_3; q, \{v_1, v_2\}) \\ = \; & 3v_1^2v_2(3v_2+v_2^2+v_1+q)q(q+2v_2+v_2^2)(1+v_2)+v_1^3v_2^2(3+v_2)q(1+v_2)^3 \\ & + (9v_2^2v_1+3v_2^2q+3v_2^3v_1+v_2^3q+6v_2v_1^2+9v_1qv_2+3v_2q^2+v_1^3 \\ & \quad + 3qv_1^2+3q^2v_1+q^3)((q+v_2)^3+(q-1)v_2^3). \end{aligned}$$

□

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