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Limit theorems for vertex-reinforced jump processes on regular trees

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Abstract

Consider a vertex-reinforced jump process defined on a regular tree, where each vertex has exactly b children, with $b \geq 3$. We prove the strong law of large numbers and the central limit theorem for the distance of the process from the root. Notice that it is still unknown if vertex-reinforced jump process is transient on the binary tree.

Key words: Reinforced random walks; strong law of large numbers; central limit theorem.

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1 Introduction

Let $\mathcal{D}$ be any graph with the property that each vertex is the end point of only a finite number of edges. Denote by Vert($\mathcal{D}$) the set of vertices of $\mathcal{D}$. The following, together with the vertex occupied at time 0 and the set of positive numbers $\{a_v : v \in \text{Vert}(\mathcal{D})\}$, defines a right-continuous process $\mathbf{X} = \{X_s, s \geq 0\}$. This process takes as values the vertices of $\mathcal{D}$ and jumps only to nearest neighbors, i.e. vertices one edge away from the occupied one. Given X_s, $0 \leq s \leq t$, and $\{X_t = x\}$, the conditional probability that, in the interval $(t, t + \mathrm{d}t)$, the process jumps to the nearest neighbor y of x is $L(y,t)\mathrm{d}t$, with

$$L(y,t) := a_y + \int_0^t 1\!\!1_{\{X_s = y\}} \mathrm{d}s, \qquad a_y > 0,$$

where $1\!\!1_A$ stands for the indicator function of the set A. The positive numbers $\{a_v : v \in \text{Vert}(\mathcal{D})\}$ are called initial weights, and we suppose $a_v \equiv 1$, unless specified otherwise. Such a process is said to be a Vertex Reinforced Jump Process (VRJP) on $\mathcal{D}$.

Consider VRJP defined on the integers, which starts from 0. With probability $1/2$ it will jump either to 1 or -1. The time of the first jump is an exponential random variable with mean $1/2$, and is independent on the direction of the jump. Suppose the walk jumps towards 1 at time z. Given this, it will wait at 1 an exponential amount of time with mean $1/(2+z)$. Independently of this time, the jump will be towards 0 with probability $(1+z)/(2+z)$.

In this paper we define a process to be recurrent if it visits each vertex infinitely many times a.s., and to be transient otherwise. VRJP was introduced by Wendelin Werner, and its properties were first studied by Davis and Volkov (see [8] and [9]). This reinforced walk defined on the integer lattice is studied in [8] where recurrence is proved. For fixed $b \in \mathbb{N} := \{1, 2, \ldots\}$, the b-ary tree, which we denote by $\mathcal{G}_b$, is the infinite tree where each vertex has $b+1$ neighbors with the exception of a single vertex, called the root and designated by ρ, that is connected to b vertices. In [9] is shown that VRJP on the b-ary tree is transient if $b \geq 4$. The case $b = 3$ was dealt in [4], where it was proved that the process is still transient. The case $b = 2$ is still open.

Another process which reinforces the vertices, the so called Vertex-Reinforced Random Walk (VRRW), shows a completely different behaviour. VRRW was introduced by Pemantle (see [17]). Pemantle and Volkov (see [19]) proved that this process, defined on the integers, gets stuck in at most five points. Tarrès (see [23]) proved that it gets stuck in exactly 5 points. Volkov (in [24]) studied this process on arbitrary trees.

The reader can find in [18] a survey on reinforced processes. In particular, we would like to mention that little is known regarding the behaviour of these processes on infinite graphs with loops. Merkl and Rolles (see [14]) studied the recurrence of the original reinforced random walk, the so-called linearly bond-reinforced random walk, on two-dimensional graphs. Sellke (see [21]) proved than once-reinforced random walk is recurrent on the ladder.

We define the distance between two vertices as the number of edges in the unique self-avoiding path connecting them. For any vertex v, denote by $|v|$ its distance from the root. Level i is the set of vertices v such that $|v| = i$. The main result of this paper is the following.

Theorem 1.1. *Let* $\mathbf{X}$ *be VRJP on* $\mathcal{G}_b$*, with* $b \geq 3$*. There exist constants* $K_b^{(1)} \in (0, \infty)$ *and* $K_b^{(2)} \in [0, \infty)$

such that

$$\lim_{t\to\infty}\frac{|X_t|}{t}=K_b^{(1)} \qquad a.s., \tag{1.1}$$

$$\frac{|X_t|-K_b^{(1)}t}{\sqrt{t}}\Longrightarrow Normal(0,K_b^{(2)}), \tag{1.2}$$

where we took the limit as $t\to\infty$, $\Rightarrow$ stands for weak convergence and $Normal(0,0)$ stands for the Dirac mass at 0*.*

Durrett, Kesten and Limic have proved in [11] an analogous result for a bond-reinforced random walk, called one-time bond-reinforced random walk, on $\mathcal{G}_b$, $b\geq 2$. To prove this, they break the path into independent identically distributed blocks, using the classical method of cut points. We also use this approach. Our implementation of the cut point method is a strong improvement of the one used in [3] to prove the strong law of large numbers for the original reinforced random walk, the so-called linearly bond-reinforced random walk, on $\mathcal{G}_b$, with $b\geq 70$. Aidékon, in [1] gives a sharp criteria for random walk in a random environment, defined on Galton-Watson tree, to have positive speed. He proves the strong law of large numbers for linearly bond-reinforced random walk on $\mathcal{G}_b$, with $b\geq 2$.

2 Preliminary definitions and properties

From now on, we consider VRJP $\mathbf{X}$ defined on the regular tree $\mathcal{G}_b$, with $b\geq 3$. For $v\neq\rho$, define $\text{par}(v)$, called the parent of v, to be the unique vertex at level $|v|-1$ connected to v. A vertex v_0 is a child of v if $v=\text{par}(v_0)$. We say that a vertex v_0 is a descendant of the vertex v if the latter lies on the unique self-avoiding path connecting v_0 to ρ, and $v_0\neq v$. In this case, v is said to be an ancestor of v_0. For any vertex μ, let Λ_μ be the subtree consisting of μ, its descendants and the edges connecting them, i.e. the subtree rooted at μ. Define

$$T_i \quad := \quad \inf\{t\geq 0\colon |X_t|=i\}.$$

We give the so-called Poisson construction of VRJP on a graph $\mathcal{D}$ (see [20]). For each ordered pair of neighbors (u,v) assign a Poisson process $P(u,v)$ of rate 1, the processes being independent. Call $h_i(u,v)$, with $i\geq 1$, the inter-arrival times of $P(u,v)$ and let $\xi_1:=\inf\{t\geq 0\colon X_t=u\}$. The first jump after ξ_1 is at time $c_1:=\xi_1+\min_v h_1(u,v)\big(L(v,\xi_1)\big)^{-1}$, where the minimum is taken over the set of neighbors of u. The jump is towards the neighbor v for which that minimum is attained. Suppose we defined $\{(\xi_j,c_j),1\leq j\leq i-1\}$, and let

$$\begin{aligned}&\xi_i:=\inf\big\{t>c_{i-1}\colon X_t=u\big\},\ \text{and}\\ &j_v-1=j_{u,v}-1:=\ \text{number of times } \mathbf{X} \text{ jumped from } u \text{ to } v \text{ by time } \xi_i.\end{aligned}$$

The first jump after ξ_i happens at time $c_i:=\xi_i+\min_v h_{j_v}(u,v)\big(L(v,\xi_i)\big)^{-1}$, and the jump is towards the neighbor v which attains that minimum.

Definition 2.1. *A vertex μ, with $|\mu|\geq 2$, is* **good** *if it satisfies the following*

$$h_1(\mu_0,\mu)<\frac{h_1\big(\mu_0,\text{par}(\mu_0)\big)}{1+h_1\big(\text{par}(\mu_0),\mu_0\big)} \qquad where\ \mu_0=\text{par}(\mu). \tag{2.3}$$

By virtue of our construction of VRJP, (2.3) can be interpreted as follows. When the process **X** visits the vertex μ_0 for the first time, if this ever happens, the weight at its parent is exactly $1 + h_1\big(\mathrm{par}(\mu_0), \mu_0\big)$ while the weight at μ is 1. Hence condition (2.3) implies that when the process visits μ_0 (if this ever happens) then it will visit μ before it returns to $\mathrm{par}(\mu_0)$, if this ever happens.

The next Lemma gives bounds for the probability that VRJP returns to the root after the first jump.

Lemma 2.2. *Let*

$$\alpha_b := \mathbb{P}\big(X_t = \rho \text{ for some } t \geq T_1\big),$$

and let β_b be the smallest among the positive solutions of the equation

$$x = \sum_{k=0}^{b} x^k p_k, \tag{2.4}$$

where, for $k \in \{0, 1, \ldots, b\}$,

$$p_k := \sum_{j=0}^{k} \binom{b}{k} \binom{k}{j} (-1)^j \int_0^\infty \frac{1+z}{j+b-k+1+z} \mathrm{e}^{-z} \mathrm{d}z. \tag{2.5}$$

We have

$$\int_0^\infty \frac{1+z}{b+1+z} b\mathrm{e}^{-bz} \mathrm{d}z \leq \alpha_b \leq \beta_b. \tag{2.6}$$

Proof. First we prove the lower bound in (2.6). The left-hand side of this inequality is the probability that the process returns to the root with exactly two jumps. To see this, notice that $L(\rho, T_1)$ is equal $1 + \min_{\nu \colon |\nu|=1} h_1(\rho, \nu)$. Hence $T_1 = L(\rho, T_1) - 1$ is distributed like an exponential with mean $1/b$. Given that $T_1 = z$, the probability that the second jump is from X_{T_1} to ρ is equal to $(1+z)/(b+1+z)$. Hence the probability that the process returns to the root with exactly two jumps is

$$\int_0^\infty \frac{1+z}{b+1+z} b\mathrm{e}^{-bz} \mathrm{d}z.$$

As for the upper bound in (2.6) we reason as follows. We give an upper bound for the probability that there exists an infinite random tree which is composed only of good vertices and which has root at one of the children of X_{T_1}. If this event holds, then the process does not return to the root after time T_1 (see the proof of Theorem 3 in [4]). We prove that a particular cluster of good vertices is stochastically larger than a branching process which is supercritical. We introduce the following color scheme. The only vertex at level 1 to be *green* is X_{T_1}. A vertex ν, with $|\nu| \geq 2$, is *green* if and only if it is good and its parent is green. All the other vertices are uncolored. Fix a vertex μ. Let C be any event in

$$\mathscr{H}_\mu := \sigma(h_i(\eta_0, \eta_1) : i \geq 1, \text{ with } \eta_0 \sim \eta_1 \text{ and both } \eta_0 \text{ and } \eta_1 \notin \Lambda_\mu), \tag{2.7}$$

that is the σ-algebra that contains the information about X_t observed outside Λ_μ. Next we show that given $C \cap \{\mu \text{ is green}\}$, the distribution of $h_1(\mathrm{par}(\mu), \mu)$ is stochastically dominated by an exponential(1). To see this, first notice that $h_1(\mathrm{par}(\mu), \mu)$ is independent of C. Let $D := \{\mathrm{par}(\mu) \text{ is green}\} \in \mathscr{H}_\mu$ and set

$$W := \frac{h_1\big(\mu_0, \mathrm{par}(\mu_0)\big)}{1 + h_1\big(\mathrm{par}(\mu_0), \mu_0\big)} \qquad \text{where } \mu_0 = \mathrm{par}(\mu). \tag{2.8}$$

The random variable W is independent of $h_1(\mathrm{par}(\mu),\mu)$ and is absolutely continuous with respect the Lebesgue measure. By the definition of good vertices we have

$$\{\mu \text{ is green}\} = \{h_1(\mathrm{par}(\mu),\mu) < W\} \cap D.$$

Denote by f_W the conditional density of W given $D \cap C \cap \{h_1(\mathrm{par}(\mu),\mu) < W\}$. We have

$$\begin{aligned}
& \mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x \;\Big|\; \{\mu \text{ is green}\} \cap C\Big) \\
= \;& \mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x \;\Big|\; \{h_1(\mathrm{par}(\mu),\mu) < W\} \cap C \cap D\Big) \\
= \;& \int_0^\infty \mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x \;\Big|\; \{h_1(\mathrm{par}(\mu),\mu) < w\} \cap C \cap D \cap \{W = w\}\Big) f_W(w)\mathrm{d}w
\end{aligned} \tag{2.9}$$

Using the facts that $h_1(\mathrm{par}(\mu),\mu)$ is independent of W, C and D and

$$\mathbb{P}(h_1(\mathrm{par}(\mu),\mu) \geq x \mid h_1(\mathrm{par}(\mu),\mu) < w) \leq \mathbb{P}(h_1(\mathrm{par}(\mu),\mu) \geq x),$$

we get that the expression in (2.9) is less or equal to $\mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x\Big)$. Summarising

$$\mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x \mid \{\mu \text{ is green}\} \cap C\Big) \geq \mathbb{P}\Big(h_1(\mathrm{par}(\mu),\mu) \geq x\Big). \tag{2.10}$$

The inequality (2.9) implies that if μ_1 is a child of μ and $C \in \mathscr{H}_\mu$ we have

$$\mathbb{P}\Big(\mu_1 \text{ is green } \mid \{\mu \text{ is green}\} \cap C\Big) \geq \mathbb{P}\Big(\mu_1 \text{ is green}\Big). \tag{2.11}$$

To see this, it is enough to integrate over the value of $h_1(\mathrm{par}(\mu),\mu)$ and use the fact that, conditionally on $h_1(\mathrm{par}(\mu),\mu)$, the events $\{\mu_1$ is green$\}$ and $\{\mu$ is green$\} \cap C$ are independent. The probability that μ_1 is good conditionally on $\{h_1(\mathrm{par}(\mu),\mu) = x\}$ is a non-increasing function of x, while the distribution of $h_1(\mathrm{par}(\mu),\mu)$ is stochastically smaller than the conditional distribution of $h_1(\mathrm{par}(\mu),\mu)$ given $\{\mu$ is green$\} \cap C$, as shown in (2.10).

Hence the cluster of green vertices is stochastically larger than a Galton–Watson tree where each vertex has k offspring, $k \in \{0, 1, \ldots, b\}$, with probability p_k defined in (2.5). To see this, fix a vertex μ and let μ_i, with $i \in \{0, 1, \ldots, b\}$ be its children. It is enough to realize that p_k is the probability that exactly k of the $h_1(\mu,\mu_i)$, with $i \in \{0, 1, \ldots, b\}$, are smaller than $\big(1+h_1(\mathrm{par}(\mu),\mu)\big)^{-1} h_1\big(\mu, \mathrm{par}(\mu)\big)$. As the random variables $h_1(\mu,\mu_i), h_1\big(\mu,\mathrm{par}(\mu)\big)$ and $h_1(\mathrm{par}(\mu),\mu)$ are independent exponentials with parameter one, we have

$$\begin{aligned}
p_k &= \binom{b}{k} \int_0^\infty \int_0^\infty \mathbb{P}\big(h_1(\mu_0,\mu) < \frac{y}{1+z}\big)^k \mathbb{P}\big(h_1(\mu_0,\mu) \geq \frac{y}{1+z}\big)^{b-k} \mathrm{e}^{-y} \mathrm{e}^{-z} \mathrm{d}y\, \mathrm{d}z \\
&= \binom{b}{k} \int_0^\infty \int_0^\infty \big(1 - \mathrm{e}^{-\frac{y}{1+z}}\big)^k \mathrm{e}^{-\frac{y}{1+z}(b-k)} \mathrm{e}^{-y} \mathrm{e}^{-z} \mathrm{d}y\, \mathrm{d}z \\
&= \sum_{j=0}^k \int_0^\infty \int_0^\infty \binom{b}{k}\binom{k}{j} (-1)^j \mathrm{e}^{-y(j+b-k+1+z)/(1+z)} \mathrm{e}^{-z} \mathrm{d}y\, \mathrm{d}z \\
&= \sum_{j=0}^k \binom{b}{k}\binom{k}{j} (-1)^j \int_0^\infty \frac{1+z}{j+b-k+1+z} \mathrm{e}^{-z} \mathrm{d}z.
\end{aligned} \tag{2.12}$$

From the basic theory of branching processes we know that the probability that this Galton–Watson tree is finite (i.e. extinction) equals the smallest positive solution of the equation

$$x - \sum_{k=0}^{b} x^k p_k = 0. \tag{2.13}$$

The proof of (2.6) follows from the fact that $1-\beta_b \leq 1-\alpha_b$. This latter inequality is a consequence of the fact that the cluster of green vertices is stochastically larger than the Galton-Watson tree, hence its probability of non-extinction is not smaller. As $b \geq 3$, the Galton-Watson tree is supercritical (see [4]),hence $\beta_b < 1$. □

For example, if we consider VRJP on $\mathcal{G}_3$, Lemma 2.2 yields

$$0.3809 \leq \alpha_3 \leq 0.8545.$$

Definition 2.3. *Level $j \geq 1$ is a* **cut level** *if the first jump after T_j is towards level $j+1$, and after time T_{j+1} the process never goes back to X_{T_j}, and*

$$L(X_{T_j}, \infty) < 2 \quad \text{and} \quad L(\text{par}(X_{T_j}), \infty) < 2.$$

Define l_1 to be the cut level with minimum distance from the root, and for $i > 1$,

$$l_i := \min\{j > l_{i-1} \colon j \text{ is a cut level}\}.$$

Define the i-th **cut time** *to be $\tau_i := T_{l_i}$. Notice that $l_i = |X_{\tau_i}|$.*

3 l_1 has an exponential tail

For any vertex $v \in \text{Vert}(\mathcal{G}_b)$, we define $\text{fc}(v)$, which stands for **first child** of v, to be the (a.s.) unique vertex connected to v satisfying

$$h_1(v, \text{fc}(v)) = \min\{h_1(v, \mu) \colon \text{par}(\mu) = v\}. \tag{3.14}$$

For definiteness, the root ρ is not a first child. Notice that condition (3.14) does not imply that the vertex $\text{fc}(v)$ is visited by the process. If $\mathbf{X}$ visits it, then it is the first among the children of v to be visited.

For any pair of distributions f and g, denote by $f \,\overline{*}\, g$ the distribution of $\sum_{k=1}^{V} M_k$, where

- V has distribution f, and
- $\{M_k,\, k \in \mathbb{N}\}$ is a sequence of i.i.d random variables, independent of V, each with distribution g.

Recall the definition of p_i, $i \in \{0, \dots, b\}$, given in (2.5). Denote by $\mathbf{p}^{(1)}$ the distribution which assigns to $i \in \{0, \dots, b\}$ probability p_i. Define, by recursion, $\mathbf{p}^{(j)} := \mathbf{p}^{(j-1)} \,\overline{*}\, \mathbf{p}^{(1)}$, with $j \geq 2$. The distribution $\mathbf{p}^{(j)}$ describes the number of elements, at time j, in a population which evolves like a branching process generated by one ancestor and with offspring distribution $\mathbf{p}^{(1)}$. If we let

$$m := \sum_{j=1}^{b} j p_j,$$

then the mean of $\mathbf{p}^{(j)}$ is m^j. The probability that a given vertex μ is good is, by definition,

$$\mathbb{P}\Big(h_1(\mu_0,\mu) < \frac{h_1\big(\mu_0,\mathrm{par}(\mu_0)\big)}{1+h_1\big(\mathrm{par}(\mu_0),\mu_0\big)}\Big) \qquad \text{where } \mu_0 = \mathrm{par}(\mu).$$

As the $h_1\big(\mathrm{par}(\mu_0),\mu_0\big)$ is exponential with parameter 1, conditioning on its value and using independence between different Poisson processes, we have that the probability above equals

$$\mathbb{P}\Big(h_1(\mu_0,\mu) < \frac{1}{1+z}h_1\big(\mu_0,\mathrm{par}(\mu_0)\big)\Big)\mathrm{e}^{-z}\mathrm{d}z = \int_0^\infty \frac{1}{2+z}\mathrm{e}^{-z}\mathrm{d}z = 0.36133\ldots. \tag{3.15}$$

Hence

$$m = b\cdot 0.36133 > 1,$$

because we assumed $b \geq 3$.

Let $q_0 = p_0+p_1$, and for $k \in \{1,2,\ldots,b-1\}$ set $q_k = p_{k+1}$. Set $\mathbf{q}$ to be the distribution which assigns to $i \in \{0,\ldots,b-1\}$ probability q_i. For $j \geq 2$, let $\mathbf{q}^{(j)} := \mathbf{p}^{(j-1)} \overline{*}\, \mathbf{q}$. Denote by $q_i^{(j)}$ the probability that the distribution $\mathbf{q}^{(j)}$ assigns to $i \in \{0,\ldots,(b-1)b^{j-1}\}$. The mean of $\mathbf{q}^{(j)}$ is $m^{j-1}(m-1)$. From now on, ζ denotes the smallest positive integer in $\{2,3,\ldots,\}$ such that

$$m^{\zeta-1}(m-1) > 1. \tag{3.16}$$

Next we want to define a sequence of events which are independent and which are closely related to the event that a given level is a cut level. For any vertex ν of $\mathcal{G}_b$ let Θ_ν be the set of vertices μ such that

- μ is a descendant of ν,
- the difference $|\mu|$ - $|\nu|$ is a multiple of ζ,
- μ is a first child.

By subtree rooted at ν we mean a subtree of Λ_ν that contains ν. Set $\widetilde{\nu} = \mathrm{fc}(\nu)$ and let

$$\begin{aligned} A(\nu) := \{&\exists \text{ an infinite subtree of } \mathcal{G}_b \text{ root at a child of } \widetilde{\nu}, \text{ which is composed only by} \\ &\text{good vertices and which contains none of the vertices in } \Theta_\nu\} \end{aligned} \tag{3.17}$$

For $i \in \mathbb{N}$, let $A_i := A\big(X_{T_i}\big)$. Notice that if the process reaches the first child of ν and if $A(\nu)$ holds, then the process will never return to ν. Hence if A_i holds, and if $X_{T_{i+1}} = X_{T_i}+1$, then i is a cut level, provided that the total weights at X_{T_i} and its parent are less than 2.

Proposition 3.1. *The events $A_{i\zeta}$, with $i \in \mathbb{N}$, are independent.*

Proof. We recall that $\zeta \geq 2$. We proceed by backward recursion and show that the events $A_{i\zeta}$ depend on disjoint Poisson processes collections. Choose integers $0 < i_1 < i_2 < \ldots < i_k$, with $i_j \in \zeta\mathbb{N} := \{\zeta, 2\zeta, 3\zeta, \ldots\}$ for all $j \in \{1,2,\ldots,k\}$. It is enough to prove that

$$\mathbb{P}\Big(\bigcap_{j=1}^k A_{i_j}\Big) = \prod_{j=1}^k \mathbb{P}\big(A_{i_j}\big). \tag{3.18}$$

Fix a vertex v at level i_k. The set $A(v)$ belongs to the sigma-algebra generated by $\{P(u,w):\ u,w \in \mathrm{Vert}(\Lambda_v)\}$. On the other hand, the set $\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = v\}$ belongs to $\{P(u,w):\ u \notin \mathrm{Vert}(\Lambda_v)\}$. As the two events belong to disjoint collections of independent Poisson processes, they are independent. As $\mathbb{P}(A(v)) = \mathbb{P}(A(\rho))$, we have

$$\begin{aligned}
&\mathbb{P}\Big(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j}\Big) = \sum_{v:\,|v|=i_k} \mathbb{P}\Big(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = v\}\Big) \\
&= \sum_{v:\,|v|=i_k} \mathbb{P}\Big(A(v) \cap \bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = v\}\Big) = \sum_{v:\,|v|=i_k} \mathbb{P}\big(A(v)\big)\mathbb{P}\Big(\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = v\}\Big) \qquad (3.19) \\
&= \mathbb{P}\big(A(\rho)\big) \sum_{v:\,|v|=i_k} \mathbb{P}\Big(\bigcap_{j=1}^{k-1} A_{i_j} \cap \{X_{T_{i_k}} = v\}\Big) = \mathbb{P}\big(A(\rho)\big)\mathbb{P}\Big(\bigcap_{j=1}^{k-1} A_{i_j}\Big).
\end{aligned}$$

The events $A(v)$ and $\{X_{T_{i_k}} = v\}$ are independent, and by virtue of the self-similarity property of the regular tree we get $\mathbb{P}\big(A(\rho)\big) = \mathbb{P}\big(A_{i_k}\big)$. Hence

$$\mathbb{P}\Big(A_{i_k} \cap \bigcap_{j=1}^{k-1} A_{i_j}\Big) = \mathbb{P}\big(A_{i_k}\big)\mathbb{P}\Big(\bigcap_{j=1}^{k-1} A_{i_j}\Big). \qquad (3.20)$$

Reiterating (3.20) we get (3.18). □

Lemma 3.2. *Define γ_b to be the smallest positive solution of the equation*

$$x = \sum_{k=0}^{b-1} x^k q_k^{(\zeta)}, \qquad (3.21)$$

where ζ and $(q_k^{(n)})$ have been defined at the beginning of this section. We have

$$\mathbb{P}(A_i) \geq 1 - \gamma_b > 0, \qquad \forall i \in \mathbb{N}. \qquad (3.22)$$

Proof. Fix $i \in \mathbb{N}$ and let $v^* = X_{T_i}$. We adopt the following color scheme. The vertex $\mathrm{fc}\big(X_{T_i}\big)$ is colored *blue*. A descendant μ of v^* is colored blue if it is good, its parent is *blue*, and either

- $|\mu| - |v^*|$ is not a multiple of ζ, or
- $\frac{1}{\zeta}\big(|\mu| - |v^*|\big) \in \mathbb{N}$ and μ is not a first child.

Vertices which are not descendants of v^* are not colored. Following the reasoning given in the proof of Lemma 2.2, we can conclude that the number of blue vertices at levels $|v^*| + j\zeta$, with $j \geq 1$, is stochastically larger than the number of individuals in a population which evolves like a branching process with offspring distribution $\mathbf{q}^{(\zeta)}$, introduced at the beginning of this section. Again, from the basic theory of branching processes we know that the probability that this tree is finite equals the smallest positive solution of the equation (3.21). By virtue of (3.16) we have that $\gamma_b < 1$. □

The proof of the following Lemma can be found in [10] pages 26-27 and 35.

Lemma 3.3. *Suppose U_n is Bin(n,p). For $x \in (0,1)$ consider the entropy*

$$H(x\,|\,p) := x \ln \frac{x}{p} + (1-x)\ln\frac{1-x}{1-p}.$$

We have the following large deviations estimate, for $s \in [0,1]$,

$$\mathbb{P}\left(U_n \le sn\right) \le \exp\{-n \inf_{x\in[0,s]} H(x\,|\,p)\}.$$

Proposition 3.4.

i) Let v be a vertex with $|v| \ge 1$. The quantity

$$\mathbb{P}\big(A(v)\mid h_1(v,\mathrm{fc}(v)) = x\big)$$

is a decreasing function of x, with $x \ge 0$.

ii) $\mathbb{P}\big(A(v)\mid h_1(v,\mathrm{fc}(v)) \le x\big) \ge \mathbb{P}\big(A(v)\big)$, for any $x \ge 0$.

Proof. Suppose $\{\mathrm{fc}(v) = \overline{v}\}$. Given $\{h_1(v,\overline{v}) = x\}$, the set of good vertices in $\Lambda_{\overline{v}}$ is a function of x. Denote this function by $\mathscr{T} : \mathbb{R}^+ \to \{$subset of vertices of $\Lambda_{\overline{v}}\}$. A child of $\overline{v}$, say v_1, is good if and only if

$$h_1(\overline{v},v_1) < \frac{h_1(\overline{v},v)}{1+x}.$$

Hence the smaller x is, the more likely v_1 is good. This is true for any child of $\overline{v}$. As for descendants of $\overline{v}$ at level strictly greater than $|v|+2$, their status of being good is independent of $h_1(v,\mathrm{fc}(v))$. Hence $\mathscr{T}(x) \supset \mathscr{T}(y)$ for $x < y$. This implies that the connected component of good vertices contining $\overline{v}$ is larger if $\{h_1(v,\overline{v}) = x\}$ rather than $\{h_1(v,\overline{v}) = y\}$, for $x < y$. Hence

$$\mathbb{P}\big(A(v)\mid h_1(v,\mathrm{fc}(v)) = x,\ \mathrm{fc}(v) = \overline{v}\big) \ge \mathbb{P}\big(A(v)\mid h_1(v,\mathrm{fc}(v)) = y,\ \mathrm{fc}(v) = \overline{v}\big), \qquad \text{for } x < y.$$

Using symmetry we get i). In order to prove ii), use i) and the fact that the distribution of $h_1(v,\mathrm{fc}(v))$ is stochastically larger that the conditional distribution of $h_1(v,\mathrm{fc}(v))$ given $\{h_1(v,\mathrm{fc}(v)) \le x\}$. □

Denote by $[x]$ the largest integer smaller than x.

Theorem 3.5. *For VRJP defined on $\mathscr{G}_b$, with $b \ge 3$, and $s \in (0,1)$, we have*

$$\mathbb{P}\big(l_{[sn]} \ge n\big) \le \exp\Big\{-[n/\zeta]\inf_{x\in[0,s]} H\Big(x\,\Big|\,(1-\gamma_b)\varphi_b\Big)\Big\}, \tag{3.23}$$

where γ_b was defined in Lemma 3.2, and

$$\varphi_b := \left(1-\mathrm{e}^{-b}\right)\left(1-\mathrm{e}^{-(b+1)}\right)\frac{b}{b+2}. \tag{3.24}$$

Proof. By virtue of Proposition 3.1 the sequence $1\!\!1_{A_{k\zeta}}$, with $k \in \mathbb{N}$, consists of i.i.d. random variables. The random variable $\sum_{j=1}^{[n/\zeta]} 1\!\!1_{A_{j\zeta}}$ has binomial distribution with parameters $\big(\mathbb{P}(A(\rho)), [n/\zeta]\big)$. We define the event

$$\begin{aligned} B_j := \{&\text{the first jump after } T_j \text{ is towards level } j+1 \text{ and } L\big(X_{T_j}, T_{j+1}\big) < 2,\\ &\text{and } L\big(\mathrm{par}(X_{T_j}), T_{j+1}\big) < 2\}. \end{aligned}$$

Let $\mathscr{F}_t$ be the smallest sigma-algebra defined by the collection $\{X_s,\, 0 \le s \le t\}$. For any stopping time S define $\mathscr{F}_S := \{A\colon A \cap \{S \le t\} \in \mathscr{F}_t\}$. Now we show

$$\mathbb{P}\left(B_j \mid \mathscr{F}_{T_{i-1}}\right) \ge \left(1 - \mathrm{e}^{-b}\right)\left(1 - \mathrm{e}^{-(b+1)}\right)\frac{b}{b+2} = \varphi_b, \tag{3.25}$$

where the inequality holds a.s.. In fact, by time T_i the total weight of the parent of X_{T_i} is stochastically smaller than 1+ an exponential of parameter b, independent of $\mathscr{F}_{T_{i-1}}$. Hence the probability that this total weight is less than 2 is larger than $1 - \mathrm{e}^{-b}$. Given this, the probability that the first jump after T_i is towards level $i+1$ is larger than $b/(b+2)$. Finally, the conditional probability that $T_{i+1} - T_i < 1$ is larger than $1 - \mathrm{e}^{-(b+1)}$. This implies, together with $\zeta \ge 2$, that the random variable $\sum_{j=1}^{[n/\zeta]} 1\!\!1_{B_j}$ is stochastically larger than a binomial(n, φ_b). For any $i \in \mathbb{N}$, and any vertex v with $|v| = i\zeta$, set

$$\begin{aligned} Z &:= \min\left(1, \frac{h_1(v, \mathrm{par}(v))}{1 + h_1(\mathrm{par}(v), v)}\right) \\ E &:= \{X_{T_{i\zeta}} = v\} \cap \{L(\mathrm{par}(v), T_{i\zeta}) < 2\}. \end{aligned}$$

We have

$$B_{i\zeta} \cap \{X_{T_{i\zeta}} = v\} = \{h_1(v, \mathrm{fc}(v)) < Z\} \cap E.$$

Moreover, the random variable Z and the event E are both measurable with respect the sigma-algebra

$$\widetilde{\mathscr{H}_v} := \sigma\Big\{P(\mathrm{par}(v), v), \{P(u, w)\colon\ u, w \notin \mathrm{Vert}(\Lambda_v)\}\Big\}.$$

Let f_Z be the density of Z given $\{h_1(v, \mathrm{fc}(v)) < \mathrm{Z}\} \cap \mathrm{E}$. Using 3.4, ii), and the independence between $h_1(v, \mathrm{fc}(v))$ and $\widetilde{\mathscr{H}_v}$, we get

$$\begin{aligned} \mathbb{P}\big(A_{i\zeta}\big|B_{i\zeta} \cap \{X_{T_{i\zeta}} = v\}\big) &= \mathbb{P}\big(A(v)\big|\{h_1(v, \mathrm{fc}(v)) < Z\} \cap E\big) \\ &= \int_0^\infty \mathbb{P}\big(A(v)\big|\{h_1(v, \mathrm{fc}(v)) < z\}\big) f_Z(z)\mathrm{d}z \ge \mathbb{P}\big(A(v)\big) \\ &= \sum_{v\,:\,|v| = i\zeta} \mathbb{P}\big(A(v) \cap \{X_{T_{i\zeta}} = v\}\big) = \mathbb{P}(A_{i\zeta}). \end{aligned} \tag{3.26}$$

The first equality in the last line of (3.26) is due to symmetry. Hence

$$\mathbb{P}(A_{i\zeta}\big|B_{i\zeta}) \ge \mathbb{P}(A_{i\zeta}). \tag{3.27}$$

If $A_k \cap B_k$ holds then k is a cut level. In fact, on this event, when the walk visits level k for the first time it jumps right away to level $k+1$ and never visits level k again. This happens because $X_{T_{k+1}} = \mathrm{fc}(X_{T_k})$ has a child which is the root of an infinite subtree of good vertices. Moreover the total weights at X_{T_k} and its parent are less than 2. Define

$$e_n := \sum_{i=1}^{[n/\zeta]} 1\!\!1_{A_{i\zeta} \cap B_{i\zeta}}.$$

By virtue of (3.22), (3.25), (3.27) and Proposition 3.1 we have that e_n is stochastically larger than a bin$([n/\zeta], (1-\gamma_b)\varphi_b)$. Applying Lemma 3.3, we have

$$\mathbb{P}\big(l_{[sn]} \ge n\big) \le \mathbb{P}\big(e_n \le [sn]\big) \le \exp\Big\{-[n/\zeta]\inf_{x \in [0,s]} H\Big(x\,\Big|\,(1-\gamma_b)\varphi_b\Big)\Big\}. \qquad \square$$

The function $H\big(x \,\big|\, (1-\gamma_b)\varphi_b\big)$ is decreasing in the interval $(0,(1-\gamma_b)\varphi_b)$. Hence for $n > 1/\big((1-\gamma_b)\varphi_b\big)$, we have $\inf_{x\in[0,1/n]} H\big(x \,\big|\, (1-\gamma_b)\varphi_b\big) = H\big(1/n \,\big|\, (1-\gamma_b)\varphi_b\big)$.

Corollary 3.6. *For $n > 1/\big((1-\gamma_b)\phi_b\big)$, by choosing $s = 1/n$ in Theorem 3.5, we have*

$$\begin{aligned}\mathbb{P}\big(l_1 \geq n\big) &\leq \exp\Big\{-[n/\zeta] \inf_{x\in[0,1/n]} H\Big(x \,\Big|\, (1-\gamma_b)\varphi_b\Big)\Big\} \\ &= \exp\Big\{-[n/\zeta] H\Big(\frac{1}{n} \,\Big|\, (1-\gamma_b)\varphi_b\Big)\Big\},\end{aligned} \tag{3.28}$$

where, from the definition of H we have

$$\lim_{n\to\infty} H\Big(\frac{1}{n} \,\Big|\, (1-\gamma_b)\varphi_b\Big) = \ln \frac{1}{1-(1-\gamma_b)\varphi_b} > 0.$$

4 τ_1 has finite $(2+\delta)$-moment

The goal of this section is to prove the finiteness of the $11/5$ moment of the first cut time. We adopt the following strategy

- first we prove the finiteness of all moments for the number of vertices visited by time τ_1, then
- we prove that the total time spent at each of these sites has finite $12/5$-moment.

Fix $n \in \mathbb{N}$ and let

$$\begin{aligned}\Pi_n &:= \text{ number of distinct vertices that } \mathbf{X} \text{ visits by time } T_n, \\ \Pi_{n,k} &:= \text{number of distinct vertices that } \mathbf{X} \text{ visits at level } k \text{ by time } T_n.\end{aligned}$$

Let $T(v) := \inf\{t \geq 0 \colon X_t = v\}$. For any subtree E of $\mathcal{G}_b$, $b \geq 1$, define

$$\delta(a,E) := \sup\left\{t \colon \int_0^t 1\!\!1_{\{X_s\in E\}} \mathrm{d}s \leq a\right\}.$$

The process $X_{\delta(t,E)}$ is called the **restriction** of $\mathbf{X}$ to E.

Proposition 4.1 (Restriction principle (see [8])). *Consider VRJP $\mathbf{X}$ defined on a tree $\mathcal{J}$ rooted at ρ. Assume this process is recurrent, i.e. visits each vertex infinitely often, a.s.. Consider a subtree $\widetilde{\mathcal{J}}$ rooted at v. Then the process $X_{\delta(t,\widetilde{\mathcal{J}})}$ is VRJP defined on $\widetilde{\mathcal{J}}$. Moreover, for any subtree $\mathcal{J}^*$ disjoint from $\widetilde{\mathcal{J}}$, we have that $X_{\delta(t,\widetilde{\mathcal{J}})}$ and $X_{\delta(t,\mathcal{J}^*)}$ are independent.*

Proof. This principle follows directly from the Poisson construction and the memoryless property of the exponential distribution. □

Definition 4.2. *Recall that $P(x,y)$, with $x,y \in Vert\big(\mathcal{G}_b\big)$ are the Poisson processes used to generate $\mathbf{X}$ on $\mathcal{G}_b$. Let $\mathcal{J}$ be a subtree of $\mathcal{G}_b$. Consider VRJP $\mathbf{V}$ on $\mathcal{J}$ which is generated by using $\big\{P(u,v)\colon u,v \in Vert(\mathcal{J})\big\}$, which is the same collection of Poisson processes used to generate the jumps of $\mathbf{X}$ from the vertices of $\mathcal{J}$. We say that $\mathbf{V}$ is the **extension** of $\mathbf{X}$ in $\mathcal{J}$. The processes V_t and $X_{\delta(t,\mathcal{J})}$ coincide up to a random time, that is the total time spent by $\mathbf{X}$ in $\mathcal{J}$.*

We construct an upper bound for $\Pi_{n,k}$, with $2 \leq k \leq n-1$. . Let $G(k)$ be the finite subtree of $\mathscr{G}_b$ composed by all the vertices at level i with $i \leq k-1$, and the edges connecting them. Let $\mathbf{V}$ be the extension of $\mathbf{X}$ to $G(k)$. This process is recurrent, because is defined on a finite graph. The total number of first children at level $k-1$ is b^{k-2}, and we order them according to when they are visited by $\mathbf{V}$, as follows. Let η_1 be the first vertex at level $k-1$ to be visited by $\mathbf{V}$. Suppose we have defined $\eta_1, \dots, \eta_{m-1}$. Let η_m be the first child at level $k-1$ which does not belong to the set $\{\eta_1, \eta_2, \dots, \eta_{m-1}\}$, to be visited. The vertices η_i, with $1 \leq i \leq b^{k-2}$ are determined by $\mathbf{V}$. All the other quantities and events such as $T(v)$ and $A(v)$, with v running over the vertices of $\mathscr{G}_b$, refer to the process $\mathbf{X}$. Define

$$f_n(k) := 1 + b^2 \inf\{m \geq 1 \colon \mathbb{1}_{A(\mathrm{par}(\eta_m))} = 1\}.$$

Let $J := \inf\{n \colon T(\eta_n) = \infty\}$, if the infimum is over an empty set, let $J = \infty$. Suppose that $A(\eta_m)$ holds, then $\mathbf{X}$, after time $T(\eta_m)$, is forced to remain inside Λ_{η_m}, and never visits $\mathrm{fc}(\eta_m)$ again. This implies that $T(\eta_{m+1}) = \infty$. Hence, if $J = m$ then $\bigcap_{i=1}^{m-1}(A(\mathrm{par}(\eta_i)))^c$ holds, and $f_n(k) \geq 1 + b^2(m-1)$. Similarly if $J = \infty$ then $f_n(k) = 1 + b^2 b^{k-2} = 1 + b^k$, which is an obvious upper bound for the number of vertices at level k which are visited by $\mathbf{X}$. On the other hand, if $J = m$ then the number of vertices at level k which are visited by $\mathbf{X}$ is at most $1+(m-1)b^2$. In fact, the processes $\mathbf{X}$ and $\mathbf{V}$ coincide up to the random time when the former process leaves $G(k)$ and never returns to it. Hence if $T(\eta_i) < \infty$ then $\mathbf{X}$ visited exactly $i-1$ distinct first children at level $k-1$ before time $T(\eta_i)$. On the event $\{J = m\}$ we have that $\{T(\eta_{m-1} < \infty\} \cap \{T(\eta_m) = \infty\}$, hence exactly $m-1$ first children are visited at level $k-1$. This implies that at most $1+(m-1)b^2$ vertices at level k are visited.

We conclude that $f_n(k)$ overcounts the number of vertices at level k which are visited, i.e. $\Pi_{n,k} \leq f_n(k)$.

Recall that $h_1(v, \mathrm{fc}(v))$, being the minimum over a set of b independent exponentials with rate 1, is distributed as an exponential with mean $1/b$.

Lemma 4.3. *For any $m \in \mathbb{N}$, we have*

$$\mathbb{P}\big(f_n(k) \,>\, 1 + mb^2\,\big) \leq (\gamma_b)^m.$$

Proof. Given $\bigcap_{i=1}^{m-1}(A(\mathrm{par}(\eta_i))^c$ the distribution of $h(\mathrm{par}(\eta_m), \eta_m)$ is stochastically smaller than an exponential with mean $1/b$. Fix a set of vertices v_i with $1 \leq i \leq m-1$ at level $k-1$ and each with a different parent. Given $\eta_i = v_i$ for $i \leq m-1$, consider the restriction of $\mathbf{V}$ to the finite subgraph obtained from $G(k)$ by removing each of the v_i and $\mathrm{par}(v_i)$, with $i \leq m-1$. The restriction of $\mathbf{V}$ to this subgraph is VRJP, independent of $\bigcap_{i=1}^{m-1}(A(\mathrm{par}(\eta_i))^c$, and the total time spent by this process in level $k-2$ is exponential with mean $1/b$. This total time is an upper bound for $h(\mathrm{par}(\eta_m), \eta_m)$. This conclusion is independent of our choice of the vertices v_i with $1 \leq i \leq m-1$. Finally, using Proposition 3.4 i), we have

$$\begin{aligned}\mathbb{P}\big(f_n(k) > 1 + mb^2 \mid f_n(k) > 1 + (m-1)b^2\big) &= \mathbb{P}\Big((A(\mathrm{par}(\eta_m)))^c \mid \bigcap_{i=1}^{m-1}(A(\mathrm{par}(\eta_i))^c\Big) \\ &\leq \mathbb{P}\big((A(\mathrm{par}(\eta_m)))^c\big) \leq \gamma_b.\end{aligned} \tag{4.29}$$

□

Let a_n, c_n be numerical sequences. We say that $c_n = O(a_n)$ if c_n/a_n is bounded.

Lemma 4.4. *For $p \geq 1$, we have $\mathbb{E}\left[\Pi_n^p\right] = O(n^p)$.*

Proof. Consider first the case $p > 1$. Notice that $\Pi_{n,0} = \Pi_{n,n} = 1$. By virtue of Lemma 4.3, we have that $\sup_n \mathbb{E}[f_n^p] < \infty$. By Jensen's inequality

$$\mathbb{E}[\Pi_n^p] = \mathbb{E}\left[\left(2 + \sum_{k=1}^{n-1} \Pi_{n,k})\right)^p\right] \leq n^p\, \mathbb{E}\left[\sum_{k=1}^{n-1} \frac{\Pi_{n,k}^p}{n} + \frac{2^p}{n}\right] \leq n^p\, \mathbb{E}\left[\sum_{k=1}^{n-1} \frac{f_n^p(k)}{n} + \frac{2^p}{n}\right] = O(n^p). \tag{4.30}$$

As for the case $p = 1$,

$$\mathbb{E}[\Pi_n] \leq 2 + \sum_{k=1}^{n-1} \mathbb{E}[f_n(k)] = O(n).$$

□

Let

$$\Pi := \sum_v 1\!\!1_{\{v \text{ is visited before time } \tau_1\}}.$$

where the sum is over the vertices of $\mathcal{G}_b$. In words, Π is the number of vertices visited before τ_1.

Lemma 4.5. *For any $p > 0$ we have $\mathbb{E}[\Pi^p] < \infty$.*

Proof. By virtue of Lemma 4.4, $\sqrt{\mathbb{E}\left[\Pi_n^{2p}\right]} \leq C_{b,p}^{(1)} n^p$, for some positive constant $C_{b,p}^{(1)}$. Hence using Cauchy-Schwartz,

$$\begin{aligned}\mathbb{E}\left[\Pi^p\right] = \sum_{n=1}^{\infty} \mathbb{E}\left[\Pi_n^p 1\!\!1_{\{l_1 = n\}}\right] &\leq \sum_{n=1}^{\infty} \sqrt{\mathbb{E}\left[\Pi_n^{2p}\right] \mathbb{P}(l_1 \geq n)} \\ &\leq C_{b,p}^{(1)} \sum_{n=1}^{\infty} n^p \exp\left\{-\frac{1}{2}[n/\zeta] H\left(\frac{1}{n}\,\middle|\,(1-\gamma_b)\varphi_b\right)\right\} < \infty.\end{aligned}$$

In the last inequality we used Corollary 3.6. □

Next, we want to prove that the 12/5-moment of $L(\rho, \infty)$ is finite. We start with three intermediate results. The first two can be found in [9]. We include the proofs here for the sake of completeness.

Lemma 4.6. *Consider VRJP on $\{0, 1\}$, which starts at 1, and with initial weights $a_0 = c$ and $a_1 = 1$. Define*

$$\xi(t) := \inf\left\{s : L(1, s) = t\right\}.$$

We have

$$\sup_{t \geq 1} \mathbb{E}\left[\left(\frac{L(0, \xi(t))}{t}\right)^3\right] = c^3 + 3c^2 + 3c. \tag{4.31}$$

Proof. We have $L(0, \xi(t + \mathrm{d}t)) = L(0, \xi(t)) + \chi\eta$, where χ is a Bernoulli which takes value 1 with probability $L(0, \xi(t))\mathrm{d}t$, and η is exponential with mean $1/t$. Given $L\big(0, \xi(t)\big)$, the random variables χ and η are independent. Hence

$$\mathbb{E}\left[L(0, \xi(t + \mathrm{d}t))\right] - \mathbb{E}\left[L(0, \xi(t))\right] = \frac{\mathbb{E}[L(0, \xi(t))]}{t}\mathrm{d}t,$$

i.e. $\mathbb{E}[L(0,\xi(t))]$ is solution of the equation $y^{'}(t) = y(t)/t$, with initial condition $y(1) = c$ (see [8]). Hence

$$\mathbb{E}[L(0,\xi(t))] = ct.$$

Similarly

$$\begin{aligned}
&\mathbb{E}\Big[L(0,\xi(t+\mathrm{d}t))^2\Big] \\
&\quad = \mathbb{E}\Big[L(0,\xi(t))^2\Big] + 2\mathbb{E}\Big[L(0,\xi(t))\mathbb{E}\Big[\chi \mid L(0,\xi(t))\Big]\Big]\mathbb{E}[\eta] + \mathbb{E}\Big[\chi^2 \mid L(0,\xi(t))\Big]\mathbb{E}[\eta^2] \\
&\quad = \mathbb{E}\Big[L(0,\xi(t))^2\Big] + (2/t)\mathbb{E}\Big[L(0,\xi(t))^2\Big]\mathrm{d}t + (2/t^2)\mathbb{E}\Big[L(0,\xi(t))\Big]\mathrm{d}t \\
&\quad = \mathbb{E}\Big[L(0,\xi(t))^2\Big] + (2/t)\mathbb{E}\Big[L(0,\xi(t))^2\Big]\mathrm{d}t + (2c/t)\mathrm{d}t.
\end{aligned}$$

Thus $\mathbb{E}\Big[L(0,\xi(t))^2\Big]$ satisfies the equation $y^{'} = (2/t)y + (2c/t)$, with $y(1) = c^2$. Then,

$$\mathbb{E}\Big[L(0,\xi(t))^2\Big] = -c + \left(c^2 + c\right)t^2.$$

Finally, reasoning in a similar way, we get that $\mathbb{E}\Big[L(0,\xi(t))^3\Big]$ satisfies the equation $y^{'} = (3/t)y + 6(c^2 + c)$, with $y(1) = c^3$. Hence,

$$\mathbb{E}\Big[L(0,\xi(t))^3\Big] = -3(c^2+c)t + \left(c^3 + 3c^2 + 3c\right)t^3.$$

Divide both sides by t^3, and use the fact that $c > 0$ to get (4.31). □

A ray σ is a subtree of $\mathcal{G}_b$ containing exactly one vertex of each level of $\mathcal{G}_b$. Label the vertices of this ray using $\{\sigma_i,\, i \geq 0\}$, where σ_i is the unique vertex at level i which belongs to σ. Denote by $\mathfrak{S}$ the collection of all rays of $\mathcal{G}_b$.

Lemma 4.7. *For any ray σ, consider VRJP $\mathbf{X}^{(\sigma)} := \{X_t^{(\sigma)},\, t \geq 0\}$, which is the extension of $\mathbf{X}$ to σ. Define*

$$\begin{aligned}
T_n^{(\sigma)} &:= \inf\{t > 0 \colon X_t^{(\sigma)} = \sigma_n\}, \\
L^{(\sigma)}(\sigma_i, t) &:= 1 + \int_0^t \mathbb{1}_{\{X_s^{(\sigma)} = \sigma_i\}}\mathrm{d}s.
\end{aligned}$$

We have that

$$\mathbb{E}\left[L^{(\sigma)}(\sigma_0, T_n^{(\sigma)})^3\right] \leq (37)^n. \tag{4.32}$$

Proof. By the tower property of conditional expectation,

$$\mathbb{E}\left[\left(L^{(\sigma)}(\sigma_0,T_n^{(\sigma)})\right)^3\right] = \mathbb{E}\left[\left(L^{(\sigma)}(\sigma_1,T_n^{(\sigma)})\right)^3\mathbb{E}\left[\left(\frac{L^{(\sigma)}\left(\sigma_0,T_n^{(\sigma)}\right)}{L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)}\right)^3 \middle| L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)\right]\right]. \tag{4.33}$$

At this point we focus on the process restricted to $\{0,1\}$. This restricted process is VRJP which starts at 1, with initial weights $a_1 = 1$, and $a_0 = 1 + h_1(\sigma_0,\sigma_1)$ and $\sigma_0 = \rho$. By applying Lemma 4.6, and using the fact that $h_1(\sigma_0,\sigma_1)$ is exponential with mean 1, we have

$$\mathbb{E}\left[\left(\frac{L^{(\sigma)}\left(\sigma_0,T_n^{(\sigma)}\right)}{L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)}\right)^3\Big|L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)\right]\le\mathbb{E}\left[3(1+h_1(\sigma_0,\sigma_1))+(1+h_1(\sigma_0,\sigma_1))^2+(1+h_1(\sigma_0,\sigma_1))^3\right]$$
$$=37. \tag{4.34}$$

Then

$$\begin{aligned}\mathbb{E}\left[\left(L(\sigma_0,T_n)\right)^3\right]&=\mathbb{E}\left[\mathbb{E}\left[\left(\frac{L^{(\sigma)}\left(\sigma_0,T_n^{(\sigma)}\right)}{L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)}\right)^3\Big|L^{(\sigma)}\left(\sigma_1,T_n^{(\sigma)}\right)\right]\left(L^{(\sigma)}(\sigma_1,T_n^{(\sigma)})\right)^3\right]\\&\le 37\,\mathbb{E}\left[\left(L^{(\sigma)}(\sigma_1,T_n^{(\sigma)})\right)^3\right].\end{aligned} \tag{4.35}$$

The Lemma follows by recursion and restriction principle. □

Next, we prove that

$$L(\rho,T(\sigma_n))\le L^{(\sigma)}(\sigma_0,T_n^{(\sigma)}). \tag{4.36}$$

In fact, we have equality if $T(\sigma_n)<\infty$, because the restriction and the extension of $\mathbf{X}$ to σ coincide during the time interval $[0,T(\sigma_n)]$. If $T(\sigma_n)=\infty$, it means that $\mathbf{X}$ left the ray σ at a time $s<T_n^{(\sigma)}$. Hence

$$L(\rho,T(\sigma_n))=L^{(\sigma)}(\sigma_0,s)\le L^{(\sigma)}(\sigma_0,T_n^{(\sigma)}).$$

Hence, for any ν, with $|\nu|=n$, we have

$$\mathbb{E}\left[L(\rho,T(\nu))^3\right]\le(37)^n. \tag{4.37}$$

Lemma 4.8. $\mathbb{E}\left[\left(L(\rho,\infty)\right)^{12/5}\right]<\infty.$

Proof. Recall the definition of $A(\nu)$ from (3.17) and set

$$D_k:=\bigcup_{\nu:\,|\nu|=k-2}A(\nu).$$

If $A(\nu)$ holds, after the first time the process hits the first child of ν, if this ever happens, it will never visit ν again, and will not increase the local time spent at the root. Roughly, our strategy is to use the extensions on paths to give an upper bound of the total time spent at the root by time T_k and show that the probability that $\bigcap_{i=1}^k D_i^c$ decreases quite fast in k.

Using the independence between disjoint collections of Poisson processes, we infer that $A(\nu)$, with $|\nu|=k-2$ are independent. In fact each $A(\nu)$ is determined by the Poisson processes attached to pairs of vertices in Λ_ν. Hence

$$\mathbb{P}(D_k^c)\le(\gamma_b)^{b^{k-2}} \tag{4.38}$$

Define $d=\inf\{n\ge1\colon \mathbb{1}_{D_n}=1\}$. Fix $k\in\mathbb{N}$. On the set $\{d=k\}$, define $\overline{\mu}$ to be one of the first children at level $k-1$ such that $A(\mathrm{par}(\mu))$ holds. On $\{T(\overline{\mu})<\infty\}\cap\{d=k\}$, we clearly have $L(\rho,\infty)=L(\rho,T(\overline{\mu}))$. On the other hand, on $\{T(\overline{\mu})<\infty\}\cap\{d=k\}$, we have that, after the process reaches $\overline{\mu}$ it will never return to the root. Hence

$$L(\rho,\infty)=1+\int_0^{T(\overline{\mu})}\mathbb{1}_{\{X_u=\rho\}}du+\int_{T(\overline{\mu})}^\infty\mathbb{1}_{\{X_u=\rho\}}du=1+\int_0^{T(\overline{\mu})}\mathbb{1}_{\{X_u=\rho\}}du=L(\rho,T(\overline{\mu})).$$

Using this fact, combined with

$$L(\rho, T(\overline{\mu})) \le \sum_{v\,:\,|v|=k-2} L(\rho, T(\mathrm{fc}(v)),$$

and $\mathbb{1}_{\{d=k\}} \le \mathbb{1}_{\{d>k-1\}} \le \mathbb{1}_{D^c_{k-1}}$, we have

$$\begin{aligned} L(\rho,\infty)\mathbb{1}_{\{d=k\}} = L(\rho, T(\overline{\mu}))\mathbb{1}_{\{d=k\}} &\le \Big(\sum_{v\,:\,|v|=k-2} L\big(\rho, T(\mathrm{fc}(v))\big)\Big)\mathbb{1}_{\{d=k\}} \\ &\le \Big(\sum_{v\,:\,|v|=k-2} L\big(\rho, T(\mathrm{fc}(v))\big)\Big)\mathbb{1}_{D^c_{k-1}}. \end{aligned} \tag{4.39}$$

Using (4.39), Holder's inequality (with $p = 5/4$) and (4.38) we have

$$\begin{aligned} &\mathbb{E}\left[\big(L(\rho,\infty)\big)^{12/5}\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[\big(L(\rho,\infty)\big)^{12/5}\mathbb{1}_{\{d=k\}}\right] = \sum_{k=1}^{\infty}\mathbb{E}\left[\big(L(\rho,\infty)\mathbb{1}_{\{d=k\}}\big)^{12/5}\right] \\ &\le \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\sum_{v\,:\,|v|=k-2} L\big(\rho, T(\mathrm{fc}(v))\big)\mathbb{1}_{D^c_{k-1}}\right)^{12/5}\right] \le \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\sum_{v\,:\,|v|=k-2} L\big(\rho, T(\mathrm{fc}(v))\big)\right)^{3}\right]^{4/5} (\gamma_b)^{b^{k-3}/5} \\ &\le \sum_{k=1}^{\infty}\mathbb{E}\left[\left(\sum_{v\,:\,|v|=k-2} L\big(\rho, T(\mathrm{fc}(v))\big)\right)^{3}\right] (\gamma_b)^{b^{k-3}/5} \qquad \text{(using } L(\rho,t)\ge 1\text{)} \\ &\le \sum_{k=1}^{\infty}\left(b^{2k}\sum_{v\,:\,|v|=k-2}\mathbb{E}\big[L\big(\rho, T(\mathrm{fc}(v))\big)^3\big]\right)(\gamma_b)^{b^{k-3}/5} \qquad \text{(by Jensen)} \\ &\le \sum_{k=1}^{\infty} b^{3k}(37)^k(\gamma_b)^{b^{k-3}/5} < \infty. \end{aligned}$$

□

Lemma 4.9. *For $v \ne \rho$, there exists a random variable Δ_v which is $\sigma\{P(u,v): u,v \in Vert(\Lambda_v)\}$-measurable, such that*

i) $L(v,\infty) \le \Delta_v$, and

ii) Δ_v and $L(\rho,\infty)$ are identically distributed.

Proof. Let $\widetilde{\mathbf{X}} := \{\widetilde{X}_t,\, t \ge 0\}$ be the extension of $\mathbf{X}$ on Λ_v. Define

$$\Delta_v := 1 + \int_0^\infty \mathbb{1}_{\{\widetilde{X}_t = v\}}\mathrm{d}t.$$

By construction, this random variable satisfies i) and ii) and is $\sigma\{P(u,v): u,v \in \mathrm{Vert}(\Lambda_v)\}$-measurable. □

Theorem 4.10. $\mathbb{E}\left[(\tau_1)^{11/5}\right] < \infty$.

Proof. Suppose we relabel the vertices that have been visited by time τ_1, using $\theta_1, \theta_2, \dots, \theta_\Pi$, where vertex v is labeled θ_k if there are exactly $k-1$ distinct vertices that have been visited before v. Notice that Δ_v and $\{\theta_k = v\}$ are independent, because they are determined by disjoint non-random sets of Poisson processes (Δ_v is $\sigma\{P(u,v)\colon u,v \in \mathrm{Vert}(\Lambda_v)\}$-measurable). As the variables Δ_v, with $v \in \mathrm{Vert}(\mathscr{G}_b)$, share the same distribution, for any $p > 0$, we have

$$\mathbb{E}[\Delta_{\theta_k}^p] = \mathbb{E}[\Delta_v^p] = \mathbb{E}[L(\rho,\infty)^p].$$

By Jensen's and Holder's (with $p = 12/11$) inequalities, Lemma 4.9 i) and ii), and Lemma 4.8, we have

$$\begin{aligned}
\mathbb{E}\left[(\tau_1)^{11/5}\right] &\le \mathbb{E}\left[\left(\sum_{k=1}^{\Pi}\Delta_{\theta_k}\right)^{11/5}\right] \le \mathbb{E}\left[\Pi^{(11/5)-1}\sum_{k=1}^{\Pi}(\Delta_{\theta_k})^{11/5}\right] \\
&= \mathbb{E}\left[\sum_{k=1}^{\infty}\Delta_{\theta_k}^{11/5}\,\Pi^{6/5}\,\mathbb{1}_{\{\Pi\ge k\}}\right] \le \sum_{k=1}^{\infty}\mathbb{E}\left[\Delta_{\theta_k}^{12/5}\right]^{11/12}\mathbb{E}\left[\Pi^{72/5}\,\mathbb{1}_{(\Pi\ge k)}\right]^{1/12} \\
&\le C_b^{(3)}\sum_{k=1}^{\infty}\mathbb{E}\left[\Pi^{144/5}\right]^{1/24}\mathbb{P}(\Pi\ge k)^{1/24} \qquad \text{(by Cauchy-Schwartz and Lemma 4.8)} \\
&\le C_b^{(4)}\sum_{k=1}^{\infty}\mathbb{P}(\Pi\ge k)^{1/24}, \qquad \text{(by Lemma 4.5)},
\end{aligned}$$

for some positive constants $C_b^{(3)}$ and $C_b^{(4)}$. It remains to prove the finiteness of the last sum. We use the fact

$$\lim_{k\to\infty} k^{48}\mathbb{P}(\Pi \ge k) = 0. \tag{4.40}$$

The previous limit is a consequence of the well-known formula

$$\sum_{k=1}^{\infty} k^{48}\mathbb{P}(\Pi\ge k) = \mathbb{E}[\Pi^{49}], \tag{4.41}$$

and the finiteness of $\mathbb{E}[\Pi^{49}]$ by virtue of Lemma 4.5.

$$\sum_{k=1}^{\infty}\mathbb{P}(\Pi\ge k)^{1/24} = \sum_{k=1}^{\infty}\frac{1}{k^2}\Big(k^{48}\mathbb{P}(\Pi\ge k)\Big)^{1/24} < \infty.$$

□

Lemma 4.11. $\sup_{x\in[1,2]}\mathbb{E}[(L(\rho,\infty))^{12/5} \mid L(\rho,T_1)=x] < \infty.$

Proof. Using 4.9, and the fact that $\Delta_{X_{T_1}}$ is independent of $L(\rho,T_1)$, we have

$$\begin{aligned}
\sup_{x\in[1,2]}\mathbb{E}[(L(X_{T_1},\infty))^{12/5} \mid L(\rho,T_1)=x] &\le \sup_{x\in[1,2]}\mathbb{E}[(\Delta_{X_{T_1}})^{12/5} \mid L(\rho,T_1)=x] \\
&= \mathbb{E}[(\Delta_{X_{T_1}})^{12/5}] = \mathbb{E}[(L(\rho,\infty))^{12/5}] < \infty.
\end{aligned} \tag{4.42}$$

Given $L(\rho,T_1)=x$, the process $\mathbf{X}$ restricted to $\{\rho, X_{T_1}\}$ is VRJP which starts from X_{T_1}, with initial weights $a_\rho = x$ and 1 on X_{T_1}. This process runs up to the last visit of $\mathbf{X}$ to one of these two vertices.

Using Lyapunov inequality, i.e. $\mathbb{E}[Z^q]^{1/q} \leq \mathbb{E}[Z^p]^{1/p}$ whenever $0 < q \leq p$, Lemma 4.7, and the fact $x \geq 1$, we have

$$\begin{aligned}
&\mathbb{E}\Big[\Big(\frac{L(\rho, T_n)}{L(X_{T_1}, T_n)}\Big)^{12/5} \mid L(X_{T_1}, T_n), \{L(\rho, T_1) = x\}\Big] \\
&\leq \mathbb{E}\Big[\Big(\frac{L(\rho, T_n)}{L(X_{T_1}, T_n)}\Big)^{3} \mid L(X_{T_1}, T_n), \{L(\rho, T_1) = x\}\Big]^{4/5} \\
&\leq (x^3 + 3x^2 + 3x)^{4/5} \leq x^3 + 3x^2 + 3x.
\end{aligned} \tag{4.43}$$

Finally

$$\begin{aligned}
\mathbb{E}[(L(\rho, T_n))^{12/5} \mid L(\rho, T_1) = x] &= \mathbb{E}\Big[\Big(\frac{L(\rho, T_n)}{L(X_{T_1}, T_n)}\Big)^{12/5} (L(X_{T_1}, T_n))^{12/5} \mid L(\rho, T_1) = x\Big] \\
&\leq (x^3 + 3x^2 + 3x)\mathbb{E}\Big[(L(X_{T_1}, T_n))^{12/5} \mid L(\rho, T_1) = x\Big] \\
&\leq (x^3 + 3x^2 + 3x)\mathbb{E}\Big[(L(X_{T_1}, \infty))^{12/5} \mid L(\rho, T_1) = x\Big] \\
&\leq (x^3 + 3x^2 + 3x)\mathbb{E}[(L(\rho, \infty))^{12/5}].
\end{aligned} \tag{4.44}$$

By sending $n \to \infty$ and taking the suprema over $x \in [1, 2]$ we get

$$\sup_{x \in [1,2]} \mathbb{E}[(L(\rho, T_n))^{12/5} \mid L(\rho, T_1) = x] \leq 26\mathbb{E}[(L(\rho, \infty))^{12/5}] < \infty.$$

□

Theorem 4.12. $\sup_{x \in [1,2]} \mathbb{E}\left[(\tau_1)^{11/5} \mid L(\rho, T_1) = x\right] < \infty.$

Proof. Label the vertices at level 1 by $\mu_1, \mu_2, \dots, \mu_b$. Let $\tau_1(\mu_i)$ be the first cut time of the extension of $\mathbf{X}$ on Λ_{μ_i}. This extension is VRJP on Λ_{μ_i} with initial weights 1, hence we can apply Theorem 4.10 to get

$$\mathbb{E}[(\tau_1(\mu_i))^{11/5}] < \infty. \tag{4.45}$$

Hence, it remains to prove that for $x \in [1, 2]$

$$\begin{aligned}
&\mathbb{E}\left[(\tau_1)^{11/5} | L(\rho, T_1) = x\right] \leq \mathbb{E}\Big[\Big(L(\rho, \infty) + \max_i \tau_1(\mu_i)\Big)^{11/5} \Big| L(\rho, T_1) = x\Big] \\
&\leq \mathbb{E}\Big[\Big(L(\rho, \infty) + \sum_{i=1}^{b} \tau_1(\mu_i)\Big)^{11/5} \Big| L(\rho, T_1) = x\Big] \\
&\leq (b+1)^{11/5-1}\mathbb{E}\Big[\Big(L(\rho, \infty)\Big)^{11/5} | L(\rho, T_1) = x\Big] + (b+1)^{11/5}\mathbb{E}\Big[((\tau_1(\mu_1))^{11/5}\Big] < \infty,
\end{aligned}$$

where we used Jensen's inequality, the independence of $\tau(\mu_i)$ and T_1 and Lemma 4.11. In fact, as $L(\rho, \infty) \geq 1$, we have

$$\mathbb{E}[(L(\rho, \infty))^{11/5} \mid L(\rho, T_1) = x] \leq \mathbb{E}[(L(\rho, \infty))^{12/5} \mid L(\rho, T_1) = x] < \infty.$$

□

5 Splitting the path into one-dependent pieces

Define $Z_i = L(X_{\tau_i}, \infty)$, with $i \geq 1$.

Lemma 5.1. *The process Z_i, with $i \geq 1$ is a homogenous Markov chain with state space $[1,2]$.*

Proof. Fix $n \geq 1$. On $\{Z_n = x\} \cap \{X_{\tau_n} = v\}$ the random variable Z_{n+1} is determined by the variables $\{P(u,v), u, v \in \Lambda_v, u \neq v\}$. In fact these Poisson processes, on the set $\{Z_n = x\} \cap \{X_{\tau_n} = v\}$, are the only ones used to generate the jumps of the process $\{X_{T(\mathrm{fc}(v))+\mathrm{t}}\}_{t\geq 0}$. Let $E_1, E_2, \ldots, E_{n-1}, E_{n+1}$ be Borel subsets of $[0,1]$. Conditionally on $\{Z_n = x\} \cap \{X_{\tau_n} = v\}$, the two events $\{Z_{n+1} \in E_{n+1}\}$ and $\{Z_1 \in E_1, Z_2 \in E_2, \ldots Z_{n-1} \in E_{n-1}\}$ are independent because are determined by disjoint collections of Poisson processes. By symmetry

$$\mathbb{P}(Z_{n+1} \in E_{n+1} \mid \{Z_n = x\} \cap \{X_{\tau_n} = v\})$$

does not depend on v. Hence

$$\begin{aligned}
&\mathbb{P}(Z_{n+1} \in E_{n+1} \mid Z_1 \in E_1, Z_2 \in E_2, \ldots Z_{n-1} \in E_{n-1}, Z_n = x) \\
&= \sum_v \mathbb{P}(Z_{n+1} \in E_{n+1} \mid Z_1 \in E_1, \ldots, Z_{n-1} \in E_{n-1}, Z_n = x, X_{\tau_n} = v)\mathbb{P}(X_{\tau_n} = v \mid Z_1 \in E_1, \ldots, Z_n = x) \\
&= \mathbb{P}(Z_{n+1} \in E_{n+1} \mid Z_n = x, X_{\tau_n} = v) = \mathbb{P}(Z_{n+1} \in E_{n+1} \mid Z_n = x).
\end{aligned}$$

This implies that $\mathbf{Z}$ is a Markov chain. The self-similarity property of $\mathcal{G}_b$ and $\mathbf{X}$ yields the homegeneity. □

From the previous proof, we can infer that given $Z_i = x$, the random vectors $(\tau_{i+1} - \tau_i, l_{i+1} - l_i)$ and $(\tau_i - \tau_{i-1}, l_i - l_{i-1})$, are independent.

Proposition 5.2.

$$\sup_{i\in\mathbb{N}} \sup_{x\in[1,2]} \mathbb{E}\Big[\big(\tau_{i+1} - \tau_i\big)^{11/5} \Big| Z_i = x\Big] < \infty \tag{5.46}$$

$$\sup_{i\in\mathbb{N}} \sup_{x\in[1,2]} \mathbb{E}\Big[\big(l_{i+1} - l_i\big)^{11/5} \mid Z_i = x\Big] < \infty. \tag{5.47}$$

Proof. We only prove (5.46), the proof of (5.47) being similar. Define $C := \{X_t \neq \rho, \forall t > T_1\}$ and fix a vertex v. Notice that by the self-similarity property of $\mathcal{G}_b$, we have

$$\mathbb{E}\Big[(\tau_{i+1} - \tau_i)^{11/5} \mid \{Z_i = x\} \cap \{X_{\tau_i} = v\}\Big] = \mathbb{E}\Big[(\tau_1)^{11/5} | \{L(\rho, T_1) = x\} \cap C\Big].$$

By the proof of Lemma 2.2, we have that

$$\inf_{1\leq x\leq 2} \mathbb{P}\big(C \,\big|\, L(\rho, T_1) = x\big) \geq \frac{b}{b+x}\mathbb{P}(A_1) \geq (1-\gamma_b)\frac{b}{b+2} > 0. \tag{5.48}$$

Hence

$$\begin{aligned}\sup_{x:\, x\in[1,2]} & \mathbb{E}\Big[(\tau_1)^{11/5}\,\Big|\, L(\rho, T_1)=x\Big] \\ & \geq \sup_{x:\, x\in[1,2]} \mathbb{E}\Big[(\tau_1)^{11/5}\,\Big|\{L(\rho,T_1)=x\}\cap C\Big]\mathbb{P}(C\,|\,L(\rho,T_1)=x) \\ & \geq (1-\gamma_b)\frac{b}{b+2}\sup_{x:\, x\in[1,2]} \mathbb{E}\Big[(\tau_1)^{11/5}\,\Big|\{L(\rho,T_1)=x\}\cap C\Big] \\ & \geq (1-\gamma_b)\frac{b}{b+2}\sup_{x:\, x\in[1,2]} \mathbb{E}\Big[(\tau_{i+1}-\tau_i)^{11/5}\,|\,\{Z_i=x\}\cap\{X_{\tau_i}=\nu\}\Big]\end{aligned}$$

Hence

$$\mathbb{E}\Big[(\tau_{i+1}-\tau_i)^{11/5}\,|\,\{Z_i=x\}\cap\{X_{\tau_i}=\nu\}\Big] \leq \frac{b+2}{b(1-\gamma_b)}\sup_{1\leq x\leq 2}\mathbb{E}\Big[(\tau_1)^{11/5}\,|\,\{L(\rho,T_1=x\}\Big].$$

□

Next we prove that **Z** satisfies the Doeblin condition.

Lemma 5.3. *There exists a probability measure $\phi(\cdot)$ and $0<\lambda\leq 1$, such that for every Borel subset B of $[1,2]$, we have*

$$\mathbb{P}\big(Z_{i+1}\in B\ \mid Z_i=z\big)\ \geq\ \lambda\,\phi(B)\qquad \forall\, z\in[1,2]. \tag{5.49}$$

Proof. As Z_i is homogeneous, it is enough to prove (5.49) for $i=1$. In this proof we show that the distribution of Z_2 is absolutely continuous and we compare it to 1+ an exponential with parameter 1 conditionated on being less than 1. The analysis is technical because Z_i depend on the behaviour of the whole process **X**. Our goal is to find a lower bound for

$$\mathbb{P}\Big(Z_2\ \in\ (x,y)\,\Big|\,Z_1=z\Big),\qquad \text{with } z\in[1,2]. \tag{5.50}$$

Moreover, we require that this lower bound is independent of $z\in[1,2]$.

Fix $\varepsilon\in(0,1)$. Our first goal is to find a lower bound for the probability of the event $\{Z_2\in(x,y),\ Z_1\in I_\varepsilon(z)\}$, where $I_\varepsilon(z):=(z-\varepsilon,z+\varepsilon)$. Fix $z\in[1,2]$ and consider the function

$$\mathrm{e}^{-(b+u)(t-1)}-(b+1)\mathrm{e}^{-(b+2)}\mathrm{e}^{-(t-1)}. \tag{5.51}$$

Its derivative with respect t is

$$(b+1)\mathrm{e}^{-(b+2)-(t-1)}-(b+u)\mathrm{e}^{-(b+u)(t-1)},$$

which is non-positive for $t\in[1,2]$ and $u\in[1,2]$. In fact

$$\begin{aligned}(b+1)\mathrm{e}^{-(b+2)-(t-1)}-(b+u)\mathrm{e}^{-(b+u)(t-1)} &\leq (b+1)\mathrm{e}^{-(b+2)-(1-1)}-(b+u)\mathrm{e}^{-(b+u)(2-1)} \\ &= (b+1)\mathrm{e}^{-(b+2)}-(b+u)\mathrm{e}^{-(b+u)}\leq 0.\end{aligned}$$

Hence for fixed $u\in[1,2]$, the function in (5.51) is non-increasing for $t\in[1,2]$. For $1\leq x<y\leq 2$, we have

$$\mathrm{e}^{-(b+u)(x-1)}-\mathrm{e}^{-(b+u)(y-1)}\geq(b+1)\mathrm{e}^{-(b+2)}\left(\mathrm{e}^{-(x-1)}-\mathrm{e}^{-(y-1)}\right). \tag{5.52}$$

We use this inequality to get a lower bound for the probability of the event $\{Z_2\in(x,y),\ Z_1\in I_\varepsilon(z)\}$. Our strategy is to calculate the probability of a suitable subset of the latter set. Consider the following event. Suppose that

a) $T_1 < 1$, then

b) the process spends at X_{T_1} an amount of time enclosed in $(z-1-\varepsilon, z-1+\varepsilon)$, then

c) it jumps to a vertex at level 2, spends there an amount of time t where $t+1 \in (x,y)$, and

d) it jumps to level 3 and never returns to X_{T_2}.

In the event just described, levels 1 and 2 are the first two cut levels, and $\{Z_2 \in (x,y),\ Z_1 \in I_\varepsilon(z)\}$ holds. The probability that a) holds is exactly e^{-b}. Given $T_1 = s-1$, the time spent in X_{T_1} before the first jump is exponential with parameter $(b+s)$. Hence b) occurs with probability larger than

$$\inf_{s\in[1,2]} \left(\mathrm{e}^{-(b+s)(z-\varepsilon)} - \mathrm{e}^{-(b+s)(z+\varepsilon)}\right).$$

Given a) and b), the process jumps to level 2 and then to level 3 with probability larger than $\big(b/(b+2)\big)\big(b/(b+z+\varepsilon)\big)$. The conditional probability, given a) and b), that the time gap between these two jumps lies in $(x-1, y-1)$ is larger than

$$\inf_{u\in I_\varepsilon(z)} \left(\mathrm{e}^{-(b+u)(x-1)} - \mathrm{e}^{-(b+u)(y-1)}\right).$$

At this point, a lower bound for the conditional probability that the process never returns to X_{T_2} is

$$\frac{b}{b+y}(1-\alpha_b) \ge \frac{b}{b+2}(1-\alpha_b).$$

We have

$$\begin{aligned}
&\mathbb{P}\Big(Z_2 \in (x,y),\ Z_1 \in I_\varepsilon(z)\Big) \\
&\ge \mathrm{e}^{-b}\frac{b^3}{(b+2)^2(b+z+\varepsilon)} \inf_{s\in[1,2]} \left(\mathrm{e}^{-(b+s)(z-\varepsilon)} - \mathrm{e}^{-(b+s)(z+\varepsilon)}\right) \\
&\qquad\qquad \inf_{u\in I_\varepsilon(z)} \left(\mathrm{e}^{-(b+u)(x-1)} - \mathrm{e}^{-(b+u)(y-1)}\right)(1-\alpha_b) \\
&\ge (1-\alpha_b)\mathrm{e}^{-b}\frac{b^3(b+1)}{(b+2)^2(b+z+\varepsilon)}\mathrm{e}^{-(b+2)}\left(\mathrm{e}^{-(x-1)} - \mathrm{e}^{-(y-1)}\right) \inf_{s\in[1,2]} \left(\mathrm{e}^{-(b+s)(z-\varepsilon)} - \mathrm{e}^{-(b+s)(z+\varepsilon)}\right),
\end{aligned} \tag{5.53}$$

where in the last inequality we used (5.52). Notice that there exists a constant $C_b^{(4)} > 0$ such that

$$\inf_{\varepsilon\in(0,1)} \inf_{z,s\in[1,2]} \frac{1}{\varepsilon}\left(\mathrm{e}^{-(b+s)(z-\varepsilon)} - \mathrm{e}^{-(b+s)(z+\varepsilon)}\right) \ge C_b^{(4)}. \tag{5.54}$$

Summarizing, we have

$$\mathbb{P}\Big(Z_2 \in (x,y),\ Z_1 \in I_\varepsilon(z)\Big) \ge C_b^{(5)}\left(\mathrm{e}^{-(x-1)} - \mathrm{e}^{-(y-1)}\right)\varepsilon, \tag{5.55}$$

where $C_b^{(5)}$ depends only on b.
In order to find a lower bound for (5.50) we need to prove that

$$\sup_{\varepsilon\in(0,1)} \frac{1}{\varepsilon}\mathbb{P}\Big(Z_1 \in I_\varepsilon(z)\Big) \le C_b^{(6)}, \tag{5.56}$$

for some positive constant $C_b^{(6)}$. To see this, recall the definition of B_j from the proof of Theorem 3.5, and ζ from (3.16). The event that level i is not a cut level is subset of $(B_i \cap A_i)^c$ (see the proof of Theorem 3.5). Denote by $m_i = h_1\big(X_{T_i}, \mathrm{fc}(X_{T_i})\big)$, which is exponential with mean $1/b$. Then

$$\begin{aligned}\mathbb{P}(Z_1 \in I_\varepsilon(z)) &\le \sum_i^\infty \mathbb{P}\Big(m_i \in I_\varepsilon(z)\Big)\mathbb{P}\Big(\bigcap_{k=1}^{i-1}(B_k \cap A_k)^c \mid m_i \in I_\varepsilon(z)\Big)\\ &\le C\varepsilon \sum_i^\infty \mathbb{P}\Big(\bigcap_{k=1}^{i-1}(B_k \cap A_k)^c \mid m_i \in I_\varepsilon(z)\Big),\end{aligned}$$

where the constant C is independent of ε and z. It remains to prove that the sum in the right-hand side is bounded by a constant independent of ε. Notice that, for $i > \zeta$, $A_{i-\zeta}$ and $B_{i-\zeta}$ are independent of m_i. Moreover the events

$$A_{i-\zeta} \cap B_{i-\zeta},\, A_{i-2\zeta} \cap B_{i-2\zeta},\, A_{i-3\zeta} \cap B_{i-3\zeta}, \dots$$

are independent by the proof of Proposition 3.1. Hence

$$\begin{aligned}\mathbb{P}(Z_1 \in I_\varepsilon(z)) &\le C\varepsilon \sum_i^\infty \mathbb{P}\Big(\bigcap_{k=1}^{[(i-1)/\zeta]} (B_{i-k\zeta} \cap A_{i-k\zeta})^c \mid m_i \in I_\varepsilon(z)\Big)\\ &= C\varepsilon \sum_i^\infty \mathbb{P}\Big(\bigcap_{k=1}^{[(i-1)/\zeta]} (B_{i-k\zeta} \cap A_{i-k\zeta})^c\Big) \qquad \text{(by independence)}\\ &= C\varepsilon \sum_i^\infty \mathbb{P}\Big((B_{i-k\zeta} \cap A_{i-k\zeta})^c\Big)^{[(i-1)/\zeta]} < \infty.\end{aligned}$$

Combining (5.53), (5.54) and (5.56), we get

$$\begin{aligned}\mathbb{P}\Big(Z_2 \in (x,y) \,\Big|\, Z_1 = z\Big) &= \lim_{\varepsilon \downarrow 0} \frac{1}{\mathbb{P}\Big(Z_1 \in I_\varepsilon(z)\Big)} \mathbb{P}\Big(Z_2 \in (x,y)\,,\, Z_1 \in I_\varepsilon(z)\Big)\\ &\ge \lambda \frac{\Big(\mathrm{e}^{-(x-1)} - \mathrm{e}^{-(y-1)}\Big)}{\big(1-\mathrm{e}^{-1}\big)},\end{aligned} \tag{5.57}$$

for some $\lambda > 0$. . A finite measure defined on field $\mathscr{A}$ can be extended uniquely to the sigma-field generated by $\mathscr{A}$, and this extension coincides with the outer measure. We apply this result to prove that (5.57) holds for any Borel set $C \subset [1,2]$, using the fact that it holds in the field of finite unions of intervals. For any interval E, the right-hand side of (5.57) can be written in an integral form as

$$\lambda \int_E \frac{e^{-x+1}}{(1-\mathrm{e}^{-1})} \mathrm{d}x.$$

Fix a Borel set $C \subset [1,2]$ and $\varepsilon > 0$ choose a countable collection of disjoint intervals $E_i \subset [1,2]$,

$i \geq 1$, with $C \subset \bigcup_{i=1}^{\infty} E_i$, such that

$$\begin{aligned}
\mathbb{P}(Z_2 \in C \bigm| Z_1 = z) &\geq \sum_{i=1}^{\infty} \mathbb{P}(Z_2 \in E_i \bigm| Z_1 = z) - \varepsilon \\
&\geq \lambda \sum_{i=1}^{\infty} \int_{E_i} \frac{e^{-x+1}}{(1-\mathrm{e}^{-1})} \mathrm{d}x - \varepsilon \\
&\geq \lambda \int_C e^{-x+1} / \left(1-\mathrm{e}^{-1}\right) \mathrm{d}x - \varepsilon.
\end{aligned}$$

The first inequality is true because of the extension theorem, and the fact that the right-hand side is a lower bound for the outer measure, for a suitable choice of the E_is. The inequality (5.49), with $\phi(C) = \int_C e^{-x+1} / \left(1-\mathrm{e}^{-1}\right) \mathrm{d}x$, follows by sending ε to 0. □

The proof of the following Proposition can be found in [2].

Proposition 5.4. *There exists a constant $\varrho \in (0,1)$ and a sequence of random times $\{N_k, k \geq 0\}$, with $N_0 = 0$, such that*

- *the sequence $\{Z_{N_k}, k \geq 1\}$ consists of independent and identically distributed random variables with distribution $\phi(\cdot)$*
- *$N_i - N_{i-1}$, $i \geq 1$, are i.i.d. with a geometric distribution(ρ), i.e.*

$$\mathbb{P}(N_2 - N_1 = j) = (1-\varrho)^{j-1}\varrho, \qquad \text{with } j \geq 1.$$

Lemma 5.5. $\sup_{i \in \mathbb{N}} \mathbb{E}[(\tau_{N_{i+1}} - \tau_{N_i})^2] < \infty$.

Proof. It is enough to prove $\mathbb{E}[(\tau_{N_2} - \tau_{N_1})^2] < \infty$. By virtue of Jensen's inequality, we have that

$$\begin{aligned}
\mathbb{E}\left[(\tau_k - \tau_m)^{11/5}\right] &= \mathbb{E}\left[(\sum_{j=1}^{k-m} \tau_{m+j} - \tau_{m+j-1})^{11/5}\right] \\
&\leq (k-m)^{11/5} \mathbb{E}[(\tau_2 - \tau_1)^{11/5}].
\end{aligned} \tag{5.58}$$

Using Holder with $p = 11/10$, we have

$$\begin{aligned}
&\mathbb{E}[(\tau_{N_2} - \tau_{N_1})^2] = \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \mathbb{E}\left[(\tau_k - \tau_m)^2 \mathbb{1}_{\{N_1 = m, N_2 = k\}}\right] \\
&\leq \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \mathbb{E}\left[\left(\tau_k - \tau_m\right)^{11/5}\right]^{10/11} \mathbb{P}(N_1 = m, N_2 - N_1 = k - m)^{1/11} \\
&= \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} \mathbb{E}\left[\left(\tau_k - \tau_m\right)^{11/5}\right]^{10/11} \mathbb{P}(N_1 = m)^{1/11} \mathbb{P}(N_2 - N_1 = k - m)^{1/11} \\
&\leq \sum_{k=2}^{\infty} \sum_{m=1}^{k-1} (k-m)^3 \mathbb{E}[(\tau_2 - \tau_1)^{11/5}]^{10/11} \varrho^{2/11} \left(1-\varrho\right)^{(k-2)/11} \\
&\leq \varrho^{2/11} \mathbb{E}[(\tau_2 - \tau_1)^{11/5}]^{10/11} \sum_{k=2}^{\infty} k^4 \left(1-\varrho\right)^{(k-2)/11} < \infty,
\end{aligned}$$

where we used the fact that $0 < \varrho < 1$. □

With a similar proof we get the following result.

Lemma 5.6. $\sup_{i\in\mathbb{N}} \mathbb{E}\left[\left(l_{N_{i+1}} - l_{N_i}\right)^2\right] < \infty$.

Definition 5.7. *A process $\{Y_k, k \geq 1\}$, is said to be* ***one-dependent*** *if Y_{i+2} is independent of $\{Y_j$, with $1 \leq j \leq i\}$.*

Lemma 5.8. *Let $\Upsilon_i := \left(\tau_{N_{i+1}} - \tau_{N_i}, l_{N_{i+1}} - l_{N_i}\right)$, for $i \geq 1$. The process $\Upsilon := \left\{\Upsilon_i, i \geq 1\right\}$ is one-dependent. Moreover Υ_i, $i \geq 1$, are identically distributed.*

Proof. Given $Z_{N_{i-1}}$, Υ_i is independent of $\{\Upsilon_j, j \leq i-2\}$. Thus, it is sufficient to prove that Υ_i is independent of $Z_{N_{i-1}}$. To see this, it is enough to realize that given Z_{N_i}, Υ_i is independent of $Z_{N_{i-1}}$, and combine this with the fact that Z_{N_i} and $Z_{N_{i-1}}$ are independent. The variables Z_{N_i} are i.i.d., hence $\{\Upsilon_i, i \geq 2\}$, are identically distributed. □

The Strong Law of Large Numbers holds for one-dependent sequences of identically distributed variables bounded in $\mathscr{L}^1$. To see this, just consider separately the sequence of random variables with even and odd indices and apply the usual Strong Law of Large Numbers to each of them.

Hence, for some constants $0 < C_b^{(7)}, C_b^{(8)} < \infty$, we have

$$\lim_{i\to\infty} \frac{\tau_{N_i}}{i} \to C_b^{(7)}, \qquad \text{and} \qquad \lim_{i\to\infty} \frac{l_{N_i}}{i} \to C_b^{(8)}, \qquad a.s.. \tag{5.59}$$

Proof of Theorem 1. If $\tau_{N_i} \leq t < \tau_{N_{i+1}}$, then by the definition of cut level, we have

$$l_{N_i} \leq |X_t| < l_{N_{i+1}}.$$

Hence

$$\frac{l_{N_i}}{\tau_{N_{i+1}}} \leq \frac{|X_t|}{t} < \frac{l_{N_{i+1}}}{\tau_{N_i}}.$$

Let

$$K_b^{(1)} = \frac{\mathbb{E}[l_{N_2} - l_{N_1}]}{\mathbb{E}[\tau_{N_2} - \tau_{N_1}]}, \tag{5.60}$$

which are the constants in (5.59). Then

$$\limsup_{t\to\infty} \frac{|X_t|}{t} \leq \lim_{i\to\infty} \frac{l_{N_{i+1}}}{\tau_{N_i}} = \lim_{i\to\infty} \frac{l_{N_{i+1}}}{i+1} \frac{i}{\tau_{N_i}} = K_b^{(1)}, \text{ a.s..}$$

Similarly, we can prove that

$$\liminf_{t\to\infty} \frac{|X_t|}{t} \geq K_b^{(1)}, \text{ a.s..}$$

Now we turn to the proof of the central limit theorem. First we prove that there exists a constant $C \geq 0$ such that

$$\frac{l_{N_m} - K_b^{(1)} \tau_{N_m}}{\sqrt{m}} \Longrightarrow \text{Normal}(0, C), \tag{5.61}$$

where Normal$(0,0)$ stands for the Dirac mass at 0. To prove (5.61) we use a theorem from [12]. The reader can find the statement of this theorem in the Appendix, Theorem 6.1, (see also [22]). In order to apply this result we first need to prove that the quantity

$$\frac{1}{m}\mathbb{E}\Big[\big(l_{N_m}-K_b^{(1)}\tau_{N_m}\big)^2\Big]=\mathbb{E}\Big[\Big(\frac{l_{N_m}-K_b^{(1)}\tau_{N_m}}{\sqrt{m}}\Big)^2\Big] \tag{5.62}$$

converges. Call $Y_1=l_{N_1}-K_b^{(1)}(\tau_{N_1}$ and let $Y_i=l_{N_i}-l_{N_{i-1}}-K_b^{(1)}(\tau_{N_i}-\tau_{N_{i-1}})$, with $i\geq 2$. The quantity in (5.62) can be written as

$$\frac{1}{m}\mathbb{E}\Big[\big(\sum_{i=1}^m Y_i\big)^2\Big].$$

The random variables Y_i are identically distributed with the exception of Y_1. From the definition of $K_b^{(1)}$ given in (5.60), we have

$$\mathbb{E}[Y_i]=\mathbb{E}[l_{N_2}-l_{N_1}]-\mathbb{E}[l_{N_2}-l_{N_1}]=0.$$

Hence Y_i, with $i\geq 1$, is a zero-mean one-dependent process, and we get

$$\begin{aligned}\mathbb{E}\Big[\big(l_{N_m}-K_b^{(1)}\tau_{N_m}\big)^2\Big]&=\mathbb{E}\Big[\Big(\sum_{i=1}^m Y_i\Big)^2\Big]\\&=(m-1)\mathbb{E}[Y_2^2]+2(m-2)\mathbb{E}[Y_3Y_2]+\mathbb{E}[Y_1^2]+2\mathbb{E}[Y_1Y_2].\end{aligned} \tag{5.63}$$

This proves that the limit in (5.62) exists and is equal to $\mathbb{E}[Y_2^2]+2\mathbb{E}[Y_3Y_2]$. Now we face two options. If the limit is equal to zero, then using Chebishev we get that

$$\lim_{m\to\infty}\mathbb{P}\Big(\Big|\frac{l_{N_m}-C\tau_{N_m}}{\sqrt{m}}\Big|>\varepsilon\Big)=\lim_{m\to\infty}\mathbb{P}\Big(\Big|\frac{1}{\sqrt{m}}\sum_{i=1}^m Y_i\Big|>\varepsilon\Big)\leq\lim_{m\to\infty}\frac{1}{\varepsilon}\mathbb{E}\Big[\Big(\frac{\sum_{i=1}^m Y_i}{\sqrt{m}}\Big)^2\Big]=0.$$

If the limit of the quantity in (5.62) is positive, then we can apply Theorem 6.1 and deduce central limit theorem for $Y_i, i\geq 1$, yielding (5.61).

Now we use (5.61) to prove the central limit theorem for $|X_t|$. If $\tau_{N_m}\leq t<\tau_{N_{m+1}}$, then

$$\begin{aligned}\frac{|X_t|-K_b^{(1)}t}{K_b^{(2)}\sqrt{t}}\geq\frac{l_{N_m}-K_b^{(1)}\tau_{N_{m+1}}}{K_b^{(2)}\sqrt{\tau_{N_{m+1}}}}&=\sqrt{\frac{m}{\tau_{N_{m+1}}}}\Big(\frac{l_{N_m}-K_b^{(1)}\tau_{N_m}}{\sqrt{m}}+\frac{K_b^{(1)}}{\sqrt{m}}(\tau_{N_m}-\tau_{N_{m+1}})\Big)\\&=\sqrt{\frac{m}{\tau_{N_{m+1}}}}\Big(\frac{\sum_{i=1}^m Y_i}{\sqrt{m}}-\frac{Y_mK_1^b}{\sqrt{m}}\Big).\end{aligned} \tag{5.64}$$

The last expression converges, by virtue of the Slutzky's lemma, either to a Normal distribution or to a Dirac mass at 0, depending on whether the limit in (5.62) is positive or is zero. To see this, notice that

$$\begin{aligned}\lim_{m\to\infty}\sqrt{\frac{m}{\tau_{N_{m+1}}}}&=\sqrt{\frac{1}{\mathbb{E}[\tau_{N_2}-\tau_{N_1}]}},\qquad\text{a.s.}\\\frac{\sum_{i=1}^m Y_i}{\sqrt{m}}&\Longrightarrow\text{Normal}(0,C)\\\lim_{m\to\infty}\frac{Y_mK_1^b}{\sqrt{m}}&=0,\qquad\text{a.s..}\end{aligned}$$

Similarly

$$\frac{|X_t| - K_b^{(1)} t}{K_b^{(2)} \sqrt{t}} \leq \sqrt{\frac{m+1}{\tau_{N_m}}} \Big(\frac{\sum_{i=1}^{m+1} Y_i}{\sqrt{m+1}} + \frac{Y_{m+1} K_1^b}{\sqrt{m}} \Big),$$

and the right-hand side converges to the same limit of the right-hand side of (5.64). □

6 Appendix

We include a corollary to a result of Hoeffding and Robbins (see [12] or [22]).

Theorem 6.1 (Hoeffding-Robbins). *Suppose* $\mathbf{Y} := \{Y_i,\ i \geq 1\}$ *is a one-dependent process whose components are identically distributed with mean 0. If*

- $\mathbb{E}[Y_i^{2+\delta}] < \infty$, *for some* $\delta > 0$,
- $\lim_{n\to\infty} \frac{1}{n} \mathrm{Var}(\sum_{i=1}^n Y_i)$ *converges to a positive finite constant K, then*

$$\frac{\sum_{i=1}^n Y_i - n\mathbb{E}[Y_1]}{K\sqrt{n}} \Longrightarrow \mathrm{Normal}(0,1).$$

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