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Intermittence and nonlinear parabolic stochastic partial differential equations*

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Abstract

We consider nonlinear parabolic SPDEs of the form $\partial_t u = \mathscr{L} u + \sigma(u)\dot{w}$, where $\dot{w}$ denotes space-time white noise, $\sigma : \mathbf{R} \to \mathbf{R}$ is [globally] Lipschitz continuous, and $\mathscr{L}$ is the L^2-generator of a Lévy process. We present precise criteria for existence as well as uniqueness of solutions. More significantly, we prove that these solutions grow in time with at most a precise exponential rate. We establish also that when σ is globally Lipschitz and asymptotically sublinear, the solution to the nonlinear heat equation is "weakly intermittent," provided that the symmetrization of $\mathscr{L}$ is recurrent and the initial data is sufficiently large.

Among other things, our results lead to general formulas for the upper second-moment Liapounov exponent of the parabolic Anderson model for $\mathscr{L}$ in dimension $(1+1)$. When $\mathscr{L} = \kappa\partial_{xx}$ for $\kappa > 0$, these formulas agree with the earlier results of statistical physics [28; 32; 33], and also probability theory [1; 5] in the two exactly-solvable cases. That is when $u_0 = \delta_0$ or $u_0 \equiv 1$; in those cases the moments of the solution to the SPDE can be computed [1].

Key words: Stochastic partial differential equations, Lévy processes, Liapounov exponents, weak intermittence, the Burkholder–Davis–Gundy inequality.

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1 Introduction

Let $\{\dot{w}(t,x)\}_{t\ge 0, x\in\mathbf{R}}$ denote space-time white noise, and $\sigma:\mathbf{R}\to\mathbf{R}$ be a fixed Lipschitz function. Presently we study parabolic stochastic partial differential equations [SPDEs] of the following type:

$$\left|\begin{aligned}&\partial_t u(t,x) = (\mathscr{L}u)(t,x) + \sigma(u(t,x))\dot{w}(t,x),\\ &u(0,x) = u_0(x),\end{aligned}\right. \tag{1.1}$$

where $t\ge 0$, $x\in\mathbf{R}$, u_0 is a measurable and nonnegative initial function, and $\mathscr{L}$ is the $L^2(\mathbf{R})$-generator of a Lévy process $X:=\{X_t\}_{t\ge 0}$; and $\mathscr{L}$ acts only on the variable x. We follow Walsh [39] and interpret (1.1) as an Itô-type stochastic PDE. Also, we normalize X so that $\mathrm{E}\exp(i\xi X_t) = \exp(-t\Psi(\xi))$ for all $t\ge 0$ and $\xi\in\mathbf{R}$; $\mathscr{L}$ is described via its Fourier multiplier as $\hat{\mathscr{L}}(\xi) = -\Psi(\xi)$ for all $\xi\in\mathbf{R}$. See the books by Bertoin [2] and Jacob [26] for pedagogic accounts.

Our principal aim is to study the mild solutions of (1.1), when they exist. At this point in time, we understand (1.1) only when its linearization with vanishing inital data has a strong solution. Together with E. Nualart [19], we have investigated precisely those linearized equations. That is,

$$\left|\begin{aligned}&\partial_t u(t,x) = (\mathscr{L}u)(t,x) + \dot{w}(t,x),\\ &u(0,x) = 0.\end{aligned}\right. \tag{1.2}$$

And we proved among other things that (1.2) has a strong solution if and only if Paul Lévy's symmetrization $\bar{X}$ of the process X has local times, where

$$\bar{X}_t := X_t - X'_t \qquad \text{for all } t\ge 0, \tag{1.3}$$

and $X' := \{X'_t\}_{t\ge 0}$ is an independent copy of X. In fact, much of the local-time theory of symmetric 1-dimensional Lévy processes can be embedded within the analysis of SPDEs defined by (1.2); see [19] for details. We also proved in [19] that, as far as matters of existence and regularity are concerned, one does not encounter new phenomena if one adds to (1.2) Lipschitz-continuous additive nonlinearities [that is, if $\mathscr{L}u$ were replaced by $\mathscr{L}u + b(u)$ for a Lipschitz-continuous and bounded function $b:\mathbf{R}\to\mathbf{R}$]. This is why we consider only multiplicative nonlinearities in (1.1).

Let Lip_σ denote the Lipschitz constant of σ, and recall that u_0 is the initial data in (1.1). Here and throughout we assume, without further mention, that:

(i) $0 < \mathrm{Lip}_\sigma < \infty$, so that σ is [globally] Lipschitz and nontrivial; and

(ii) u_0 is bounded, nonnegative, and measurable.

Under these conditions, we prove that the SPDE (1.1) has a mild solution $u := \{u(t,x)\}_{t\ge 0, x\in\mathbf{R}}$ that is unique up to a modification. More significantly, we show that the growth of $t\mapsto u(t,x)$ is tied closely with the existence of u. With this aim in mind we choose and fix some $x_0\in\mathbf{R}$ define the *upper pth-moment Liapounov exponent* $\bar{\gamma}(p)$ of u [at x_0] as

$$\bar{\gamma}(p) := \limsup_{t\to\infty} \frac{1}{t}\ln \mathrm{E}\left(\left|u(t,x_0)\right|^p\right) \quad \text{for all } p\in(0,\infty). \tag{1.4}$$

It is possible to prove that when u_0 is a constant, $\bar{\gamma}$ does not depend on the value of x_0. But it does, in general. However, we suppress the dependence of $\bar{\gamma}$ on x_0, since we plan to derive inequalities that hold uniformly over all $x_0\in\mathbf{R}$.

Let us say that u is *weakly intermittent*[1] if, regardless of the value of x_0,

$$\bar{\gamma}(2) > 0 \quad \text{and} \quad \bar{\gamma}(p) < \infty \quad \text{for all } p > 2. \tag{1.5}$$

We are interested primarily in establishing weak intermittence. However, let us mention also that weak intermittence can sometimes imply the much better-known notion of *full intermittency* [5, Definition III.1.1, p. 55]; the latter is the property that, regardless of the value of x_0,

$$p \mapsto \frac{\bar{\gamma}(p)}{p} \quad \text{is strictly increasing for all } p \geq 2. \tag{1.6}$$

Here is a brief justification: Evidently, $\bar{\gamma}$ is convex and zero at zero, and hence $p \mapsto \bar{\gamma}(p)/p$ is nondecreasing. Convexity implies readily that if in addition $\bar{\gamma}(1) = 0$, then (1.5) implies (1.6).[2] On the other hand, a sufficient condition for $\bar{\gamma}(1) = 0$ is that $u(t\,,x) \geq 0$ a.s. for all $t > 0$ and $x \in \mathbf{R}$; for then, (3.5) below shows immediately that $\mathrm{E}(|u(t\,,x)|) = \mathrm{E}[u(t\,,x)]$ is bounded uniformly in t. We have proved the following: "*Whenever one has a comparison principle—such as that of Mueller [36] in the case that $\mathscr{L} = \kappa\partial_{xx}$ and $\sigma(x) = \lambda x$—weak intermittence necessarily implies full intermittency*."

Here, we do not pursue comparison principles. Rather, the principal goal of this note is to demonstrate that under various nearly-optimal conditions on σ and u_0, the solution u to (1.1) is weakly intermittent.

There is a big literature on intermittency that investigates the special case of (1.1) with $\mathscr{L} = \kappa\partial_{xx}$ and $\sigma(z) = \lambda z$ for constants $\kappa > 0$ and $\lambda \in \mathbf{R}$; that is the *parabolic Anderson model*. See, for example, [1; 5; 28; 32; 33; 35], together with their sizable combined references. The existing rigorous intermittency results all begin with a probabilistic formulation of (1.1) in terms of the Feynman–Kac formula. Presently, we introduce an analytic method that shows clearly that weak intermittence is connected intimately with the facts that: (i) (1.1) has a strong solution; and (ii) σ has linear growth, in one form or another. Our method is motivated very strongly by the theory of optimal regularity for analytic semigroups [34].

We would like to mention also that there is an impressive body of recent mathematical works on other Anderson models and $L^p(\mathrm{P})$ intermittency, as well as almost-sure intermittency [7; 6; 8; 10; 11; 16; 18; 20; 21; 24; 22; 27; 31; 38, and their combined references].

A brief outline follows: In §2 we state the main results of the paper; these results are proved subsequently in §4, after we establish some a priori bounds in §3. Finally, we show in Appendix A that if the initial data is continuous, then the solution to (1.1) is continuous in probability, in fact continuous in $L^p(\mathrm{P})$ for all $p > 0$. Consequently, if u_0 is continuous, then u has a separable modification. As an immediate byproduct of our proof we find that when $\mathscr{L}$ is the fractional Laplacian of index $\alpha \in (1\,,2]$ and u_0 is continuous, u has a jointly Hölder-continuous modification (Example A.6).

2 Main results

We combine a result of Dalang [14, Theorem 13] with a theorem of Hawkes [25] to deduce that (1.2) has a strong solution if and only if $\Upsilon(\beta) < \infty$ for some $\beta > 0$, where

$$\Upsilon(\beta) := \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{\beta + 2\mathrm{Re}\,\Psi(\xi)} \qquad \text{for all } \beta > 0. \tag{2.1}$$

[1] Our notion (1.5) of weak intermittence differs from that of [20, Definition 1.2].

[2] Inspect the proof of Theorem III.1.2 in Carmona and Molchanov [5, p. 55] for example.

See also [19]. Furthermore, $\Upsilon(\beta)$ is finite for some $\beta > 0$ if and only if it is finite for all $\beta > 0$. And under this integrability condition, (1.2) has a unique solution as well. For related results, see Brzeźniak and van Neerven [3].

Motivated by the preceding remarks, we consider only the case that the linearized equation (1.2) has a strong solution. That is, we suppose here and throughout that $\Upsilon(\beta) < \infty$ for all $\beta > 0$. We might note that Υ is decreasing, $\Upsilon(\beta) > 0$ for all $\beta > 0$, and $\lim_{\beta\uparrow\infty}\Upsilon(\beta) = 0$.

Our next result establishes natural conditions for: (i) the existence and uniqueness of a solution to (1.1); and (ii) u to grow at most exponentially with a sharp exponent. It is possible to adapt the Hilbert-space methods of Peszat and Zabczyk [37] to derive existence and uniqueness. See also Da Prato [12] and Da Prato and Zabczyk [13]. The theory of Dalang [14] produces the desired existence and uniqueness in the case that u_0 is a constant. And Dalang and Mueller [15] establish existence and uniqueness when u_0 is in a suitable Sobolev space.

Presently, we devise a method that shows very clearly that exponential growth is a consequence of the existence of a solution, provided that u_0 is bounded and measurable. Moreover, our method yields constants that will soon be shown to be essentially unimproveable.

Henceforth, by a "solution" to (1.1) we mean a mild solution u that satisfies the following:

$$\sup_{x\in\mathbf{R}}\sup_{t\in[0,T]} \mathrm{E}\left(|u(t\,,x)|^2\right) < \infty \quad \text{for all } T > 0. \tag{2.2}$$

It turns out that solutions to (1.1) have better *a priori* integrability features. The following quantifies this remark.

Theorem 2.1. *Equation* (1.1) *has a solution u that is unique up to a modification. Moreover, for all even integers $p \ge 2$,*

$$\bar{\gamma}(p) \le \inf\left\{\beta > 0 : \Upsilon\left(\frac{2\beta}{p}\right) < \frac{1}{(z_p \mathrm{Lip}_\sigma)^2}\right\} < \infty, \tag{2.3}$$

where z_p denotes the largest positive zero of the Hermite polynomial He_p.

Remark 2.2. We recall that $He_p(x) = 2^{-p/2}H_p(x/2^{1/2})$ for all $p > 0$ and $x \in \mathbf{R}$, where $\exp(-2xt - t^2) = \sum_{k=0}^{\infty} H_k(x)t^k/k!$ for all $t > 0$ and $x \in \mathbf{R}$. It is not hard to verify that

$$z_2 = 1 \quad \text{and} \quad z_4 = \sqrt{3+\sqrt{6}} \approx 2.334. \tag{2.4}$$

This is valid simply because $He_2(x) = x^2 - 1$ and $He_4(x) = x^4 - 6x^2 + 3$. In addition, $z_p \sim 2p^{1/2}$ as $p \to \infty$, and $\sup_{p\ge1}(z_p/p^{1/2}) = 2$; see Carlen and Kree [4, Appendix]. □

Before we explore the sharpness of (2.3), let us examine two cases that exhibit nonintermittence, in fact *subexponential growth*. The first concerns *subdiffusive growth*.

Proposition 2.3. *If u_0 and σ are bounded and measurable, then for all integers $p \ge 2$,*

$$\mathrm{E}\left(|u(t\,,x)|^p\right) = o\left(t^{p/2}\right) \qquad \textit{as } t \to \infty. \tag{2.5}$$

Remark 2.4. The preceding is close to optimal; for instance, when $p = 2$, the "$o(t)$" cannot in general be improved to "$o(t^\rho)$" for any $\rho < 1/2$. Indeed, consider the case that $\mathscr{L} = -(-\Delta)^{\alpha/2}$ is the fractional Laplacian. It is easy to see that $\Upsilon(\beta) < \infty$ for some $\beta > 0$ iff $\alpha \in (1\,,2]$. If $0 < \inf_{z\in\mathbf{R}}|\sigma(z)| \le \sup_{z\in\mathbf{R}}|\sigma(z)| < \infty$, then $\mathrm{E}(|u(t\,,x)|^2)$ is bounded above and below by constant multiples of $t^{(\alpha-1)/\alpha}$. We omit the details. □

For our second proposition we first recall the symmetrized Lévy process $\bar{X}$ from (1.3).

Proposition 2.5. *If $\bar{X}$ is transient, then for all integers $p \geq 2$ there exists $\delta(p) > 0$ such that $\bar{\gamma}(p) = 0$ whenever* $\mathrm{Lip}_\sigma < \delta(p)$*.*

Example 2.6. The conditions of Proposition 2.5 are not vacuous. For instance, $\Psi(\xi) = |\xi|^\alpha + |\xi|^\rho$ is the exponent of a symmetric Lévy process $\bar{X}$. Moreover, if $\alpha \in (0,1)$ and $\rho \in (1,2]$, then $\bar{X}$ is transient and has local times. □

Our next result addresses the sharpness of (2.3), and establishes an easy-to-check sufficient criterion for u to be weakly intermittent. Throughout, Υ^{-1} denotes the inverse to Υ in the following sense:

$$\Upsilon^{-1}(t) := \sup\{\beta > 0 : \ \Upsilon(\beta) > t\}, \tag{2.6}$$

where $\sup\varnothing := 0$.

Theorem 2.7. *If $\inf_{z\in\mathbf{R}} u_0(z) > 0$ and $q := \inf_{x\neq 0} |\sigma(x)/x| > 0$, then*

$$\bar{\gamma}(2) \geq \Upsilon^{-1}\left(\frac{1}{q^2}\right) > 0. \tag{2.7}$$

Our next result is a ready corollary of Theorems 2.1 and 2.7; see Carmona and Molchanov [5, p. 59], Cranston and Molchanov [9], and Gärtner and den Hollander [20] for phenomenologically-similar results. It might help to recall (1.3).

Corollary 2.8. *If $\sigma(x) := \lambda x$ and $\inf_{x\in\mathbf{R}} u_0(x) > 0$, then:*

1. *If $\bar{X}$ is recurrent, then u is weakly intermittent;*
2. *If $\bar{X}$ is transient, then u is weakly intermittent if and only if $\Upsilon(\beta) \geq \lambda^{-2}$ for some $\beta > 0$; and*
3. *In all the cases that u is weakly intermittent, $\bar{\gamma}(2) = \Upsilon^{-1}(\lambda^{-2})$.*

Even though Corollary 2.8 is concerned with a very special case of (1.1), that special case has a rich history. Indeed, Corollary 2.8 contains a moment analysis of the socalled *parabolic Anderson model* for $\mathscr{L}$. When $\mathscr{L} = \kappa\partial_{xx}$, that equation arises in the analysis of branching processes in random environment [5; 35]. If the spatial motion is a Lévy process with generator $\mathscr{L}$, then we arrive at (1.1) with $\sigma(x) = \lambda x$. For somewhat related—though not identical—reasons, the parabolic Anderson model also paves the way for a mathematical understanding of the socalled "KPZ equation" in dimension $(1+1)$. For further information see the original paper by Kardar, Parisi, and Zhang [29], Chapter 5 of Krug and Spohn [32], and the Introduction by Carmona and Molchanov [5].

Example 2.9. If the conditions of Corollary 2.8 hold, then the solution to (1.1) with $\mathscr{L} = -\kappa(-\Delta)^{\alpha/2}$ is weakly intermittent with

$$\bar{\gamma}(2) = \left(\frac{\nu^\alpha \lambda^{2\alpha}}{\kappa}\right)^{1/(\alpha-1)} \text{where} \quad \nu := \frac{\mathrm{cosec}(\pi/\alpha)}{2^{1/\alpha}\alpha}. \tag{2.8}$$

Of course, we need $\alpha \in (1,2]$, and this implies that $\bar{X}$ is recurrent; see Remark 2.4. In order to derive (2.8), we first recall that $\int_0^\infty \mathrm{d}x/(1+x^\alpha) = (\pi/\alpha)\mathrm{cosec}(\pi/\alpha)$ [23, **3.222**#2, p. 337]. Thus, a direct computation yields $\Upsilon(\beta) = \nu\kappa^{-1/\alpha}\beta^{-1+(1/\alpha)}$ for all $\beta > 0$. Corollary 2.8, and a few more simple calculations, together imply (2.8). A similar argument shows that

$$\bar{\gamma}(p) \le \frac{p}{2}\left(\frac{\nu^\alpha}{\kappa}\left(z_p\lambda\right)^{2\alpha}\right)^{1/(\alpha-1)} \qquad \text{for all even integers } p \ge 2. \tag{2.9}$$

We can use this in conjunction with the Carlen–Kree inequality [$z_p \le 2\sqrt{p}$; see Remark 2.2] to obtain explicit numerical bounds. □

In the special case that $\mathscr{L} = \kappa\partial_{xx}$, Example 2.9 tells that $\bar{\gamma}(2) = \lambda^4/(8\kappa)$, regardless of the value of $x_0 \in \mathbf{R}$. This formula is anticipated by the earlier investigations of Lieb and Liniger [33] and Kardar [28, Eq. (2.9)] in statistical physics; it can also be deduced upon combining the results of Bertini and Cancrini [1], in the exact case $u_0 \equiv 1$, with Mueller's comparison principle [36]. Carmona and Molchanov [5, p. 59] study a closely-related parabolic Anderson model in which $\dot{w}(t\,,x)$ is white noise over $(t\,,x) \in \mathbf{R}_+ \times \mathbf{Z}^d$.

It is also easy to see that the bound furnished by (2.9) is nearly sharp in the case that $\alpha = 2$ and $p > 2$. For example, (2.9) and the Carlen–Kree inequality [$z_p \le 2\sqrt{p}$] together yield $\bar{\gamma}(p) \le \vartheta(p) := (p^3\lambda^4)/\kappa$, valid for all even integers $p \ge 2$. When $p \ge 2$ is an arbitrary integer, the exact answer is $\bar{\gamma}(p) = p(p^2-1)\lambda^4/(48\kappa)$ [1; 28; 33], and the lim sup in the definition of $\bar{\gamma}$ is a bona fide limit. Our bound $\vartheta(p)$ agrees well with the exact answer in this special case. Indeed,

$$1 \le \frac{\vartheta(p)}{\bar{\gamma}(p)} \le 48\left(1 + \frac{1}{p^2-1}\right), \tag{2.10}$$

uniformly for all even integers $p \ge 2$, as well as all $\lambda \in \mathbf{R}$ and $\kappa \in (0\,,\infty)$.

We close with a result that states roughly that if σ is asymptotically linear and $\bar{X}$ is recurrent, then a sufficiently large initial data will ensure intermittence. More precisely, we have the following.

Theorem 2.10. *Suppose $\bar{X}$ is recurrent, and $q := \liminf_{|x|\to\infty} |\sigma(x)/x| > 0$. Then, there exists $\eta_0 > 0$ such that whenever $\eta := \inf_{x\in\mathbf{R}} u_0(x) \ge \eta_0$, the solution u is weakly intermittent.*

We believe this result presents a notable improvement on the content of Theorem 2.7 in the case that $\bar{X}$ is recurrent.

3 A priori bounds

Before we prove the mathematical assertions of §2, let us develop some of the required background. Throughout we note the following elementary bound:

$$|\sigma(x)| \le |\sigma(0)| + \mathrm{Lip}_\sigma|x| \qquad \text{for all } x \in \mathbf{R}. \tag{3.1}$$

Define $\{\mathscr{P}_t\}_{t\ge 0}$ as the semigroup associated with $\mathscr{L}$. According to Lemma 8.1 of Foondun et al. [19], there exist transition densities $\{p_t\}_{t>0}$, whence we have

$$(\mathscr{P}_t g)(x) = \int_{-\infty}^{\infty} p_t(y-x)g(y)\,\mathrm{d}y \qquad (t > 0). \tag{3.2}$$

For us, the following relation is also significant: $(\mathscr{P}_t^* g)(x) = (\check{p}_t * g)(x)$, where $\mathscr{P}_t^*$ denotes the adjoint of $\mathscr{P}_t$ in $L^2(\mathbf{R})$, and $\check{p}_t(x) := p_t(-x)$.

Consider

$$(\mathscr{G}u_0)(t\,,x) := (\mathscr{P}_t u_0)(x) = (\check{p}_t * u_0)(x) \qquad \text{for all } t > 0 \text{ and } x \in \mathbf{R}. \tag{3.3}$$

We define also $(\mathscr{G}u_0)(0\,,x) := u_0(x)$ for all $x \in \mathbf{R}$. The function $v = \mathscr{G}u_0$ solves the nonrandom integro-differential equation

$$\left|\begin{array}{ll} \partial_t v = \mathscr{L} v & \text{on } (0\,,\infty) \times \mathbf{R}, \\ v(0\,,x) = u_0(x) & \text{for all } x \in \mathbf{R}. \end{array}\right. \tag{3.4}$$

Thus, we can follow the terminology and methods of Walsh [39] closely to deduce that (1.1) admits a mild solution u if and only if u is a predictable process that solves

$$u(t\,,x) = (\mathscr{G}u_0)(t\,,x) + \int_{-\infty}^{\infty} \int_0^t \sigma(u(s\,,y)) p_{t-s}(y-x)\, w(\mathrm{d}s\,\mathrm{d}y). \tag{3.5}$$

We begin by making two simple computations. The first is a basic potential-theoretic bound.

Lemma 3.1. *For all $\beta > 0$,*

$$\sup_{t>0} e^{-\beta t} \int_0^t \|p_s\|_{L^2(\mathbf{R})}^2\, \mathrm{d}s \leq \int_0^\infty e^{-\beta s} \|p_s\|_{L^2(\mathbf{R})}^2\, \mathrm{d}s = \Upsilon(\beta). \tag{3.6}$$

Proof. The inequality is obvious; we apply Plancherel's theorem to find that

$$\|p_s\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2s \operatorname{Re} \Psi(\xi)}\, \mathrm{d}\xi \qquad \text{for all } s > 0. \tag{3.7}$$

Therefore, Tonelli's theorem implies the remaining equality. □

Next we present our second elementary estimate.

Lemma 3.2. *For all $a, b \in \mathbf{R}$ and $\epsilon > 0$,*

$$(a+b)^2 \leq (1+\epsilon)a^2 + \left(1 + \epsilon^{-1}\right) b^2. \tag{3.8}$$

Proof. Define $h(\epsilon)$ to be the upper bound of the lemma. Then, $h : (0\,,\infty) \to \mathbf{R}_+$ is minimized at $\epsilon = |b/a|$, and the minimum value of h is $a^2 + 2|ab| + b^2$, which is in turn $\geq (a+b)^2$. □

Now we proceed to establish the remaining required estimates.

For every positive t and all Borel sets $A \subset \mathbf{R}$, we set $w_t(A) := \dot{w}([0\,,t] \times A)$, and let $\mathscr{F}_t$ denote the σ-algebra generated by all Wiener integrals of the form $\int g(x)\, w_s(\mathrm{d}x)$, as the function g ranges over $L^2(\mathbf{R})$ and the real number s ranges over $[0\,,t]$. Without loss of too much generality we may assume that the resulting filtration $\mathscr{F} := \{\mathscr{F}_t\}_{t\geq 0}$ satisfies the usual conditions, else we enlarge each $\mathscr{F}_t$ in the standard way. Here and throughout, a process is said to be *predictable* if it is predictable with respect to $\mathscr{F}$; see also Walsh [39, p. 292].

Given a predictable random field f, we define

$$(\mathscr{A}f)(t\,,x) := \int_{-\infty}^{\infty}\int_0^t \sigma(f(s\,,y))p_{t-s}(y-x)\,w(\mathrm{d}s\,\mathrm{d}y), \tag{3.9}$$

for all $t \geq 0$ and $x \in \mathbf{R}$, provided that the stochastic integral exists in the sense of Walsh [39]. We also define a family of p-norms $\{\|f\|_{p,\beta}\}_{\beta>0}$, one for each integer $p \geq 2$, via

$$\|f\|_{p,\beta} := \left\{\sup_{t\geq 0}\sup_{x\in\mathbf{R}} e^{-\beta t}\,\mathrm{E}\left(\left|f(t\,,x)\right|^p\right)\right\}^{1/p}. \tag{3.10}$$

Variants of these norms appear in several places in the SPDE literature. See, in particular, Peszat and Zabczyk [37]. However, there is a subtle [but very important!] novelty here: The supremum is taken over all time.

Recall the definition of z_p from Theorem 2.1.

Lemma 3.3. *If f is predictable and $\|f\|_{p,\beta} < \infty$ for a real $\beta > 0$ and an even integer $p \geq 2$, then*

$$\|\mathscr{A}f\|_{p,\beta} \leq z_p\left(|\sigma(0)| + \mathrm{Lip}_\sigma\|f\|_{p,\beta}\right)\sqrt{\Upsilon\left(\frac{2\beta}{p}\right)}. \tag{3.11}$$

Proof. In his seminal 1976 paper [17], Burgess Davis found the optimal constants in the Burkholder–Davis–Gundy [BDG] inequality. In particular, Davis proved that for all $t \geq 0$ and $p \geq 2$,

$$z_p = \sup\left\{\frac{\|N_t\|_{L^p(\mathrm{P})}}{\|\langle N\,,N\rangle_t\|_{L^{p/2}(\mathrm{P})}^{1/2}} : N \in \mathfrak{M}_p\right\}, \tag{3.12}$$

where $0/0 := 0$ and $\mathfrak{M}_p$ denotes the collection of all continuous $L^p(\mathrm{P})$-martingales. We apply Davis's form of the BDG inequality [*loc. cit.*], and find that

$$\begin{aligned}&\|(\mathscr{A}f)(t\,,x)\|_{L^p(\mathrm{P})}^p\\ &\quad\leq z_p^p\,\mathrm{E}\left(\left|\int_{-\infty}^{\infty}\mathrm{d}y\int_0^t\mathrm{d}s\,\left|\sigma(f(s\,,y))\right|^2\left|p_{t-s}(y-x)\right|^2\right|^{p/2}\right).\end{aligned} \tag{3.13}$$

Since $p/2$ is a positive integer, the preceding expectation can be written as

$$\mathrm{E}\left(\prod_{j=1}^{p/2}\int_{-\infty}^{\infty}\mathrm{d}y_j\int_0^t\mathrm{d}s_j\,\left|\sigma(f(s_j\,,y_j))\right|^2\left|p_{t-s_j}(y_j-x)\right|^2\right). \tag{3.14}$$

The generalized Hölder inequality tells us that

$$\mathrm{E}\left(\prod_{j=1}^{p/2}\left|\sigma(f(s_j\,,y_j))\right|^2\right) \leq \prod_{j=1}^{p/2}\left\|\sigma(f(s_j\,,y_j))\right\|_{L^p(\mathrm{P})}^2. \tag{3.15}$$

Therefore, a little algebra shows us that

$$\|(\mathscr{A}f)(t\,,x)\|_{L^p(\mathrm{P})}^2 \le z_p^2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s\, \left\|\sigma(f(s\,,y))\right\|_{L^p(\mathrm{P})}^2 \left|p_{t-s}(y-x)\right|^2. \tag{3.16}$$

Owing to (3.1) and Minkowski's inequality,

$$\begin{aligned} &\|(\mathscr{A}f)(t\,,x)\|_{L^p(\mathrm{P})}^2 \\ &\quad\le z_p^2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s\, \left(c_0 + c_1\|f(s\,,y)\|_{L^p(\mathrm{P})}\right)^2 \left|p_{t-s}(y-x)\right|^2, \end{aligned} \tag{3.17}$$

where $c_0 := |\sigma(0)|$ and $c_1 := \mathrm{Lip}_\sigma$, for brevity. Therefore, Lemmas 3.1 and 3.2 together imply the following bound, valid for all $\epsilon, \beta > 0$:

$$\begin{aligned} \|(\mathscr{A}f)(t\,,x)\|_{L^p(\mathrm{P})}^2 \le &\left(1+\epsilon^{-1}\right) z_p^2 c_0^2 e^{2\beta t/p}\Upsilon\left(\frac{2\beta}{p}\right) \\ &+ (1+\epsilon) z_p^2 c_1^2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s\, \|f(s\,,y)\|_{L^p(\mathrm{P})}^2 \left|p_{t-s}(y-x)\right|^2. \end{aligned} \tag{3.18}$$

Because $\|f(s\,,y)\|_{L^p(\mathrm{P})}^2 \le \exp(2\beta s/p)\|f\|_{p,\beta}^2$, it follows that

$$\begin{aligned} &\|(\mathscr{A}f)(t\,,x)\|_{L^p(\mathrm{P})}^2 \\ &\le \left(1+\epsilon^{-1}\right) z_p^2 c_0^2 e^{2\beta t/p}\Upsilon\left(\frac{2\beta}{p}\right) \\ &\quad + (1+\epsilon) z_p^2 c_1^2 e^{2\beta t/p}\|f\|_{p,\beta}^2 \int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s\, e^{-2\beta s/p} \left|p_s(y-x)\right|^2 \\ &\le \left(1+\epsilon^{-1}\right) z_p^2 c_0^2 e^{2\beta t/p}\Upsilon\left(\frac{2\beta}{p}\right) + (1+\epsilon) z_p^2 c_1^2 e^{2\beta t/p}\|f\|_{p,\beta}^2 \Upsilon\left(\frac{2\beta}{p}\right). \end{aligned} \tag{3.19}$$

See Lemma 3.1 for the final inequality. We multiply both sides by $\exp(-2\beta t/p)$ and optimize over $t \ge 0$ and $x \in \mathbf{R}$ to deduce the estimate

$$\|\mathscr{A}f\|_{p,\beta}^2 \le z_p^2 \left\{ \left(1+\epsilon^{-1}\right) |\sigma(0)|^2 + (1+\epsilon)\mathrm{Lip}_\sigma^2 \|f\|_{p,\beta}^2 \right\} \Upsilon\left(\frac{2\beta}{p}\right). \tag{3.20}$$

The preceding is valid for all $\epsilon > 0$. Now we choose

$$\epsilon := \begin{cases} |\sigma(0)|/(\mathrm{Lip}_\sigma \|f\|_{p,\beta}) & \text{if } |\sigma(0)|\cdot\|f\|_{p,\beta} > 0, \\ 0 & \text{if } \sigma(0) = 0, \\ \infty & \text{if } \|f\|_{p,\beta} = 0, \end{cases} \tag{3.21}$$

to arrive at the statement of the lemma. Of course, "$\epsilon = \infty$" means "send $\epsilon \to \infty$" in the preceding. □

We plan to carry out a fixed-point argument in order to prove Theorem 2.1. The following result shows that the stochastic-integral operator $f \mapsto \mathscr{A}f$ is a contraction on suitably-chosen spaces.

Lemma 3.4. *Choose and fix an even integer $p \geq 2$. For every $\beta > 0$, and all predictable random fields f and g that satisfy $\|f\|_{p,\beta} + \|g\|_{p,\beta} < \infty$,*

$$\|\mathscr{A}f - \mathscr{A}g\|_{p,\beta} \leq z_p \mathrm{Lip}_\sigma \sqrt{\Upsilon\left(\frac{2\beta}{p}\right)} \|f - g\|_{p,\beta}. \tag{3.22}$$

Proof. The proof is a variant of the preceding argument. Namely,

$$\begin{aligned} &\mathrm{E}\left(\left|(\mathscr{A}f)(t,x) - (\mathscr{A}g)(t,x)\right|^p\right) \\ &\leq z_p^p \mathrm{E}\left(\left|\int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s \left|\sigma(f(s,y)) - \sigma(g(s,y))\right|^2 \left|p_{t-s}(y-x)\right|^2\right|^{p/2}\right) \\ &\leq \left(z_p \mathrm{Lip}_\sigma\right)^p \mathrm{E}\left(\left|\int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s \left|f(s,y) - g(s,y)\right|^2 \left|p_{t-s}(y-x)\right|^2\right|^{p/2}\right). \end{aligned} \tag{3.23}$$

We write the expectation as

$$\mathrm{E}\left(\prod_{j=1}^{p/2} \int_{-\infty}^{\infty} \mathrm{d}y_j \int_0^t \mathrm{d}s_j \left|f(s_j, y_j) - g(s_j, y_j)\right|^2 \left|p_{t-s_j}(y_j - x)\right|^2\right), \tag{3.24}$$

and apply (3.15) to obtain the bound

$$\begin{aligned} &\mathrm{E}\left(\left|(\mathscr{A}f)(t,x) - (\mathscr{A}g)(t,x)\right|^p\right) \\ &\leq \left(z_p \mathrm{Lip}_\sigma\right)^p \left(\int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s \, \|f(s,y) - g(s,y)\|_{L^p(\mathrm{P})}^2 \left|p_{t-s}(y-x)\right|^2\right)^{p/2} \\ &\leq \left(z_p \mathrm{Lip}_\sigma\right)^p \|f - g\|_{p,\beta}^p e^{\beta t} \left(\int_{-\infty}^{\infty} \mathrm{d}y \int_0^t \mathrm{d}s \, e^{-2\beta s/p} \left|p_s(y-x)\right|^2\right)^{p/2}. \end{aligned} \tag{3.25}$$

This has the desired effect; see Lemma 3.1. □

4 Proofs of the main results

Proof of Theorem 2.1. Define $v_0(t,x) := u_0(x)$ for all $(t,x) \in \mathbf{R}_+ \times \mathbf{R}$. Since u_0 is assumed to be bounded, $\|v_0\|_{p,\beta} < \infty$ for all $\beta > 0$ and all even integers $p \geq 2$. Now we iteratively set

$$v_{n+1}(t,x) := (\mathscr{A}v_n)(t,x) + (\mathscr{G}u_0)(t,x) \qquad \text{for all } n \geq 0. \tag{4.1}$$

If we set $\mathscr{A}v_{-1} := v_0$, then thanks to Lemma 3.3, for all $n \geq -1$,

$$\|\mathscr{A}v_{n+1}\|_{p,\beta} \leq z_p \left(|\sigma(0)| + \mathrm{Lip}_\sigma \|\mathscr{A}v_n\|_{p,\beta}\right) \sqrt{\Upsilon\left(\frac{2\beta}{p}\right)}. \tag{4.2}$$

Since $\lim_{\beta\to\infty}\Upsilon(\beta)=0$, we can always choose and fix $\beta>0$ such that

$$z_p^2\mathrm{Lip}_\sigma^2\,\Upsilon\left(\frac{2\beta}{p}\right)<1. \tag{4.3}$$

Given such a β we find, after a few lines of computation, that

$$\sup_{n\geq 0}\|\mathscr{A}v_n\|_{p,\beta}\leq\frac{z_p|\sigma(0)|\sqrt{\Upsilon(2\beta/p)}}{1-z_p\mathrm{Lip}_\sigma\sqrt{\Upsilon(2\beta/p)}}. \tag{4.4}$$

Because $\mathscr{G}u_0$ is bounded uniformly by $\sup_{z\in\mathbf{R}}u_0(z)$, the preceding yields

$$\sup_{k\geq 1}\|v_k\|_{p,\beta}\leq\frac{z_p|\sigma(0)|\sqrt{\Upsilon(2\beta/p)}}{1-z_p\mathrm{Lip}_\sigma\sqrt{\Upsilon(2\beta/p)}}+\sup_{z\in\mathbf{R}}|u_0(z)|, \tag{4.5}$$

which is finite. Consequently, Lemma 3.4 assures us that all $n\geq 1$,

$$\begin{aligned}\|v_{n+1}-v_n\|_{p,\beta}&=\|\mathscr{A}v_n-\mathscr{A}v_{n-1}\|_{p,\beta}\\&\leq z_p\mathrm{Lip}_\sigma\sqrt{\Upsilon\left(\frac{2\beta}{p}\right)}\,\|v_n-v_{n-1}\|_{p,\beta}.\end{aligned} \tag{4.6}$$

Because of (4.3), this proves the existence of a predictable random field u such that $\lim_{n\to\infty}\|v_n-u\|_{p,\beta}=\lim_{n\to\infty}\|\mathscr{A}v_n-\mathscr{A}u\|_{p,\beta}=0$. Consequently, $\|u\|_{p,\beta}<\infty$, $\|u-\mathscr{A}u-\mathscr{G}u_0\|_{p,\beta}=0$, and

$$\mathrm{E}\left(\left|u(t\,,x)-(\mathscr{A}u)(t\,,x)-(\mathscr{G}u_0)(t\,,x)\right|^p\right)=0\quad\text{for all }(t\,,x)\in\mathbf{R}_+\times\mathbf{R}. \tag{4.7}$$

These remarks prove all but one of the assertions of the theorem; we still need to establish that u is unique up to a modification. For that we follow the methods of Da Prato [12], Da Prato and Zabczyk [13], and especially Peszat and Zabczyk [37]: Suppose there are two solutions u and $\bar u$ to (1.1). Define for all predictable random fields f, and $T>0$,

$$\|f\|_{2,\beta,T}:=\left\{\sup_{t\in[0,T]}\sup_{x\in\mathbf{R}}e^{-\beta t}\mathrm{E}\left(\left|f(t\,,x)\right|^2\right)\right\}^{1/2}. \tag{4.8}$$

Then, we can easily modify the proof of Lemma 3.4, using also the fact that $z_2=1$ [Remark 2.2], to deduce that if u and $\bar u$ are two solutions to (1.1), then the following holds for all $T>0$:

$$\begin{aligned}\|u-\bar u\|_{2,\beta,T}&=\|\mathscr{A}u-\mathscr{A}\bar u\|_{2,\beta,T}\\&\leq\mathrm{Lip}_\sigma\sqrt{\Upsilon(\beta)}\,\|u-\bar u\|_{2,\beta,T}.\end{aligned} \tag{4.9}$$

Because $\Upsilon(\beta)$ vanishes as β tends to infinity, this proves that $\|u-\bar u\|_{2,\beta,T}=0$ for all $T>0$ and all sufficiently large $\beta>0$. This implies that u and $\bar u$ are modifications of one another. ☐

Proof of Proposition 2.3. Because $c_4:=\sup_{x\in\mathbf{R}}(|\sigma(x)|\vee|u_0(x)|)<\infty$, the Burkholder–Davis–Gundy implies that

$$\|u(t\,,x)\|_{L^p(\mathrm{P})}\leq c_4+c_4z_p\left(\int_0^t\|p_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s\right)^{1/2}. \tag{4.10}$$

Therefore, it suffices to prove that

$$\int_0^t \|p_s\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s = o(t) \quad \text{as } t \to \infty. \tag{4.11}$$

The left-most term is equal to $t\int_0^1 \|p_{st}\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s$. According to (3.7), the map $s \mapsto \|p_s\|_{L^2(\mathbf{R})}^2$ is non-increasing, and $\lim_{s\to\infty}\|p_s\|_{L^2(\mathbf{R})} = 0$ by the dominated convergence theorem. Therefore, a second appeal to the dominated convergence theorem yields (4.11) and hence the theorem. □

Proof of Theorem 2.7. We aim to prove that

$$\int_0^\infty e^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)\mathrm{d}t = \infty \quad \text{provided that } \Upsilon(\beta) \geq q^{-2}. \tag{4.12}$$

This implies (2.7), as the following argument shows: Suppose, to the contrary, that $\mathrm{E}(|u(t\,,x)|^2) = O(\exp(\alpha t))$ as $t \to \infty$, where $\Upsilon(\alpha) > q^{-2}$ and $x \in \mathbf{R}$. It follows from this that

$$\int_0^\infty e^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)\mathrm{d}t \leq \text{const}\cdot\int_0^\infty e^{-(\beta-\alpha)t}\,\mathrm{d}t, \tag{4.13}$$

and this is finite for every $\beta \in (\alpha\,,\Upsilon^{-1}(q^{-2}))$. Our finding contradicts (4.12), and thence follows (2.7). It remains to establish (4.12).

Let us introduce the following notation:

$$\begin{aligned} F_\beta(x) &:= \int_0^\infty e^{-\beta t}\mathrm{E}\left(|u(t\,,x)|^2\right)\mathrm{d}t \\ G_\beta(x) &:= \int_0^\infty e^{-\beta t}\left|(\tilde{p}_t * u_0)(x)\right|^2\mathrm{d}t \\ H_\beta(x) &:= \int_0^\infty e^{-\beta t}\left|p_t(x)\right|^2\mathrm{d}t. \end{aligned} \tag{4.14}$$

Because

$$\begin{aligned} &\mathrm{E}\left(|u(t\,,x)|^2\right) \\ &\quad= \left|(\tilde{p}_t * u_0)(x)\right|^2 + \int_{-\infty}^\infty \mathrm{d}y\int_0^t \mathrm{d}s\,\mathrm{E}\left(\left|\sigma(u(s\,,y))\right|^2\right)\left|p_{t-s}(y-x)\right|^2, \end{aligned} \tag{4.15}$$

we may apply Laplace transforms to both sides, and then deduce that for all $\beta > 0$ and $x \in \mathbf{R}$,

$$F_\beta(x) = G_\beta(x) + \int_{-\infty}^\infty \mathrm{d}y\,H_\beta(x-y)\int_0^\infty \mathrm{d}s\,e^{-\beta s}\mathrm{E}\left(\left|\sigma(u(s\,,y))\right|^2\right). \tag{4.16}$$

Because $|\sigma(z)|^2 \geq q^2|z|^2$ for all $z \in \mathbf{R}$, we are led to the following:

$$F_\beta(x) \geq G_\beta(x) + q^2(F_\beta * H_\beta)(x). \tag{4.17}$$

This is a “renewal inequation,” and can be solved by standard methods. We will spell that argument out carefully, since we need an enhanced version shortly: If we define the linear operator $\mathscr{H}$ by

$$(\mathscr{H}f)(x) := q^2 \left(H_\beta * f\right)(x), \tag{4.18}$$

then we can deduce that $\mathscr{H}^n F_\beta - \mathscr{H}^{n+1} F_\beta \geq \mathscr{H}^n G_\beta$, pointwise, for all integers $n \geq 0$. We sum this inequality from $n = 0$ to $n = N$ and find that

$$\begin{aligned} F_\beta(x) &\geq \left(\mathscr{H}^{N+1} F_\beta\right)(x) + \sum_{n=0}^{N} \left(\mathscr{H}^n G_\beta\right)(x) \\ &\geq \sum_{n=0}^{N} \left(\mathscr{H}^n G_\beta\right)(x). \end{aligned} \tag{4.19}$$

It follows, upon letting N tend to infinity, that

$$F_\beta(x) \geq \sum_{n=0}^{\infty} (\mathscr{H}^n G_\beta)(x). \tag{4.20}$$

If $\eta := \inf_x u_0(x)$, then $(\check{p}_t * u_0)(x) \geq \eta$ pointwise, and hence $G_\beta(x) \geq \eta^2/\beta$. Consequently,

$$\begin{aligned} (\mathscr{H} G_\beta)(x) &\geq \frac{q^2\eta^2}{\beta} \cdot \int_{-\infty}^{\infty} H_\beta(x)\,\mathrm{d}x \\ &= \frac{q^2\eta^2}{\beta} \cdot \Upsilon(\beta); \end{aligned} \tag{4.21}$$

consult Lemma 3.1 for the identity. We can iterate the preceding argument to deduce that $F_\beta(x) \geq \eta^2\beta^{-1} \sum_{n=0}^{\infty} (q^2\Upsilon(\beta))^n$, whence $F_\beta(x) = \infty$ as long as $\Upsilon(\beta) \geq q^{-2}$. This verifies (4.12), and concludes our proof. □

Proof of Proposition 2.5. We recall the well-known fact that

$$\bar{X} \text{ is recurrent if and only if } \quad \int_{-1}^{1} \frac{\mathrm{d}\xi}{\mathrm{Re}\,\Psi(\xi)} = \infty. \tag{4.22}$$

Otherwise, $\bar{X}$ is transient; see Exercise V.6 of Bertoin [2, p. 152]. Because $\Upsilon(\beta) < \infty$ for all $\beta > 0$, and since $\mathrm{Re}\,\Psi(\xi) \geq 0$, it is manifest that (4.22) is equivalent to the following:

$$\bar{X} \text{ is recurrent if and only if } \quad \lim_{\beta \downarrow 0} \Upsilon(\beta) = \infty. \tag{4.23}$$

Consequently, when $\bar{X}$ is transient, $\sup_{\beta>0} \Upsilon(\beta) = \lim_{\beta\downarrow 0} \Upsilon(\beta) < \infty$, and the proposition follows immediately from Theorem 2.1. In fact, we can choose $\delta(p)$ to be the reciprocal of $z_p\{\sup_{\beta>0} \Upsilon(\beta)\}^{1/2}$. □

Proof of Corollary 2.8. Thanks to (4.22), when $\bar{X}$ is recurrent, we can find $\beta > 0$ such that $\Upsilon(\beta) > 1/\lambda^2$. Theorem 2.7 implies the exponential growth of u, and the formula for $\bar{\gamma}(2)$ follows upon combining the quantitative bounds of Theorems 2.1 and 2.7. The case where $\bar{X}$ is transient is proved similarly. □

We close the paper with the following.

Proof of Theorem 2.10. We modify the proof of Theorem 2.7, and point out only the requisite changes. First of all, let us note that for all $q_0 \in (0,q)$ there exists $A = A(q_0) \in [0,\infty)$ such that $|\sigma(z)| \geq q_0|z|$ provided that $|z| > A$. Consequently, for all $s \in \mathbf{R}_+$ and $y \in \mathbf{R}$,

$$\begin{aligned}\mathrm{E}\left(\left|\sigma(u(s\,,y))\right|^2\right) &\geq q_0^2\mathrm{E}\left(\left|u(s\,,y)\right|^2;\, |u(s\,,y)| > A\right)\\ &\geq q_0^2\mathrm{E}\left(\left|u(s\,,y)\right|^2\right) - q_0^2A^2.\end{aligned} \tag{4.24}$$

Eq. (4.15) implies that $\mathrm{E}(|u(t\,,x)|^2)$ is bounded below by

$$\begin{aligned}\left|(\check{p}_t * u_0)(x)\right|^2 + q_0^2\int_{-\infty}^{\infty}\mathrm{d}y\int_0^t \mathrm{d}s\,\mathrm{E}\left(\left|u(s\,,y)\right|^2\right)\left|p_{t-s}(y-x)\right|^2&\\ - q_0^2A^2\int_0^t \left\|p_s\right\|_{L^2(\mathbf{R})}^2\,\mathrm{d}s.&\end{aligned} \tag{4.25}$$

We multiply both sides of the preceding display by $\exp(-\beta t)$, for a fixed $\beta > 0$, and integrate $[\mathrm{d}t]$ to find that

$$F_\beta(x) \geq G_\beta(x) + (\mathscr{H}F_\beta)(x) - \frac{q_0^2A^2}{\beta}\Upsilon(\beta), \tag{4.26}$$

where the notation is borrowed from the proof of Theorem 2.7. We apply $\mathscr{H}^n$ to both sides to deduce the following: For all integers $n \geq 0$ and $x \in \mathbf{R}$,

$$\begin{aligned}(\mathscr{H}^nF_\beta)(x) &\geq (\mathscr{H}^nG_\beta)(x) + (\mathscr{H}^{n+1}F_\beta)(x) - \frac{q_0^2A^2}{\beta}\Upsilon(\beta)\cdot\left|q_0^2\Upsilon(\beta)\right|^n\\ &\geq \frac{\eta^2}{\beta}\cdot\left|q_0^2\Upsilon(\beta)\right|^n + (\mathscr{H}^{n+1}F_\beta)(x) - \frac{A^2}{\beta}\cdot\left|q_0^2\Upsilon(\beta)\right|^{n+1},\end{aligned} \tag{4.27}$$

thanks to the tautological bound $u_0 \geq \eta$. We collect terms to obtain the following key estimate for the present proof:

$$(\mathscr{H}^nF_\beta)(x) - (\mathscr{H}^{n+1}F_\beta)(x) \geq \frac{\eta^2 - A^2q_0^2\Upsilon(\beta)}{\beta}\times\left|q_0^2\Upsilon(\beta)\right|^n, \tag{4.28}$$

valid for all integers $n \geq 0$ and $x \in \mathbf{R}$. Because $\bar{X}$ is recurrent, (4.23) ensures that we can choose $\beta > 0$ sufficiently small that $q_0\Upsilon(\beta) > 1$. Consequently, $F_\beta(x) \equiv \infty$ as long as η is greater than $Aq_0\Upsilon(\beta)$. This proves the theorem; confer with the paragraph immediately following (4.12). □

5 Final Remarks

We close the bulk of this paper with a few remarks. These remarks are motivated by some of the thoughtful questions of the anonymous referees.

5.1

Consider the formal SPDE (1.1) when $x \in \mathbf{R}^d$ for $d \geq 2$ and $\mathscr{L}$ denotes the generator of a d-dimensional Lévy process. In the linearized case [$\sigma \equiv 1$], that SPDE does not have a random-field solution [14; 19]. Therefore, it is not known how one describes the analogue of (1.1) for general multiplicative nonlinearities σ. When $\sigma(u) \equiv u$, some authors have studied the analogue of (1.1) where $x \in \mathbf{Z}^d$ and $\mathscr{L} :=$ the generator of a continuous-time random walk on $\mathbf{Z}^d$; consult the bibliography. Under various conditions on the noise term, full intermittency is shown to hold. The methods of the present paper can be extended to establish weak intermittency for fully-nonlinear discrete-space versions of (1.1), but we will not develop such a theory here.

5.2

One might wish to improve weak intermittency [i.e., existence of finite lim sups for the moments] to full intermittency in the fully nonlinear setting. We do not know how to do that. In fact, it is highly likely that such limits do not exist, as the following heuristic argument might suggest.

Consider a function σ such that $\sigma(u) = u$ for a "positive density of $u \in \mathbf{R}$," and $\sigma(u) = 2u$ for the remaining values of $u \in \mathbf{R}$. Then one might imagine that the solution $u(t\,,x)$ to (1.1) divides its time equally on $\{u \in \mathbf{R} : \ \sigma(u) = u\}$ and $\{u \in \mathbf{R} : \ \sigma(u) = 2u\}$. The existing literature on the parabolic Anderson model then might suggest that $\limsup_{t\to\infty} t^{-1} \ln \mathrm{E}(|u(t\,,x)|^2)$ is equal to $\lim_{t\to\infty} t^{-1} \ln \mathrm{E}(|\rho(t\,,x)|^2)$ where ρ solves (1.1) with $\sigma(u) = 2u$. And one might imagine equally well that $\liminf_{t\to\infty} t^{-1} \ln \mathrm{E}(|u(t\,,x)|^2)$ is identical to $\lim_{t\to\infty} t^{-1} \ln \mathrm{E}(|r(t\,,x)|^2)$, where r solves (1.1) with $\sigma(u) = u$. If these heuristic arguments are in fact correct, then one does not expect to have second-moment Liapounov exponents; only an upper exponent—defined in terms of a lim sup—and a typically-different lower exponent—defined in terms of a lim inf. At present, we are not able to make these arguments rigorous. Nor can we construct counter-examples.

5.3

Some of the central estimates of this paper require the assumption that u_0 is bounded below. This condition is quite natural [1; 5]. But the physics literature on the parabolic Anderson model [28] suggests that one might expect similar phenomena when u_0 has compact support [and is, possibly, sufficiently smooth]. At this time, we do not know how to study the fully-nonlinear case wherein u_0 has compact support.

A Regularity

The goal of this appendix is to show that one can produce a nice modification of the solution to (1.1). We recall that $\sigma : \mathbf{R} \to \mathbf{R}$ is assumed to be Lipschitz continuous.

Theorem A.1. *If u_0 is continuous, then the solution to* (1.1) *is continuous in $L^p(\mathrm{P})$ for all $p > 0$. Consequently, u has a separable modification. If, in addition, u_0 is uniformly continuous, then for all $T, p > 0$,*

$$\lim_{\delta,\rho\downarrow 0} \sup_{\substack{|s-t|\leq\delta \\ 0\leq s,t\leq T}} \sup_{\substack{|x-y|\leq\rho \\ x,y\in\mathbf{R}}} \left\| u(t\,,x) - u(s\,,y) \right\|_{L^p(\mathrm{P})} = 0. \tag{A.1}$$

This theorem is a ready consequence of the following series of Lemmas A.2, A.3, A.4, and A.5, together with successive applications of the triangle inequality. Many of the methods of this section expand on those of the earlier sections.

Let ϖ denote the uniform modulus of continuity of u_0. That is,

$$\varpi(\delta) := \sup_{\substack{|a-b|<\delta\\ a,b\in\mathbf{R}}} \left|u_0(a) - u_0(b)\right|. \tag{A.2}$$

Lemma A.2. *If u_0 is continuous, then so is $(t\,,x)\mapsto(\mathscr{P}_t u_0)(x)$. If u_0 is uniformly continuous, then so is $(t\,,x)\mapsto(\mathscr{P}_t u_0)(x)$; in fact for all $\delta,\rho>0$,*

$$\sup_{t\ge 0}\sup_{|x-z|\le\delta}\left|(\mathscr{P}_t u_0)(x) - (\mathscr{P}_t u_0)(z)\right| \le \varpi(\delta), \tag{A.3}$$

and

$$\sup_{|t-s|<\rho}\sup_{x\in\mathbf{R}}\left|(\mathscr{P}_t\, u_0)(x) - (\mathscr{P}_s u_0)(x)\right| \le \inf_{a>0}\left[\varpi(a) + A\rho\sup_{0<\xi<1/a}|\Psi(\xi)|\right], \tag{A.4}$$

with $A := 14\sup_{z\in\mathbf{R}}|u_0(z)|$.

Proof. We note that

$$(\mathscr{P}_t u_0)(x) - (\mathscr{P}_s u_0)(y) = \mathrm{E}\left(u_0(X_t + x) - u_0(X_s + y)\right). \tag{A.5}$$

Because u_0 is bounded, if it were continuous also, then $(t\,,x)\mapsto(\mathscr{P}_t u_0)(x)$ is continuous by the dominated convergence theorem. Henceforth, we assume that u_0 is uniformly continuous. Inequality (A.3) follows again from the dominated convergence theorem. As regards (A.4), we note that

$$\sup_{x\in\mathbf{R}}\left|(\mathscr{P}_t\, u_0)(x) - (\mathscr{P}_s u_0)(x)\right| \le \mathrm{E}\left[\varpi\left(\left|X_t - X_s\right|\right)\wedge 2\sup_{z\in\mathbf{R}}|u_0(z)|\right]. \tag{A.6}$$

Because $|1-\mathrm{E}\exp(i\xi(X_t - X_s))| \le |t-s|\cdot|\Psi(\xi)|$, Paul Lévy's characteristic-function inequality [30, Exercise 7.9, p. 112] shows that for all $a>0$,

$$\begin{aligned}\mathrm{P}\left\{|X_t - X_s| > a\right\} &\le 7a\int_0^{1/a}\left|1-\mathrm{E}e^{i\xi(X_t - X_s)}\right|\mathrm{d}\xi\\ &\le 7|t-s|\sup_{0<\xi<1/a}|\Psi(\xi)|.\end{aligned} \tag{A.7}$$

This completes our proof readily. □

Lemma A.3. *For all even integers $p\ge 2$, $x,z\in\mathbf{R}$, $t\ge 0$, and $\beta>0$,*

$$\begin{aligned}&\|(\mathscr{A}u)(t\,,x) - (\mathscr{A}u)(t\,,z)\|_{L^p(\mathrm{P})}\\ &\quad\le\left(\frac{p}{\pi}\right)^{1/2}\|\sigma\circ u\|_{p,\beta}\, e^{t\beta/p}\left\{\int_{-\infty}^{\infty}\frac{1-\cos(\xi|x-z|)}{\beta + 2\mathrm{Re}\,\Psi(\xi)}\,\mathrm{d}\xi\right\}^{1/2},\end{aligned} \tag{A.8}$$

where $(\sigma\circ u)(t\,,x) := \sigma(u(t\,,x))$.

Proof. We follow the pattern of the proof of Lemma 3.4. By the Burkholder–Davis–Gundy inequality,

$$\|(\mathscr{A}u)(t\,,x)-(\mathscr{A}u)(t\,,z)\|_{L^p(\mathrm{P})}^p \le z_p^p \mathrm{E}\left(\left|\int_{-\infty}^{\infty}\mathrm{d}y\int_0^t \mathrm{d}s\,\sigma(u(s\,,y))^2\left[p_{t-s}(y-x)-p_{t-s}(y-z)\right]^2\right|^{p/2}\right). \tag{A.9}$$

We write the $(p/2)$ power of the integral as a product and apply the generalized Hölder inequality to deduce from the preceding that

$$\|(\mathscr{A}u)(t\,,x)-(\mathscr{A}u)(t\,,z)\|_{L^p(\mathrm{P})}^p \le z_p^p\left|\int_{-\infty}^{\infty}\mathrm{d}y\int_0^t \mathrm{d}s\,\|\sigma(u(s\,,y))\|_{L^p(\mathrm{P})}^2\left[p_{t-s}(y-x)-p_{t-s}(y-z)\right]^2\right|^{p/2}. \tag{A.10}$$

Since $\|\sigma(u(s\,,y))\|_{L^p(\mathrm{P})}^2 \le \exp(2s\beta/p)\|\sigma\circ u\|_{p,\beta}^2$ for all $\beta>0$, the preceding and the Carlen–Kree inequality (Remark 2.2) together yield

$$\|(\mathscr{A}u)(t\,,x)-(\mathscr{A}u)(t\,,z)\|_{L^p(\mathrm{P})}^p \le 2^p p^{p/2}\,\|\sigma\circ u\|_{p,\beta}^2\,e^{\beta t}\left|\int_{-\infty}^{\infty}\mathrm{d}y\int_0^t \mathrm{d}s\,e^{-2s\beta/p}\left[p_s(y-x)-p_s(y-z)\right]^2\right|^{p/2}. \tag{A.11}$$

In accord with Plancherel's theorem,

$$\int_{-\infty}^{\infty}\left[p_s(y-x)-p_s(y-z)\right]^2\mathrm{d}y = \frac{1}{\pi}\int_{-\infty}^{\infty}(1-\cos(\xi|x-z|))\,e^{-2s\mathrm{Re}\,\Psi(\xi)}\,\mathrm{d}\xi. \tag{A.12}$$

The lemma follows from this and a few more lines of computation. □

Choose $x\in\mathbf{R}$ and $0\le t\le T$. We can write

$$(\mathscr{A}u)(T,x)-(\mathscr{A}u)(t\,,x)=D_1+D_2, \tag{A.13}$$

where

$$\begin{aligned} D_1 &:= \int_{-\infty}^{\infty}\int_0^t \sigma(u(s\,,y))\left[p_{T-s}(y-x)-p_{t-s}(y-x)\right]\,w(\mathrm{d}s\,\mathrm{d}y),\\ D_2 &:= \int_{-\infty}^{\infty}\int_t^T \sigma(u(s\,,y))p_{T-s}(y-x)\,w(\mathrm{d}s\,\mathrm{d}y). \end{aligned} \tag{A.14}$$

Lemma A.4. *For all even integers $p\ge 2$ and $\beta>0$,*

$$\|D_1\|_{L^p(\mathrm{P})} \le e^{\beta t/p}\left(\frac{p}{\pi}\right)^{1/2}\|\sigma\circ u\|_{p,\beta}\left(\int_{-\infty}^{\infty}\frac{\left|1-e^{-(T-t)\Psi(\xi)}\right|^2}{(\beta/p)+\mathrm{Re}\,\Psi(\xi)}\,\mathrm{d}\xi\right)^{1/2}. \tag{A.15}$$

Proof. We adjust the beginning portion of the preceding proof, and after a few lines, arrive at the following:

$$\mathrm{E}(D_1^p) \le 2^p p^{p/2} \|\sigma \circ u\|_{p,\beta}^p \left(\int_0^t e^{2\beta s/p} \left\| p_{T-s} - p_{t-s} \right\|_{L^2(\mathbf{R})}^2 \mathrm{d}s \right)^{p/2}. \tag{A.16}$$

Thanks to Plancherel's theorem,

$$\left\| p_{T-s} - p_{t-s} \right\|_{L^2(\mathbf{R})}^2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-2(t-s)\mathrm{Re}\,\Psi(\xi)} \left| 1 - e^{-(T-t)\Psi(\xi)} \right|^2 \mathrm{d}\xi. \tag{A.17}$$

Therefore, the Lebesgue integral in (A.16) is bounded above by

$$\frac{e^{\beta t}}{2\pi} \int_{-\infty}^{\infty} \frac{\left| 1 - e^{-(T-t)\Psi(\xi)} \right|^2}{(2\beta/p) + 2\mathrm{Re}\,\Psi(\xi)} \mathrm{d}\xi. \tag{A.18}$$

Solve to finish. □

Lemma A.5. *For all even integers $p \ge 2$ and $\beta > 0$,*

$$\|D_2\|_{L^p(\mathrm{P})} \le \sqrt{8p}\, e^{\beta T/p} \|\sigma \circ u\|_{p,\beta} \sqrt{\Upsilon\left(\frac{1}{T-t}\right)}. \tag{A.19}$$

Proof. We adapt the proof of the preceding lemma, to the present setting, and deduce that

$$\|D_2\|_{L^p(\mathrm{P})} \le 2\sqrt{p}\, \|\sigma \circ u\|_{p,\beta}\, e^{\beta T/p} \left(\int_0^{T-t} \|p_s\|_{L^2(\mathbf{R})}^2 \mathrm{d}s \right)^{1/2}. \tag{A.20}$$

But for all $\rho > 0$,

$$\begin{aligned} \int_0^\rho \|p_s\|_{L^2(\mathbf{R})}^2 \mathrm{d}s &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1 - e^{-2\rho\,\mathrm{Re}\,\Psi(\xi)}}{2\mathrm{Re}\,\Psi(\xi)} \mathrm{d}\xi \\ &\le \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{d}\xi}{(1/\rho) + 2\mathrm{Re}\,\Psi(\xi)} \\ &= 2\Upsilon(1/\rho). \end{aligned} \tag{A.21}$$

We have used the elementary fact that $(1 - e^{-\rho\theta})/\theta \le 2/(\rho^{-1} + \theta)$ for all $\theta > 0$. The lemma follows easily from these observations. □

One can often combine the preceding proof of Theorem A.1 with methods of Gaussian analysis, and produce an *almost-surely continuous* modification of u. We conclude this paper with an example of this method.

Example A.6. Suppose $1 < \alpha \le 2$ and $\mathscr{L} = -\kappa(-\Delta)^{\alpha/2}$. Suppose also that u_0 is uniformly Hölder continuous; that is, $\varpi(a) = O(a^\theta)$ as $a \to 0^+$ for a fixed $\theta > 0$. We claim that in this case u has a modification that is continuous almost surely. We prove this claim by working out the estimates produced by Lemmas A.2, A.3, A.4, and A.5. Indeed, (A.3) and (A.4) together show that $(t, x) \mapsto (\mathscr{P}_t u_0)(x)$ is uniformly jointly Hölder continuous with respective Hölder indices $\upsilon := \theta/(\theta + \alpha)$

[for t] and θ [for x]. A few more simple calculations show that: (i) The right-hand side of (A.8) is $O(|x-z|^{\mu})$ with $\mu := \min(1/2\,,\alpha-1)$; and (ii) the right-hand sides of (A.15) and (A.19) are both $O((T-t)^{\eta})$ with $\eta := (\alpha-1)/(2\alpha)$. In other words, we can choose and fix $\beta > 0$ that yields the following estimate: For all $T, p > 0$ there exists $a = a(p\,,T,\beta) \in (0\,,\infty)$ such that for all $s,t \in [0\,,T]$ and $x,y \in \mathbf{R}$,

$$\|u(t\,,x) - u(s\,,y)\|_{L^p(\mathrm{P})} \le a\left(|t-s|^{\upsilon\wedge\eta} + |x-y|^{\theta\wedge\mu}\right). \tag{A.22}$$

A suitable form of the Kolmogorov continuity theorem yields the desired Hölder-continuous modification. □

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