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# Ends in Uniform Spanning Forests[1]

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**Abstract**

It has hitherto been known that in a transitive unimodular graph, each tree in the wired spanning forest has only one end a.s. We dispense with the assumptions of transitivity and unimodularity, replacing them with a much broader condition on the isoperimetric profile that requires just slightly more than uniform transience.

**Key words:** Spanning trees, Cayley graphs.

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## §1. Introduction.

The area of uniform spanning forests has proved to be very fertile. It has important connections to several areas, such as random walks, sampling algorithms, domino tilings, electrical networks, and potential theory. It led to the discovery of the SLE processes, which are a major theme of contemporary research in planar stochastic processes. Although much is known about uniform spanning forests, several important questions remain open. We answer some of them here.

Given a finite connected graph, $G$, let $\mathsf{UST}(G)$ denote the uniform measure on spanning trees of $G$. If an infinite connected graph $G$ is exhausted by a sequence of finite connected subgraphs $G_n$, then the weak limit of $\langle \mathsf{UST}(G_n) \rangle$ exists. (This was conjectured by R. Lyons and proved by Pemantle (1991).) However, it may happen that the limit measure is not supported on trees, but only on forests. This limit measure is now called the **free (uniform) spanning forest** on $G$, denoted $\mathsf{FSF}$ or $\mathsf{FUSF}$. If $G$ is itself a tree, then this measure is trivial, namely, it is concentrated on $\{G\}$. Therefore, Häggström (1998) formally introduced another limit that had been considered on $\mathbb{Z}^d$ more implicitly by Pemantle (1991) and explicitly by Häggström (1995), namely, the weak limit of the uniform spanning tree measures on $G_n^*$, where $G_n^*$ is the graph $G_n$ with its boundary identified ("wired") to a single vertex. As Pemantle (1991) showed, this limit also exists on any graph and is now called the **wired (uniform) spanning forest**, denoted $\mathsf{WSF}$ or $\mathsf{WUSF}$. It is clear that both $\mathsf{FSF}$ and $\mathsf{WSF}$ are concentrated on the set of spanning forests[5] of $G$ all of whose trees are infinite. Both $\mathsf{FSF}$ and $\mathsf{WSF}$ are important in their own right; see Lyons (1998) for a survey and Benjamini, Lyons, Peres, and Schramm (2001), later referred to as BLPS (2001), for a comprehensive treatment.

A very basic global topological invariant of a tree is the number of ends it has. The ends of a tree can be defined as equivalence classes of infinite simple paths in the tree, where two paths are equivalent if their symmetric difference is finite. Trees that have a single end are infinite trees that are in some sense very close to being finite. For example, a tree with one end is recurrent for simple random walk and has critical percolation probability $p_{\mathrm{c}} = 1$. In an attempt to better understand the properties of the $\mathsf{WSF}$ and $\mathsf{FSF}$, it is therefore a natural problem to determine the number of ends in their trees. In fact, this was one of the very first questions asked about infinite uniform spanning forests.

Pemantle (1991) proved that a.s., each tree of the $\mathsf{WSF}$ has only one end in $\mathbb{Z}^d$ with $d = 3, 4$ and also showed that there are at most two ends per tree for $d \geq 5$. BLPS (2001) completed and extended this to all unimodular transitive networks, showing that

[5] By a "spanning forest", we mean a subgraph without cycles that contains every vertex.

each tree has only one end. In fact, each tree of the $\mathsf{WSF}$ has only one end in every quasi-transitive transient network, as well as in a host of other natural networks, as we show in Theorems 7.1 and 7.4 below. Our proof is simpler even for $\mathbb{Z}^d$, besides having the advantage of greater generality. Instead of transitivity, the present proof is based on a form of uniform transience arising from an isoperimetric profile. The proof relies heavily on electrical networks and not at all on random walks, in contrast to the previous proofs of results on the number of ends in the $\mathsf{WSF}$. We use the fact discovered by Morris (2003) that with an appropriate setup, the conductance to infinity is a martingale with respect to a filtration that examines edges sequentially and discovers if they are in the tree of the origin. Our results answer positively Questions 15.3 and 15.5 of BLPS (2001).

The statement that each tree has one end is a qualitative statement. However, our method of proof can provide quantitative versions of this result. To illustrate this point, we show that in $\mathbb{Z}^d$, the Euclidean diameter of the past of the origin satisfies the tail estimate $\mathbf{P}[\mathrm{diam} > t] \le C_d\, t^{-\beta_d}$, where $\beta_d = \frac{1}{2} - \frac{1}{d}$ and the **past** of a vertex $x$ is the union of the finite connected components of $T_x \backslash x$, where $T_x$ is the $\mathsf{WSF}$ tree containing $x$. However, this tail estimate is not optimal in $\mathbb{Z}^d$.

For an application of the property of one end to the abelian sandpile model, see Járai and Redig (2008); more precisely, we prove in Lemma 3.2 a property they use that is equivalent to one end. For an illustration of the usefulness of the one-end property in the context of the euclidean minimal spanning tree, see Krikun (2007).

## §2. Background, Notations and Terminology.

We shall now introduce some notations and give a brief background on electrical networks and uniform spanning forests. For a more comprehensive account of the background, please see BLPS (2001).

### Graphs and networks.

A **network** is a pair $(G, c)$, where $G = (\mathsf{V}, \mathsf{E})$ is a graph and $c : \mathsf{E} \to (0, \infty)$ is a positive function, which is often called the weight function or the edge conductance. If no conductance is specified for a graph, then we take $c \equiv 1$ as the default conductance; thus, any graph is also a network. Let $G = (\mathsf{V}, \mathsf{E}, c)$ be a network, and let $\vec{\mathsf{E}} = \big\{ \langle x, y \rangle \,;\, [x, y] \in \mathsf{E} \big\}$ denote the set of oriented edges. For $e \in \vec{\mathsf{E}}$, let $-e$ denote its reversal, let $e^-$ denote its tail, and let $e^+$ denote its head.

For a set of vertices $K \subset \mathsf{V}$, let $\partial_{\mathsf{E}} K$ be its **edge boundary** that consists of edges exactly one of whose endpoints is in $K$. Sometimes $K$ is a subset of two graphs; when we need to indicate in which graph $G$ we take the edge boundary, we write $\partial_{\mathsf{E}(G)}(K)$. Let $G\backslash K$ denote the graph $G$ with the vertices $K$ and all the edges incident with them removed, and let $G/K$ denote the graph $G$ with the vertices in $K$ identified (wired) to a single point and any resulting loops (edges with $e^+ = e^-$) dropped. (Note that it may happen that $G/K$ is a multigraph even if $G$ is a simple graph; that is, $G/K$ may contain multiple edges joining a pair of vertices even if $G$ does not.) If $H$ is a subgraph of $G$, then $G\backslash H$ and $G/H$ mean the same as $G\backslash\mathsf{V}(H)$ and $G/\mathsf{V}(H)$, respectively. However, when $e$ is an edge (or a set of edges), $G\backslash e$ means $G$ with $e$ removed but no vertices removed.

Write $|F|_c := \sum_{e\in F} c(e)$ for any set of edges $F$. If $K$ is a set of vertices, we also write

$$\pi(K) := \sum \Big\{ c(e)\,;\ e \in \vec{\mathsf{E}},\, e^- \in K \Big\}.$$

We shall generally consider only networks where $\pi(x) < \infty$ for every $x \in \mathsf{V}$.

On occasion, we shall need to prove statements about infinite networks from corresponding statements about finite networks. For this purpose, the concept of an exhaustion is useful. An **exhaustion** of $G$ is an increasing sequence of finite connected subnetworks $H_0 \subset H_1 \subset H_2 \subset \cdots$ such that $\bigcup_{n\ge 1} H_n = G$.

Given two graphs $G = (\mathsf{V}, \mathsf{E})$ and $G' = (\mathsf{V}', \mathsf{E}')$, call a function $\phi : \mathsf{V} \to \mathsf{V}'$ a **rough isometry** if there are positive constants $\alpha$ and $\beta$ such that for all $x, y \in \mathsf{V}$,

$$\alpha^{-1} d(x,y) - \beta \le d'\big(\phi(x), \phi(y)\big) \le \alpha d(x,y) + \beta \tag{2.1}$$

and such that every vertex in $G'$ is within distance $\beta$ of the image of $\mathsf{V}$. Here, $d$ and $d'$ denote the usual graph distances on $G$ and $G'$. The same definition applies to metric spaces, with "vertex" replaced by "point".

**Effective resistance and effective conductance.**

Let $(G, c)$ be a connected network. The **resistance** of an edge $e$ is $1/c(e)$ and will be denoted by $r(e)$. A function $\theta : \vec{\mathsf{E}} \to \mathbb{R}$ is called **antisymmetric** if $\theta(e) = -\theta(-e)$ for all $e \in \vec{\mathsf{E}}$. For antisymmetric functions $\theta, \theta' : \vec{\mathsf{E}} \to \mathbb{R}$, set

$$(\theta, \theta')_r := \frac{1}{2} \sum_{e \in \vec{\mathsf{E}}} r(e)\, \theta(e)\, \theta'(e)\,.$$

The **energy** of $\theta$ is given by $\mathcal{E}(\theta) := (\theta, \theta)_r$. The **divergence** of $\theta$ is the function $\nabla \cdot \theta : \mathsf{V} \to \mathbb{R}$ defined by

$$\nabla \cdot \theta(x) := \sum \Big\{ \theta(e)\,;\ e \in \vec{\mathsf{E}},\, e^- = x \Big\}.$$

(In some papers the definition of divergence differs by a factor of $\pi(x)^{-1}$ from our current definition.)

If $f : \mathsf{V} \to \mathbb{R}$ is any function, its **gradient** $\nabla f$ is the antisymmetric function on $\vec{\mathsf{E}}$ defined by $\nabla f(e) := c(e)\big(f(e^+) - f(e^-)\big)$. The **Dirichlet energy** of $f$ is defined as $D(f) := (\nabla f, \nabla f)_r$, and its **laplacian** is $\Delta f := \nabla \cdot \nabla f$. The function $f$ is **harmonic** on a set $A \subset \mathsf{V}$ if $\Delta f = 0$ on $A$.

Consider now the case where $G$ is finite. Let $f : \mathsf{V} \to \mathbb{R}$ and let $\theta : \vec{\mathsf{E}} \to \mathbb{R}$ be antisymmetric. The useful identity

$$(\nabla f, \theta)_r = -\sum_{v \in \mathsf{V}} f(v)\, \nabla \cdot \theta(v) \tag{2.2}$$

follows by gathering together the terms involving $f(v)$ in the definition of $(\nabla f, \theta)_r$. Let $A$ and $B$ be nonempty disjoint subsets of $\mathsf{V} = \mathsf{V}(G)$. A unit **flow** from $A$ to $B$ is an antisymmetric $\theta : \vec{\mathsf{E}} \to \mathbb{R}$ satisfying $\nabla \cdot \theta(x) = 0$ when $x \notin A \cup B$, and $\sum_{x \in A} \nabla \cdot \theta(x) = 1$. Since, clearly, $\sum_{x \in \mathsf{V}} \nabla \cdot \theta(x) = 0$, it follows that $\sum_{x \in B} \nabla \cdot \theta(x) = -1$. The **effective resistance** between $A$ and $B$ in $(G, c)$ is the infimum of $(\theta, \theta)_r$ over all unit flows $\theta$ from $A$ to $B$, and will be denoted by $\mathcal{R}(A \leftrightarrow B)$. This minimum is achieved by the **unit current flow**. The **effective conductance** between $A$ and $B$, denoted $\mathcal{C}(A \leftrightarrow B)$, is the infimum of $D(f)$ over all functions $f : \mathsf{V} \to \mathbb{R}$ satisfying $f = 0$ on $A$ and $f = 1$ on $B$. It is well known (and follows from (2.2)) that $\mathcal{C}(A \leftrightarrow B) = \mathcal{R}(A \leftrightarrow B)^{-1}$. Furthermore, $D(f)$ is minimized for a function $f$ that is harmonic except on $A \cup B$ and whose gradient is proportional to the unit current flow.

Now suppose that the network $(G, c)$ is infinite. The effective conductance from a finite set $A \subset \mathsf{V}$ to $\infty$ is defined as the infimum of $D(f)$ over all $f : \mathsf{V} \to \mathbb{R}$ such that $f = 0$ on $A$ and $f = 1$ except on finitely many vertices. This will be denoted by $\mathcal{C}(A \leftrightarrow \infty)$, naturally. The effective resistance to $\infty$ can be defined as $\mathcal{R}(A \leftrightarrow \infty) := \mathcal{C}(A \leftrightarrow \infty)^{-1}$, or, equivalently, as the infimum energy of any unit flow from $A$ to $\infty$, where a unit flow from $A$ to $\infty$ is an antisymmetric $\theta : \vec{\mathsf{E}} \to \mathbb{R}$ that satisfies $\nabla \cdot \theta = 0$ outside of $A$ and $\sum_{x \in A} \nabla \cdot \theta(x) = 1$. When $B \subset \mathsf{V}$, we define $\mathcal{R}(A \leftrightarrow B \cup \{\infty\})$ as the infimum energy of any antisymmetric $\theta : \vec{\mathsf{E}} \to \mathbb{R}$ whose divergence vanishes in $\mathsf{V} \setminus (A \cup B)$ and which satisfies $\sum_{x \in A} \nabla \cdot \theta(x) = 1$, and define $\mathcal{C}(A \leftrightarrow B \cup \{\infty\})$ as $\mathcal{R}(A \leftrightarrow B \cup \{\infty\})^{-1}$, or, equivalently, as the infimum of $D(f)$, where $f$ ranges over all functions that are 0 on $A$, 1 on $B$, and different from 1 on a finite set of vertices.

When $A$ or $B$ belong to two different networks under consideration, we use the notations $\mathcal{R}(A \leftrightarrow B; G)$ and $\mathcal{C}(A \leftrightarrow B; G)$ in order to specify that the the effective resistance or conductance is with respect to the network $G$.

**Uniform spanning trees and forests.**

If $(G, c)$ is finite, the corresponding **uniform spanning tree** is the measure on spanning trees of $G$ such that the probability of a spanning tree $T$ is proportional to $\prod_{e \in T} c(e)$.

The following relation between spanning trees and electrical networks is due to Kirchhoff (1847).

PROPOSITION 2.1. *Let $T$ be a uniform spanning tree of a finite network $G$ and $e$ an edge of $G$. Then*

$$\mathbf{P}[e \in T] = i(e) = c(e)\, \mathcal{R}(e^- \leftrightarrow e^+)\,,$$

*where $i$ is the unit current from $e^-$ to $e^+$.*

If $e$ is an edge in $G$, it is easy to see that the conditional law of the uniform spanning tree $T$ given $e \in T$ is the same (considered as a set of edges) as the law of the uniform spanning tree of $G/e$ union with $\{e\}$. Also, the conditional law of $T$ given $e \notin T$ is the same as the law of the uniform spanning tree of $G \backslash e$. These facts will be very useful for us.

Now assume that $(G, c)$ is infinite, and let $H_n$ be an exhaustion of $G$. Let $H_n^*$ denote the graph $G/(G \backslash H_n)$, namely, $G$ with the complement of $H_n$ identified to a single vertex (and loops dropped). Let $T_n$ denote the uniform spanning tree on the network $(H_n, c)$, and let $T_n^*$ denote the uniform spanning tree on $(H_n^*, c)$. Monotonicity of effective resistance and Proposition 2.1 imply that for every $e \in \mathsf{E}$ the limit of $\mathbf{P}[e \in T_n]$ exists as $n \to \infty$. By the previous paragraph, it is easy to conclude that the weak limit of the law of $T_n$ exists (as a measure on the Borel subsets of $2^{\mathsf{E}}$). This measure is the **free uniform spanning forest** on $G$, and is denoted by $\mathsf{FSF}$. Likewise, the weak limit of the law of $T_n^*$ exists (this time the monotonicity goes in the opposite direction, though), and is called the **wired uniform spanning forest** on $G$, which is denote by $\mathsf{WSF}$. In $\mathbb{Z}^d$ and many other networks, we have $\mathsf{WSF} = \mathsf{FSF}$. However, there are some interesting cases where $\mathsf{WSF} \neq \mathsf{FSF}$ (an example is when $G$ is a transient tree and $c \equiv 1$).

In the sequel, stochastic domination of probability measures on spanning forests will refer to the partial order induced by inclusion when forests are regarded as sets of edges.

## §3. $\mathsf{WSF}_o$.

In this section, $G$ is an arbitrary connected network and $o$ is some vertex in $G$. We now define a probability measure on spanning forests of $G$, which is the wired spanning forest with $o$ wired to $\infty$. Suppose that $G$ is exhausted by finite subgraphs $\langle G_n \rangle$. Let $\widehat{G}_n$ be the graph obtained from $G$ by identifying $o$ and the exterior of $G_n$ to a single point. Then the wired spanning forest on $G$ with $o$ wired to $\infty$ is defined as the weak limit (as a set of edges) of the uniform spanning tree on $\widehat{G}_n$ as $n \to \infty$, and will be denoted by $\mathsf{WSF}_o$. The existence of the limit follows from monotonicity by the same argument that gives the existence of the $\mathsf{WSF}$.

PROPOSITION 3.1. *Let $G$ be a transient network and $o \in G$. Given a forest $\mathfrak{F}$ in $G$, let $\mathfrak{F}(o)$ be its component that contains $o$. Then $|\mathfrak{F}(o)| < \infty$ $\mathsf{WSF}_o$-a.s. iff $\mathsf{WSF}$-a.s., there do not exist two edge-disjoint infinite paths in $\mathfrak{F}(o)$ starting at $o$.*

The "if" direction of this proposition was proved in Járai and Redig (2008). The "only if" direction (which is the direction we shall use) is an immediate consequence of the following lemma in which we take $x$ to be $o$ and $y$ to be the wired vertex in $G/\big(\mathsf{V}(G \setminus G_n) \setminus \{o\}\big)$, and then take the weak limits:

LEMMA 3.2. *Let $G$ be a finite connected network and $x, y \in \mathsf{V}$ be distinct vertices. Given a spanning tree $T$ of $G$, let $L(T)$ be the path in $T$ that connects $x$ to $y$. The uniform spanning tree in $G/\{x, y\}$ stochastically dominates $T \setminus L(T)$ when $T$ is a uniform spanning tree of $G$.*

Before presenting the proof, we recall that for any tree $T_0 \subset G$, the set of edges in the uniform spanning tree $T$ conditioned on $T_0 \subset T$ has the same distribution as the union of $\mathsf{E}(T_0)$ with the set of edges of the uniform spanning tree of $G/\mathsf{V}(T_0)$. Here, $\mathsf{V}(T_0)$ and $\mathsf{E}(T_0)$ denote the vertices and edges of $T_0$, respectively.

*Proof.* Indeed, with the proper identification of edges, conditioned on $L(T)$, $T \setminus L(T)$ is the uniform spanning tree of $G/\mathsf{V}\big(L(T)\big)$. Since $\mathsf{V}\big(L(T)\big)$ includes $x$ and $y$, it follows from a repeated application of the well-known negative association theorem of Feder and Mihail (1992) (see, e.g., Theorem 4.4 of BLPS (2001)) that the stochastic domination holds when we condition on $L(T)$. By averaging we conclude that it also holds unconditionally. ∎

Informally, the following lemma says that conductance to infinity is a martingale. When $\mathfrak{F}$ is a spanning forest of a graph $G$ and $E$ is a set of edges of $G$, we denote the graph $\big(\mathsf{V}(G), \mathsf{E}(\mathfrak{F}) \cap E\big)$ by $\mathfrak{F} \cap E$.

LEMMA 3.3. *Let $\mathfrak{F}$ be a sample from $\mathsf{WSF}_o$ on $(G, c)$. Let $E_0 \subset E_1$ be finite sets of edges in $G$, and for $j = 0, 1$, let $S_j$ be the set of vertices of the connected component of $o$ in $\mathfrak{F} \cap E_j$,* *and let $M^j$ be the effective conductance from $S_j$ to $\infty$ in $G\backslash E_j$. On the event that every edge in $E_1$ has at least one endpoint in $S_0$, we have*

$$\mathbf{E}\big[M^1 \mid \mathfrak{F} \cap E_0\big] = M^0 \,.$$

*Proof.* Suppose that $G$ is exhausted by finite subgraphs $\langle G_n \rangle$, with $E_1 \subset \mathsf{E}(G_n)$ for all $n$. Let $\widehat{G}_n$ be defined as in the definition of $\mathsf{WSF}_o$ and let $T_n$ be a random spanning tree of $\widehat{G}_n$. For $j = 0, 1$ and $n \geq 1$, let $\widehat{G}_n^j$ be the graph obtained from $\widehat{G}_n$ by contracting or deleting the edges in $E_j$ according to whether they are in $T_n$ or not and let $G^j$ be the graph obtained from $G$ by contracting or deleting the edges in $E_j$ according to whether they are in $\mathfrak{F}$ or not; let $M_n^j$ be the effective conductance from $o$ to the exterior of $G_n$ in $G^j$, and let $\bar{M}^j$ be the effective conductance between $o$ and $\infty$ in $G^j$. Theorem 7 of Morris (2003) implies that

$$\mathbf{E}\big[M_n^1 \mid \widehat{G}_n^0\big] = M_n^0 \,. \tag{3.1}$$

Since the weak limit of the $T_n$ is $\mathsf{WSF}_o$, taking the limit of both sides of equation (3.1) as $n \to \infty$ gives $\mathbf{E}\big[\bar{M}^1 \mid G^0\big] = \bar{M}^0$. Note that $\sigma(G^0) = \sigma(\mathfrak{F} \cap E_0)$.

Since on the event that every edge in $E_1$ has at least one endpoint in $S_0$, the graph obtained from $G$ by identifying the vertices in $S_j$ and deleting the edges in $E_j$ is $G^j$ for $j = 0, 1$, we also have that $\bar{M}^j = M^j$ on that event, whence the lemma follows. ∎

## §4. The case of $\mathbb{Z}^d$.

This section is devoted to proving the following theorem.

THEOREM 4.1. *Let $d > 2$, $d \in \mathbb{N}$, and let $\mathfrak{F}$ denote a sample from the* $\mathsf{WSF}$ *on $\mathbb{Z}^d$. Let $\mathfrak{F}(\mathbf{0})$ denote the connected component of $\mathbf{0}$ in $\mathfrak{F}$. Then $\mathfrak{F}(\mathbf{0})$ a.s. has one end, and therefore $\mathfrak{F}(\mathbf{0})\backslash\{\mathbf{0}\}$ has just one infinite connected component a.s. Moreover, if $Q$ denotes the union of the finite connected components of $\mathfrak{F}(\mathbf{0})\backslash\{\mathbf{0}\}$, then for all $t > 0$,*

$$\mathbf{P}\big[\mathrm{diam}(Q) > t\big] \leq C_d \, t^{-\beta_d},$$

*where $\beta_d := \frac{1}{2} - \frac{1}{d}$ and $C_d$ is some constant depending only on $d$. Here,* diam *means the Euclidean diameter.*

Of course, the first statement, namely that $\mathfrak{F}(\mathbf{0})$ has one end, is not new, but the proof we give is rather different from the proof in BLPS (2001). Our proof, without the quantitative estimate, is significantly shorter than that in BLPS (2001).

The tail estimate on $\operatorname{diam}(Q)$ that the theorem provides is not optimal. It is possible to show that $\mathbf{P}[\operatorname{diam}(Q) > t]$ behaves like $t^{-2}$ in $\mathbb{Z}^d$, $d > 4$, with possible polylogarithmic corrections, but we shall not prove this in the present paper. However, the proof we have in mind for the stronger estimates relies heavily on the fact that $\mathbb{Z}^d$ is transitive, and would not work for non-transitive networks that are similar to $\mathbb{Z}^d$, such as $\mathbb{N} \times \mathbb{Z}^{d-1}$ or $\mathbb{Z}^d$ with variable edge conductances bounded between two positive constants. In contrast, it is not too hard to see that the proof of Theorem 4.1 given below easily extends to these settings.

Throughout this section, $O_d(1)$ will stand for an unspecified positive constant depending only on the dimension $d$.

LEMMA 4.2. *Let $d \geq 3$ and let $G$ be the graph $\mathbb{N} \times \mathbb{Z}^{d-1} \subset \mathbb{Z}^d$. If $S \subset \mathsf{V}(G)$ is finite and nonempty and contained in $\{0\} \times \mathbb{Z}^{d-1}$, then the effective conductance from $S$ to $\infty$ in $G$ is at least $|S|^{\frac{d-2}{d-1}}/O_d(1)$.*

*Proof.* We consider first $\mathcal{C}(S \leftrightarrow \infty; \mathbb{Z}^d)$. For $v \in \mathbb{Z}^d$, let $g_v$ be the Green function in $\mathbb{Z}^d$, that is, $g_v(x)$ is the expected number of visits to $x$ for simple random walk started at $v$. Set $\theta_v := -\nabla g_v/(2d)$, that is, $\theta_v(e) = \big(g_v(e^-) - g_v(e^+)\big)/(2d)$ for oriented edges $e$. It is easy to verify that the divergence of $\theta_v$ is zero at every $u \in \mathbb{Z}^d \setminus \{v\}$ and that $g_v(v) = 1 + g_u(v) = 1 + g_v(u)$ holds whenever $u$ neighbors $v$. Therefore, $\theta_v$ is a unit flow from $v$ to $\infty$ in $\mathbb{Z}^d$. If we formally apply (2.2) with $f := g_u$ and $\theta := \theta_v$, we get $(\theta_v, \theta_u)_1 = g_u(v)/(2d)$. However, since in this case the summation corresponding to (2.2) is infinite, we need to be more careful. Let $G$ be exhausted by $\langle G_n \rangle$ and $G_n^{\mathrm{W}} := G/(G \backslash G_n)$. We may assume that $v \in \mathsf{V}(G_n)$ for all $n$. Let $g_{v,n}$ be the Green function in $G_n^{\mathrm{W}}$ for the walk on $G_n^{\mathrm{W}}$ killed when it first leaves $G_n$, so that it takes the value 0 off of $G_n$. Clearly $\lim_{n\to\infty} g_{v,n} = g_v$ pointwise. Also, $g_{v,n}$ is harmonic on $\mathsf{V}(G_n) \setminus \{v\}$. Thus, $\theta_{v,n} = -\nabla g_{v,n}/(2d)$ is the unit current flow on $G_n^{\mathrm{W}}$ from $v$ to the complement of $G_n$. Because $g_{v,n} \to g_v$, we have $\theta_{v,n} \to \theta_v$ on each edge. Fatou's lemma implies that $\mathcal{E}(\theta_v) \leq \liminf_{n\to\infty} \mathcal{E}\big(\theta_{v,n}\big)$. If $\mathcal{E}(\theta_v) < \limsup_{n\to\infty} \mathcal{E}\big(\theta_{v,n}\big)$, then for some $n$, the restriction of $\theta_v$ to $\mathsf{E}(G_n^{\mathrm{W}})$ would give a unit flow from $v$ to the complement of $G_n$ with smaller energy than $\theta_{v,n}$, a contradiction, whence $\mathcal{E}(\theta_v) = \lim_{n\to\infty} \mathcal{E}\big(\theta_{v,n}\big)$. It follows that $\mathcal{E}(\theta_{v,n} - \theta_v) \to 0$ as $n \to \infty$. Since $(\theta_{v,n}, \theta_{u,n})_1 = g_{u,n}(v)/(2d)$, the result follows by taking limits.

Define $\theta := |S|^{-1} \sum_{v \in S} \theta_v$. Then $\theta$ is a unit flow in $\mathbb{Z}^d$ from $S$ to $\infty$. Consequently,

$$\mathcal{C}(S \leftrightarrow \infty; \mathbb{Z}^d) \geq (\theta, \theta)_1^{-1} = \Big(|S|^{-2} \sum_{v,u \in S} (\theta_v, \theta_u)_1\Big)^{-1} = 2d|S|^2 \Big(\sum_{v,u \in S} g_v(u)\Big)^{-1}.$$

Recall that $g_v(x) \leq O_d(1)\,|v - x|^{2-d}$. (See, e.g., Theorem 1.5.4 of Lawler (1991).) Since

$S \subset \{0\} \times \mathbb{Z}^{d-1}$, it follows that for every $v \in \mathbb{Z}^d$ we have

$$\sum_{u \in S} g_v(u) \le O_d(1)\, |S|^{1/(d-1)}$$

since our upper bound on $g_v$ is monotone decreasing in distance from $v$, whence the sum of the bounds is maximized by the set $S$ closest to $v$. This gives $\mathcal{C}(S \leftrightarrow \infty; \mathbb{Z}^d) \ge |S|^{(d-2)/(d-1)}/O_d(1)$.

The corresponding result now follows for $G$, since we may restrict $\theta$ to $\mathsf{E}(G)$ and double it on edges that are not contained in the hyperplane $\{0\} \times \mathbb{Z}^{d-1}$. The result is a unit flow from $S$ to $\infty$ in $G$, and its energy is at most 4 times the energy of $\theta$ in $\mathbb{Z}^d$. The lemma follows. ∎

As a warm up, we start with a non-quantitative version of Theorem 4.1, proving that a.s. the connected components of the $\mathsf{WSF}$ in $\mathbb{Z}^d$ ($d \ge 3$) all have one end. Let $\mathfrak{F}$ be a sample from $\mathsf{WSF}_{\mathbf{0}}$. By Proposition 3.1, it suffices to show that $\mathsf{WSF}_{\mathbf{0}}$-a.s. the connected component of $\mathbf{0}$ in $\mathfrak{F}$ is finite.

Set $B_r := \{z \in \mathbb{R}^d ;\ \|z\|_\infty \le r\}$. We inductively construct an increasing sequence $E_0 \subset E_1 \subset E_2 \subset \cdots$ of sets of edges. Put $E_0 := \varnothing$. Assuming that $E_n$ has been defined, we let $S_n$ be the vertices of the connected component of $\mathbf{0}$ in $\mathfrak{F} \cap E_n$. If all the edges of $\mathbb{Z}^d$ incident with $S_n$ are in $E_n$, then set $E_{n+1} := E_n$. Otherwise, we choose some edge $e \notin E_n$ incident with $S_n$ and set $E_{n+1} := E_n \cup \{e\}$. Among different possible choices for $e$, we take one that minimizes $\min\{r ;\ e \subset B_r\}$, with ties broken in some arbitrary but fixed manner. Let $M_n := \mathcal{C}(S_n \leftrightarrow \infty; \mathbb{Z}^d \setminus E_n)$ denote the effective conductance from $S_n$ to $\infty$ in $\mathbb{Z}^d \setminus E_n$. Let $\mathcal{F}_n$ denote the $\sigma$-field generated by $E_n$ and $E_n \cap \mathfrak{F}$. By Lemma 3.3, $M_n$ is a martingale with respect to the filtration $\mathcal{F}_n$, that is, $\mathbf{E}\big[M_{n+1} \bigm| \mathcal{F}_n\big] = M_n$. Since $M_n \ge 0$, it follows that $\sup_n M_n < \infty$ a.s.

Let $\mathcal{A}_n$ be the event that $E_{n+1} = E_n$, and set $\mathcal{A} := \bigcup_{n \in \mathbb{N}} \mathcal{A}_n$, which is the event that the component of $\mathbf{0}$ in $\mathfrak{F}$ is finite. For $r \ge 1$, let $n_r$ denote the first $n$ such that $E_n$ contains all the edges inside $B_r$ that are incident with $S_n$. (Thus we have $E_{n_r} \subset B_r$ and either $E_{n_r+1} = E_{n_r}$ or $E_{n_r+1} \not\subset B_r$.) We claim that for every $c > 0$ there is a $\delta_c > 0$ such that for every $r \in \mathbb{N}$

$$\mathbf{P}\big[\mathcal{A}_{n_{r+1}} \bigm| \mathcal{F}_{n_r}\big] \ge \delta_c \, \mathbf{1}_{\{M_{n_r} \le c\}}. \tag{4.1}$$

To prove this, fix some $c > 0$ and $r \in \mathbb{N}$, set $m := n_r$, and suppose that $M_m \le c$. Let $\ell$ be the number of edges that connect $S_m$ to a vertex in $[r+1, \infty) \times \mathbb{Z}^{d-1}$, and let $k_r$ be the number of edges that connect $S_m$ to a vertex outside $B_r$. Because of the assumption $M_m \le c$, Lemma 4.2 gives a finite upper bound $K_c$ for $\ell$, which depends only on $c$ and $d$.

By symmetry, it follows that $k_r \le 2\,d\,K_c$. If $k_r = 0$, then $\mathcal{A}_{n_{r+1}}$ occurs, so (4.1) certainly holds for every $\delta_c \le 1$. Otherwise, suppose that $k_r \ge 1$. Fix some $j \in \mathbb{N} \cap [m+1, m+k_r]$. If it so happens that $E_{j-1} \cap \mathfrak{F} \subset E_m$, then the edge $e_j$ in $E_j \setminus E_{j-1}$ is one of those $k_r$ edges that connect $S_m$ to the complement of $B_r$. Suppose that this is the case, and let $v$ be the endpoint of $e_j$ that is not in $S_m$. There is a universal lower bound $\alpha > 0$ on the conductance from $v$ to $\infty$ in $\mathbb{Z}^d \backslash S_m$. Thus, the effective conductance between $v$ and $S_m \cup \{\infty\}$ in the complement of $E_{j-1}$ is at least $1 + \alpha$ (seen, e.g., by minimizing Dirichlet energy). This gives

$$\mathsf{WSF}_{\mathbf{0}}\big[e_j \in \mathfrak{F} \bigm| \mathcal{F}_{j-1}\big]\,\mathbf{1}_{E_{j-1}\cap\mathfrak{F}\subset E_m} \le (1+\alpha)^{-1}$$

by Proposition 2.1. Induction on $j$ yields

$$\mathsf{WSF}_{\mathbf{0}}\big[E_j \cap \mathfrak{F} \subset E_m \bigm| \mathcal{F}_m\big] \ge \big(1 - (1+\alpha)^{-1}\big)^j$$

for $j \in \mathbb{N} \cap [m+1, m+k_r]$. Since $\mathcal{A}_{n_{r+1}}$ is the event $E_{m+k_r} \cap \mathfrak{F} \subset E_m$ and since $k_r \le 2\,d\,K_c$, we therefore get (4.1) with

$$\delta_c = \big(1 - (1+\alpha)^{-1}\big)^{2\,d\,K_c}.$$

Induction on $r$ and (4.1) give

$$\mathbf{P}\Big[\sup_{n \le n_r} M_n \le c,\, \neg\mathcal{A}_{n_r}\Big] \le (1-\delta_c)^r.$$

Hence, $\mathbf{P}\big[\sup_n M_n \le c,\, \neg\mathcal{A}\big] = 0$. Because $\sup_n M_n < \infty$ a.s., this clearly implies that $\mathsf{WSF}_{\mathbf{0}}[\mathcal{A}] = 1$, which proves that all the connected components of the $\mathsf{WSF}$ in $\mathbb{Z}^d$ have one end a.s.

The above proof can easily be made quantitative, but the bound it provides is rather weak. We now proceed to establish a more reasonable bound.

*Proof of Theorem 4.1.* Here, we shall again use the sequence $E_n$ constructed above, as well as the notations $S_n, M_n, B_r$, etc. However, for the following argument to work, we need to be more specific about the way in which $E_{n+1}$ is chosen given $E_n$ and $\mathfrak{F} \cap E_n$. When $n = n_r$ for some $r$ and the set of edges adjacent to $S_n$ outside of $B_r$ is nonempty, we now require that $E_{n+1} = E_n \cup \{\tilde{e}_r\}$, where $\tilde{e}_r$ is an edge along which the unit current flow from $S_n$ to $\infty$ in the complement of $E_n$ is maximal.

Fix some $r \in \mathbb{N}$ and let $m := n_r$. Suppose that $E_{m+1} = E_m \cup \{\tilde{e}_r\}$, that is, $\mathcal{A}_{n_r}$ does not hold. Let $\theta$ denote the unit current flow from $S_m$ to $\infty$ in the complement of $E_m$. Then $M_m = \mathcal{E}(\theta)^{-1}$. Clearly, $|\theta(\tilde{e}_r)| \ge 1/k_r$ (where $k_r$ is the number of edges not in $E_m$

that are incident to $S_m$). If we have $\tilde{e}_r \in \mathfrak{F}$, then the restriction of $\theta$ to the complement of $\tilde{e}_r$ is a unit flow from $S_{m+1}$ to $\infty$ in the complement of $E_{m+1}$. Therefore, in this case,

$$M_{m+1} \geq \big(\mathcal{E}(\theta) - \theta(\tilde{e}_r)^2\big)^{-1} \geq \big(M_m^{-1} - k_r^{-2}\big)^{-1} \geq M_m \Big(1 + \frac{M_m}{k_r^2}\Big).$$

We also know that $k_r \leq O_d(1)\, M_m^{\frac{d-1}{d-2}}$ from Lemma 4.2. Consequently, on the event $\tilde{e}_r \in \mathfrak{F}$, we have

$$M_{m+1} \geq M_m + M_m^{-2/(d-2)}/O_d(1)\,. \tag{4.2}$$

By Proposition 2.1,

$$\mathbf{P}\big[\tilde{e}_r \in \mathfrak{F} \bigm| \mathcal{F}_{n_r}\big] \geq (2\,d)^{-1} \qquad \text{on the event } \neg\mathcal{A}_{n_r}\,, \tag{4.3}$$

since the conductance of $\tilde{e}_r$ is 1 and the effective conductance in $\mathbb{Z}^d$ between the endpoint $x$ of $\tilde{e}_r$ outside of $S_m$ and $S_m \cup \{\infty\}$ is at most $(2\,d)^{-1}$, as there are $2\,d$ edges coming out of $x$ and we may raise the effective conductance by identifying all vertices other than $x$.

Fix some $a \in \mathbb{R}$ and set

$$f(x) := f_a(x) := a\,x - x^{2+\frac{4}{d-2}}\,.$$

Since $f$ is concave, $f(M_n)$ is an $\mathcal{F}_n$-supermartingale. Set $X_n := 1$ if $n = n_r$ for some $r \in \mathbb{N}$ and $\neg\mathcal{A}_n$ holds. Otherwise, set $X_n := 0$. We claim that $Y_n := f(M_n) + b\sum_{j=0}^{n-1} X_j$ is an $\mathcal{F}_n$-supermartingale, where $b > 0$ is a certain constant depending only on $d$. This will be established once we show that

$$\mathbf{E}\big[f(M_{n_r+1}) \bigm| \mathcal{F}_{n_r}\big] \leq f(M_{n_r}) - b \qquad \text{on the event } \neg\mathcal{A}_{n_r}\,. \tag{4.4}$$

Set $m := n_r$ and assume that $\neg\mathcal{A}_m$ holds. Note that since $\neg\mathcal{A}_m$ holds, we have $M_{m+1} \geq M_m$. Set $L := f(M_m) + f'(M_m)\,(M_{m+1} - M_m)$. Then

$$L - f(M_{m+1}) = \int_{M_m}^{M_{m+1}} \big(f'(M_m) - f'(x)\big)\,dx = \int_{M_m}^{M_{m+1}} \int_{M_m}^{x} -f''(y)\,dy\,dx\,. \tag{4.5}$$

Since $f''(x) < 0$, we get $L \geq f(M_{m+1})$. Observe that there is a constant $C' = C'_d \geq 1$ such that $M_{n+1} \leq C'\,M_n$. (Given a unit flow from $S_{n+1}$ to $\infty$ in $\mathbb{Z}^d \setminus E_{n+1}$, one can produce from it a unit flow from $S_n$ to $\infty$ in $\mathbb{Z}^d \setminus E_n$ by setting the flow appropriately on $E_{n+1} \setminus E_n$. Since there is at most one edge in $E_{n+1} \setminus E_n$ and the degrees in $\mathbb{Z}^d$ are bounded, it is easy to see that the $\ell^2$ norm of the new flow is bounded by some constant times the $\ell^2$ norm of the former flow.) Therefore, $f''(M_m)/f''(x)$ is bounded on $[M_m, M_{m+1}]$ when

$M_{m+1} \geq M_m$. Recall that (4.2) holds when $\tilde{e}_r \in \mathfrak{F}$. By our choice of $f$ and by (4.5), we therefore have

$$O_d(1)\left(L - f(M_{m+1})\right) \geq 1 \qquad\qquad \text{when } \tilde{e}_r \in \mathfrak{F}\,.$$

By (4.3), we therefore get $O_d(1)\,\mathbf{E}\big[L - f(M_{m+1}) \bigm| \mathcal{F}_m\big] \geq 1$ (when $m = n_r$ and $\neg\mathcal{A}_{n_r}$ holds). Since $M_n$ is a martingale, $\mathbf{E}[L \mid \mathcal{F}_m] = f(M_m)$, by the choice of $L$. This proves (4.4).

Now fix some $\bar{M} > C'\,M_0$. Let $\tilde{n} := \inf\{n\,;\ M_n \geq \bar{M}/C'\}$, with the usual convention that $\tilde{n} := \infty$ if $\forall n\ M_n < \bar{M}/C'$. We now choose the constant $a$ in the definition of $f$ so that $f \geq 0$ on $[0, \bar{M}]$ and $f = 0$ at the endpoints of this interval, namely, $a := \bar{M}^{\frac{d+2}{d-2}}$. Set $Z_n := \sum_{0 \leq j < n} X_j$, where $n \in \mathbb{N} \cup \{\infty\}$. Then

$$\mathbf{E}[Z_{n\wedge\tilde{n}}] = b^{-1}\,\mathbf{E}[Y_{n\wedge\tilde{n}}] - b^{-1}\,\mathbf{E}[f(M_{n\wedge\tilde{n}})]\,.$$

By our choice of $a$ and $\tilde{n}$, $f(M_{n\wedge\tilde{n}}) \geq 0$. Since $Y_n$ is a supermartingale, we get

$$\mathbf{E}[Z_{n\wedge\tilde{n}}] \leq b^{-1}\,Y_0 = b^{-1}\,f(M_0) \leq O_d(1)\,\bar{M}^{\frac{d+2}{d-2}}.$$

The monotone convergence theorem implies that the same bound applies to $\mathbf{E}[Z_{\tilde{n}}]$. Therefore, for every $t > 0$, we have

$$\begin{aligned}
\mathbf{P}[Z_\infty > t] &\leq \mathbf{P}[Z_{\tilde{n}} > t] + \mathbf{P}[\tilde{n} \neq \infty] \\
&\leq t^{-1}\,\mathbf{E}[Z_{\tilde{n}}] + \mathbf{P}\big[\sup\{M_n\,;\ n \in \mathbb{N}\} \geq \bar{M}/C'\big] \\
&\leq O_d(1)\,t^{-1}\,\bar{M}^{\frac{d+2}{d-2}} + C' M_0/\bar{M}\,.
\end{aligned}$$

(The final inequality uses, say, Doob's optional stopping theorem.) We now choose $\bar{M} := t^{(d-2)/(2d)}$ for large $t$ and get $\mathbf{P}[Z_\infty > t] \leq O_d(1)\,t^{(2-d)/(2d)}$. Observe that $O_d(1)\,Z_\infty$ bounds the Euclidean diameter of the component of $o$ in $\mathfrak{F}$. Note that (by taking a limit along an exhaustion) Lemma 3.2 implies that the diameter of the connected component of $o$ in $\mathfrak{F}$ under $\mathsf{WSF}_{\mathbf{0}}$ stochastically dominates the diameter of the union of the finite connected components of $\mathfrak{F}\backslash\{\mathbf{0}\}$ under the $\mathsf{WSF}$. This completes the proof. ∎

## §5. Conductance Criterion for One End.

In this section we consider a network $(G, c)$, and prove a rather general sufficient condition for the $\mathsf{WSF}$ on $(G, c)$ to have one end for every tree a.s.

We recall that for a set of edges $F \subset \mathsf{E} = \mathsf{E}(G)$, we write $|F|_c$ for $\sum_{e \in F} c(e)$, and for a set of vertices $K \subset \mathsf{V} = \mathsf{V}(G)$, we write $\pi(K)$ for the sum of $c(e)$ over all edges $e \in \vec{\mathsf{E}}$ having at least one endpoint in $K$.

Proposition 5.1. *Let $(G, c)$ be a transient connected network and let $V_0 \subset V_1 \subset \cdots$ be finite sets of vertices satisfying $\bigcup_{j=0}^{\infty} V_j = \mathsf{V}$, where $\mathsf{V} = \mathsf{V}(G)$. Suppose that*

$$\inf \left\{ \mathcal{C}(v \leftrightarrow \infty; G \backslash V_n) \,;\, n \in \mathbb{N},\, v \in \mathsf{V} \setminus V_n \right\} > 0 \tag{5.1}$$

*and*

$$\lim_{t \to \infty} \inf \left\{ \mathcal{C}(K \leftrightarrow \infty; G \backslash V_n) \,;\, n \in \mathbb{N},\, K \subset \mathsf{V} \setminus V_n,\, K \text{ finite},\, \pi(K) > t \right\} = \infty\,. \tag{5.2}$$

*Then $\mathsf{WSF}$-a.s. every tree has one end.*

*Proof.* Let $H_n$ be the subgraph of $G$ spanned by $V_n$. Let $o \in \mathsf{V}$. With no loss of generality, we assume that $o \in V_0$ and that $\partial_{\mathsf{E}} V_n \subset H_{n+1}$ for each $n \in \mathbb{N}$ (since we may take a subsequence of the exhaustion). As in Section 4, let $\mathfrak{F}$ be a sample from $\mathsf{WSF}_o$ on $G$ and let $S_n$ be the set of vertices of the connected component of $\mathfrak{F} \cap H_n$ containing $o$. Let $E_n := \mathsf{E}(H_n)$, and let $E'_n$ be the set of edges in $E_n$ that have at least one endpoint in $S_n$. By Lemma 3.3, we know that the effective conductance $M_n$ from $S_n$ to $\infty$ in $G \backslash E'_n$ is a non-negative martingale and therefore bounded a.s. (Since in the formulation of the lemma it is assumed that every edge of $E_1$ has an endpoint in $S_0$, in order to deduce the above statement one has to first go through a procedure of examining edges one-by-one, as we have done in Section 4, but with the graphs $H_n$ used in place of the balls $B_r$.)

Fix some $n \in \mathbb{N}$. Let $Z_n$ be a set of vertices such that $\mathsf{V} \setminus Z_n$ is finite and contains $S_n$ and $\mathcal{C}(S_n \leftrightarrow Z_n; G \backslash E'_n) \le 2\,M_n$. By the definition of $M_n$, there is such a $Z_n$. For every vertex $v \in \mathsf{V}$, let $h(v)$ denote the probability that the network random walk on $G \backslash E'_n$ started at $v$ hits $Z_n$ before hitting $S_n$. Let $F_0^n$ denote the set of edges in $\mathsf{E} \setminus E'_n$ that join vertices in $S_n$ to vertices in $U_0^n := \{v \in \mathsf{V} \setminus S_n \,;\, h(v) \in [0, 1/2]\}$, and let $F_1^n$ denote the set of edges in $\mathsf{E} \setminus E'_n$ that join vertices in $S_n$ to vertices in $\{v \in \mathsf{V} \setminus S_n \,;\, h(v) > 1/2\}$. We claim that a.s.

$$\sup_{n \in \mathbb{N}} |F_0^n \cup F_1^n|_c < \infty\,. \tag{5.3}$$

We start by estimating $|F_0^n|_c$. Set $H(v) := \max\{0,\, 2\,h(v) - 1\}$. The Dirichlet energy $D(H)$ of $H$ is at most 4 times the Dirichlet energy of $h$, which is bounded by $2\,M_n$. Since $H = 1$

on $Z_n$, it follows that $\mathcal{C}(H^{-1}(0) \leftrightarrow Z_n; G\backslash E'_n) \le D(H) \le 8\,M_n$. Since $U_0^n \subseteq H^{-1}(0)$ and $E'_n \subset E_n$, it follows that $\mathcal{C}(U_0^n \leftrightarrow \infty; G\backslash V_n) \le 8\,M_n$. Since $M_n$ is a.s. bounded, it follows that $\sup_n \mathcal{C}(U_0^n \leftrightarrow \infty; G\backslash V_n) < \infty$ a.s., and by (5.2), $\sup_n |F_0^n|_c \le \sup_n \pi(U_0^n) < \infty$ a.s. Since

$$2\,M_n \ge D(h) = \sum_e c(e)\,\big(h(e^-) - h(e^+)\big)^2 \ge \sum_{e \in F_1^n} c(e)\,(1/2)^2\,,$$

we get $|F_1^n|_c \le 8\,M_n$, and so (5.3) follows.

Next, suppose that $e_1, e_2, \ldots, e_k$ are all the edges in $\mathsf{E} \setminus E'_n$ joining $S_n$ to $\mathsf{V} \setminus V_n$, that is, $F_0^n \cup F_1^n = \{e_1, e_2, \ldots, e_k\}$. Let $v_j$ denote the vertex of $e_j$ outside of $S_n$ for $j = 1, \ldots, k$. We now show that for $j = 1, \ldots, k$,

$$\mathbf{P}\big[e_j \notin \mathfrak{F} \bigm| e_1, \ldots, e_{j-1} \notin \mathfrak{F}, S_n\big] \ge \exp\big(-c(e_j)/a\big)\,, \tag{5.4}$$

where $a > 0$ is the left-hand side of (5.1). Let $\tilde{G}$ denote the graph $G\backslash(E'_n \cup \{e_1, \ldots, e_{j-1}\})$. Then 1 minus the left-hand side of (5.4) is equal to

$$\frac{c(e_j)}{\mathcal{C}(v_j \leftrightarrow \{\infty\} \cup S_n; \tilde{G})} \le \frac{c(e_j)}{\mathcal{C}(v_j \leftrightarrow \infty; G\backslash V_n) + c(e_j)} \le \frac{c(e_j)}{a + c(e_j)}\,,$$

which implies (5.4) (using the inequality $(1+x)^{-1} \ge e^{-x}$, valid for $x > -1$). Now, (5.4) and induction on $j$ imply

$$\mathbf{P}\big[S_{n+1} = S_n \bigm| S_n\big] \ge \exp\big(-|F_0^n \cup F_1^n|_c/a\big)\,.$$

Since by (5.3) the right-hand side is a.s. bounded away from zero, it follows that a.s. there is some $n \in \mathbb{N}$ such that $S_{n+1} = S_n$. Consequently, the connected component of $o$ in $\mathfrak{F}$ is finite a.s., and this completes the proof by Lemma 3.2. ∎

## §6. Isoperimetric Profile and Transience.

Our next goal is to prove that the $\mathsf{WSF}$ has one end a.s. in networks with a "reasonable isoperimetric profile", but first, we must discuss the relationship of the profile to transience, which is the subject of this section. Define

$$\kappa(G, A, t) := \inf\big\{|\partial_{\mathsf{E}} K|_c\,;\ A \subseteq K,\ K \text{ is finite and connected}, t \le \pi(K)\big\}\,.$$

We also abbreviate $\kappa(G, t) := \kappa(G, \varnothing, t)$. Write

$$\partial_{\mathsf{E}}^{\infty} K := \big\{(x, y) \in \mathsf{E}\,;\ x \in K,\ y \text{ belongs to an infinite component of } G\backslash K\big\}\,.$$

Then

$$\kappa(G,A,t) = \inf\big\{|\partial_{\mathsf{E}}^{\infty} K|_c\,;\ A\subseteq K,\ K \text{ is finite and connected}, t\le \pi(K)\big\}\,. \tag{6.1}$$

The following result refines Thomassen (1992) and is adapted from a similar result of He and Schramm (1995). Our proof is simpler than that of Thomassen, but we do not obtain his conclusion of the existence of a transient subtree.

THEOREM 6.1. *Let $A$ be a finite set of vertices in a network $G$ with $\pi\big(\mathsf{V}(G)\big)=\infty$. Let $\kappa(t) := \kappa(G,A,t)$. Define $s_0 := |\partial_{\mathsf{E}}^{\infty}(A)|_c$ and $s_{k+1} := s_k + \kappa(s_k)/2$ recursively for $k\ge 0$. Then*

$$\mathcal{R}(A\leftrightarrow\infty) \le \sum_{k\ge 0}\frac{2}{\kappa(s_k)}\,.$$

This is an immediate consequence of the following analogue for finite networks.

LEMMA 6.2. *Let $a$ and $z$ be two distinct vertices in a finite connected network $G$. Define*

$$\kappa(t) := \min\big\{|\partial_{\mathsf{E}} K|_c\,;\ a\in K,\ z\notin K,\quad K \textit{ is connected, } t\le\pi(K)\big\}$$

*for $t\le \pi\big(\mathsf{V}(G)\setminus\{z\}\big)$ and $\kappa(t):=\infty$ for $t>\pi\big(\mathsf{V}(G)\setminus\{z\}\big)$. Define $s_0 := \pi(a)$ and $s_{k+1} := s_k+\kappa(s_k)/2$ recursively for $k\ge 0$. Then*

$$\mathcal{R}(a\leftrightarrow z) \le \sum_{k=0}^{\infty}\frac{2}{\kappa(s_k)}\,.$$

*Proof.* Let $g : \mathsf{V}(G)\to\mathbb{R}$ be the function that is harmonic in $\mathsf{V}(G)\setminus\{a,z\}$ and satisfies $g(a)=0$, $g(z)=\mathcal{R}(a\leftrightarrow z)$. Recall from Section 2 that $\nabla g$ is a unit flow from $a$ to $z$. (To connect with electrical network terminology, note that $-\nabla g$ is the unit current from $z$ to $a$ and $g$ is its voltage.) For $t\ge 0$, let $W(t) := \big\{x\in\mathsf{V}\,;\ g(x)\le t\big\}$, and for $t'>t\ge 0$ let $E(t,t')$ be the set of directed edges from $W(t)$ to $\big\{x\in\mathsf{V}\,;\ g(x)\ge t'\big\}$. Define $t_0 := 0$ and inductively,

$$t_{k+1} := \sup\big\{t\ge t_k\,;\ |E(t_k,t)|_c \ge |\partial_{\mathsf{E}} W(t_k)|_c/2\big\}.$$

Set $\bar{k} := \min\{j\,;\ z\in W(t_j)\} = \min\{j\,;\ t_{j+1}=\infty\}$. Fix some $k<\bar{k}$. Note that $\nabla g(e)\ge 0$ for every $e\in\partial_{\mathsf{E}} W(t_k)$ (where edges in $\partial_{\mathsf{E}} W(t_k)$ are oriented away from $W(t_k)$). Now

$$\begin{aligned}1 = \sum_{e\in\partial_{\mathsf{E}} W(t_k)}\nabla g(e) &\ge \sum_{e\in E(t_k,t_{k+1})} c(e)\,\big(g(e^+)-g(e^-)\big)\\ &\ge \sum_{e\in E(t_k,t_{k+1})} c(e)\,(t_{k+1}-t_k) \ge (t_{k+1}-t_k)\,\frac{|\partial_{\mathsf{E}} W(t_k)|_c}{2}\,,\end{aligned}$$

where the last inequality follows from the definition of $t_{k+1}$.

Thus

$$t_{k+1} - t_k \leq 2/\kappa\big(\pi(W_k)\big)\,, \tag{6.2}$$

where we abbreviate $W_k := W(t_k)$. Clearly,

$$\begin{aligned}\pi(W_{k+1}) = \pi(W_k) + \pi(W_{k+1} \setminus W_k) &\geq \pi(W_k) + |\partial_{\mathsf{E}} W_k|_c - \sup_{t > t_{k+1}} |E(t_k, t)|_c \\ &\geq \pi(W_k) + \frac{1}{2}\,|\partial_{\mathsf{E}} W_k|_c \geq \pi(W_k) + \frac{1}{2}\,\kappa\big(\pi(W_k)\big)\,.\end{aligned}$$

Since $\kappa$ is a non-decreasing function, it follows by induction that $\pi(W_k) \geq s_k$ for $k < \bar{k}$ and (6.2) gives

$$\mathcal{R}(a \leftrightarrow z) = g(z) = t_{\bar{k}} - t_0 \leq \sum_{k=0}^{\bar{k}-1} \frac{2}{\kappa\big(\pi(W_k)\big)} \leq \sum_{k=0}^{\bar{k}-1} \frac{2}{\kappa(s_k)}\,. \qquad \blacksquare$$

In the setting of Theorem 6.1, it is commonly the case that $\kappa(t) = \kappa(G, A, t) \geq f(t)$ for some increasing function $f$ on $\big[\pi(A), \infty\big)$ that satisfies $0 < f(t) \leq t$ and $f(2\,t) \leq \alpha f(t)$ for some constant $\alpha$. In this case, define $t_0 := \pi(A)$ and $t_{k+1} := t_k + f(t_k)/2$ recursively. We have that $s_k \geq t_k$ and $t_k \leq t_{k+1} \leq 2\,t_k$, whence for $t_k \leq t \leq t_{k+1}$, we have $f(t) \leq f(2\,t_k) \leq \alpha f(t_k)$, so that

$$\begin{aligned}\int_{\pi(A)}^{\infty} \frac{4\,\alpha^2}{f(t)^2}\,dt &= \sum_{k \geq 0} \int_{t_k}^{t_{k+1}} \frac{4\,\alpha^2}{f(t)^2}\,dt \geq \sum_{k \geq 0} \int_{t_k}^{t_{k+1}} \frac{4}{f(t_k)^2}\,dt = \sum_{k \geq 0} \frac{4(t_{k+1} - t_k)}{f(t_k)^2} \\ &= \sum_{k \geq 0} \frac{2f(t_k)}{f(t_k)^2} \geq \sum_{k \geq 0} \frac{2}{\kappa(t_k)} \geq \sum_{k \geq 0} \frac{2}{\kappa(s_k)} \geq \mathcal{R}(A \leftrightarrow \infty)\,.\end{aligned} \tag{6.3}$$

This bound on the effective resistance is usually easier to estimate than the one of Theorem 6.1.

We shall need the following fact, which states that a good isoperimetric profile is inherited by some exhaustion.

LEMMA 6.3. *Let $(G, c)$ be a connected locally finite network such that $\lim_{t \to \infty} \kappa(G, o, t) = \infty$ for some fixed $o \in \mathsf{V}$ and such that every infinite connected subset $K \subset \mathsf{V}(G)$ satisfies $\pi(K) = \infty$. Then the network $G$ has an exhaustion $\langle G_n \rangle$ by finite connected subgraphs such that*

$$|\partial_{\mathsf{E}} U \setminus \partial_{\mathsf{E}} \mathsf{V}(G_n)|_c \geq |\partial_{\mathsf{E}} U|_c/2 \tag{6.4}$$

*for all $n$ and all finite $U \subset \mathsf{V}(G) \setminus \mathsf{V}(G_n)$ and*

$$\kappa\big(G \backslash \mathsf{V}(G_n), t\big) \geq \kappa(G, t)/2 \tag{6.5}$$

*for all $n$ and $t > 0$.*

*Proof.* Given a finite connected $K \subset \mathsf{V}(G)$ containing $o$, let $W(K)$ minimize $|\partial_{\mathsf{E}} L|_c$ over all finite sets $L \subset \mathsf{V}(G)$ that contain $K \cup \partial_{\mathsf{V}} K$ (here $\partial_{\mathsf{V}} K$ denotes the vertices outside of $K$ neighboring some vertex in $K$); such a set $W(K)$ exists by our two hypotheses on $(G, c)$. Moreover, $W(K)$ is connected since $K$ is. Let $G' := G \backslash W(K)$ and write $\partial'_{\mathsf{E}}$ for the edge-boundary operator in $G'$. If $U$ is a finite subset of vertices in $G'$, then $|\partial'_{\mathsf{E}} U|_c \geq |\partial_{\mathsf{E}} U|_c/2$, since if not, we would have $|\partial_{\mathsf{E}}(W(K) \cup U)|_c < |\partial_{\mathsf{E}} W(K)|_c$, which contradicts the definition of $W(K)$. Thus, $\kappa(G', t) \geq \kappa(G, t)/2$ for all $t > 0$. It follows that we may define an exhaustion $G_n$ as the subgraphs induced by a sequence $K_n$ defined recursively by $K_1 := W\big(\{o\}\big)$ and $K_{n+1} := W(K_n)$. ∎

## §7. Isoperimetric Criterion for One End.

We now state and prove our general condition for the $\mathsf{WSF}$ trees to have one end a.s. After the proof, the range of its applicability will be discussed.

THEOREM 7.1. *Suppose that $G$ is an infinite connected locally finite network. Let $\kappa(t) := \kappa(G, t)$. Suppose that $s_0 := \inf_{s>0} \kappa(s) > 0$ and*

$$\sum_{k \geq 0} \frac{1}{\kappa(s_k)} < \infty\,, \tag{7.1}$$

*where $s_k$ is defined recursively by $s_{k+1} := s_k + \kappa(s_k)/2$ for $k \in \mathbb{N}$. Then $\mathsf{WSF}$-a.s. every tree has only one end.*

*Proof.* The proof will be based on an appeal to Proposition 5.1. Note that the hypothesis (7.1) certainly guarantees $\lim_{t\to\infty} \kappa(G, o, t) \geq \lim_{t\to\infty} \kappa(G, t) = \infty$ for every $o \in \mathsf{V}$. Also, $s_0 > 0$ guarantees that $\pi(K) = \infty$ for $|K| = \infty$. Thus, there exists an exhaustion $\langle G_n \rangle$ satisfying the conclusion of Lemma 6.3, specifically, (6.5). For $n \geq 0$, define $L_n := G \backslash \mathsf{V}(G_n)$. Consider some nonempty finite $A \subset \mathsf{V}(L_n)$. Define $r_0 := |\partial^{\infty}_{\mathsf{E}(L_n)}(A)|_c$ and $r_{k+1} := r_k + \kappa(L_n, A, r_k)/2$ recursively. By (6.5), $r_{k+1} \geq r_k + s_0/4$. Therefore, we get in particular $r_4 \geq s_0$. We now prove by induction that if $r_k \geq s_m$ for some $k, m \in \mathbb{N}$, then $r_{k+2\ell} \geq s_{m+\ell}$ for every $\ell \in \mathbb{N}$. Indeed, by (6.5), we have $\kappa(L_n, r) \geq \kappa(G, r)/2$ for all $r > 0$. Apply this to $r := r_{k+2\ell+1}, r_{k+2\ell}$ in turn to obtain

$$\begin{aligned} r_{k+2\ell+2} &\geq r_{k+2\ell+1} + \kappa(G, r_{k+2\ell+1})/4 \\ &\geq r_{k+2\ell} + \kappa(G, r_{k+2\ell})/4 + \kappa(G, r_{k+2\ell+1})/4 \\ &\geq s_{m+\ell} + \kappa(G, s_{m+\ell})/2 = s_{m+\ell+1} \end{aligned}$$

if $r_{k+2\ell} \ge s_{m+\ell}$, which completes the induction step. The above results in particular give $r_k \ge s_\ell$ when $k \ge 4 + 2\,\ell$.

Given $\epsilon > 0$, choose $m$ so that $\sum_{k=m}^{\infty} 8/\kappa(G, s_k) < \epsilon$ and choose $t$ so that $\kappa(G, t) \ge 2s_m$. Given any $n$ and any finite $A \subset \mathsf{V}(L_n)$ with $\pi(A) > t$, we claim that $\mathcal{R}(A \leftrightarrow \infty; L_n) < \epsilon$, thereby establishing (5.2). To see this, note that by (6.1) and $\kappa(L_n, r) \ge \kappa(G, r)/2$ we have

$$r_0 = |\partial^{\infty}_{\mathsf{E}(L_n)}(A)|_c \ge \kappa(L_n, t) \ge \kappa(G, t)/2 \ge s_m \,.$$

Therefore, Theorem 6.1 and the above yield that

$$\mathcal{R}(A \leftrightarrow \infty; L_n) \le \sum_{k=0}^{\infty} \frac{2}{\kappa(L_n, r_k)} \le \sum_{k=0}^{\infty} \frac{4}{\kappa(G, r_k)} \le \sum_{k=m}^{\infty} \frac{8}{\kappa(G, s_k)} < \epsilon \,.$$

A similar argument implies (5.1) and completes the proof: Let $A \subset \mathsf{V}(L_n)$ be a singleton. Since we have $r_4 \ge s_0$, we obtain

$$\mathcal{R}(A \leftrightarrow \infty; L_n) \le \sum_{k=0}^{\infty} \frac{2}{\kappa(L_n, r_k)} \le \sum_{k=0}^{\infty} \frac{4}{\kappa(G, r_k)} \le \frac{16}{s_0} + \sum_{k=0}^{\infty} \frac{8}{\kappa(G, s_k)} \,. \qquad \blacksquare$$

Which networks satisfy the hypothesis of Theorem 7.1? Of course, all non-amenable transitive networks do. (By definition, a network $G$ is non-amenable if $\inf_{t>0} \kappa(G, t)/t > 0$. It is transitive if its automorphism group acts transitively on its set of vertices. It is quasi-transitive if the vertex set breaks up into only finitely many orbits under the action of the automorphism group.) Although it is not obvious, so do all quasi-transitive transient graphs. To show this, we begin with the following slight extension of a result due to Coulhon and Saloff-Coste (1993) and Saloff-Coste (1995). Define the **internal vertex boundary** of a set $K$ as $\partial^{\mathrm{int}}_{\mathsf{V}} K := \{x \in K \,;\, \exists y \notin K \;\; y \sim x\}$.

A locally compact group is called **unimodular** if its left Haar measure is also right invariant. We call a graph $G$ **unimodular** if its automorphism group $\mathrm{Aut}(G)$ is unimodular, where $\mathrm{Aut}(G)$ is given the weak topology generated by its action on $G$. See Benjamini, Lyons, Peres, and Schramm (1999) for more details on unimodular graphs.

Lemma 7.2. *Let $G$ be an infinite unimodular transitive graph. Let $\rho(m)$ be the smallest radius of a ball in $G$ that contains at least $m$ vertices. Then for all finite $K \subset \mathsf{V}$, we have*

$$\frac{|\partial^{\mathrm{int}}_{\mathsf{V}} K|}{|K|} \ge \frac{1}{2\,\rho(2\,|K|)} \,.$$

*Proof.* Fix a finite set $K$ and let $\rho := \rho(2\,|K|)$. Let $B'(x, r)$ be the ball of radius $r$ about $x$ excluding $x$ itself and let $b := |B'(x, \rho)|$. For $x, y, z \in \mathsf{V}(G)$, define $f_k(x, y, z)$ as the proportion of shortest paths from $x$ to $z$ whose $k$th vertex is $y$. Let $S(x,r)$ be the sphere of radius $r$ about $x$. Write $q_r := |S(x,r)|$. Let $F_{r,k}(x,y) := \sum_{z\in S(x,r)} f_k(x,y,z)$. Clearly, $\sum_y F_{r,k}(x,y) = q_r$ for every $x \in \mathsf{V}(G)$ and $r \geq 1$. Since $F_{r,k}$ is invariant under the diagonal action of the automorphism group of $G$, the Mass-Transport Principle (Benjamini, Lyons, Peres, and Schramm (1999)) gives $\sum_x F_{r,k}(x,y) = q_r$ for every $y \in \mathsf{V}(G)$ and $r \geq 1$. Now we consider the sum

$$Z_r := \sum_{x\in K} \sum_{z\in S(x,r)\setminus K} \sum_{y\in\partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} f_k(x,y,z)\,.$$

If we fix $x \in K$ and $z \in S(x,r) \setminus K$, then the inner double sum is at least 1, since if we fix any shortest path from $x$ to $z$, it must pass through $\partial_{\mathsf{V}}^{\mathrm{int}} K$. It follows that

$$Z_r \geq \sum_{x\in K} |S(x,r) \setminus K|\,,$$

whence, by the definitions of $\rho$ and $b$,

$$Z := \sum_{r=1}^{\rho} Z_r \geq \sum_{x\in K} |B'(x,\rho) \setminus K| \geq \sum_{x\in K} |B'(x,\rho)|/2 = |K|b/2\,.$$

On the other hand, if we do the summation in another order, we find

$$\begin{aligned}
Z_r &= \sum_{y\in\partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x\in K} \sum_{z\in S(x,r)\setminus K} f_k(x,y,z) \\
&\leq \sum_{y\in\partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x\in \mathsf{V}(G)} \sum_{z\in S(x,r)} f_k(x,y,z) \\
&= \sum_{y\in\partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} \sum_{x\in \mathsf{V}(G)} F_{r,k}(x,y) \\
&= \sum_{y\in\partial_{\mathsf{V}}^{\mathrm{int}} K} \sum_{k=0}^{r-1} q_r = |\partial_{\mathsf{V}}^{\mathrm{int}} K|\, r\, q_r\,.
\end{aligned}$$

Therefore,

$$Z \leq \sum_{r=1}^{\rho} |\partial_{\mathsf{V}}^{\mathrm{int}} K|\, r\, q_r \leq |\partial_{\mathsf{V}}^{\mathrm{int}} K|\, \rho\, b\,.$$

Comparing these upper and lower bounds for $Z$, we get the desired result. ∎

An immediate consequence is the following bound:

Corollary 7.3. *If $G$ is a connected quasi-transitive graph with balls of radius $n$ having at least $c\,n^3$ vertices for some constant $c > 0$, then*

$$\kappa(G, t) \geq c' t^{2/3}$$

*for some constant $c' > 0$ and all $t \geq 1$.*

*Proof.* First assume that $G$ is transitive. If $G$ is also amenable, then it is unimodular by Soardi and Woess (1990). Thus, the inequality follows from Lemma 7.2. If $G$ is not amenable, then the inequality is trivial by definition.

Now, suppose that $G$ is only quasi-transitive. Pick some vertex $o \in \mathsf{V}(G)$, and let $\mathsf{V}'$ denote the orbit of $o$ under the automorphism group of $G$. Let $r \in \mathbb{N}$ be such that every vertex in $G$ is within distance $r$ of some vertex in $\mathsf{V}'$. Let $G'$ be the graph on $\mathsf{V}'$ where two vertices are adjacent if and only if the distance between them in $G$ is at most $2\,r + 1$. It is easy to verify that $G'$ satisfies the assumptions of the corollary and is also transitive. Consequently, we have $\kappa(G', t) \geq c''\,t^{2/3}$ for some $c'' > 0$. The result now easily follows for $G$ as well. ∎

Since all quasi-transitive transient graphs have at least cubic volume growth by a theorem of Gromov (1981) and Trofimov (1985), we may use Corollary 7.3, (6.3) and Theorem 7.1 to obtain:

Theorem 7.4. *If $G$ is a transient quasi-transitive network or is a non-amenable network with $\inf_{x \in \mathsf{V}(G)} \pi(x) > 0$, then* WSF*-a.s. every tree has only one end.*

In particular, we arrive at the following results that extend Theorem 12.7 of BLPS (2001):

Theorem 7.5. *Suppose that $G$ is a bounded-degree graph that is roughly isometric to $\mathbb{H}^d$ for some $d \geq 2$. Then the* WSF *of $G$ has infinitely many trees a.s., each having one end a.s. If $d = 2$ and $G$ is planar, then the* FSF *of $G$ has one tree with infinitely many ends a.s.*

Recall that when $d > 2$ in the above setting, we have FSF = WSF. (This follows from Theorems 7.3 and 12.6 in BLPS (2001).)

*Proof.* Rough isometry preserves non-amenability (Woess (2000), Theorem 4.7) and $\mathbb{H}^d$ is non-amenable when $d \geq 2$. Hence, $G$ is also non-amenable, and Theorem 7.4 implies that the WSF has one end per tree a.s. The fact that the WSF has infinitely many trees a.s. follows from Theorem 9.4 of BLPS (2001) and the exponential decay of the return probabilities for random walks on non-amenable bounded-degree graphs. (Stronger results are proved in Theorems 13.1 and 13.7 of BLPS (2001).)

When $d = 2$ and $G$ is planar, it is not hard to see that the planar dual of $G$ also has bounded degree. This implies that it is roughly isometric to $G$, hence to $\mathbb{H}^2$. Thus, the above conclusions apply also to the WSF of the dual of $G$. Therefore, the claims about the FSF in the planar setting follow from Proposition 12.5 in BLPS (2001), which relates the properties of the WSF on the dual to the properties of the FSF on $G$. ∎

Finally, we conclude with some new questions that arise from our results.

QUESTION 7.6. If $G$ and $G'$ are roughly isometric graphs and the wired spanning forest in $G$ has only one end in each tree a.s., then is the same true in $G'$?

QUESTION 7.7. Is the probability that each tree has only one end equal to either 0 or 1 for both WSF and FSF?

Consider the subgraph $G$ of $\mathbb{Z}^6$ spanned by the vertices

$$\big(\mathbb{Z}^5 \times \{0\}\big) \cup \big(\{(0,0,0,0,0),(2,0,0,0,0)\} \times \mathbb{N}\big)\,.$$

This graph is obtained from $\mathbb{Z}^5$ by adjoining two copies of $\mathbb{N}$. Let $\mathfrak{F}$ denote a sample from the WSF on $G$. With positive probability, $x := (0,0,0,0,0,0)$ and $y := (2,0,0,0,0,0)$ are in the same component of $\mathfrak{F}$. In that case, a.s. that component has 3 ends while all other trees in $\mathfrak{F}$ have one end. Also with positive probability, $x$ and $y$ are in two distinct components of $\mathfrak{F}$. In that case, a.s. each of these components have two ends while all other components have one end. Thus, in particular, there are graphs such that the existence of a tree in the WSF with precisely two ends has probability in $(0,1)$. This example naturally leads to the following question.

QUESTION 7.8. Define the **excess** of a tree as the number of ends minus one. Is the sum of the excesses of the components of the WSF equal to some constant a.s.?

Note that the total number of ends of all trees is tail measurable, hence an a.s. constant by Theorem 8.3 of BLPS (2001). Since the number of trees is also a.s. constant by Theorem 9.4 of BLPS (2001), it follows that Question 7.8 has a positive answer when the number of trees is finite.

QUESTION 7.9. Does our main result, Theorem 7.1, hold when $\kappa(G,t)$ is replaced by $\kappa(G,o,t)$ for some fixed basepoint $o$? In particular, does it hold when $\inf_t \kappa(G,o,t)/t > 0$? If so, this would provide a new proof that Question 15.4 of BLPS (2001) has a positive answer in the case of bounded-degree Galton-Watson trees (by Corollary 1.3 of Chen and

Peres (2004)). This case of Galton-Watson trees has already been established by Aldous and Lyons (2007).

QUESTION 7.10. Suppose that $G$ is a quasi-transitive recurrent graph that is not roughly isometric to $\mathbb{Z}$. Then BLPS (2001) proves that the uniform spanning tree of $G$ has only one end a.s. That proof is rather long; can it be simplified and the result generalized?

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