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# Bounding a Random Environment for Two-dimensional Edge-reinforced Random Walk

Franz Merkl
Mathematical Institute
University of Munich
Theresienstr. 39, D-80333 Munich, Germany
Email: merkl@mathematik.uni-muenchen.de
http://www.mathematik.uni-muenchen.de/~merkl

Silke W.W. Rolles
Zentrum Mathematik, Bereich M5,
Technische Universität München
D-85747 Garching bei München, Germany
Email: srolles@ma.tum.de
http://www-m5.ma.tum.de/pers/srolles

**Abstract**

We consider edge-reinforced random walk on the infinite two-dimensional lattice. The process has the same distribution as a random walk in a certain strongly dependent random environment, which can be described by random weights on the edges. In this paper, we show some decay properties of these random weights. Using these estimates, we derive bounds for some hitting probabilities of the edge-reinforced random walk .

**Key words:** Reinforced random walk, random environment.

**AMS 2000 Subject Classification:** Primary 82B41; Secondary: 60K35, 60K37.

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# 1 Introduction

**Definition of the model.** Linearly edge-reinforced random walk (ERRW) on $\mathbb{Z}^2$ is the following model: Consider the two-dimensional integer lattice $\mathbb{Z}^2$ as a graph with edge set

$$E = \{\{x, y\} \in \mathbb{Z}^2 \times \mathbb{Z}^2 : \; |x - y| = 1\}. \tag{1.1}$$

Here, $|\cdot|$ denotes the Euclidean norm. In particular, the edges are undirected. Fix a vertex $v_0 \in \mathbb{Z}^2$ and a positive number $a > 0$. A non-Markovian random walker starts in $X_0 = v_0$. At every discrete time $t \in \mathbb{N}_0$, it jumps from its current position $X_t$ to a neighboring vertex $X_{t+1}$ in $\mathbb{Z}^2$, $|X_{t+1} - X_t| = 1$. The law $P_{v_0,a}$ of the random walker is defined in terms of the time-dependent weights

$$w_e(t) = a + \sum_{s=0}^{t-1} 1_{\{e=\{X_s, X_{s+1}\}\}}, \quad e \in E. \tag{1.2}$$

The weight $w_e(t)$ of edge $e$ at time $t$ equals the number of traversals of $e$ up to time $t$ plus the initial weight $a$. Thus, $w_e(t)$ increases linearly in the number of crossings of $e$. The transition probability $P_{v_0,a}[\{X_t, X_{t+1}\} = e \mid X_0, \dots, X_t]$ is proportional to $w_e(t)$ for all edges $e \ni X_t$:

$$P_{v_0,a}[\{X_t, X_{t+1}\} = e \mid X_0, \dots, X_t] = \frac{w_e(t)}{\sum_{e' \ni X_t} w_{e'}(t)} 1_{\{e \ni X_t\}}. \tag{1.3}$$

This model was introduced by Diaconis [Dia88]. Throughout this paper, "edge-reinforced random walk" will always mean "*linearly* edge-reinforced random walk", as defined above. We do not consider any other reinforcement scheme.

**Main open problem and previous results for $\mathbb{Z}^d$.** As Pemantle [Pem88] remarks, Diaconis asked whether this process on $\mathbb{Z}^2$ and more generally on $\mathbb{Z}^d$, $d \geq 2$, is recurrent. This question is still open almost twenty years after it was raised. Even worse, proving *anything* about edge-reinforced random walk on $\mathbb{Z}^2$ tends to be hard. To our best knowledge, almost nothing is known about edge-reinforced random walk on $\mathbb{Z}^d$ for any $d \geq 2$, with only a few exceptions:

1. Sellke [Sel94] has proved for edge-reinforced random walk on $\mathbb{Z}^d$ in any dimension $d$, that almost surely the range is infinite, and that the random walker hits each coordinate plane $x_k = 0$, $k = 1, \dots, d$ almost surely infinitely often.

2. Recently, in [MR07c], we have shown that the edge-reinforced random walk on any locally finite graph has the same distribution as a random walk in a random environment given by random, time-independent, strictly positive weights $(x_e)_e$ on the edges. As a consequence, the edge-reinforced random walk on any infinite, locally finite, connected graph visits all vertices infinitely often with probability one if and only if it returns to its starting point almost surely.

Up to now, except for the present paper, there seem to be no further rigorous results about edge-reinforced random walk on $\mathbb{Z}^2$. All attempts to apply coupling methods or monotonicity arguments to this problem have failed so far. Monte Carlo simulations seem to indicate non-rigorously that - at least for some initial weights - edge-reinforced random walk on $\mathbb{Z}^2$ typically does not stay very close to its starting point. Rather, the random walker spends most of its time at some, possibly infinitely many, small islands located at varying distance from the starting point. This indicates that the weights $(x_e)_e$ in (b) might vary in rather small regions over many orders of magnitude.

**Intention of this paper.** In this paper, we show for an edge-reinforced random walk on $\mathbb{Z}^2$ that at least in a weak, stochastic sense, the weights $x_e$ in (b) converge to 0 as $e$ gets infinitely far from the origin. This provides the first analysis of the global behaviour of the random environment over $\mathbb{Z}^2$. Bounds on the weights $x_e$ induce information for the law of the edge-reinforced random walk. For example, our bounds allow for the first time the estimation of hitting probabilities of the paths of edge-reinforced random walk on $\mathbb{Z}^2$.

The present paper is independent of our papers [MR07a], [MR05], [Rol06] on edge-reinforced random walk on ladders, as the transfer operator technique used there breaks down in two dimensions.

**Previous results on other graphs.** For graphs where the edge-reinforced random walk is recurrent, a representation different from the one in [MR07c] follows from the paper [DF80] by Diaconis and Freedman, which was written before edge-reinforced random walk was introduced. For finite graphs, Coppersmith and Diaconis [CD86] discovered an explicit, but complicated formula for the joint law of the fraction of time spent on the edges. In fact, this joint law just equals the law of the weights $(x_e)_e$ normalized appropriately; see [Rol03]. A proof of the formula describing the joint law of $(x_e)_e$ is published in [KR00]. The random environment is strongly dependent, unless the graph is a tree-graph.

Pemantle [Pem88] examined the model on infinite trees; in particular, he showed that the process *on tree graphs* has the same distribution as a random walk in an *independent* random environment.

On infinite ladders, for sufficiently large initial weights $a$, the random environment can be described as an infinite-volume Gibbs measure. It arises as an infinite-volume limit of finite-volume Gibbs measures; see [MR05], [Rol06], [MR07a]. The finite-volume Gibbs measures are just a reinterpretation of the formula of Coppersmith and Diaconis in the version described by Keane and Rolles. This reinterpretation has been used in the above references to prove recurrence of the process on ladders and also to analyze the asymptotic behavior of the edge-reinforced random walk in more detail.

In [MR07b], we have recently proven recurrence of the edge-reinforced random walk on some fully two-dimensional graphs different from $\mathbb{Z}^2$. In the present paper, we push the method used for these graphs to its limits: Unlike in [MR07b], in the present paper, we prove bounds for the random weights in infinite volume, and we prove strong bounds for the expected logarithm of the random weights. These bounds require more sophisticated deformations than in [MR07b]. However, the present paper can be read independently of [MR07b].

More information about the history of linearly edge-reinforced random walk can be found in [MR06b].

# 2 Results

We present results for edge-reinforced random walk on $\mathbb{Z}^2$ ("infinite volume"), but also on finite boxes in $\mathbb{Z}^2$ with periodic boundary conditions ("finite volume"). The latter results are used in the proof of the former ones.

All constants like $\beta(a), c_1, c_2, \ldots$ keep their meaning throughout the whole article.

## 2.1 Hitting probabilities for ERRW

For $v \in \mathbb{Z}^2$, let $\tau_v := \inf\{n \geq 1 : X_n = v\}$ denote the first time $\geq 1$ when the random walker visits $v$.

**Theorem 2.1** (Hitting probabilities for ERRW)**.** *For all $a > 0$, there are $c_1(a) > 0$ and $\beta(a) > 0$, such that for all $v \in \mathbb{Z}^2 \setminus \{0\}$, the following hold:*

1. *The probability to reach the vertex $v$ before returning to the vertex $0$ is bounded by*
$$P_{0,a}[\tau_v < \tau_0] \leq c_1(a)|v|^{-\beta(a)}. \tag{2.1}$$
2. *For all $n \geq 1$, the probability that the random walker visits the vertex $v$ at time $n$ satisfies the same bound*
$$P_{0,a}[X_n = v] \leq c_1(a)|v|^{-\beta(a)}. \tag{2.2}$$

Although these estimates are weak if compared with what is known for *simple* random walk on $\mathbb{Z}^2$, to our knowledge, no bounds of any similar type were known before.

If we knew $\beta(a) > 1$, the estimate (2.1) would imply recurrence. However, our estimates yield only

$$\beta(a) = \left[1024e(1+4a)(1+(e \cdot \max\{\lceil 1/\sqrt{a}\rceil, 2\} \log 2)^{-1})\right]^{-1}. \tag{2.3}$$

In particular, $\beta(a)$ is a decreasing function of $a$ with

$$\beta(a) \stackrel{a\to 0}{\longrightarrow} \frac{1}{1024e} \qquad \text{and} \qquad \beta(a) \stackrel{a\to\infty}{\longrightarrow} 0. \tag{2.4}$$

Thus recurrence is still an open problem.

Furthermore, our estimates yield

$$c_1(a) = 2 \cdot (6\sqrt{2} \max\{\lceil 1/\sqrt{a}\rceil, 2\})^{\beta(a)}. \tag{2.5}$$

This fulfills

$$c_1(a) \stackrel{a\to 0}{\longrightarrow} \infty \qquad \text{and} \qquad c_1(a) \stackrel{a\to\infty}{\longrightarrow} 2. \tag{2.6}$$

### 2.2 Bounds in infinite volume

Set $\Omega = (0,\infty)^E$. For $x \in \Omega$, let $Q_{v_0,x}$ denote the law of a Markovian nearest neighbor random walk on $(\mathbb{Z}^2, E)$ with starting point $v_0$ in a time-independent environment given by the edge weights $x$. Given this random walk is at $u$ and given the past of the random walk, the probability it jumps to the neighboring vertex $v$ is proportional to the weight $x_e$ of the edge $e = \{u, v\}$.

Edge-reinforced random walk on $\mathbb{Z}^2$ can be represented as a random walk in a random environment (RWRE). More precisely:

**Theorem 2.2** (ERRW as RWRE, Theorem 2.2 in [MR07c])**.** *Let $a > 0$. There is a probability measure $\mathbb{Q}_{0,a}$ on $\Omega$, such that for all events $A \subseteq (\mathbb{Z}^2)^{\mathbb{N}_0}$, one has*

$$P_{0,a}[A] = \int_\Omega Q_{0,x}[A]\, \mathbb{Q}_{0,a}(dx). \tag{2.7}$$

It is not known whether $\mathbb{Q}_{0,a}$ is unique up to normalization of the edge weights $x$. In this article, we prove some decay properties of the random environment. These results hold for at least one choice of $\mathbb{Q}_{0,a}$. For $x \in \Omega$ and $v \in \mathbb{Z}^2$, set

$$x_v = \sum_{e \ni v} x_e. \tag{2.8}$$

The following two theorems show that in some weak probabilistic sense, the ratios of weights $x_v/x_0$ tend to zero as $|v| \to \infty$. We phrase two formal versions of this statement. The first version only cares about the expected logarithms of ratios $x_v/x_0$. In this version, we get a fast divergence to $-\infty$ as $|v| \to \infty$ if only $a > 0$ is small enough.

**Theorem 2.3** (Decay of the expected log weights)**.** *There exist functions $c_2, c_3$ : $(0,\infty) \to (0,\infty)$ with $c_2(a) \to \infty$ as $a \downarrow 0$, such that the following holds: For all $a > 0$ and all $v \in \mathbb{Z}^2 \setminus \{0\}$,*

$$E_{\mathbb{Q}_{0,a}}\left[\log \frac{x_v}{x_0}\right] \le c_3(a) - c_2(a) \log |v|. \tag{2.9}$$

Such a fast decay of the logarithms of the weights in a stochastic sense has not been proven before on *any* fully two-dimensional graph. The methods used in [MR07b] do not suffice to prove such a bound.

However, Theorem 2.3 does not imply weak convergence of $x_v/x_0$ to zero. Weak convergence is stated among others in the following theorem, but our bound for the rate of convergence is not as strong as in the preceding theorem.

**Theorem 2.4** (Decay of the expected weight power)**.** *For all $a > 0$, there are constants $c_1(a) > 0$ and $\beta(a) > 0$, such that for all $v \in \mathbb{Z}^2 \setminus \{0\}$, the following holds:*

$$E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \le c_1(a)|v|^{-\beta(a)}. \tag{2.10}$$

*In particular, $x_v/x_0$ converges weakly to zero as $|v| \to \infty$.*

The estimates for the hitting probabilities of the edge-reinforced random walk in Theorem 2.1 are derived from the bounds in the preceding theorem.

Let $e, f \in E$ be two neighboring edges, i.e. two edges containing a common vertex $v$. The random variables $\log(x_e/x_f)$ with respect to $\mathbb{Q}_{0,a}$ are tight with exponential tails, uniformly in the choice of the edges $e$ and $f$. This is stated formally in the following theorem:

**Theorem 2.5** (Tightness of ratios of neighboring edge weights)**.** *For all $a > 0$ and all $\alpha \in (0, a/2)$, one has*

$$\sup_{\substack{e,f\in E\\ e\cap f\neq\emptyset}} E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_e}{x_f}\right)^{\alpha}\right] < \infty. \tag{2.11}$$

## 2.3 Uniform bounds in finite volume

All our infinite-volume results are derived from uniform finite-volume analogs for edge-reinforced random walk on finite boxes. "Uniform" here means "uniform in the size of the finite box".

We consider a $(2N+1)\times(2N+1)$ box

$$V^{(N)} = \mathbb{Z}^2/(2N+1)\mathbb{Z}^2 \tag{2.12}$$

with periodic boundary conditions. For $v \in \mathbb{Z}^2$, let $v^{(N)} = v + (2N+1)\mathbb{Z}^2$ denote the class of $v$ in $V^{(N)}$. If there is no risk of confusion, we identify $V^{(N)}$ with the subset $\tilde{V}^{(N)} = [-N, N]^2 \cap \mathbb{Z}^2$ of $\mathbb{Z}^2$. Let

$$E^{(N)} = \{\{u^{(N)}, v^{(N)}\} :\ \{u, v\} \in E\}. \tag{2.13}$$

For an equivalence class $v^{(N)} = v + (2N+1)\mathbb{Z}^2$, set

$$\left|v^{(N)}\right| = \min\{|v + (2N+1)z| :\ z \in \mathbb{Z}^2\}. \tag{2.14}$$

This is just the Euclidean distance of $v^{(N)}$ to the origin, viewed as an element of $\tilde{V}^{(N)}$.

Just as in the infinite-volume case, for $v_0 \in V^{(N)}$ and $a > 0$, let $P^{(N)}_{v_0,a}$ denote the law of edge-reinforced random walk on $(V^{(N)}, E^{(N)})$ with starting point $v_0$ and constant initial weights $a$. Set $\Omega^{(N)} = (0,\infty)^{E^{(N)}}$. For $x \in \Omega^{(N)}$, let $Q^{(N)}_{v_0,x}$ denote the law of a random walk on $(V^{(N)}, E^{(N)})$ with starting point $v_0$ in a time-independent environment given by weights $x$. Note that multiplying all components of $x$ by the same (possibly $x$-dependent) scaling factor $\alpha$ does not change the law of the corresponding random walk. The following finite-volume analog of Theorem 2.2 is well-known:

**Theorem 2.6** (ERRW as RWRE on finite boxes, Theorem 3.1 in [Rol03])**.**
*Let $a > 0$ and $v_0 \in V^{(N)}$. There is a probability measure $\mathbb{Q}^{(N)}_{v_0,a}$ on $\Omega^{(N)}$, such that for all events $A \subseteq (V^{(N)})^{\mathbb{N}_0}$, one has*

$$P^{(N)}_{v_0,a}[A] = \int_{\Omega^{(N)}} Q^{(N)}_{v_0,x}[A]\, \mathbb{Q}^{(N)}_{v_0,a}(dx). \tag{2.15}$$

*Up to an arbitrary normalization of the random edge weights $x \in \Omega^{(N)}$, the law $\mathbb{Q}^{(N)}_{v_0,a}$ of the random environment is unique.*

In [Dia88] and [KR00], the distribution $\mathbb{Q}^{(N)}_{v_0,a}$ is described explicitly; see also Lemma 4.1, below. The weaker statement that the edge-reinforced random walk on $(V^{(N)}, E^{(N)})$ is a mixture of Markov chains follows already from [DF80].

The following two theorems are finite-volume analogs of Theorems 2.3 and 2.4. The bounds are uniform in the size of a finite box.

**Theorem 2.7** (Decay of the expected log weights in finite volume)**.** *There exist functions $c_2, c_3 : (0,\infty) \to (0,\infty)$ with $c_2(a) \to \infty$ as $a \downarrow 0$, such that the following holds: For all $a > 0$, all $N \in \mathbb{N}$, and all $v \in V^{(N)} \setminus \{0\}$,*

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log \frac{x_v}{x_0}\right] \leq c_3(a) - c_2(a) \log |v|. \tag{2.16}$$

**Theorem 2.8** (Decay of the expected weight power in finite volume)**.** *For all initial weights $a > 0$, there are $c_1(a) > 0$ and $\beta(a) > 0$, such that for all $N \in \mathbb{N}$ and all $v \in V^{(N)} \setminus \{0\}$, the following holds:*

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \leq c_1(a)|v|^{-\beta(a)}. \tag{2.17}$$

The analog of Theorem 2.5 holds also in finite volume, uniformly in the size $N$ of the box:

**Lemma 2.9.** *For all $a > 0$ and all $\alpha \in (0, a/2)$, one has*

$$\sup_{\substack{N\in\mathbb{N}\\ N>1}} \max_{\substack{e,f\in E^{(N)}\\ e\cap f\neq\emptyset}} E_{\mathbb{Q}^{(N)}_{0,a}}\left[\left(\frac{x_e}{x_f}\right)^{\alpha}\right] < \infty. \tag{2.18}$$

*Furthermore,*

$$c_4(a) := \sup_{\substack{N\in\mathbb{N}\\ N>1}} \max_{\substack{v,w\in V^{(N)}\\ |v-w|=1}} E_{\mathbb{Q}^{(N)}_{0,a}}\left[\left|\log \frac{x_v}{x_w}\right|\right] < \infty. \tag{2.19}$$

# 3 The big picture – intuitively explained

The main work in the paper consists of showing the bounds of Theorems 2.7 and 2.8, which concern finite boxes. Before giving detailed proofs, let us first explain roughly and informally the ideas coming from statistical mechanics. The purpose of this section is only to give an overview and some intuitive picture. Detailed proofs are given in Sections 4–8; they can be checked even when one ignores this intuition completely.

We start Section 4 by reviewing an explicit description of the mixing measure $\mathbb{Q}^{(N)}_{v_0,a}$ from equation (2.15), (Lemma 4.1, below). For two different starting points 0 and $\ell$ of the random walk paths, the mixing measures are mutually absolutely continuous with the Radon-Nikodym derivative

$$\frac{d\mathbb{Q}^{(N)}_{\ell,a}}{d\mathbb{Q}^{(N)}_{0,a}} = \sqrt{\frac{x_\ell}{x_0}} =: \exp \Sigma_\ell \qquad ((4.8) \text{ below}). \tag{3.1}$$

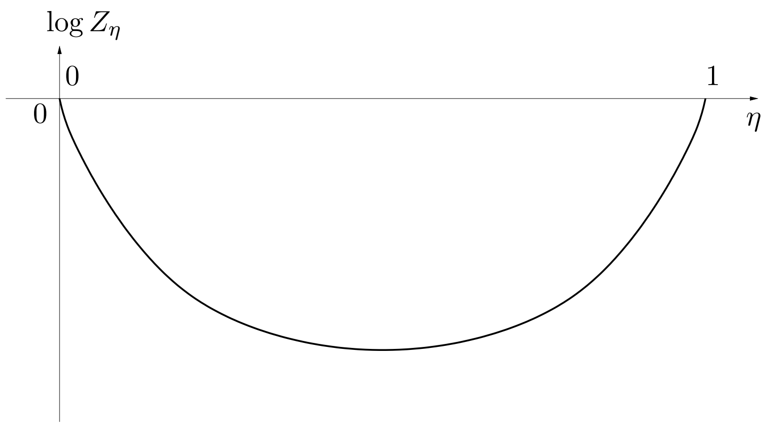
We interpolate between $\mathbb{Q}_{0,a}^{(N)}$ and $\mathbb{Q}_{\ell,a}^{(N)}$ by an exponential family, in other words, by a family of Gibbs distributions

$$d\mathbb{P}_\eta = \frac{1}{Z_\eta} e^{\eta \Sigma_\ell}\, d\mathbb{Q}_{0,a}^{(N)}, \quad 0 \leq \eta \leq 1 \qquad \text{((4.9) below)} \tag{3.2}$$

with a normalizing constant (partition sum) $Z_\eta = E_{\mathbb{Q}_{0,a}^{(N)}}[\exp(\eta\Sigma_\ell)]$. By symmetry considerations, one knows $Z_\eta = Z_{1-\eta}$ and $E_{\mathbb{P}_\eta}[\Sigma_\ell] = -E_{\mathbb{P}_{1-\eta}}[\Sigma_\ell]$; see Figure 1 and formula (8.10), below. Note that

$$\frac{d}{d\eta} \log Z_\eta = E_{\mathbb{P}_\eta}[\Sigma_\ell] \qquad \text{and} \qquad \frac{d^2}{d\eta^2} \log Z_\eta = \mathrm{Var}_{\mathbb{P}_\eta}(\Sigma_\ell). \tag{3.3}$$

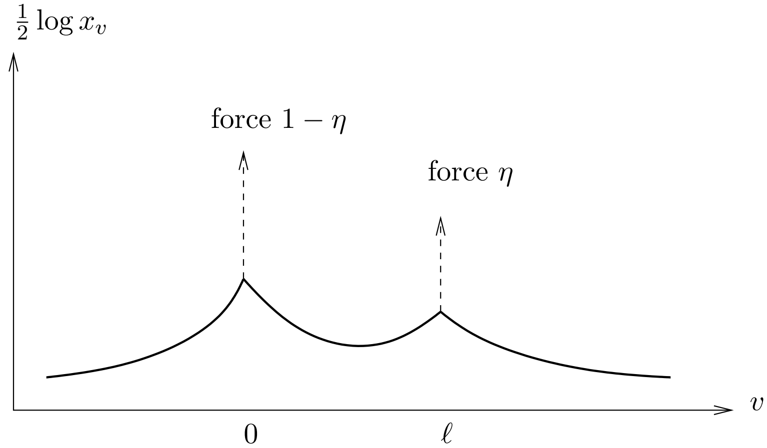
Figure 1: A qualitative picture of $\log Z_\eta$ as a function of $\eta$.




In the language of statistical physics, the random environment described by $\mathbb{P}_\eta$ may be imagined as a rubber membrane, exposed to thermal fluctuations. For a vertex $v$, $\frac{1}{2}\log x_v$ is interpreted as the height of the membrane at the location $v$; thus $\Sigma_\ell = \frac{1}{2}(\log x_\ell - \log x_0)$ gets the interpretation of a height difference.

The parameter $\eta$ may be imagined as an external force: The "membrane" described by $\mathbb{P}_\eta$ is exposed to a force $\eta$ at $\ell$ and force $1-\eta$ at 0; see formula (4.10), below, and figure 2.

Figure 2: An intuitive picture for a rubber membrane described by $\mathbb{P}_\eta$




We are interested in the elastic properties of the rubber membrane:

- On the one hand, this involves how much the expected height difference $E_{\mathbb{P}_\eta}[\Sigma_\ell]$ changes as the force $\eta$ varies from 0 to 1. Indeed, the expected logarithm in (2.16) can be interpreted in terms of this change:

$$E_{\mathbb{Q}_{0,a}^{(N)}}\left[\log\frac{x_\ell}{x_0}\right] = 2E_{\mathbb{P}_0}[\Sigma_\ell] = -2E_{\mathbb{P}_1}[\Sigma_\ell] = E_{\mathbb{P}_0}[\Sigma_\ell] - E_{\mathbb{P}_1}[\Sigma_\ell]. \tag{3.4}$$

- On the other hand, it also involves the physical work (change of the free energy, where the free energy is given by $-\log Z_\eta$) acting on the membrane as the force $\eta$ varies from 1/2 to 0. Indeed, the logarithm of the quantity bounded in (2.17) gets the interpretation of this physical work:

$$\log E_{\mathbb{Q}_{0,a}^{(N)}}\left[\left(\frac{x_\ell}{x_0}\right)^{1/4}\right] = \log Z_{1/2} = \log Z_{1/2} - \log Z_0. \tag{3.5}$$

Both items can be bounded by the same method (Section 6), using a general variational principle for free energies. Let us first explain this variational principle abstractly: Consider two mutually absolutely continuous probability measures, a reference measure $\sigma$ and a thermal measure $\mu$, where

$$d\mu = \frac{1}{Z}e^{-\beta H}d\sigma \tag{3.6}$$

with a Hamiltonian $H$, a temperature $T = \beta^{-1}$, and the normalizing constant

$$Z = E_\sigma[e^{-\beta H}] = \frac{1}{E_\mu[e^{\beta H}]}. \tag{3.7}$$

For probability measures $\nu$, define the free energy

$$F(\nu,\sigma) := U(\nu) - TS(\nu,\sigma) \tag{3.8}$$

with the internal energy $U(\nu) = E_\nu[H]$ and the entropy (with the sign convention used in physics) $S(\nu, \sigma) = -E_\nu[\log(d\nu/d\sigma)]$, whenever these quantities exist. Then the functional $\nu \mapsto F(\nu, \sigma)$ is minimized for $\nu = \mu$, with the minimum $F(\mu, \sigma) = -\beta^{-1} \log Z$. Indeed,

$$F(\nu, \sigma) - F(\mu, \sigma) = -\beta^{-1} S(\nu, \mu) \geq 0. \tag{3.9}$$

We apply this variational principle for $\sigma = \mathbb{P}_\eta$, $\mu = \mathbb{Q}_{0,a}^{(N)}$, $\beta = \eta$, $H = \Sigma_\ell$, and $Z = Z_\eta^{-1}$. In Subsection 5.2, we take some "deformed measure" $\nu = \Pi_{\gamma,\eta}$ ((5.13), below) of $\sigma$, with a deformation parameter $\gamma$. The deformed measure $\nu$ is intended to be close to $\mu$. The physical picture of the rubber membrane exposed to external forces motivates us to define the deformed measure $\nu$, using a deformation map $\Xi_\gamma$ (Definition 5.5, below), which changes the heights of the membrane approximately proportional to the Green's function of the Laplace operator in two dimensions (Definitions 5.2 and 5.3, below). With the right choices, the internal energy changes *linearly* in the deformation parameter,

$$U(\nu) - U(\sigma) = \text{const} \cdot \gamma \qquad \text{(expectation of (6.5), below)}, \tag{3.10}$$

while the the entropy fulfills a *quadratic* bound

$$-S(\nu, \sigma) \leq \text{const}(\ell) \cdot \gamma^2 \qquad \text{((5.15), below)}, \tag{3.11}$$

as it has a minimum at $\gamma = 0$.

The proof of this entropy bound with good estimates for the constant const$(\ell)$ is technically involved (Subsection 5.2). We have much better bounds for the constant const$(\ell)$ in the case $\eta = 1$ than for all other values of $\eta$, since $\mathbb{P}_1$ is much easier to control than general $\mathbb{P}_\eta$; this is the main obstruction against an answer to Diaconis' recurrence question.

By the variational principle, we arrive at

$$\begin{aligned}
&\eta^{-1} \log Z_\eta = -\beta^{-1} \log Z = F(\mu, \sigma) \\
&\leq F(\nu, \sigma) = U(\nu) - \eta^{-1} S(\nu, \sigma) \\
&\leq U(\sigma) + \text{const} \cdot \gamma + \text{const}(\ell) \eta^{-1} \gamma^2 = E_{\mathbb{P}_\eta}[\Sigma_\ell] + \text{const} \cdot \gamma + \text{const}(\ell) \eta^{-1} \gamma^2.
\end{aligned} \tag{3.12}$$

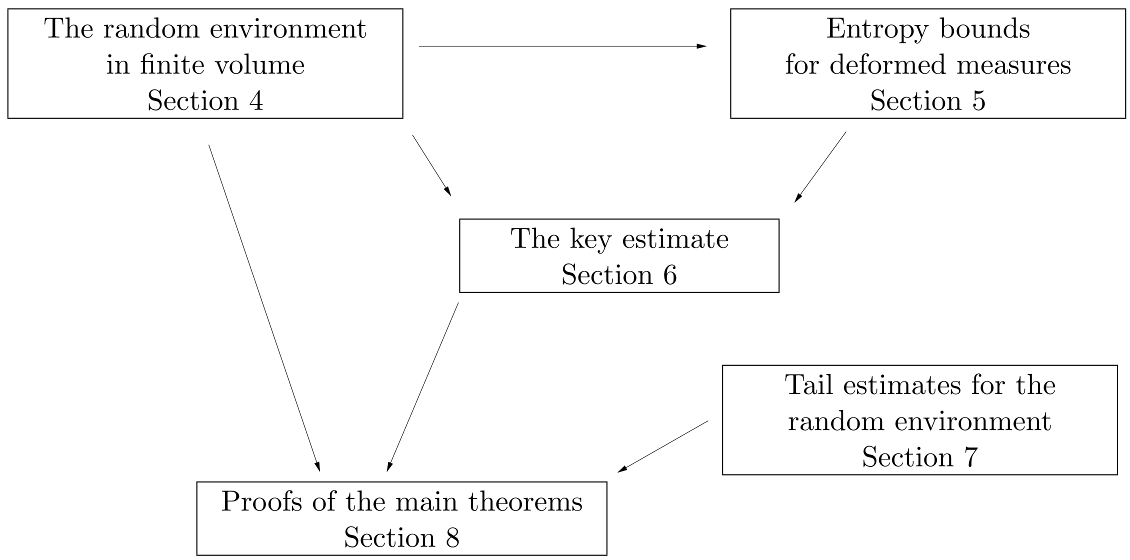
Optimizing over $\gamma$ yields the key estimate (Theorem 6.1 below)

$$\log Z_\eta - \eta E_{\mathbb{P}_\eta}[\Sigma_\ell] \leq -\text{const}'(\ell) \cdot \eta^2. \tag{3.13}$$

In the special cases $\eta = 1$ and $\eta = 1/2$, this yields the bounds for the expectations in (3.4) and (3.5) as claimed in Theorems 2.7 and 2.8, *uniformly in the size of the box* (Subsection 8.1). The infinite volume versions, Theorems 2.3 and 2.4, are then derived in Subsection 8.2 from these theorems by taking an infinite volume limit, using tightness arguments from [MR07c]. The latter are prepared in Section 7. Finally, in Subsection 8.3, we derive bounds for the hitting probabilities of the edge-reinforced random walk.

Figure 3 displays the logical dependence of the different parts of the proofs.

Figure 3: Overview




# 4 The random environment in finite volume

Throughout this section, fix a box size $N \in \mathbb{N}$ and an initial weight $a > 0$.

First, we state a description of the random environment for $G^{(N)} := (V^{(N)}, E^{(N)})$. Let $e_0 \in E^{(N)}$ be any reference edge. Later, from Section 5 on, it will be convenient to choose the edge $e_0$ adjacent to the origin. So far, the arbitrary normalization of the random edge weights $x \in \Omega^{(N)}$ in the representation (2.15) of the edge-reinforced random walk on $G^{(N)}$ as a random walk in a random environment has not been specified. However, in this section, it is convenient to choose the normalization

$$x_{e_0} = 1 \quad \mathbb{Q}^{(N)}_{v_0,a}\text{-a.s.} \tag{4.1}$$

We introduce a reference measure $\rho$ on $\Omega^{(N)}$ to be the following product measure:

$$\rho(dx) = \delta_1(dx_{e_0}) \prod_{e \in E^{(N)} \setminus \{e_0\}} \frac{dx_e}{x_e}. \tag{4.2}$$

Here $\delta_1$ denotes the Dirac measure on $(0, \infty)$ with unit mass at 1. Let $\mathcal{T}^{(N)}$ denote the set of all spanning trees of $(V^{(N)}, E^{(N)})$, viewed as subsets of the set of edges $E^{(N)}$. For a given starting point $v_0 \in V^{(N)}$ of the random walk paths, the distribution $\mathbb{Q}^{(N)}_{v_0,a}$ of the random weights can be described as follows:

**Lemma 4.1** (Random environment for a finite box)**.** *For $v_0 \in V^{(N)}$, the law $\mathbb{Q}^{(N)}_{v_0,a}$ of the random environment is absolutely continuous with respect to the reference measure $\rho$ with density*

$$\frac{d\mathbb{Q}^{(N)}_{v_0,a}}{d\rho}(x) = \frac{1}{z^{(N)}_{v_0,a}} \frac{\prod_{e \in E^{(N)}} x_e^a}{x_{v_0}^{2a} \prod_{v \in V^{(N)} \setminus \{v_0\}} x_v^{2a+1/2}} \sqrt{\sum_{T \in \mathcal{T}^{(N)}} \prod_{e \in T} x_e} \tag{4.3}$$

*with some normalizing constant $z_{v_0,a}^{(N)} > 0$, not depending on the choice of the reference edge $e_0$.*

The claim of the lemma is essentially the formula of Coppersmith and Diaconis [CD86] for the distribution of the random environment, transformed such that one has the normalization $x_{e_0} = 1$. The transformation to this normalization and thus the proof of Lemma 4.1 is given in the appendix (Section 9.1, below).

Now, we consider an interpolation between the random environments $\mathbb{Q}_{0,a}^{(N)}$ and $\mathbb{Q}_{\ell,a}^{(N)}$ associated with two different starting points 0 and $\ell$ in $V^{(N)}$. We introduce an "external force" $\eta\Sigma_\ell$ with the switching parameter $\eta \in [0,1]$. Turning the external force off ($\eta = 0$) corresponds to $\mathbb{Q}_{0,a}^{(N)}$, while turning the external force completely on ($\eta = 1$) corresponds to $\mathbb{Q}_{\ell,a}^{(N)}$. More formally, we proceed as follows:

**Definition 4.2** (Interpolated measures for the environment)**.** *For $\ell \in V^{(N)}$ and $0 \le \eta \le 1$, we define the following probability measure on $\Omega^{(N)}$:*

$$\mathbb{P}_\eta = \mathbb{P}_{\eta,0,\ell,a}^{(N)} := \frac{1}{Z_{\eta,0,\ell,a}^{(N)}} (\mathbb{Q}_{0,a}^{(N)})^{1-\eta} (\mathbb{Q}_{\ell,a}^{(N)})^{\eta} \tag{4.4}$$

*with some normalizing constant $Z_{\eta,0,\ell,a}^{(N)}$. This means:*

$$\frac{d\mathbb{P}_\eta}{d\rho} = \frac{1}{Z_{\eta,0,\ell,a}^{(N)}} \left(\frac{d\mathbb{Q}_{0,a}^{(N)}}{d\rho}\right)^{1-\eta} \left(\frac{d\mathbb{Q}_{\ell,a}^{(N)}}{d\rho}\right)^{\eta}, \tag{4.5}$$

$$Z_{\eta,0,\ell,a}^{(N)} = \int_{\Omega^{(N)}} \left(\frac{d\mathbb{Q}_{0,a}^{(N)}}{d\rho}\right)^{1-\eta} \left(\frac{d\mathbb{Q}_{\ell,a}^{(N)}}{d\rho}\right)^{\eta} d\rho. \tag{4.6}$$

By Hölder's inequality, $Z_{\eta,0,\ell,a}^{(N)}$ is finite. Note that this definition is independent of the choice of the reference measure $\rho$, as long as both $\mathbb{Q}_{0,a}^{(N)}$ and $\mathbb{Q}_{\ell,a}^{(N)}$ are absolutely continuous with respect to $\rho$.

We define the random variable

$$\Sigma_\ell = \Sigma_{0,\ell}^{(N)} := \frac{1}{2} \log \frac{x_\ell}{x_0}. \tag{4.7}$$

Then, the following identity holds:

$$\frac{d\mathbb{Q}_{\ell,a}^{(N)}}{d\mathbb{Q}_{0,a}^{(N)}} = \sqrt{\frac{x_\ell}{x_0}} = \exp \Sigma_\ell. \tag{4.8}$$

In the formula in (4.8), there appears no normalizing constant, because there is a reflection symmetry of the box $V^{(N)}$ which interchanges 0 and $\ell$. Recall that the box $V^{(N)}$ has periodic boundary conditions, and that the normalizing constant $z_{v_0,a}^{(N)}$ does not depend on the choice of the reference edge $e_0$ by Lemma 4.1.

Furthermore, using (4.8), note that $\mathbb{P}_{\eta,0,\ell,a}^{(N)}$ is absolutely continuous with respect to $\mathbb{Q}_{0,a}^{(N)}$ with the density

$$\frac{d\mathbb{P}_\eta}{d\mathbb{Q}_{0,a}^{(N)}} = \frac{1}{Z_{\eta,0,\ell,a}^{(N)}} \left(\frac{x_\ell}{x_0}\right)^{\eta/2} = \frac{\exp(\eta\Sigma_\ell)}{Z_{\eta,0,\ell,a}^{(N)}}, \qquad \text{and} \qquad Z_{\eta,0,\ell,a}^{(N)} = E_{\mathbb{Q}_{0,a}^{(N)}}[\exp(\eta\Sigma_\ell)]. \tag{4.9}$$

Together with (4.3) this implies

$$\frac{d\mathbb{P}_\eta}{d\rho}(x) = \frac{x_0^{(1-\eta)/2} x_\ell^{\eta/2}}{z_{0,a}^{(N)} Z_{\eta,0,\ell,a}^{(N)}} \frac{\prod_{e\in E^{(N)}} x_e^a}{\prod_{v\in V^{(N)}} x_v^{2a+1/2}} \sqrt{\sum_{T\in\mathcal{T}^{(N)}} \prod_{e\in T} x_e}. \tag{4.10}$$

# 5 Entropy bounds for deformed measures

In this section, we introduce a deformation map $\Xi_\gamma : \Omega^{(N)} \to \Omega^{(N)}$, which serves to introduce a deformed measure $\Pi_{\gamma,\eta}$ from $\mathbb{P}_\eta$. This deformed measure has the purpose to be a good approximation for $\mathbb{Q}_{0,a}^{(N)}$ in a variational principle, but in such a way that one can estimate its entropy with respect to $\mathbb{P}_\eta$.

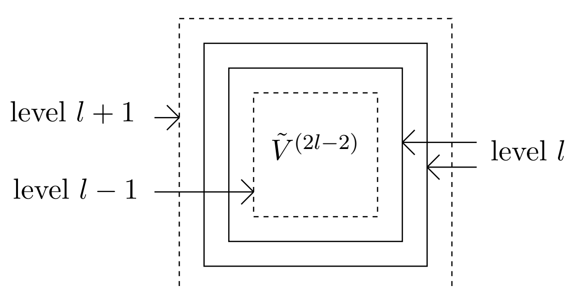
From now on, we assume that the reference edge $e_0$ is adjacent to the origin: $e_0 \ni 0$.

## 5.1 The deformation map

We prepare the definition of the map $\Xi_\gamma$ by introducing an approximation $D_e$ to the Green's function of the Laplace operator in two dimensions.

Let us first introduce some notation: Recall the definition $\tilde{V}^{(l)} = [-l,l]^2 \cap \mathbb{Z}^2$. For $l \in \mathbb{N}_0$, we say that a vertex $\ell \in \mathbb{Z}^2$ is on *level* $l$ and we write $l = \text{level}(\ell)$, if $\ell \in \tilde{V}^{(2l)} \setminus \tilde{V}^{(2(l-1))}$, where we use the convention $\tilde{V}^{(-2)} = \emptyset$. Identifying $V^{(N)}$ with the subset $\tilde{V}^{(N)} \subset \mathbb{Z}^2$, the level of $\ell$ is also defined for vertices $\ell \in V^{(N)}$. Note that the level sets are defined to have width 2 instead of width 1, see Figure 4.

Figure 4: Vertices at level $l$ are located on the solid lines




**Assumption 5.1** (Choice of parameters)**.** *In the following, we assume that*

*(i)* $a > 0$,

*(ii)* $n_a = \max\{\lceil 1/\sqrt{a}\rceil, 2\}$,

*(iii)* $N \in \mathbb{N}$ *with* $N > 6n_a$, *and*

*(iv)* $\ell \in V^{(N)}$ *is a vertex on a level* $l > 3n_a$.

In the following two definitions, we introduce a modified, truncated version of the Green's function in 2 dimensions. At first, we take the logarithm, appropriately scaled and truncated:

**Definition 5.2** (Auxiliary function 1: truncated and scaled logarithm)**.** *Define a function* $\varphi = \varphi_{l,a} : \mathbb{N}_0 \to [0,1]$ *by*

$$\varphi(n) = \begin{cases} 0 & \text{for } 0 \le n < n_a, \\ \dfrac{\log(n/n_a)}{\log((l-1)/n_a)} & \text{for } n_a \le n \le l-1, \\ 1 & \text{for } n \ge l. \end{cases} \tag{5.1}$$

Let $C^{(n)} := \{(-n,-n), (-n,n), (n,-n), (n,n)\}$ denote the set of corner points of the box $\tilde{V}^{(n)}$.

**Definition 5.3** (Auxiliary function 2: modified Green's function)**.** *For every* $e = \{u,v\} \in E^{(N)}$*, we define a map* $D_e = D^{(N)}_{e,0,\ell,a} : \Omega^{(N)} \to [0,1]$ *as follows: Let* $x \in \Omega^{(N)}$ *and* $l = \text{level}(\ell)$*. Using this* $l$*, take* $\varphi = \varphi_{l,a}$ *from Definition 5.2.*

- *If for some* $n \in \mathbb{N}_0$ *we have* $\text{level}(u) = \text{level}(v) = n$ *or* $\{u,v\} \cap C^{(2n)} \ne \emptyset$*, then we set*

$$D_e(x) = \varphi(n). \tag{5.2}$$

- *Otherwise, we set*

$$D_e(x) = \begin{cases} \varphi(\text{level}(u)) & \text{if } x_u < x_v, \\ \varphi(\text{level}(v)) & \text{if } x_v < x_u, \\ \frac{1}{2}\{\varphi(\text{level}(u)) + \varphi(\text{level}(v))\} & \text{if } x_u = x_v. \end{cases} \tag{5.3}$$

Thus $e \mapsto D_e(x)$ is an approximation to the Green's function in 2 dimensions, slightly dependent on $x$ for technical reasons. It has the property that for any vertex $v$ not being a corner point, there is at most one edge $e \ni v$ with $D_e(x) \ne \varphi(\text{level}(v))$. This property is very convenient below, and it is our reason to define the level sets having width 2 instead of width 1.

We write the set $E^{(N)}$ of edges as a disjoint union $E^{(N)} = F \cup F'$, $F \cap F' = \emptyset$, where $F$ denotes the set of all edges $e = \{u,v\} \in E^{(N)}$ where $u$ and $v$ are on the same level, and $F'$ denotes the set of all edges $e = \{u,v\} \in E^{(N)}$ with $u$ and $v$ on different levels. For $e, e' \in E^{(N)}$, we write $e \prec e'$ if and only if $e \in F$ and $e' \in F'$. Let

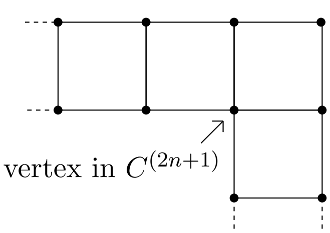
$$\mathcal{F}^{(N)}_e := \sigma(x_{e'} : e' \prec e) \tag{5.4}$$

denote the $\sigma$-field on $\Omega^{(N)}$ generated by the canonical projections on the coordinates $e' \in E^{(N)}$ with $e' \prec e$.

**Lemma 5.4.** *For any* $e \in E^{(N)}$*, the map* $D_e$ *is* $\mathcal{F}^{(N)}_e$*-measurable.*

*Proof.* Let $e = \{u,v\}$. If $\text{level}(u) = \text{level}(v)$ or $\{u,v\} \cap C^{(2n)} \ne \emptyset$ for some $n \in \mathbb{N}_0$, then $D_e$ is constant and there is nothing to show. By definition, our levels have thickness 2. Therefore, if $\{u,v\} \cap C^{(2n+1)} \ne \emptyset$ for some $n \in \mathbb{N}_0$, then $\text{level}(u) = \text{level}(v)$ (see Figure 5), and this case has already been taken care of.

Figure 5: All vertices drawn are on level $n$.

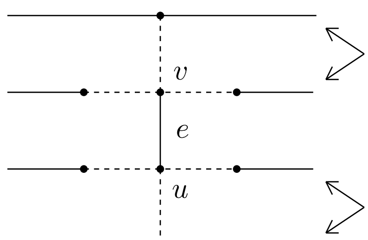



Assume that $u$ and $v$ are on different levels and $\{u,v\} \cap C^{(n)} = \emptyset$ for all $n \in \mathbb{N}_0$. Then, $D_e$ is constant on the sets $\{x_u < x_v\}$, $\{x_v < x_u\}$, and $\{x_u = x_v\}$. Observe that

$$x_u < x_v \quad \text{ if and only if } \quad \sum_{\substack{e' \ni u \\ e' \neq e}} x_{e'} < \sum_{\substack{e' \ni v \\ e' \neq e}} x_{e'}. \tag{5.5}$$

Since $u$ and $v$ are on different levels, but not corner points of any box $\tilde{V}^{(n)}$, all edges $e' \neq e$ incident to $u$ have endpoints on the same level, namely on level$(u)$, see Figure 6.

Figure 6: The edge $e$ has endpoints $u$ and $v$ on different levels. The edges $e' \neq e$ with $e' \ni u$ or $e' \ni v$ are drawn with dashed lines.




Similarly, all edges $e' \neq e$ incident to $v$ have endpoints on level$(v)$. In other words, $e' \prec e$ holds for these edges $e'$. Thus all $x_{e'}$ appearing in the two sums on the right hand side of (5.5) are $\mathcal{F}_e^{(N)}$-measurable; recall the definition (5.4) of $\mathcal{F}_e^{(N)}$. Consequently, because of (5.5), we have $\{x_u < x_v\} \in \mathcal{F}_e^{(N)}$. The same argument shows that $\{x_v < x_u\} \in \mathcal{F}_e^{(N)}$ and $\{x_u = x_v\} \in \mathcal{F}_e^{(N)}$. $\square$

With these preparations, we can now introduce the deformation map $\Xi_\gamma$. Roughly speaking, on a logarithmic scale, the random environment $x_e$ is just shifted by $\gamma$ times the approximate Green's function $D_e$:

**Definition 5.5** (Deformation). *For any $\gamma \in \mathbb{R}$, we define the deformation map $\Xi_\gamma = \Xi_{\gamma,0,\ell,a}^{(N)} : \Omega^{(N)} \to \Omega^{(N)}$ by*

$$\Xi_\gamma(x) = \left(\exp\left(\gamma D_e(x)\right) \cdot x_e\right)_{e \in E^{(N)}}. \tag{5.6}$$

In the following, we denote the composition of the map $\Xi_\gamma : \Omega^{(N)} \to \Omega^{(N)}$ with the map $x_v : \Omega^{(N)} \to (0,\infty)$, $(x_e)_{e\in E^{(N)}} \mapsto \sum_{e\ni v} x_e$ for any $v \in V^{(N)}$ by

$$x_v \circ \Xi_\gamma : \Omega^{(N)} \to (0,\infty),\ (x_v \circ \Xi_\gamma)((x_e)_{e\in E^{(N)}}) = \sum_{e\ni v} \exp\left(\gamma D_e(x)\right) \cdot x_e. \tag{5.7}$$

The following lemma collects some basic facts about this deformation map $\Xi_\gamma$.

**Lemma 5.6** (Basic facts of the deformation map)**.** *With parameters as in Assumption 5.1, for all $\gamma \in \mathbb{R}$, the map $\Xi_\gamma$ is measurable and measurably invertible. Furthermore, it has the following properties:*

1. *One has*
$$x_0 \circ \Xi_\gamma = x_0 \quad \textit{and} \quad x_\ell \circ \Xi_\gamma = e^\gamma x_\ell. \tag{5.8}$$

2. *The reference measure $\rho$ is invariant with respect to $\Xi_\gamma$.*

*Proof.* Let $\gamma \in \mathbb{R}$. The measurability of $\Xi_\gamma$ follows from its definition (5.6) and Lemma 5.4.

1. Let $x \in \Omega^{(N)}$, and let $e \in E^{(N)}$ with $0 \in e$. Then, $e \cap C^{(0)} \neq \emptyset$, and consequently, $D_e(x) = \varphi(0) = 0$. Hence, $x_e \circ \Xi_\gamma = x_e$, and it follows that $x_0 = \sum_{e'\ni 0} x_{e'} = x_0 \circ \Xi_\gamma$.

   Let $e \in E^{(N)}$ with $\ell \in e$. Then, $D_e(x)$ takes a value in the set $\{\varphi(l), \varphi(l \pm 1), (\varphi(l) + \varphi(l \pm 1))/2\} = \{1\}$. Hence, $x_\ell \circ \Xi_\gamma = e^\gamma x_\ell$. This completes the proof of (5.8).

2. We list the edges in $E^{(N)}$ in such a way that every $e = \{u, v\} \in E^{(N)}$ with $\mathrm{level}(u) = \mathrm{level}(v)$ gets a smaller index than every $e' = \{u', v'\} \in E^{(N)}$ with $\mathrm{level}(u') \neq \mathrm{level}(v')$. Thus, we get a list $e_0, e_1, \ldots, e_K$ with the property that $e$ has a smaller index than $e'$ whenever $e \prec e'$. We rewrite the definitions (5.6) of $\Xi_\gamma$ and (4.2) of the reference measure $\rho$ in logarithmic form:
$$\log\left(x_e \circ \Xi_\gamma\right) = \log x_e + \gamma D_e(x), \tag{5.9}$$
$$\rho(dx) = \delta_1(dx_{e_0}) \prod_{e\in E^{(N)}\setminus\{e_0\}} d\log x_e. \tag{5.10}$$

   Since $D_e$ is $\mathcal{F}_e^{(N)}$-measurable, we see that $\log x_{e_j}$ is just translated by a value which depends only on the components $x_{e_i}$ with $i < j$. Since $0 \in e_0$, we have $D_{e_0} = 0$, and the component $x_{e_0}$ remains unchanged. Such translations leave the reference measure $\rho$ invariant. Here we use that every measurable map $f : \mathbb{R}^d \to \mathbb{R}^d$ of the form
$$f(x_1, \ldots, x_d) = (x_i + g_i(x_j; j < i))_{i=1,\ldots,d} \tag{5.11}$$
   leaves the Lebesgue measure invariant.

One verifies that the inverse of $\Xi_\gamma$ is given by

$$x_e \circ [\Xi_\gamma]^{-1} = \exp\left\{-\gamma D_e\left([\Xi_\gamma]^{-1}(x)\right)\right\} x_e, \tag{5.12}$$

$x \in \Omega^{(N)}$, $e \in E^{(N)}$. This is a recursive system for determining the inverse, since $D_e([\Xi_\gamma]^{-1}(x))$ depends only on the components $x_f$ with $f$ earlier in the list $e_0, \ldots, e_K$ than $e$. It follows that $[\Xi_\gamma]^{-1}$ is measurable. $\square$

## 5.2 A quadratic entropy bound

The goal of this subsection is to prove the following lemma:

**Lemma 5.7** (Quadratic entropy bound)**.** *With parameters as in Assumption 5.1, for all* $\gamma \in \mathbb{R}$ *and all* $\eta \in [0,1]$*, the image measure*

$$\Pi_{\gamma,\eta} = \Pi^{(N)}_{\gamma,\eta,0,\ell,a} := \Xi^{(N)}_{\gamma,0,\ell,a}\mathbb{P}^{(N)}_{\eta,0,\ell,a} \tag{5.13}$$

*of* $\mathbb{P}_\eta$ *with respect to* $\Xi_\gamma$ *is absolutely continuous with respect to* $\mathbb{P}_\eta$*.*
*If furthermore*

$$|\gamma| \leq n_a \log \frac{l-1}{n_a} \tag{5.14}$$

*holds, one has the entropy bound*

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \leq \frac{c_5(a,\eta)\gamma^2}{n_a \log((l-1)/n_a)} \tag{5.15}$$

*with the constant*

$$c_5(a,\eta) := 32\left(2a + \frac{1}{2}\right)\left(ec_6(a,\eta)n_a + \frac{1}{\log 2}\right) > 16 \qquad \textit{and} \tag{5.16}$$

$$c_6(a,\eta) := \begin{cases} \min\{\sqrt{a}, 1\} & \textit{if } \eta = 1, \\ 1 & \textit{otherwise.} \end{cases} \tag{5.17}$$

*In the special case* $\eta = 1$*, one has* $\lim_{a\to 0} c_5(a,1) < \infty$*.*

Throughout this subsection, we assume that $a$, $n_a$, $N$, $\ell$, and $l$ are fixed according to Assumption 5.1.
For $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$, we denote by

$$\Pi^-_{\gamma,\eta} = \Pi^{(N)-}_{\gamma,\eta,0,\ell,a} := [\Xi^{(N)}_{\gamma,0,\ell,a}]^{-1}\mathbb{P}^{(N)}_{\eta,0,\ell,a} \tag{5.18}$$

the image measure of $\mathbb{P}_\eta$ under the inverse of $\Xi_\gamma$. In the following, we suppress the dependence of $D_e$ on the environment $x$. Thus, we write $D_e$ instead of $D_e(x)$. We abbreviate for $v \in V^{(N)}$, $\gamma \in \mathbb{R}$, $T \in \mathcal{T}^{(N)}$, and $x \in \Omega^{(N)}$:

$$x_{v,\gamma} := \sum_{e \ni v}\left(e^{\gamma D_e} x_e\right) \qquad \text{and} \qquad Y_{T,\gamma} = Y_{T,\gamma}(x) := \prod_{e\in T}\left(e^{\gamma D_e} x_e\right). \tag{5.19}$$

In order to estimate the entropy in formula (5.15), we use the following lemma. It provides an explicit formula for the logarithm of the relevant Radon-Nikodym derivative.

**Lemma 5.8.** *1. For any* $\gamma \in \mathbb{R}$ *and* $\eta \in [0,1]$*, we have*

$$\Pi_{\gamma,\eta} \ll \mathbb{P}_\eta \ll \Pi^-_{\gamma,\eta}, \tag{5.20}$$

*where "*$\ll$*" means that the left-hand side is absolutely continuous with respect to the right-hand side.*

2. *For any $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$, the Radon-Nikodym derivatives are bounded functions on $\Omega^{(N)}$ and fulfill*

$$\begin{aligned}\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta} \circ \Xi_\gamma =& \log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}} \\ =& -\sum_{e\in E^{(N)}} \gamma D_e a + \gamma\left(2a + \frac{1-\eta}{2}\right) \\ &+ \left(2a+\frac{1}{2}\right) \sum_{v\in V^{(N)}\setminus\{0,\ell\}} \log \frac{x_{v,\gamma}}{x_v} - \frac{1}{2}\log \frac{\sum_{T\in\mathcal{T}^{(N)}} Y_{T,\gamma}}{\sum_{T\in\mathcal{T}^{(N)}} Y_{T,0}}. \end{aligned} \tag{5.21}$$

3. *The following two entropies are finite and coincide:*

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] = E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]. \tag{5.22}$$

*Proof.* Let $\gamma \in \mathbb{R}$ and $\eta \in [0,1]$. By (4.10), $\mathbb{P}_\eta$ is absolutely continuous with respect to $\rho$ with a strictly positive Radon-Nikodym-derivative $d\mathbb{P}_\eta/d\rho$. The reference measure $\rho$ is invariant under $\Xi_\gamma$ by Lemma 5.6 (b). Consequently, we find

$$\frac{d\Pi_{\gamma,\eta}}{d\rho} = \frac{d(\Xi_\gamma\mathbb{P}_\eta)}{d(\Xi_\gamma\rho)} = \frac{d\mathbb{P}_\eta}{d\rho} \circ [\Xi_\gamma]^{-1} \quad \text{and} \quad \frac{d\Pi^-_{\gamma,\eta}}{d\rho} = \frac{d(\Xi_\gamma^{-1}\mathbb{P}_\eta)}{d(\Xi_\gamma^{-1}\rho)} = \frac{d\mathbb{P}_\eta}{d\rho} \circ \Xi_\gamma. \tag{5.23}$$

Taking ratios, this implies claim (a).

The first equality in (5.21) follows from (5.23). To prove the second equality in part (b), recall the explicit form of $d\mathbb{P}_\eta/d\rho$ in formula (4.10). By (5.8), we know that $x_0 \circ \Xi_\gamma = x_0$ and $x_\ell \circ \Xi_\gamma = e^\gamma x_\ell$. Consequently, for any $\gamma \in \mathbb{R}$, we obtain using (5.23) and (4.10),

$$\begin{aligned}\frac{d\Pi^-_{\gamma,\eta}}{d\rho}(x) =& \frac{d\mathbb{P}_\eta}{d\rho}\left(\Xi_\gamma(x)\right) \\ =& \frac{1}{z^{(N)}_{0,a} Z^{(N)}_{\eta,0,\ell,a}} \frac{\prod_{e\in E^{(N)}} \left(e^{\gamma D_e a} x_e^a\right) \sqrt{\sum_{T\in\mathcal{T}^{(N)}} \prod_{e\in T} \left(e^{\gamma D_e} x_e\right)}}{x_0^{2a+\eta/2} (e^\gamma x_\ell)^{2a+(1-\eta)/2} \prod_{v\in V^{(N)}\setminus\{0,\ell\}} \left[\sum_{e\ni v} e^{\gamma D_e} x_e\right]^{2a+1/2}}. \end{aligned} \tag{5.24}$$

Combining (4.10) and (5.24) yields the second equality in the claim (5.21).

Since $D_e$ takes only values in $[0,1]$, we have $e^{-|\gamma|}x_e \le e^{\gamma D_e} x_e \le e^{|\gamma|} x_e$ for all $e \in E^{(N)}$. Hence,

$$\begin{aligned} &e^{-|\gamma|}x_v \le x_{v,\gamma} \le e^{|\gamma|}x_v \qquad \text{and} \\ &e^{-|E^{(N)}|\cdot|\gamma|} Y_{T,0} \le Y_{T,\gamma} \le e^{|E^{(N)}|\cdot|\gamma|} Y_{T,0} \end{aligned} \tag{5.25}$$

hold. Thus, it follows from (5.21) that $x \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta}(x))$ is a bounded measurable function on $\Omega^{(N)}$. Furthermore, using the first equality in (5.21), we see that $x \mapsto d\Pi_{\gamma,\eta}/d\mathbb{P}_\eta$ is also a bounded measurable function on $\Omega^{(N)}$. Consequently, the entropies in (5.22) are both finite. Using $\Pi_{\gamma,\eta} = \Xi_\gamma \mathbb{P}_\eta$, we obtain

$$E_{\Pi_{\gamma,\eta}}\left[\log \frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] = E_{\Xi_\gamma\mathbb{P}_\eta}\left[\log\left(\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}} \circ \Xi_\gamma^{-1}\right)\right] = E_{\mathbb{P}_\eta}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]. \tag{5.26}$$

□

Recall the definitions (5.19). Fix $x \in \Omega^{(N)}$ and take $v \in V^{(N)}$ and $\gamma \in \mathbb{R}$. We define a probability measure $\mu_{v,\gamma} = \mu_{v,x,\gamma}^{(N)}$ on the set $E_v^{(N)} := \{e \in E^{(N)} : e \ni v\}$ by

$$\mu_{v,\gamma} := \sum_{e \in E_v^{(N)}} \frac{e^{\gamma D_e} x_e}{x_{v,\gamma}} \delta_e. \tag{5.27}$$

For our fixed $x \in \Omega^{(N)}$, we view $D_\bullet : E_v^{(N)} \to \mathbb{R}$, $e \mapsto D_e$, as a random variable on the probability space $(E_v^{(N)}, \mathcal{P}(E_v^{(N)}), \mu_{v,\gamma})$, again suppressing the dependence on the parameter $x$ in the notation; here $\mathcal{P}(A)$ denotes the power set of the set $A$.

Recall that we want to derive a bound for an entropy proportional to $\gamma^2$. This is done by a second order Taylor expansion with respect to $\gamma$. The relevant second derivative is estimated in the following lemma.

**Lemma 5.9.** *For all $\eta \in [0,1]$, the function $\mathbb{R} \ni \gamma \mapsto \log(d\mathbb{P}_\eta / d\Pi_{\gamma,\eta}^-)$ is twice continuously differentiable. The second derivative satisfies the bound*

$$\frac{\partial^2}{\partial^2 \gamma}\left[\log \frac{d\mathbb{P}_\eta}{d\Pi_{\gamma,\eta}^-}\right] \le \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \mathrm{Var}_{\mu_{v,\gamma}}(D_\bullet). \tag{5.28}$$

*Proof.* Fix $x \in \Omega^{(N)}$. We define a probability measure $\nu_\gamma = \nu_{x,\gamma}^{(N)}$ on the set $\mathcal{T}^{(N)}$ by

$$\nu_\gamma := \sum_{T \in \mathcal{T}^{(N)}} \frac{Y_{T,\gamma}}{\sum_{T' \in \mathcal{T}^{(N)}} Y_{T',\gamma}} \delta_T. \tag{5.29}$$

Note that

$$\frac{\partial}{\partial \gamma} x_{v,\gamma} = \sum_{e \ni v} D_e e^{\gamma D_e} x_e, \tag{5.30}$$

$$\frac{\partial}{\partial \gamma} Y_{T,\gamma} = \sum_{e \in T} D_e Y_{T,\gamma} = \Delta_T Y_{T,\gamma}, \tag{5.31}$$

where we set

$$\Delta_T = \Delta_T(x) := \sum_{e \in T} D_e. \tag{5.32}$$

For our fixed $x \in \Omega^{(N)}$, we view $\Delta_\bullet : \mathcal{T}^{(N)} \to \mathbb{R}$, $T \mapsto \Delta_T$, as a random variable on the probability space $(\mathcal{T}^{(N)}, \mathcal{P}(\mathcal{T}^{(N)}), \nu_\gamma)$; again we drop the dependence on $x$ in the notation. We calculate the first and second derivative of $\gamma \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta})$ using the representation (5.21):

$$\begin{aligned}\frac{\partial}{\partial\gamma}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] &= -\sum_{e\in E^{(N)}} D_e a + 2a + \frac{1-\eta}{2}\\ &\quad + \left(2a+\frac{1}{2}\right)\sum_{v\in V^{(N)}\setminus\{0,\ell\}} \frac{1}{x_{v,\gamma}}\sum_{e\ni v} D_e e^{\gamma D_e} x_e - \frac{1}{2}\frac{\sum_{T\in\mathcal{T}^{(N)}}\Delta_T Y_{T,\gamma}}{\sum_{T\in\mathcal{T}^{(N)}} Y_{T,\gamma}}\\ &= -\sum_{e\in E^{(N)}} D_e a + 2a + \frac{1-\eta}{2}\\ &\quad + \left(2a+\frac{1}{2}\right)\sum_{v\in V^{(N)}\setminus\{0,\ell\}} E_{\mu_{v,\gamma}}[D_\bullet] - \frac{1}{2}E_{\nu_\gamma}[\Delta_\bullet]. \end{aligned} \tag{5.33}$$

We also calculate the second derivative:

$$\frac{\partial^2}{\partial^2\gamma}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] = \left(2a+\frac{1}{2}\right)\sum_{v\in V^{(N)}\setminus\{0,\ell\}} \mathrm{Var}_{\mu_{v,\gamma}}(D_\bullet) - \frac{1}{2}\mathrm{Var}_{\nu_\gamma}(\Delta_\bullet). \tag{5.34}$$

Since $\mathrm{Var}_{\nu_\gamma}(\Delta_\bullet) \geq 0$, the claim of the lemma follows. □

The second order Taylor expansion for the entropy is derived in the following lemma.

**Lemma 5.10.** *The function*

$$f : \mathbb{R} \ni \gamma \mapsto f(\gamma) = E_{\mathbb{P}_\eta}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right] \tag{5.35}$$

*is twice continuously differentiable. The derivatives can be obtained by differentiating inside of the expectation, i.e.*

$$\frac{\partial^j}{\partial^j\gamma}\left[E_{\mathbb{P}_\eta}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right] = E_{\mathbb{P}_\eta}\left[\frac{\partial^j}{\partial^j\gamma}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right] \tag{5.36}$$

*for $j = 1, 2$. Furthermore, for any $\gamma \in \mathbb{R}$, one has*

$$f(\gamma) = \int_0^\gamma E_{\mathbb{P}_\eta}\left[\frac{\partial^2}{\partial^2\tilde{\gamma}}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\tilde{\gamma},\eta}}\right]\right](\gamma-\tilde{\gamma})\,d\tilde{\gamma}. \tag{5.37}$$

*Proof.* Note that $0 \leq D_e \leq 1$ for all $e$ and $0 \leq \Delta_T \leq |E^{(N)}|$ for all $T \in \mathcal{T}^{(N)}$. These bounds are valid for all $x \in \Omega^{(N)}$. Consequently, it follows from (5.33) and (5.34) that there exists a constant $c_7(a, N) \in (0, \infty)$ such that for $j = 1, 2$ and all $x \in \Omega^{(N)}$, we have

$$\sup_{\gamma\in\mathbb{R}}\left|\frac{\partial^j}{\partial^j\gamma}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\gamma,\eta}}\right]\right| \leq c_7(a, N). \tag{5.38}$$

By Lemma 5.9, $\gamma \mapsto \log(d\mathbb{P}_\eta/d\Pi^-_{\gamma,\eta})$ is twice continuously differentiable. Thus, by the dominated convergence theorem, the same is true for $f$, and (5.36) is valid for $j = 1, 2$.

We know $f \geq 0$ because entropies are always non-negative. Furthermore, since $\Pi^-_{0,\eta} = \mathbb{P}_\eta$, we have $f(0) = 0$. Consequently, $f'(0) = 0$. A Taylor expansion of $f$ around 0 yields

$$f(\gamma) = \int_0^\gamma f''(\tilde{\gamma})(\gamma - \tilde{\gamma})\, d\tilde{\gamma} = \int_0^\gamma E_{\mathbb{P}_\eta}\left[\frac{\partial^2}{\partial^2\tilde{\gamma}}\left[\log\frac{d\mathbb{P}_\eta}{d\Pi^-_{\tilde{\gamma},\eta}}\right]\right](\gamma - \tilde{\gamma})\, d\tilde{\gamma}; \tag{5.39}$$

note that the last integral is finite by (5.38). $\square$

We need another technical ingredient for the proof of Lemma 5.7. For $e = \{u, v\} \in E$, we define

$$L_e := \frac{x_e}{\sqrt{x_u x_v}}. \tag{5.40}$$

Recall that $\mathbb{P}_\eta = \mathbb{P}^{(N)}_{\eta,0,\ell,a}$.

**Lemma 5.11.** *For all $e \in E^{(N)}$ with $0, \ell \notin e$ and all $\eta \in [0, 1]$, we have*

$$E_{\mathbb{P}_\eta}[L_e] \leq c_6(a, \eta) \tag{5.41}$$

*with $c_6(a, \eta)$ as in (5.17).*

*Proof.* Let $e = \{u, v\}$. Since $x_e \leq x_u$ and $x_e \leq x_v$, we have $L_e \leq 1$ and the claim follows in the case $0 \leq \eta < 1$.

Assume $\eta = 1$. By Definition 4.2, $\mathbb{P}_1 = \mathbb{Q}^{(N)}_{\ell,a}$; the normalizing constant equals $Z^{(N)}_{\eta,0,\ell,a} = 1$ because $\mathbb{Q}^{(N)}_{\ell,a}$ is a probability measure. Hence, Proposition 4.6 of [DR06], specialized to the present model and rewritten in our present notation, claims the following equality:

$$E_{\mathbb{Q}^{(N)}_{\ell,a}}\left[L_e^2\right] = \frac{a(a+1)}{(4a+1)^2} \leq a. \tag{5.42}$$

Consequently, $E_{\mathbb{Q}^{(N)}_{\ell,a}}[L_e] \leq \left(E_{\mathbb{Q}^{(N)}_{\ell,a}}\left[L_e^2\right]\right)^{1/2} \leq c_6(a, 1)$. $\square$

Intuitively, equation (5.42) has the following interpretation: We view $L_e^2$ as the probability that the Markovian random walk in the environment $x$ traverses the edge $e$ twice (forward and backward) as soon as the random walker visits one of the two vertices of $e$. The left hand side of the equation in (5.42) is then the probability of the same event with respect to the mixture of Markov chains. On the other hand, the right hand side in the same equation equals the probability of the same event with respect to the reinforced random walk.

We have now all tools to prove the quadratic entropy bound.

*Proof of Lemma 5.7.* Recall Assumption 5.1. By Lemma 5.8(a), $\Pi_{\gamma,\eta}$ is absolutely continuous with respect to $\mathbb{P}_\eta$. To prove the entropy bound (5.15), we combine first Lemma 5.8(c) with (5.37), and then we insert the bound (5.28). This yields:

$$E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right] \leq \left(2a + \frac{1}{2}\right) \sum_{v \in V^{(N)} \setminus \{0,\ell\}} \int_0^\gamma E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,\tilde{\gamma}}}(D_\bullet)\right](\gamma - \tilde{\gamma})d\tilde{\gamma}. \tag{5.43}$$

Let $v \in V^{(N)} \setminus \{0, \ell\}$, and abbreviate

$$l_v := \text{level}(v). \tag{5.44}$$

Recall that the measure $\mu_{v,\gamma} = \mu_{v,x,\gamma}^{(N)}$ depends on the environment $x \in \Omega^{(N)}$. In the following, we stress this dependence by writing $\mu_{v,x,\gamma}^{(N)}$ instead of $\mu_{v,\gamma}$. To estimate the variance of $D_\bullet$ with respect to the measure $\mu_{v,x,\gamma}^{(N)}$, we distinguish three cases.

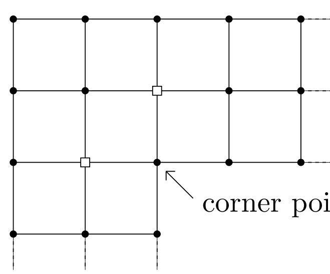
**Case 1:** Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is a corner point of a box $[-m, m]^2$, i.e. $v \in C^{(m)}$, for some $m \geq 1$.

Then, there are two possibilities (see Figure 4): If $m$ is odd, all edges incident to $v$ have both endpoints on level $l_v$. Otherwise, $m$ is even and $m = 2l_v$. In both cases, by Definition 5.3, it follows that $D_{e'} = \varphi(l_v)$ for any $e'$ incident to $v$. Hence, we have the estimate:

$$\text{Var}_{\mu_{v,x,\gamma}^{(N)}}(D_\bullet) \leq E_{\mu_{v,x,\gamma}^{(N)}}\left[(D_\bullet - \varphi(l_v))^2\right] = 0. \tag{5.45}$$

**Case 2:** Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is a neighbor of a corner point $u$ with level$(u) \neq l_v$ (see Figure 7).

Figure 7: The vertices marked with a square are the neighbors of corner points considered in case 2.




Then, three edges incident to $v$ have both endpoints on level $l_v$ and one edge has one endpoint on the level of the corner point $u$, namely on level $l_v - 1$. Hence, for any $e'$ incident to $v$, we have $D_{e'} \in \{\varphi(l_v), \varphi(l_v - 1)\}$, and consequently

$$\begin{aligned}\text{Var}_{\mu_{v,x,\gamma}^{(N)}}(D_\bullet) \leq& E_{\mu_{v,x,\gamma}^{(N)}}\left[(D_\bullet - \varphi(l_v))^2\right] \\ &\leq (\varphi(l_v - 1) - \varphi(l_v))^2 .\end{aligned} \tag{5.46}$$

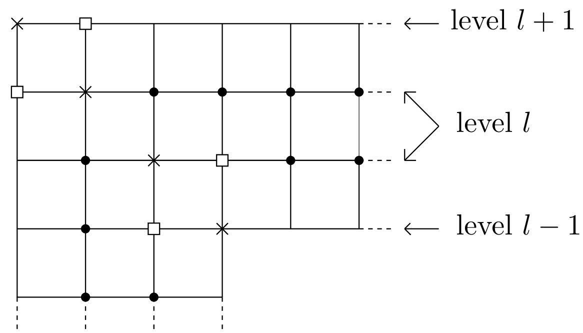
Let $I$ denote the set of all vertices $v \in V^{(N)} \setminus \{0, \ell\}$ considered in case 2. Then,

$$\sum_{v \in I} \text{Var}_{\mu_{v,x,\gamma}^{(N)}}(D_\bullet) \leq 8 \sum_{n=1}^{\infty} (\varphi(n+1) - \varphi(n))^2 . \tag{5.47}$$

The factor 8 arises since there are 8 edges connecting corner points at level $n$ to vertices at level $n+1$: Each of the 4 relevant corner points at level $n$ is connected to 2 vertices at level $n+1$.

**Case 3:** Assume that $v \in V^{(N)} \setminus \{0, \ell\}$ is not a corner point of any box and $v$ is not a neighbor of a corner point at a different level (see Figure 8).

Figure 8: Corner points are marked with a cross, neighbors of corner points at a different level (as treated in case 2) are marked with a square. The black dots are the vertices at level $l$ covered in case 3.




Then, there is precisely one vertex $u$ adjacent to $v$ with $l_u := \text{level}(u) \neq l_v$. We set $e(v) := \{u, v\}$. One has $D_{e'} = \varphi(l_v)$ for all edges $e' \ni v$ with $e' \neq e(v)$, and thus, it follows:

$$\begin{aligned}\operatorname{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq & E_{\mu^{(N)}_{v,x,\gamma}}\left[(D_\bullet - \varphi(l_v))^2\right] \\ &= \left(D_{e(v)} - \varphi(l_v)\right)^2 \mu^{(N)}_{v,x,\gamma}(e(v)).\end{aligned} \tag{5.48}$$

Furthermore, since we have excluded $v$ to be as in case 2, the definition (5.3) applies to $D_{e(v)}$. In particular, if $x_v < x_u$, then $D_{e(v)} = \varphi(l_v)$, and hence

$$\operatorname{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) = 0 \qquad \text{if } x_v < x_u. \tag{5.49}$$

For $e'$ incident to $v$, we know that the difference $D_{e(v)} - D_{e'}$ takes one of the three values 0, $\varphi(l_u) - \varphi(l_v)$, and $(\varphi(l_u) - \varphi(l_v))/2$. Consequently,

$$\begin{aligned}\mu^{(N)}_{v,x,\gamma}(\{e(v)\}) = & \frac{e^{\gamma D_{e(v)}} x_{e(v)}}{x_{v,\gamma}} = \frac{x_{e(v)}}{\sum_{e' \ni v} e^{\gamma(D_{e'} - D_{e(v)})} x_{e'}} \\ & \leq \exp\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\} \frac{x_{e(v)}}{x_v}.\end{aligned} \tag{5.50}$$

Assume that $x_u \leq x_v$. Then, combining (5.48) and (5.50) and using $\sqrt{x_u x_v} \leq x_v$ yields

$$\operatorname{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet) \leq (\varphi(l_u) - \varphi(l_v))^2 \exp\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\} \frac{x_{e(v)}}{\sqrt{x_u x_v}}. \tag{5.51}$$

Because of (5.49), this estimate is also true in the case $x_v < x_u$.

A side remark: At this point, it becomes clear why in Definition 5.3, $D_e$ was introduced in such a tricky, $x$-dependent way: If we had used a more naive definition of $D_e$ instead, formula (5.51) would have failed to hold.

Integrating both sides of (5.51) with respect to $\mathbb{P}_\eta$ and applying Lemma 5.11 gives

$$E_{\mathbb{P}_\eta}\left[\operatorname{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] \leq c_6(a, \eta)\,(\varphi(l_u) - \varphi(l_v))^2 \exp\{|\gamma| \cdot |\varphi(l_u) - \varphi(l_v)|\}. \tag{5.52}$$

We sum the preceding inequality over the different vertices $v$:

$$\begin{aligned}
&\sum_{\substack{v\in V^{(N)}\setminus\{0,\ell\}\\ v\notin\bigcup_{m=1}^{\infty}C^{(m)},v\notin I}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right]\\
&\leq 8c_6(a,\eta)\sum_{n=1}^{\infty}(4n+1)\left(\varphi(n+1)-\varphi(n)\right)^2\exp\left\{|\gamma|\cdot|\varphi(n+1)-\varphi(n)|\right\}. \qquad (5.53)
\end{aligned}$$

The factor $8(4n+1)$ arises, since there are not more than $4(4n+1)$ edges connecting level $n$ to level $n+1$. Each of these edges is counted at most twice, once for each of its two endpoints.

By definition (5.1) of $\varphi$, we have

$$\varphi(n+1)-\varphi(n)=0 \quad \text{if } 0\leq n\leq n_a-1 \quad \text{or} \quad n\geq l-1. \qquad (5.54)$$

Furthermore, for $n_a\leq n\leq l-2$, we have

$$|\varphi(n+1)-\varphi(n)|\leq \sup_{n\leq x\leq n+1}\left|\frac{\partial}{\partial x}\frac{\log(x/n_a)}{\log((l-1)/n_a)}\right|=\frac{1}{n\log((l-1)/n_a)}. \qquad (5.55)$$

Assume that $\gamma$ satisfies (5.14):

$$|\gamma|\leq n_a\log\frac{l-1}{n_a}. \qquad (5.56)$$

Then, for $n_a\leq n\leq l-2$,

$$\exp\left\{|\gamma|\cdot|\varphi(n+1)-\varphi(n)|\right\}\leq e. \qquad (5.57)$$

In the following step, we use $1+c_6(a,\eta)e\leq 4$, which follows from $c_6(a,\eta)\leq 1$. Inserting the bound (5.57) in (5.53) and using (5.45), (5.47), (5.54), and (5.55) yields:

$$\begin{aligned}
\sum_{v\in V^{(N)}\setminus\{0,\ell\}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right] &\leq 8\sum_{n=n_a}^{l-2}\left(1+c_6(a,\eta)(4n+1)e\right)\left(\varphi(n+1)-\varphi(n)\right)^2\\
&\leq \frac{8}{(\log((l-1)/n_a))^2}\sum_{n=n_a}^{l-2}\left(\frac{4c_6(a,\eta)e}{n}+\frac{1+c_6(a,\eta)e}{n^2}\right)\\
&\leq \frac{32}{(\log((l-1)/n_a))^2}\sum_{n=n_a}^{l-2}\left(\frac{c_6(a,\eta)e}{n}+\frac{1}{n^2}\right). \qquad (5.58)
\end{aligned}$$

Observe that

$$\sum_{n=n_a}^{l-2}\frac{1}{n}\leq 2\log\frac{l-1}{n_a} \qquad \text{and} \qquad \sum_{n=n_a}^{l-2}\frac{1}{n^2}\leq\frac{2}{n_a}. \qquad (5.59)$$

Hence, using that $\log((l-1)/n_a)\geq\log 2$ by (iv) in Assumption 5.1, we obtain

$$\sum_{v\in V^{(N)}\setminus\{0,\ell\}} E_{\mathbb{P}_\eta}\left[\mathrm{Var}_{\mu^{(N)}_{v,x,\gamma}}(D_\bullet)\right]\leq\frac{64}{n_a\log((l-1)/n_a)}\left(c_6(a,\eta)en_a+\frac{1}{\log 2}\right). \qquad (5.60)$$

Combining this bound with (5.43) yields

$$E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right]\leq\frac{c_5(a,\eta)\gamma^2}{n_a\log((l-1)/n_a)} \tag{5.61}$$

with

$$c_5(a,\eta):=32\left(2a+\frac{1}{2}\right)\left(ec_6(a,\eta)n_a+\frac{1}{\log 2}\right)\geq\frac{16}{\log 2}>16. \tag{5.62}$$

In the special case $\eta=1$, because of part (ii) in Assumption 5.1 and (5.17), $c_6(a,1)n_a\to 1$ as $a\to 0$. Hence, $\lim_{a\to 0}c_5(a,1)<\infty$. This completes the proof of Lemma 5.7. □

# 6 The key estimate

The following theorem is the key to all bounds in the main theorems. One might view it as a bound for the Legendre transform of the logarithm of the partition sum $E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)]$. Given the basic facts on the deformation map $\Xi_\gamma$ and the entropy bound from the previous section, its proof boils down to non-negativity of relative entropies.

Recall the definition (4.7) of $\Sigma_\ell$. We abbreviate

$$\mathbb{Q}_0:=\mathbb{Q}_{0,a}^{(N)}. \tag{6.1}$$

**Theorem 6.1** (Key estimate)**.** *Let $a>0$, $n_a$, $N\in\mathbb{N}$, $\ell\in V^{(N)}$, and $l\in\mathbb{N}$ fulfill Assumption 5.1, and let $0\leq\eta\leq 1$. Then*

$$\log E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)]-\eta E_{\mathbb{P}_\eta}[\Sigma_\ell]\leq\frac{-\eta^2 n_a\log((l-1)/n_a)}{16c_5(a,\eta)} \tag{6.2}$$

*with $c_5(a,\eta)$ given by (5.16).*

*Proof.* We set

$$\gamma=-\frac{\eta n_a\log((l-1)/n_a)}{4c_5(a,\eta)}. \tag{6.3}$$

Hypothesis (5.14) of Lemma 5.7 is satisfied for this choice, since we have $c_5(a,\eta)>16$ for all $\eta\in[0,1]$ and $a>0$. Using the entropy bound (5.15) from this lemma and positivity of entropies, we get

$$\begin{aligned}-\frac{\eta\gamma}{4}=&\frac{c_5(a,\eta)\gamma^2}{n_a\log((l-1)/n_a)}\geq E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\Pi_{\gamma,\eta}}{d\mathbb{P}_\eta}\right]\\=&E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\right]+E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\Pi_{\gamma,\eta}}{d\mathbb{Q}_0}\right]\\\geq&E_{\Pi_{\gamma,\eta}}\left[\log\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\right]=E_{\mathbb{P}_\eta}\left[\log\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\circ\Xi_\gamma\right)\right];\end{aligned} \tag{6.4}$$

here we used the definition (5.13) of the measure $\Pi_{\gamma,\eta}$ in the last step. Note that all expectations occurring in (6.4) are finite. As a consequence of the two equations (5.8), we find

$$\Sigma_\ell\circ\Xi_\gamma=\Sigma_\ell+\frac{\gamma}{2}. \tag{6.5}$$

Thus, using (4.9), we find

$$\log\left(\frac{d\mathbb{Q}_0}{d\mathbb{P}_\eta}\circ\Xi_\gamma\right) = \log Z^{(N)}_{\eta,0,\ell,a} - \eta\Sigma_\ell\circ\Xi_\gamma = \log E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)] - \eta\Sigma_\ell - \frac{\eta\gamma}{2}. \tag{6.6}$$

Consequently, it follows from (6.4):

$$-\frac{\eta\gamma}{4} \geq \log E_{\mathbb{Q}_0}[\exp(\eta\Sigma_\ell)] - \eta E_{\mathbb{P}_\eta}[\Sigma_\ell] - \frac{\eta\gamma}{2}. \tag{6.7}$$

Combining this with our choice (6.3) for $\gamma$, we obtain the claim (6.2). □

# 7 Tail estimates for the random environment

The following tail estimates are proved in [MR07c]. We need them for tightness arguments when we take the infinite-volume limit in Subsection 8.2, below. Recall that the random environment $\mathbb{Q}^{(N)}_{0,a}$ for the finite box $V^{(N)}$ is unique up to a multiplication of the edge weights by a constant.

**Theorem 7.1** (Theorems 2.3 and 2.4 in [MR07c])**.** *For all $a > 0$, there are constants $c_8(a) > 0$ and $c_9(a) > 0$, depending only on $a$, such that the following estimates hold for all $N \in \mathbb{N}$:*

1. *For all $e, f \in E^{(N)}$ with $e \cap f \neq \emptyset$, and all $M > 0$, one has*

$$\mathbb{Q}^{(N)}_{0,a}\left[\frac{x_e}{x_f} \geq M\right] \leq c_8(a) M^{-a/2}. \tag{7.1}$$

2. *For all $v \in V^{(N)}$, all $e \in E^{(N)}$ incident to $v$, and all $M > 0$, one has*

$$\mathbb{Q}^{(N)}_{0,a}\left[\frac{x_e}{x_v} \leq M\right] \leq c_9(a) M^{a/2}. \tag{7.2}$$

*Proof of Lemma 2.9.* Let $a > 0$, $N \geq 2$, and let $e, f \in E^{(N)}$ satisfy $e \cap f \neq \emptyset$. Using (7.1), we obtain for any $\alpha > 0$:

$$\begin{aligned} E_{\mathbb{Q}^{(N)}_{0,a}}\left[\left(\frac{x_e}{x_f}\right)^\alpha\right] \leq& 1 + \int_1^\infty \mathbb{Q}^{(N)}_{0,a}\left[\left(\frac{x_e}{x_f}\right)^\alpha \geq M\right] dM \\ \leq& 1 + c_8(a)\int_1^\infty M^{-a/(2\alpha)}\, dM. \end{aligned} \tag{7.3}$$

The last integral is finite whenever $\alpha \in (0, a/2)$. Since the upper bound in (7.3) is uniform in $e$, $f$, and $N$, the claim (2.18) follows.

To prove (2.19), let $v, w \in V^{(N)}$ with $|v - w| = 1$. Denote by $e := \{v, w\}$ the edge connecting $v$ and $w$. Since $x_e \leq \min\{x_v, x_w\}$, it follows that $\log(x_v/x_e) \geq 0$ and $\log(x_w/x_e) \geq 0$. Using this and (7.2), we obtain

$$\begin{aligned} E_{\mathbb{Q}^{(N)}_{0,a}}\left[\left|\log\frac{x_v}{x_w}\right|\right] \leq& E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log\frac{x_v}{x_e}\right] + E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log\frac{x_w}{x_e}\right] \\ =& \int_0^\infty \mathbb{Q}^{(N)}_{0,a}\left[\log\frac{x_v}{x_e} \geq M\right] dM + \int_0^\infty \mathbb{Q}^{(N)}_{0,a}\left[\log\frac{x_w}{x_e} \geq M\right] dM \\ \leq& 2\left(1 + c_9(a)\int_1^\infty e^{-Ma/2}\, dM\right) < \infty; \end{aligned} \tag{7.4}$$

clearly, the upper bound is uniform in $v$, $w$, and $N$. This completes the proof of (2.19). $\square$

# 8 Proofs of the main theorems

## 8.1 Uniform bounds in finite volume

Roughly speaking, Theorems 2.7 and 2.8 are just the special cases $\eta = 1$ and $\eta = 1/2$ of Theorem 6.1:

*Proof of Theorem 2.7.* Let $a > 0$, and let $n_a$ be as in part (ii) of Assumption 5.1. We set $c_2(a) = n_a/(8c_5(a, 1)) > 0$ with $c_5(a, 1)$ as in (5.16) with $\eta = 1$ and

$$c_3(a) = \max\left\{\frac{n_a \log(4n_a)}{8c_5(a, 1)}, 12n_a c_4(a) + c_2(a)\log(12n_a)\right\}, \tag{8.1}$$

where the constant $c_4(a)$ is taken from formula (2.19). Note that $c_2(a) \to \infty$ as $a \to 0$ because $n_a \to \infty$ as $a \to 0$ and $\lim_{a\to 0} c_5(a, 1) < \infty$. Now let $N \in \mathbb{N}$ and $v \in V^{(N)} \setminus \{0\}$, and set $l :=$ level$(v)$. We distinguish two cases; first finitely many exceptional cases, and then the general case.

**Case 1:** $N \le 6n_a$ or $l \le 3n_a$. In this case, there is a path $0 = v_0, v_1, \ldots, v_k = v$ in $V^{(N)}$ joining the vertices 0 and $v$ of length $k \le 12n_a$; recall that levels have width two. Taking the expectation $E_{\mathbb{Q}^{(N)}_{0,a}}$ of

$$\log\frac{x_v}{x_0} = \sum_{i=1}^{k}\log\frac{x_{v_i}}{x_{v_{i-1}}} \tag{8.2}$$

and using the formula (2.19) from Lemma 2.9 and the fact that $|v| \le 12n_a$, we obtain the bound (2.16):

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log\frac{x_v}{x_0}\right] \le 12n_a c_4(a) \le c_3(a) - c_2(a)\log(12n_a) \le c_3(a) - c_2(a)\log|v|. \tag{8.3}$$

**Case 2:** $N > 6n_a$ and $l > 3n_a$. In this case, Theorem 6.1 is applicable with $\ell = v$. Using (4.8) and $\mathbb{P}_1 = \mathbb{Q}^{(N)}_{v,a}$, we rewrite (6.2) for the special value $\eta = 1$ in the form

$$\begin{aligned}
&\frac{1}{2}E_{\mathbb{Q}^{(N)}_{v,a}}\left[\log\frac{x_0}{x_v}\right] = -E_{\mathbb{P}_1}\left[\log\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] \\
&= \log E_{\mathbb{Q}_0}\left[\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] - E_{\mathbb{P}_1}\left[\log\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] \le \frac{-n_a\log((l-1)/n_a)}{16c_5(a, 1)} \\
&\le \frac{-n_a\log(|v|/(4n_a))}{16c_5(a, 1)} \le \frac{1}{2}\left(c_3(a) - c_2(a)\log|v|\right);
\end{aligned} \tag{8.4}$$

note that $l - 1 \ge l/\sqrt{2} \ge |v|/4$ for all $l > 3n_a$. Recall that the box $V^{(N)}$ has periodic boundary conditions. Using reflection symmetry, we interchange 0 and $v$ to obtain the claim (2.16):

$$E_{\mathbb{Q}^{(N)}_{0,a}}\left[\log\frac{x_v}{x_0}\right] = E_{\mathbb{Q}^{(N)}_{v,a}}\left[\log\frac{x_0}{x_v}\right] \le c_3(a) - c_2(a)\log|v|. \tag{8.5}$$

$\square$

*Proof of Theorem 2.8.* Let $a > 0$, $n_a$ as above, and set

$$\beta(a) = \frac{n_a}{64c_5(a, 1/2)} \qquad \text{and} \qquad c_1(a) = 2 \cdot (6\sqrt{2}n_a)^{\beta(a)}. \tag{8.6}$$

Let $N \in \mathbb{N}$ and $v \in V^{(N)} \setminus \{0\}$, $l = \text{level}(v)$. We distinguish the same two cases as in the previous proof:

**Case 1:** $N \le 6n_a$ or $l \le 3n_a$. We observe that $E_{\mathbb{Q}_{0,a}^{(N)}}[(x_v/x_0)^{1/4}]$ is bounded: Using (4.8) and $z^{1/4} \le 1 + z^{1/2}$ for all $z \ge 0$, we get with the abbreviation (6.1):

$$E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \le 1 + E_{\mathbb{Q}_0}\left[\sqrt{\frac{x_v}{x_0}}\right] = 1 + E_{\mathbb{Q}_0}\left[\frac{d\mathbb{P}_1}{d\mathbb{Q}_0}\right] = 2 \le c_1(a)|v|^{-\beta(a)}; \tag{8.7}$$

in the last step, we used $|v| \le 2\sqrt{2}l \le 6\sqrt{2}n_a$.

**Case 2:** $N > 6n_a$ and $l > 3n_a$. This time, we apply Theorem 6.1 with $\eta = 1/2$ and $\ell = v$. Note that by (4.4), we have

$$\frac{d\mathbb{P}_{1/2}}{d\mathbb{Q}_0} = \frac{1}{Z_{1/2,0,v,a}^{(N)}}\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \tag{8.8}$$

and thus

$$\mathbb{P}_{1/2}(dx) = \frac{1}{Z_{1/2,0,v,a}^{(N)}}\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\mathbb{Q}_0(dx) = \frac{1}{Z_{1/2,0,v,a}^{(N)}}\left(\frac{x_0}{x_v}\right)^{\frac{1}{4}}\mathbb{Q}_{v,a}^{(N)}(dx). \tag{8.9}$$

Using reflection symmetry again, we interchange 0 and $v$ in the following computation:

$$\begin{aligned} E_{\mathbb{P}_{1/2}}[\Sigma_v] &= \frac{1}{Z_{1/2,0,v,a}^{(N)}} E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\log\sqrt{\frac{x_v}{x_0}}\right] \\ &= \frac{1}{Z_{1/2,0,v,a}^{(N)}} E_{\mathbb{Q}_{v,a}^{(N)}}\left[\left(\frac{x_0}{x_v}\right)^{\frac{1}{4}}\log\sqrt{\frac{x_0}{x_v}}\right] = -E_{\mathbb{P}_{1/2}}[\Sigma_v] \end{aligned} \tag{8.10}$$

and thus

$$E_{\mathbb{P}_{1/2}}[\Sigma_v] = 0. \tag{8.11}$$

Inserting this in the estimate (6.2) of Theorem 6.1 with $\eta = 1/2$, we obtain the claim (2.17) of Theorem 2.8:

$$\begin{aligned} &\log E_{\mathbb{Q}_0}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] = \log E_{\mathbb{Q}_0}[\exp(\Sigma_v/2)] \le \frac{-n_a\log((l-1)/n_a)}{64c_5(a,1/2)} \\ &\le \frac{-n_a\log(|v|/(4n_a))}{64c_5(a,1/2)} \le \log[c_1(a)|v|^{-\beta(a)}]. \end{aligned} \tag{8.12}$$

□

## 8.2 Bounds in infinite volume

In this subsection, we deduce the infinite-volume results Theorems 2.3, 2.4, and Theorem 2.5 from their finite volume analogs, namely Theorems 2.7, 2.8, and Lemma 2.9. First, we use a tightness argument given in the appendix to pass to the infinite volume limit.

*Proof of Theorems 2.3, 2.4, and Theorem 2.5.* Let $a > 0$, and let $v \in \mathbb{Z}^2$. By Lemma 9.2 in the appendix, there is a measure $\mathbb{Q}_{0,a}$ on $\Omega = (0, \infty)^E$ and a strictly increasing sequence $(n(k))_{k\in\mathbb{N}}$ of natural numbers with the following properties:

a) For any finite subset $F \subset E$, the $\mathbb{Q}_{0,a}^{(n(k))}$-distribution of $(x_e)_{e\in F}$ converges weakly to the $\mathbb{Q}_{0,a}$-distribution of $(x_e)_{e\in F}$.

b) The representation (2.7) of edge-reinforced random walk on $\mathbb{Z}^2$ as a mixture of Markov chains holds with the mixing measure $\mathbb{Q}_{0,a}$.

Recall from (4.1) that the weights are normalized such that $x_{e_0} = 1$ holds $\mathbb{Q}_{0,a}^{(n(k))}$-a.s. for a fixed reference edge $e_0 \in E$.

To prove Theorem 2.4, let $v \in \mathbb{Z}^2 \setminus \{0\}$. Since $(x_v/x_0)^{1/4}$ takes only positive values and is a continuous function of the finitely many weights $x_e$ with $e \ni v$ or $e \ni 0$, we conclude

$$E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] = \lim_{M\to\infty} E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \wedge M\right]$$
$$= \lim_{M\to\infty} \liminf_{k\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}} \wedge M\right] \le \liminf_{k\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[\left(\frac{x_v}{x_0}\right)^{\frac{1}{4}}\right] \le c_1(a)|v|^{-\beta(a)}; \qquad (8.13)$$

we used Theorem 2.8 in the last step. This proves Theorem 2.4. Using (2.18) from Lemma 2.9, the same argument yields Theorem 2.5.

To prove Theorem 2.3, we observe that $\log(x_v/x_0)$ is also a continuous function of the finitely many weights $x_e$ with $e \ni v$ or $e \ni 0$. Let $0 = v_0, v_1, \ldots, v_L = v$ be a path from 0 to $v$, and let $e_i := \{v_{i-1}, v_i\}$, $1 \le i \le L$. Then, for all $N$ large enough that $v_i \in V^{(N)}$ for all $1 \le i \le L$ and all $M > 0$, one finds:

$$\begin{aligned}
\mathbb{Q}_{0,a}^{(N)}\left[\left|\log\frac{x_v}{x_0}\right| \ge M\right] \le & \mathbb{Q}_{0,a}^{(N)}\left[\sum_{i=1}^{L}\left|\log\frac{x_{v_i}}{x_{v_{i-1}}}\right| \ge M\right] \\
\le & \sum_{i=1}^{L} \mathbb{Q}_{0,a}^{(N)}\left[\left|\log\frac{x_{v_i}}{x_{v_{i-1}}}\right| \ge \frac{M}{L}\right] \\
\le & \sum_{i=1}^{L}\left\{\mathbb{Q}_{0,a}^{(N)}\left[\log\frac{x_{v_i}}{x_{e_i}} \ge \frac{M}{L}\right] + \mathbb{Q}_{0,a}^{(N)}\left[\log\frac{x_{v_{i-1}}}{x_{e_i}} \ge \frac{M}{L}\right]\right\} \\
\le & 2Lc_9(a)e^{-Ma/(2L)}; \qquad (8.14)
\end{aligned}$$

in the last but one step, we used $x_{e_i} \le \min\{x_{v_i}, x_{v_{i-1}}\}$, and in the last step we used (7.2) from Theorem 7.1. Thus, $\log(x_v/x_0)$ has exponential tails, uniformly in the size $N$ of the box.

Consequently, for any fixed vertex $v$, the law of $\log(x_v/x_0)$ with respect to $\mathbb{Q}_{0,a}^{(N)}$ is uniformly integrable in $N$ for large $N$. Abbreviating $f_M(x) := (x \wedge M) \vee (-M)$ for the truncation at $M$ and $-M$, we get, using uniform integrability to interchange limits:

$$
\begin{aligned}
&E_{\mathbb{Q}_{0,a}}\left[\log \frac{x_v}{x_0}\right] = \lim_{M\to\infty} E_{\mathbb{Q}_{0,a}}\left[f_M\left(\log \frac{x_v}{x_0}\right)\right] \\
&= \lim_{M\to\infty}\lim_{k\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[f_M\left(\log \frac{x_v}{x_0}\right)\right] = \lim_{k\to\infty}\lim_{M\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[f_M\left(\log \frac{x_v}{x_0}\right)\right] \\
&= \lim_{k\to\infty} E_{\mathbb{Q}_{0,a}^{(n(k))}}\left[\log \frac{x_v}{x_0}\right] \le c_3(a) - c_2(a)\log|v|. \qquad (8.15)
\end{aligned}
$$

The last inequality follows from the bound (2.16). □

## 8.3 Hitting probabilities for ERRW

Finally, we apply our bounds for the random environment to deduce estimates for the hitting probabilities for the edge-reinforced random walk.

*Proof of Theorem 2.1.* 1. We claim first that for all $x \in \Omega$ and all $v \in \mathbb{Z}^2\setminus\{0\}$, the probability $Q_{0,x}[\tau_v < \tau_0]$ for the random walk starting in 0 in the fixed environment $x$ to visit $v$ before returning to 0 and the probability $Q_{v,x}[\tau_0 < \tau_v]$ for the random walk with exchanged roles of 0 and $v$ in the same environment are connected by the following equation:

$$Q_{0,x}[\tau_v < \tau_0] = \frac{x_v}{x_0} Q_{v,x}[\tau_0 < \tau_v]. \qquad (8.16)$$

To prove this claim, take two vertices $u \neq w$. Denote by $\Pi_{u,w}$ the set of all admissible finite paths

$$\pi = (u = v_0, v_1, \ldots, v_n = w), \qquad (8.17)$$

$n \in \mathbb{N}$, joining $u$ and $w$, which visit $u$ or $w$ precisely once, namely at the end points. For any such path $\pi$, we introduce the event

$$A_\pi := \{X_i = v_i \text{ for } i = 0, 1, \ldots, n\} \subseteq (\mathbb{Z}^2)^{\mathbb{N}_0} \qquad (8.18)$$

that $\pi$ is an initial piece of the random path. Note that the events $A_\pi$, $\pi \in \Pi_{u,w}$, are pairwise disjoint. Furthermore, let

$$\pi^{\leftrightarrow} := (v_n, \ldots, v_1, v_0) \qquad (8.19)$$

denote the reversed path. Note that the reversion defines a bijection $\cdot^{\leftrightarrow} : \Pi_{0,v} \to \Pi_{v,0}$. Moreover, for any path $\pi$ as in (8.17),

$$x_u Q_{u,x}[A_\pi] = x_u \prod_{i=0}^{n-1} \frac{x_{\{v_i,v_{i+1}\}}}{x_{v_i}} = x_w \prod_{i=1}^{n} \frac{x_{\{v_i,v_{i-1}\}}}{x_{v_i}} = x_w Q_{w,x}[A_{\pi^{\leftrightarrow}}]. \qquad (8.20)$$

Now, we take $u = 0$ and $w = v$. Summing (8.20) over all $\pi \in \Pi_{0,v}$, we obtain the claim (8.16) as follows:

$$
\begin{aligned}
x_0 Q_{0,x}[\tau_v < \tau_0] &= x_0 \sum_{\pi \in \Pi_{0,v}} Q_{0,x}[A_\pi] = x_v \sum_{\pi \in \Pi_{0,v}} Q_{v,x}[A_{\pi^\leftrightarrow}] \\
&= x_v \sum_{\pi \in \Pi_{v,0}} Q_{v,x}[A_\pi] = x_v Q_{v,x}[\tau_0 < \tau_v].
\end{aligned} \tag{8.21}
$$

From this, we conclude

$$
Q_{0,x}[\tau_v < \tau_0] \leq \frac{x_v}{x_0}. \tag{8.22}
$$

Taking the 1/4-th power and expectations yields

$$
\begin{aligned}
P_{0,a}[\tau_v < \tau_0] =& E_{\mathbb{Q}_{0,a}}[Q_{0,x}[\tau_v < \tau_0]] \leq E_{\mathbb{Q}_{0,a}}[Q_{0,x}[\tau_v < \tau_0]^{1/4}] \\
\leq& E_{\mathbb{Q}_{0,a}}\left[\left(\frac{x_v}{x_0}\right)^{1/4}\right] \leq c_1(a)|v|^{-\beta(a)};
\end{aligned} \tag{8.23}
$$

we used the representation of the edge-reinforced random walk as a random walk in random environment from Theorem 2.2 in the first step and the bound (2.10) from Theorem 2.4 in the last step. This shows part (a) of Theorem 2.1.

2. To prove part (b), let $\Sigma_{u,w}^n$ denote the set of all admissible paths

$$
\pi = (u = v_0, v_1, \ldots, v_n = w) \tag{8.24}
$$

from $u$ to $w$ of length $n$. Again, reversion yields a bijection between $\Sigma_{u,w}^n$ and $\Sigma_{w,u}^n$, and the events $A_\pi$, $\pi \in \Sigma_{u,w}^n$, are pairwise disjoint. In analogy to (8.21), we obtain

$$
x_0 Q_{0,x}[X_n = v] = x_0 \sum_{\pi \in \Sigma_{0,v}^n} Q_{0,x}[A_\pi] = x_v \sum_{\pi \in \Sigma_{v,0}^n} Q_{v,x}[A_\pi] = x_v Q_{v,x}[X_n = 0]. \tag{8.25}
$$

Using this, an analogous argument to (8.23) yields the claim (2.2).

□

# 9 Appendix

## 9.1 Proof of Lemma 4.1

In this appendix, we consider a generalization of Lemma 4.1 to arbitrary finite graphs. It essentially states the formula of Coppersmith and Diaconis [CD86] for the law of the random environment, transformed to a special normalization.

Consider edge-reinforced random walk on any finite graph $(V, E)$ with starting point $v_0 \in V$ and initial weights $a = (a_e)_{e \in E} \in (0, \infty)^E$. Recall definition (2.8) of $x_v$; we use the similar notation $a_v = \sum_{e \ni v} a_e$. For $x = (x_e)_{e \in E} \in (0, \infty)^E$, we set

$$
\phi_{v_0,a}(x) = c_{10}(v_0, a) \frac{\prod_{e \in E} x_e^{a_e - 1}}{x_{v_0}^{a_{v_0}/2} \prod_{v \in V \setminus \{v_0\}} x_v^{(a_v+1)/2}} \sqrt{\sum_{T \in \mathcal{T}} \prod_{e \in T} x_e}, \tag{9.1}
$$

where the sum is indexed by the set $\mathcal{T}$ of all spanning trees of $(V,E)$, viewed as sets of edges, and the constant $c_{10}(v_0,a)$ is defined by

$$c_{10}(v_0,a) = \frac{\Gamma(a_{v_0}/2)\prod_{v\in V\setminus\{v_0\}}\Gamma((a_v+1)/2)}{\prod_{e\in E}\Gamma(a_e)}\,\frac{2^{1-|V|+\sum_{e\in E}a_e}}{\pi^{(|V|-1)/2}}. \tag{9.2}$$

**Lemma 9.1.** *The above edge-reinforced random walk on $(V,E)$ has the same distribution as a random walk in a random environment given by random positive weights $\tilde{x} = (\tilde{x}_e)_{e\in E}$ on the edges. Normalizing $\tilde{x}$ such that $\tilde{x}_{e_0} = 1$ for a fixed reference edge $e_0$, the law of $\tilde{x}$ has the density $\phi_{v_0,a}$ with respect to the Lebesgue measure $\delta_1(d\tilde{x}_{e_0})\prod_{e\in E\setminus\{e_0\}} d\tilde{x}_e$ on the hyperplane $H = \{(\tilde{x}_e)_{e\in E}\in(0,\infty)^E \mid \tilde{x}_{e_0} = 1\}$.*

Note that the normalizing constant $c_{10}(v_0,a)$ does not depend on the choice of the reference edge $e_0$.

*Proof.* By Theorem 3.1 of [Rol03], the edge-reinforced random walk on $(V,E)$ has the same distribution as a random walk in a random environment given by random positive weights $x = (x_e)_{e\in E}$ on the edges. The law of the random environment $\mathbb{Q}^{\Delta}_{v_0,a}$, normalized such that $\sum_{e\in E} x_e = 1$, has a density with respect to the normalized surface measure on the simplex $\Delta = \{(x_e)_{e\in E}\in(0,\infty)^E \mid \sum_{e\in E} x_e = 1\}$. The density is provided by Theorem 1 in [KR00]. Combining this theorem with the matrix-tree-theorem ([Mau76], p. 145, Theorem 3', see also Theorem 3 in [KR00]), it is given by

$$\frac{d\mathbb{Q}^{\Delta}_{v_0,a}}{d\sigma}(x) = \frac{\phi_{v_0,a}(x)}{(|E|-1)!}. \tag{9.3}$$

Consider the change of normalization

$$F:\Delta\to H, \qquad F((x_e)_{e\in E}) = \left(\frac{x_e}{x_{e_0}}\right)_{e\in E}. \tag{9.4}$$

We factor $F$ as follows: $\Delta \xrightarrow{\pi} \pi[\Delta] \xrightarrow{f} (0,\infty)^{E\setminus\{e_0\}} \xrightarrow{\iota} H$,

$$(x_e)_{e\in E} \overset{\pi}{\longmapsto} (x_e)_{e\in E\setminus\{e_0\}} \overset{f}{\longmapsto} \left(\tilde{x}_e = \frac{x_e}{1-\sum_{e'\in E\setminus\{e_0\}} x_{e'}} = \frac{x_e}{x_{e_0}}\right)_{e\in E\setminus\{e_0\}} \overset{\iota}{\longmapsto} (\tilde{x}_e)_{e\in E} \tag{9.5}$$

where the first map is the canonical projection and the last map $\iota$ just includes an extra component $\tilde{x}_{e_0} = 1$. Let us calculate the Jacobi determinant of the map $f$. Using the abbreviation $x_{e_0} = 1-\sum_{e'\in E\setminus\{e_0\}} x_{e'}$, we have the Jacobi matrix

$$Df(x) = \left(\frac{\partial\tilde{x}_e}{\partial x_{e'}}\right)_{e,e'\in E\setminus\{e_0\}} = \frac{1}{x_{e_0}}\left(\delta_{ee'} + \frac{x_e}{x_{e_0}}\right)_{e,e'\in E\setminus\{e_0\}}, \tag{9.6}$$

which is $1/x_{e_0}$ times the identity matrix $I$ plus a rank 1 matrix. Since $\det(I+A) = 1+\operatorname{tr}A$ holds for rank 1 matrices $A$, we get the Jacobi determinant

$$\det Df(x) = \frac{1}{x_{e_0}^{|E|-1}}\left(1+\sum_{e\in E\setminus\{e_0\}}\frac{x_e}{x_{e_0}}\right) = \frac{1}{x_{e_0}^{|E|}}. \tag{9.7}$$

Abbreviating

$$\alpha := \sum_{e\in E} a_e - |E| - \sum_{v\in V} \frac{a_v}{2} - \frac{|V|-1}{2} = -|E| - \frac{|V|-1}{2} = -|E| - \frac{|T|}{2} \tag{9.8}$$

for any spanning tree $T \subseteq E$ in $(V, E)$, we rewrite (9.1) as

$$\phi_{v_0,a}(x) = c_{10}(v_0,a) x_{e_0}^{\alpha} \frac{\prod_{e\in E} \tilde{x}_e^{a_e-1}}{\tilde{x}_{v_0}^{a_{v_0}/2} \prod_{v\in V\setminus\{v_0\}} \tilde{x}_v^{(a_v+1)/2}} \sqrt{\sum_{T\in\mathcal{T}} x_{e_0}^{|T|} \prod_{e\in T} \tilde{x}_e} = x_{e_0}^{-|E|} \phi_{v_0,a}(\tilde{x}) \tag{9.9}$$

and thus

$$\frac{\phi_{v_0,a}(x)}{\det Df(\pi(x))} = \phi_{v_0,a}(\tilde{x}). \tag{9.10}$$

We combine this with (9.3). Using that the projected normalized surface measure $\pi\sigma$ has the density $(|E|-1)!$ with respect to the Lebesgue measure on $\pi[\Delta]$, we get that the transformed distribution $\mathbb{Q}_{v_0,a} = F\mathbb{Q}^{\Delta}_{v_0,a}$ has the density $\phi_{v_0,a}(\tilde{x})$ with respect to the Lebesgue measure $\delta_1(d\tilde{x}_{e_0}) \prod_{e\neq e_0} d\tilde{x}_e$ on the hyperplane $H$. This proves the claim. $\square$

## 9.2 A tightness argument for infinite-volume limits

In this appendix, we prove a variant of Lemma 5.3 of [MR07c] for more general graphs. Consider any infinite, locally finite, undirected, weighted graph $G_\infty = (V_\infty, E_\infty, a)$ with vertex set $V_\infty$, edge set $E_\infty$ and positive weights $a = (a_e)_{e\in E_\infty}$ on the edges. Furthermore, let $G_N = (V_N, E_N, a_N)$, $N \in \mathbb{N}$, be a sequence of finite, undirected, weighted graphs with weights $a_N = (a_{N,e})_{e\in E_N} \in (0,\infty)^{E_N}$. Finally, let $\tilde{G}_N = (\tilde{V}_N, \tilde{E}_N, \tilde{a}_N)$, $N \in \mathbb{N}$, be an increasing sequence of connected weighted subgraphs of $G_\infty$ with the weights $\tilde{a}_N$ induced by $G_\infty$ with the following properties:

1. For any $N$, $\tilde{G}_N$ is a full subgraph of $G_\infty$, i.e., for every edge $e \in E_\infty$ with $e \subseteq \tilde{V}_N$, one has $e \in \tilde{E}_N$.
2. One has $\tilde{V}_N \uparrow V_\infty$ as $N \to \infty$.
3. Furthermore, for any $N$, the graph $\tilde{G}_N$ is also a full weighted subgraph of $G_N$. In particular, restricted to $\tilde{E}_N$, the weights on $G_N$ and on $G_\infty$ coincide.

Take a starting point $0 \in \tilde{V}_0$ and a reference edge $e_0 \in \tilde{E}_0$. Consider edge-reinforced random walk on $G_N$ and on $G_\infty$ with starting point 0 and initial weights $a_N$ and $a$, respectively. Let $\mathbb{Q}_N$ denote the distribution of the random environment on $(0,\infty)^{E_N}$ in the representation of the edge-reinforced random walk on $G_N$ as in Lemma 9.1, normalized such that $x_{e_0} = 1$ holds $\mathbb{Q}_N$-almost surely. The following lemma is a variant of Lemma 5.3 in [MR07c]. In that paper, we consider the special case $\tilde{G}_N = G_N$. Boxes with periodic boundary conditions, as needed in our application, are not covered by the cited lemma.

**Lemma 9.2.** *There exist a probability measure $\mathbb{Q}_\infty$ on $(0,\infty)^{E_\infty}$ and a strictly increasing sequence $(n(k))_{k\in\mathbb{N}}$ of natural numbers such that the following hold:*

(a) *For any finite subset $F \subset E_\infty$, the $\mathbb{Q}_{n(k)}$-distribution of $(x_e)_{e\in F}$ converges weakly to the $\mathbb{Q}_\infty$-distribution of $(x_e)_{e\in F}$ as $k \to \infty$.*

(b) *Edge-reinforced random walk on $G_\infty$, started in 0, has the same law as a random walk in a random environment, where the random environment is distributed according to $\mathbb{Q}_\infty$.*

*Proof.* We construct $\mathbb{Q}_\infty$ by a tightness and diagonalization argument. We claim: For every $e \in E_\infty$ and every $N_0 \in \mathbb{N}$ with $e \in \tilde{E}_{N_0}$, the sequence of laws $([\log x_e]\mathbb{Q}_N)_{N\geq N_0}$ of $\log x_e$ with respect to $\mathbb{Q}_N$ is tight. To see this, fix a path of edges $e_0, e_1, \ldots, e_k$ starting in $e_0$ and ending in $e = e_k$. Now, Theorem 2.4 in [MR07c] claims that for all $M > 0$

$$\mathbb{Q}_N[x_e \geq Mx_{e_0}] \leq c_{11}M^{-c_{12}} \quad \text{and} \quad \mathbb{Q}_N[x_{e_0} \geq Mx_e] \leq c_{11}M^{-c_{12}} \tag{9.11}$$

holds with some positive constants $c_{11}$ and $c_{12}$, depending on the initial weights $a$ and on the path $e_0, e_1, \ldots, e_k$, but *not* depending on $N$. Since $x_{e_0} = 1$ holds $\mathbb{Q}_N$-almost surely, this implies the tightness claimed above. We recursively construct a sequence of strictly increasing sequences $(n_{k,N})_{k\in\mathbb{N}}$, $N \in \mathbb{N}$ with $n_{k,N} \geq N$ for all $k$ and $N$, with the following properties:

(i) $(n_{k,N+1})_{k\in\mathbb{N}}$ is a subsequence of $(n_{k,N})_{k\in\mathbb{N}}$ for all $N$.

(ii) The joint law of $(\log x_e)_{e\in\tilde{E}_N}$ with respect to $\mathbb{Q}_{n_{k,N}}$ converges weakly as $k \to \infty$ for all $N$.

For $N = 0$, using tightness, there exists a strictly increasing sequences $(n_{k,0})_{k\in\mathbb{N}}$ of natural numbers such that (ii) holds for $N = 0$. For the recursion step $N \rightsquigarrow N + 1$, using tightness again, choose a subsequence $(n_{k,N+1})_{k\in\mathbb{N}}$ of $(n_{k,N})_{k\in\mathbb{N}}$ such that (ii) holds also for $N+1$. Finally, take the diagonal sequence $n(k) := n_{k,k}$. Then, for all $N \in \mathbb{N}$, the law of $(x_e)_{e\in\tilde{E}_N}$ with respect to $\mathbb{Q}_{n(k)}$ converges weakly as $k \to \infty$ to some distribution $\tilde{\mathbb{Q}}_N$ on $(0,\infty)^{\tilde{E}_N}$. By construction, the projection of $\tilde{\mathbb{Q}}_{N+1}$ to $(0,\infty)^{\tilde{E}_N}$ coincides with $\tilde{\mathbb{Q}}_N$. Thus Kolmogorov's extension theorem yields a probability measure $\mathbb{Q}_\infty$ on $(0,\infty)^{E_\infty}$ with marginals $\tilde{\mathbb{Q}}_N$ for all $N$; recall that $\tilde{E}_N \uparrow E_\infty$ as $N \to \infty$. By construction, claim (a) of the lemma is true.

The proof of part (b) is very similar to the proof of the first part of Lemma 5.1 in [MR06a]. For completeness, we repeat the argument. We claim that the law $P_{0,a}$ of the edge-reinforced law on $G_\infty$ has the following representation:

$$P_{0,a}[A] = \int_{(0,\infty)^{E_\infty}} Q_{0,x}[A]\, \mathbb{Q}_\infty(dx) \tag{9.12}$$

for all events $A$ of admissible paths, where $Q_{0,x}$ denotes the law of a Markovian random walk on $G_\infty$ in the fixed environment $x$. Events of the form $A = \{(X_s)_{s=0,\ldots,m-1} = \pi\}$, ($\pi \in V^m$ an admissible path, $m \in \mathbb{N}$), together with the empty set, generate the canonical $\sigma$-field on the space of admissible paths and form a closed system with respect to intersection. Thus it suffices to check (9.12) for these events. Fix $m \in \mathbb{N}$ and $\pi \in V_\infty^m$. Without loss of generality we may assume that $\pi$ is a path in $G_\infty$ starting in 0. Let $N \in \mathbb{N}$ be so large that $\pi$ and all edges in $E_\infty$ adjacent to $\pi$ are contained in $\tilde{E}_N$. Then the probability that the edge-reinforced random walk with initial weights $a$ starting in 0 follows $\pi$ up to time $m$ is the same for the three graphs $G_\infty$,

$\tilde{G}_N$, and $G_N$. Using the representation of the edge-reinforced random walk on $G_N$ as a mixture of Markov chains $Q_{0,x}$ with mixing measure $\mathbb{Q}_N(dx)$ (Lemma 9.1), this implies

$$P_{0,a}[(X_s)_{s=0,\ldots,m-1} = \pi] = \int_{(0,\infty)^{E_N}} Q_{0,x}[(X_s)_{s=0,\ldots,m-1} = \pi]\, \mathbb{Q}_N(dx) \tag{9.13}$$

for sufficiently large $N$. Taking the limit along the sequence $(n(k))_{k\in\mathbb{N}}$ yields the claim (9.12) as follows:

$$\begin{aligned} P[(X_s)_{s=0,\ldots,m-1} = \pi] &= \lim_{k\to\infty} \int_{(0,\infty)^{E_{n(k)}}} Q_{0,x}[(X_s)_{s=0,\ldots,m-1} = \pi]\, \mathbb{Q}_{n(k)}(dx) \\ &= \int_{(0,\infty)^{E_\infty}} Q_{0,x}[(X_s)_{s=0,\ldots,m-1} = \pi]\, \mathbb{Q}_\infty(dx). \end{aligned} \tag{9.14}$$

□

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