DOI: 10.1214/EJP.v12-388

Electronic Journal of Probability

Vol. 12 (2007), Paper no. 1, pages 1–46.

Journal URL
http://www.math.washington.edu/~ejpecp/

Frequent Points for Random Walks in Two Dimensions

Richard F. Bass*
Department of Mathematics
University of Connecticut
Storrs, CT 06269-3009
bass@math.uconn.edu

Jay Rosen†
Department of Mathematics
College of Staten Island, CUNY
Staten Island, NY 10314
jrosen3@earthlink.net

Abstract

For a symmetric random walk in $\mathbb{Z}^2$ which does not necessarily have bounded jumps we study those points which are visited an unusually large number of times. We prove the analogue of the Erdős-Taylor conjecture and obtain the asymptotics for the number of visits to the most visited site. We also obtain the asymptotics for the number of points which are visited very frequently by time n. Among the tools we use are Harnack inequalities and Green's function estimates for random walks with unbounded jumps; some of these are of independent interest.

Key words: Random walks, Green's functions, Harnack inequalities, frequent points.

AMS 2000 Subject Classification: Primary 60G50; Secondary: 60F15.

Submitted to EJP on July 25 2006, final version accepted December 22 2006.

*Research partially supported by NSF grant #DMS-0601783.
†Research supported, in part, by grants from the NSF and from PSC-CUNY

1

1 Introduction

The paper (4) proved a conjecture of Erdős and Taylor concerning the number L_n^* of visits to the most visited site for simple random walk in $\mathbb{Z}^2$ up to step n. It was shown there that

$$\lim_{n\to\infty} \frac{L_n^*}{(\log n)^2} = 1/\pi \quad \text{a.s.} \tag{1.1}$$

The approach in that paper was to first prove an analogous result for planar Brownian motion and then to use strong approximation. This approach applies to other random walks, but only if they have moments of all orders. In a more recent paper (11), Rosen developed purely random walk methods which allowed him to prove (1.1) for simple random walk. A key to the approach both for Brownian motion and simple random walk is to exploit a certain tree structure with regard to excursions between nested families of disks. When we turn to random walks with jumps, this tree structure is no longer automatic, since the walk may jump across disks. In this paper we show how to extend the method of (11) to symmetric recurrent random walks X_j, $j \geq 0$, in $\mathbb{Z}^2$. Not surprisingly, our key task is to control the jumps across disks. Our main conclusion is that it suffices to require that for some $\beta > 0$

$$\mathbb{E}|X_1|^{3+2\beta} < \infty, \tag{1.2}$$

together with some mild uniformity conditions. (It will make some formulas later on look nicer if we use 2β instead of β here.) We go beyond (1.1) and study the size of the set of 'frequent points,' i.e. those points in $\mathbb{Z}^2$ which are visited an unusually large number of times, of order $(\log n)^2$. Perhaps more important than our specific results, we develop powerful estimates for our random walks which we expect will have wide applicability. In particular, we develop Harnack inequalities extending those of (10) and we develop estimates for Green's functions for random walks killed on entering a disk. The latter estimates are new even for simple random walk and are of independent interest.

We assume for simplicity that X_1 has covariance matrix equal to the identity and that X_n is strongly aperiodic. Set $p_1(x,y) = p_1(x-y) = \mathbb{P}^x(X_1 = y)$. We will say that our walk satisfies Condition A if the following holds.

Condition A. *Either X_1 is finite range, that is, $p_1(x)$ has bounded support, or else for any $s \leq R$ with s sufficiently large*

$$\inf_{y\,;\, R\leq |y|\leq R+s} \sum_{z\in D(0,R)} p_1(y,z) \geq ce^{-\beta\, s^{1/4}}. \tag{1.3}$$

Condition A is implied by

$$p_1(x) \geq ce^{-\beta\,|x|^{1/4}}, \qquad x\in\mathbb{Z}^2, \tag{1.4}$$

but (1.3) is much weaker. (1.3) is a mild uniformity condition, and is used to obtain Harnack inequalities. Recent work on processes with jumps (see (1)) indicates that without some sort of uniformity condition such as (1.3) the Harnack inequality may fail.

Let L_n^x denote the number of times that $x \in \mathbb{Z}^2$ is visited by the random walk in $\mathbb{Z}^2$ up to step n and set $L_n^* = \max_{x\in\mathbb{Z}^2} L_n^x$.

Theorem 1.1. *Let $\{X_j\,;\,j\geq 1\}$ be a symmetric strongly aperiodic random walk in $\mathbb{Z}^2$ with X_1 having the identity as the covariance matrix and satisfying Condition A and (1.2). Then*

$$\lim_{n\to\infty}\frac{L_n^*}{(\log n)^2}=1/\pi \quad a.s. \tag{1.5}$$

Theorem 1.1 is the analogue of the Erdős-Taylor conjecture for simple random walks. We also look at how many frequent points there are. Set

$$\Theta_n(\alpha)=\Big\{x\in\mathbb{Z}^2:\ \frac{L_n^x}{(\log n)^2}\geq \alpha/\pi\Big\}. \tag{1.6}$$

For any set $B\subseteq\mathbb{Z}^2$ let $T_B=\inf\{i\geq 0\,|\,X_i\in B\}$ and let $|B|$ be the cardinality of B. Let

$$\Psi_n(a)=\Big\{x\in D(0,n):\ \frac{L^x_{T_{D(0,n)^c}}}{(\log n)^2}\geq 2a/\pi\Big\} \tag{1.7}$$

Theorem 1.2. *Let $\{X_j\,;\,j\geq 1\}$ be as in Theorem 1.1. Then for any $0<\alpha<1$*

$$\lim_{n\to\infty}\frac{\log|\Theta_n(\alpha)|}{\log n}=1-\alpha \quad a.s. \tag{1.8}$$

Equivalently, for any $0<a<2$

$$\lim_{n\to\infty}\frac{\log|\Psi_n(a)|}{\log n}=2-a \quad a.s. \tag{1.9}$$

The equivalence of (1.8) and (1.9) follows from the fact that

$$\lim_{n\to\infty}\frac{\log T_{D(0,n)^c}}{\log n}=2 \quad \text{a.s.} \tag{1.10}$$

For the convenience of the reader we give a proof of this fact in the appendix.

In Section 2 we collect some facts about random walks in $\mathbb{Z}^2$, and in Section 3 we establish the Harnack inequalities we need. The upper bound of Theorem 1.2 is proved in Section 4. The lower bound is established in Section 5, subject to certain estimates which form the subject of the following three sections. An appendix gives further information about random walks in $\mathbb{Z}^2$.

There is a good deal of flexibility in our choice of Condition A. For example, if $\mathbb{E}|X_1|^{4+2\beta}<\infty$, we can replace $\beta\, s^{1/4}$ by $s^{1/2}$. On the other hand, if we merely assume that $\mathbb{E}|X_1|^{2+2\beta}<\infty$, our methods do not allow us to obtain any useful analogue of the Harnack inequalities we derive in Section 3.

2 Random Walk Preliminaries

Let X_n, $n\geq 0$, denote a symmetric recurrent random walk in $\mathbb{Z}^2$ with covariance matrix equal to the identity. We set $p_n(x,y)=p_n(x-y)=\mathbb{P}^x(X_n=y)$ and assume that for some $\beta>0$

$$\mathbb{E}|X_1|^{3+2\beta}=\sum_x |x|^{3+2\beta}p_1(x)<\infty. \tag{2.1}$$

(It will make some formulas later on look nicer if we use 2β instead of β here.) In this section we collect some facts about X_n, $n \geq 0$, which will be used in our paper. The estimates for the interior of a ball are the analogues for 2 dimensions and $3 + 2\beta$ moments of some results that are proved in (10) for random walks in dimensions 3 and larger that have 4 moments. Several results which are well known to experts but are not in the literature are given in an appendix.

We will assume throughout that X_n is strongly aperiodic. Define the potential kernel

$$a(x) = \lim_{n\to\infty} \sum_{j=0}^{n} \{p_j(0) - p_j(x)\}. \tag{2.2}$$

We have that $a(x) < \infty$ for all $x \in \mathbb{Z}^2$, $a(0) = 0$, $a(x) \geq 0$, and for $|x|$ large

$$a(x) = \frac{2}{\pi} \log |x| + k + o(|x|^{-1}) \tag{2.3}$$

with k an explicit constant. See Proposition 9.2 for a proof.

Let $D(x, r) = \{y \in \mathbb{Z}^2 \,|\, |y - x| < r\}$. For any set $A \subseteq \mathbb{Z}^2$ we define the boundary ∂A of A by $\partial A = \{y \in \mathbb{Z}^2 \,|\, y \in A^c, \text{ and } \inf_{x\in A} |y - x| \leq 1\}$ and the s-band ∂A_s of ∂A by $\partial A_s = \{y \in \mathbb{Z}^2 \,|\, y \in A^c, \text{ and } \inf_{x\in A} |y - x| \leq s\}$. For any set $B \subseteq \mathbb{Z}^2$ let $T_B = \inf\{i \geq 0 \,|\, X_i \in B\}$. For $x, y \in A$ define the Green's function

$$G_A(x, y) = \sum_{i=0}^{\infty} \mathbb{E}^x \left(X_i = y,\, i < T_{A^c}\right). \tag{2.4}$$

Our first goal is to obtain good estimates for the Green's function in the interior of a disk and for the exit distributions of annuli.

For some $c < \infty$

$$\mathbb{E}^x(T_{D(0,n)^c}) \leq cn^2, \qquad x \in D(0, n), \qquad n \geq 1. \tag{2.5}$$

This is proved in Lemma 9.3. In particular,

$$\sum_{y\in D(0,n)} G_{D(0,n)}(x, y) \leq cn^2. \tag{2.6}$$

Define the hitting distribution of A by

$$H_A(x, y) = \mathbb{P}^x(X_{T_A} = y). \tag{2.7}$$

As in Proposition 1.6.3 of (8), with A finite, by considering the first hitting time of A^c we have that for $x, z \in A$

$$G_A(x, z) = \left\{ \sum_{y\in A^c} H_{A^c}(x, y)a(y - z) \right\} - a(x - z). \tag{2.8}$$

In particular

$$\begin{aligned} G_{D(0,n)}(0, 0) \quad &= \sum_{n\leq|y|\leq n+n^{3/4}} H_{D(0,n)^c}(0, y)a(y) \\ &\quad + \sum_{|y|>n+n^{3/4}} H_{D(0,n)^c}(0, y)a(y). \end{aligned} \tag{2.9}$$

Using the last exit decomposition

$$H_{D(0,n)^c}(0,y) = \sum_{z\in D(0,n)} G_{D(0,n)}(0,z)p_1(z,y) \tag{2.10}$$

together with (2.6) and (2.1) we have for any $k \geq 1$

$$\begin{aligned}\sum_{|y|\geq n+kn^{3/4}} H_{D(0,n)^c}(0,y) &\leq \sum_{z\in D(0,n)} G_{D(0,n)}(0,z)P(|X_1| \geq kn^{3/4}) \\ &\leq cn^2/(kn^{3/4})^{3+2\beta} \leq c/(k^3 n^{1/4+\beta}). \end{aligned}\tag{2.11}$$

Using this together with (2.3) we can bound the last term in (2.9) by

$$\begin{aligned}&\sum_{|y|>n+n^{3/4}} H_{D(0,n)^c}(0,y)a(y) \\ &\leq \sum_{k=1}^{\infty} \frac{c}{k^3 n^{1/4+\beta}} \sup_{n+kn^{3/4}\leq|x|\leq n+(k+1)n^{3/4}} a(x) \\ &\leq \frac{C}{n^{1/4+\beta}} \sum_{k=1}^{\infty} \frac{1}{k^3} \log(n+(k+1)n^{3/4}) = O(n^{-1/4}).\end{aligned}\tag{2.12}$$

Using (2.3) for the first term in (2.9) then gives

$$G_{D(0,n)}(0,0) = \frac{2}{\pi}\log n + k + O(n^{-1/4}). \tag{2.13}$$

Let $\eta = \inf\{i \geq 1 \,|\, X_i \in \{0\} \cup D(0,n)^c\}$. Applying the optional sampling theorem to the martingale $a(X_{j\wedge\eta})$ and letting $j \to \infty$, we have that for any $x \in D(0,n)$

$$a(x) = \mathbb{E}^x(a(X_\eta)) = \mathbb{E}^x(a(X_\eta)\,;\, X_\eta \in D(0,n)^c). \tag{2.14}$$

To justify taking the limit as $j \to \infty$, note that $|a(X_{j\wedge\eta})|^2$ is a submartingale, so $\mathbb{E}|a(X_{j\wedge\eta})|^2 \leq \mathbb{E}|a(X_\eta)|^2$, which is finite by (2.3) and (2.11); hence the family of random variables $\{a(X_{j\wedge\eta})\}$ is uniformly integrable. Using (2.3) and the analysis of (2.12) we find that

$$\begin{aligned}&\mathbb{E}^x\left(a(X_\eta)\,;\, X_\eta \in D(0,n)^c\right) \\ &= \sum_{y\in\partial D(0,n)_{n^{3/4}}} a(y)\mathbb{P}^x\left(X_\eta = y\right) + \sum_{y\in D(0,n+n^{3/4})^c} a(y)\mathbb{P}^x\left(X_\eta = y\right) \\ &= \left(\frac{2}{\pi}\log n + k + O(n^{-1/4})\right)\mathbb{P}^x\left(X_\eta \in D(0,n)^c\right) + O(n^{-1/4}).\end{aligned}\tag{2.15}$$

Using this and (2.3) we find that for $0 < |x| < n$,

$$\begin{aligned}\mathbb{P}^x\left(T_0 < T_{D(0,n)^c}\right) &= \frac{(2/\pi)\log(n/|x|) + O(|x|^{-1/4})}{(2/\pi)\log n + k + O(n^{-1/4})} \\ &= \frac{\log(n/|x|) + O(|x|^{-1/4})}{\log(n)}(1 + O((\log n)^{-1}).\end{aligned}\tag{2.16}$$

By the strong Markov property

$$G_{D(0,n)}(x,0) = \mathbb{P}^x\left(T_0 < T_{D(0,n)^c}\right) G_{D(0,n)}(0,0). \tag{2.17}$$

Using this, (2.13) and the first line of (2.16) we obtain

$$G_{D(0,n)}(x,0) = \frac{2}{\pi}\log\left(\frac{n}{|x|}\right) + O(|x|^{-1/4}). \tag{2.18}$$

Hence

$$G_{D(0,n)}(x,y) \le G_{D(x,2n)}(x,y) = G_{D(0,2n)}(0,y-x) \le c\log n. \tag{2.19}$$

Let $r < |x| < R$ and $\zeta = T_{D(0,R)^c} \wedge T_{D(0,r)}$. Applying the argument leading to (2.16), but with the martingale $a(X_{j\wedge\zeta}) - k$ we can obtain that uniformly in $r < |x| < R$

$$\mathbb{P}^x\left(T_{D(0,R)^c} < T_{D(0,r)}\right) = \frac{\log(|x|/r) + O(r^{-1/4})}{\log(R/r)} \tag{2.20}$$

and

$$\mathbb{P}^x\left(T_{D(0,r)} < T_{D(0,R)^c}\right) = \frac{\log(R/|x|) + O(r^{-1/4})}{\log(R/r)}. \tag{2.21}$$

Using a last exit decomposition, for any $0 < \delta < \epsilon < 1$, uniformly in $x \in D(0,n) \setminus D(0,\epsilon n)$

$$\begin{aligned}
\mathbb{P}^x(|X_{T_{D(0,\epsilon n)} \wedge T_{D(0,n)^c}}| \le \delta n) &= \sum_{z \in D(0,n)\setminus D(0,\epsilon n)} \sum_{w \in D(0,\delta n)} G_{D(0,n)}(x,z) p_1(z,w) \\
&\le cn^2 \log n \mathbb{P}(|X_1| > (\epsilon - \delta)n) \\
&\le c_{\epsilon,\delta} n^2 \log n \frac{1}{n^{3+2\beta}} = c_{\epsilon,\delta} \log n \frac{1}{n^{1+2\beta}}.
\end{aligned} \tag{2.22}$$

Here we used (2.19) and the fact that $|z - w| \ge (\epsilon - \delta)n$.

We need a more precise error term when x is near the outer boundary. Let $\rho(x) = n - |x|$. We have the following lemma.

Lemma 2.1. *For any $0 < \delta < \epsilon < 1$ we can find $0 < c_1 < c_2 < \infty$, such that for all $x \in D(0,n) \setminus D(0,\epsilon n)$ and all n sufficiently large*

$$c_1 \frac{\rho(x) \vee 1}{n} \le \mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) \le c_2 \frac{\rho(x) \vee 1}{n}. \tag{2.23}$$

Proof: Upper bound: By looking at two lines, one tangential to $\partial D(0,n)$ and perpendicular to the ray from 0 to x, and the other parallel to the first but at a distance δn from 0, the upper bound follows from the gambler's ruin estimates of (10, Lemma 2.1).

Lower bound: We first show that for any $\zeta > 0$ we can find a constant $c_\zeta > 0$ such that

$$c_\zeta \frac{\rho(x)}{n} \le \mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) \tag{2.24}$$

for all $x \in D(0, n-\zeta) \setminus D(0,\epsilon n)$.

Let $T = \inf\{t \mid X_t \in D(0,\delta n) \cup D(0,n)^c\}$ and $\gamma \in (0, 2\beta)$. Let $\overline{a}(x) = \frac{2}{\pi}(a(x) - k)$, where k is the constant in (2.3) so that $\overline{a}(x) = \log|x| + o(1/|x|)$. Clearly $\overline{a}(X_{j\wedge T})$ is a martingale, and by the optional sampling theorem,

$$\begin{aligned} \overline{a}(x) = {} & \mathbb{E}^x(\overline{a}(X_T);\, X_T \in D(0,n)^c) \\ & + \mathbb{E}^x(\overline{a}(X_T);\, X_T \in D(0,\delta n) \setminus D(0,\delta n/2)) \\ & + \mathbb{E}^x(\overline{a}(X_T);\, X_T \in D(0,\delta n/2)). \end{aligned} \tag{2.25}$$

It follows from (2.22) that

$$\mathbb{E}^x(\overline{a}(X_T);\, X_T \in D(0,\delta n/2)) = O(n^{-1-\gamma}) \tag{2.26}$$

and

$$\mathbb{P}^x(X_T \in D(0,\delta n/2)) = O(n^{-1-\gamma}). \tag{2.27}$$

From (2.25) we see that

$$\begin{aligned} \log|x| \geq {} & \log n\, \mathbb{P}^x(T_{D(0,n)^c} < T_{D(0,\delta n)}) \\ & + \log(\delta n/2)\mathbb{P}^x(T_{D(0,n)^c} > T_{D(0,\delta n)}) + o(1/n) \\ = {} & \log n[1 - \mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c})] + \log(\delta n/2)\mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c})] \\ & + o(1/n) \\ = {} & \log n + (\log(\delta n/2) - \log n)\mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) + o(1/n). \end{aligned} \tag{2.28}$$

Note that for some $c > 0$

$$\log(1-z) \leq -cz, \qquad 0 \leq z \leq 1-\epsilon. \tag{2.29}$$

Hence for $x \in D(0, n-\zeta) \setminus D(0,\epsilon n)$

$$\log(n/|x|) = -\log\Big(1 - \frac{(n-|x|)}{n}\Big) \geq c\frac{\rho(x)}{n}$$

Solving (2.28) for $\mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c})$ and using the fact that and $\rho(x) \geq \zeta$ to control the $o(1/n)$ term completes the proof of (2.24).

Let $A = D(0, n-\zeta) \setminus D(0,\epsilon n)$. Then by the strong Markov property and (2.24), for any $x \in D(0,n) \setminus D(0,n-\zeta)$

$$\begin{aligned} & \mathbb{P}^x(T_{D(0,\delta n)} < T_{D(0,n)^c}) \\ & \geq \mathbb{P}^x(T_{D(0,\delta n)} \circ \theta_{T_A} < T_{D(0,n)^c} \circ \theta_{T_A};\, T_A < T_{D(0,n)^c}) \\ & = \mathbb{E}^x(\mathbb{P}^{X_{T_A}}(T_{D(0,\delta n)} < T_{D(0,n)^c});\, T_A < T_{D(0,n)^c}) \\ & \geq c_\zeta\, \frac{\zeta}{n}\, \mathbb{P}^x(T_A < T_{D(0,n)^c}). \end{aligned} \tag{2.30}$$

(2.23) then follows if we can find $\zeta > 0$ such that uniformly in n

$$\inf_{x \in D(0,n)\setminus D(0,n-\zeta)} \mathbb{P}^x(T_A < T_{D(0,n)^c}) > 0. \tag{2.31}$$

To prove (2.31) we will show that we can find $N < \infty$ and $\zeta, c > 0$ such that for any $x \in \mathbb{Z}^2$ with $|x|$ sufficiently large we can find $y \in \mathbb{Z}^2$ with

$$p_1(x,y) \geq c \qquad \text{and} \qquad |x| - N \leq |y| \leq |x| - \zeta. \tag{2.32}$$

Let C_x be the cone with vertex at x that contains the origin, has aperture $9\pi/10$ radians, and such that the line through 0 and x bisects the cone. If $|x|$ is large enough, $C_x \cap D(x,N)$ will be contained in $D(0,|x|)$. We will show that there is a point $y \in \mathbb{Z}^2 \cap C_x \cap D(x,N)$, $y \neq x$, which satisfies (2.32). Note that for any $y \in \mathbb{Z}^2 \cap C_x \cap D(x,N)$, $y \neq x$, we have

$$1 \leq |y-x| \leq N. \tag{2.33}$$

Furthermore, if α denotes the angle between the line from x to the origin and the line from x to y, by the law of cosines,

$$|y|^2 = |x|^2 + |y-x|^2 - 2\cos(\alpha)\,|x|\,|y-x|. \tag{2.34}$$

Then for $|x|$ sufficiently large, using (2.33),

$$\begin{aligned} |y| &= \sqrt{|x|^2 + |y-x|^2 - 2\cos(\alpha)\,|x|\,|y-x|} \\ &= |x|\sqrt{1+\frac{|y-x|^2}{|x|^2} - \frac{2\cos(\alpha)\,|y-x|}{|x|}} \\ &= |x|\left(1-\frac{\cos(\alpha)\,|y-x|}{|x|} + O(\frac{1}{|x|^2})\right) \\ &= |x| - \cos(\alpha)\,|y-x| + O(\frac{1}{|x|}) \\ &\leq |x| - \cos(9\pi/20)/2. \end{aligned} \tag{2.35}$$

Setting $\zeta = \cos(9\pi/20)/2 > 0$ we see that the second condition in (2.32) is satisfied for all $y \in \mathbb{Z}^2 \cap C_x \cap D(x,N)$, $y \neq x$, and it suffices to show that we can find such a y with $p_1(x,y) \geq c$. (c, N remain to be chosen).

By translating by $-x$ it suffices to work with cones having their vertex at the origin. We let $C(\theta,\theta')$ denote the cone with vertex at the origin whose sides are the half lines making angles $\theta < \theta'$ with the positive x-axis. Set $C(\theta,\theta',N) = C(\theta,\theta') \cap D(0,N)$. It suffices to show that for any θ we can find $y \in C(\theta, \theta+9\pi/10, N)$, $y \neq 0$, with $p_1(0,y) \geq c$. Let $j_\theta = \inf\{j \geq 0 \,|\, j\pi/5 \geq \theta\}$. Then it is easy to see that

$$C(j_\theta\pi/5, j_\theta\pi/5 + 2\pi/3, N) \subseteq C(\theta, \theta+9\pi/10, N).$$

It now suffices to show that for each $0 \leq j \leq 9$ we can find $y_j \in C(j\pi/5, j\pi/5+2\pi/3)$, $y_j \neq 0$, with $p_1(0,y_j) > 0$, since we can then take

$$c = \inf_j p_1(0,y_j) \qquad \text{and} \qquad N = 2\sup_j |y_j|. \tag{2.36}$$

First consider the cone $C(-\pi/3,\pi/3)$, and recall our assumption that the covariance matrix of $X_1 = (X_1^{(1)}, X_1^{(2)})$ is I. If

$$\mathbb{P}(X_1 \in C(-\pi/3,\pi/3),\ X_1 \neq 0) = 0$$

then by symmetry, $\mathbb{P}(X_1 \in -C(-\pi/3,\pi/3),\ X_1 \neq 0) = 0$. Therefore

$$\mathbb{P}\Big(|X_1^{(2)}| > |X_1^{(1)}| \,\Big|\, X_1 \neq 0\Big) = 1.$$

But then $1 = \mathbb{E}|X_1^{(2)}|^2 > \mathbb{E}|X_1^{(1)}|^2 = 1$, a contradiction. So there must be a point $y \in C(-\pi/3, \pi/3)$, $y \neq 0$, with $p_1(0, y) > 0$.

Let A_j be the rotation matrix such that the image of $C(j\pi/5, j\pi/5 + 2\pi/3)$ under A_j is the cone $C(-\pi/3, \pi/3)$ and let $Y_1 = A_j X_1$. Then

$$\begin{aligned} &\mathbb{P}(X_1 \in C(j\pi/5, j\pi/5 + 2\pi/3),\ X_1 \neq 0) \\ &= \mathbb{P}(Y \in C(-\pi/3, \pi/3),\ Y_1 \neq 0). \end{aligned} \tag{2.37}$$

Note Y_1 is mean 0, symmetric, and since A_j is a rotation matrix, the covariance matrix of Y_1 is the identity. Hence the argument of the last paragraph shows that the probability in (2.37) is non-zero. This completes the proof of of our lemma.
□

Lemma 2.2. *For any $0 < \delta < \epsilon < 1$ we can find $0 < c_1 < c_2 < \infty$, such that for all $x \in D(0, n) \setminus D(0, \epsilon n), y \in D(0, \delta n)$ and all n sufficiently large*

$$c_1 \frac{\rho(x) \vee 1}{n} \leq G_{D(0,n)}(y, x) \leq c_2 \frac{\rho(x) \vee 1}{n}. \tag{2.38}$$

Proof: Upper bound: Choose $\delta < \gamma < \epsilon$ and let $T' = \inf\{t \,|\, X_t \in D(0, \gamma n) \cup D(0, n)^c\}$. By the strong Markov property,

$$\begin{aligned} G_{D(0,n)}(x, y) &= \sum_{z \in D(0,\gamma n)} \mathbb{P}^x(X_{T'} = z) G_{D(0,n)}(z, y) \\ &= \sum_{z \in D(0,\delta n)} \mathbb{P}^x(X_{T'} = z) G_{D(0,n)}(z, y) \\ &+ \sum_{z \in D(0,\gamma n) \setminus D(0,\delta n)} \mathbb{P}^x(X_{T'} = z) G_{D(0,n)}(z, y) \\ &\leq c\mathbb{P}^x(X_{T'} \in D(0, \delta n)) \log n + c\mathbb{P}^x(X_{T'} \in D(0, \gamma n)). \end{aligned} \tag{2.39}$$

Here we used (2.19) and the bound $G_{D(0,n)}(z, y) \leq G_{D(y,2n)}(z, y) \leq c$ uniformly in n which follows from (2.18) and the fact that $|z - y| \geq (\gamma - \delta)n$. The upper bound in (2.38) then follows from (2.22) and Lemma 2.1.

Lower bound: Let $T = \inf\{t \,|\, X_t \in D(0, \delta n) \cup D(0, n)^c\}$. By the strong Markov property,

$$G_{D(0,n)}(x, y) = \sum_{z \in D(0,\delta n)} \mathbb{P}^x(X_T = z) G_{D(0,n)}(z, y). \tag{2.40}$$

The lower bound in (2.38) follows from Lemma 2.1 once we show that

$$\inf_{y,z \in D(0,\delta n)} G_{D(0,n)}(z, y) \geq a > 0 \tag{2.41}$$

for some $a > 0$ independent of n. We first note that

$$\begin{aligned} &\inf_{y,z \in D(0,\delta n)} \inf_{z \in D(y,(\epsilon-\delta)n/2)} G_{D(0,n)}(z, y) \\ &\geq \inf_{y \in D(0,\delta n)} \inf_{z \in D(y,(\epsilon-\delta)n/2)} G_{D(y,(\epsilon-\delta)n)}(z, y) > 0 \end{aligned} \tag{2.42}$$

uniformly in n by (2.18). But by the invariance principle, there is a positive probability independent of n that the random walk starting at $z \in D(0,\delta n)$ will enter $D(y,(\epsilon-\delta)n/2)\cap D(0,\delta n)$ before exiting $D(0,n)$. (2.41) then follows from (2.42) and the strong Markov property.
□

We obtain an upper bound for the probability of entering a disk by means of a large jump.

Lemma 2.3.

$$\sup_{x\in D(0,n/2)} \mathbb{P}^x(T_{D(0,n)^c} \neq T_{\partial D(0,n)_s}) \leq c(s^{-1-2\beta} \vee n^{-1-2\beta}\log n). \tag{2.43}$$

Proof of Lemma 2.3: We begin with the last exit decomposition

$$\begin{aligned} &\sup_{x\in D(0,n/2)} \mathbb{P}^x(T_{D(0,n)^c} \neq T_{\partial D(0,n)_s}) \\ &= \sup_{x\in D(0,n/2)} \sum_{\substack{y\in D(0,n)\\ w\in D(0,n+s)^c}} G_{D(0,n)}(x,y)p_1(y,w) \\ &= \sup_{x\in D(0,n/2)} \sum_{\substack{|y|\leq 3n/4\\ n+s\leq |w|}} G_{D(0,n)}(x,y)p_1(y,w) \\ &+ \sup_{x\in D(0,n/2)} \sum_{\substack{3n/4<|y|<n\\ n+s\leq |w|}} G_{D(0,n)}(x,y)p_1(y,w). \end{aligned} \tag{2.44}$$

Using (2.19) and (2.1)

$$\begin{aligned} &\sup_{x\in D(0,n/2)} \sum_{\substack{|y|\leq 3n/4\\ n+s\leq |w|}} G_{D(0,n)}(x,y)p_1(y,w) \\ &\leq c\log n \sum_{|y|\leq 3n/4} \mathbb{P}(|X_1|\geq n/4) \\ &\leq c\log n \sum_{|y|\leq 3n/4} \frac{1}{|n|^{3+2\beta}} \leq cn^{-1-2\beta}\log n. \end{aligned} \tag{2.45}$$

Using (2.38) and (2.1) we have

$$\begin{aligned} &\sup_{x\in D(0,n/2)} \sum_{\substack{3n/4<|y|<n\\ n+s<|w|}} G_{D(0,n)}(x,y)p_1(y,w) \\ &\leq cn^{-1} \sum_{3n/4<|y|\leq n} (n-|y|)\mathbb{P}(|X_1|\geq s+n-|y|) \\ &\leq cn^{-1} \sum_{3n/4<|y|\leq n} \frac{n-|y|}{(s+n-|y|)^{3+2\beta}} \\ &\leq cn^{-1} \sum_{3n/4<|y|<n} \frac{1}{(s+n-|y|)^{2+2\beta}} \leq cs^{-1-2\beta}. \end{aligned} \tag{2.46}$$

Here, we bounded the last sum by an integral and used polar coordinates:

$$\sum_{3n/4<|y|<n} \frac{1}{(s+n-|y|)^{2+2\beta}} \leq 2\pi \int_{3n/4}^{n} \frac{1}{(s+n-u)^{2+2\beta}}\, u\, du \tag{2.47}$$

$$\leq cn \int_0^{n/4} \frac{1}{(s+u)^{2+2\beta}}\, du \leq cns^{-1-2\beta}.$$

□

Combining (2.16) and (2.43) we obtain, provided $n^{-1-2\beta}\log n \leq s^{-1-2\beta}$ and $|x| \leq n/2$,

$$\mathbb{P}^x\left(T_{D(0,n)^c} < T_0\,;\, T_{D(0,n)^c} = T_{\partial D(0,n)_s}\right) \tag{2.48}$$
$$= 1 - \frac{(2/\pi)\log(n/|x|) + O(|x|^{-1/4})}{(2/\pi)\log n + k + O(n^{-1/4})} + O(s^{-1-2\beta}).$$

Using (2.20) and (2.43) we then obtain, provided $R^{-1-2\beta}\log R \leq s^{-1-2\beta}$ and $|x| \leq R/2$,

$$\mathbb{P}^x\left(T_{D(0,R)^c} < T_{D(0,r)}\,;\, T_{D(0,R)^c} = T_{\partial D(0,R)_s}\right) \tag{2.49}$$
$$= \frac{\log(|x|/r) + O(r^{-1/4})}{\log(R/r)} + O(s^{-1-2\beta}).$$

Lemma 2.4. *For any $s < r < R$ sufficiently large with $R \leq r^2$ we can find $c < \infty$ and $\delta > 0$ such that for any $r < |x| < R$*

$$\mathbb{P}^x(T_{D(0,r)} < T_{D(0,R)^c}; X_{T_{D(0,r)}} \in D(0, r-s)) \leq cr^{-\delta} + cs^{-1-2\beta}. \tag{2.50}$$

Proof. Let $A(R, r)$ denote the annulus $D(0, R) \setminus D(0, r)$. Using a last exit decomposition we have

$$\mathbb{P}^x(T_{D(0,r)} < T_{D(0,R)^c}; X_{T_{D(0,r)}} \in D(0, r-s))$$
$$= \sum_{w\in D(0,r-s)} \sum_{y\in A} G_A(x,y)p_1(y,w)$$
$$= \sum_{w\in D(0,r-s)} \sum_{r<|y|\leq r+r^{1/(2+\beta)}} G_A(x,y)p_1(y,w)$$
$$+ \sum_{w\in D(0,r-s)} \sum_{r+r^{1/(2+\beta)}<|y|<R} G_A(x,y)p_1(y,w). \tag{2.51}$$

By (2.1), for $y \in A$

$$\sum_{w\in D(0,r-s)} p_1(y,w) \leq \frac{c}{(|y|-(r-s))^{3+2\beta}}.$$

Let $U_k = \{y \in \mathbb{Z}^2 : r+k-1 < |y| \leq r+k\}$. We show below that we can find $c < \infty$ such that for all $1 \leq k \leq r^{1/(2+\beta)}$,

$$\sum_{y\in U_k} G_A(x,y) \leq ck, \qquad x \in A. \tag{2.52}$$

For the first sum in (2.51), we then obtain

$$\begin{aligned}
&\sum_{w\in D(0,r-s)}\ \sum_{r<|y|\le r+r^{1/(2+\beta)}} G_A(x,y)p_1(y,w)\\
&\le c\sum_{k=1}^{r^{1/(2+\beta)}}\ \sum_{y\in U_k} G_A(x,y)\frac{1}{(|y|-(r-s))^{3+2\beta}}\\
&\le c\sum_{k=1}^{r^{1/(2+\beta)}} \frac{k}{(k-1+s)^{3+2\beta}} \le c\sum_{j=0}^{\infty}\frac{c}{(j+s)^{2+2\beta}}\\
&\le \frac{c}{s^{1+2\beta}}. \qquad (2.53)
\end{aligned}$$

For the second sum in (2.51), we use (2.19) to bound $G_A(x,y)$ by $c\log R$ and the fact that the cardinality of U_k is bounded by $c(r+k)$ to obtain

$$\begin{aligned}
&\sum_{w\in D(0,r-s)}\ \sum_{r+r^{1/(2+\beta)}<|y|\le R} G_A(x,y)p_1(y,w)\\
&\le c\sum_{k=r^{1/(2+\beta')}}^{\infty}\ \sum_{y\in U_k} G_A(x,y)\frac{1}{(|y|-(r-s))^{3+2\beta}}\\
&\le c(\log R)\sum_{k=r^{1/(2+\beta)}}^{\infty}\frac{r+k}{k^{3+2\beta}}\\
&\le c(\log R)[r(r^{1/(2+\beta)})^{-(2+2\beta)}+(r^{1/(2+\beta)})^{-(1+2\beta)}]. \qquad (2.54)
\end{aligned}$$

By our assumptions on R, r and s we have our desired estimate, and it only remains to prove (2.52).

We divide the proof of (2.52) into two steps.

Step 1. Suppose $1\le k\le r^{1/(2+\beta)}$ and $x\in D(0,r+k-2)$. Then there exists $0<c<1$ not depending on k, r, or x such that

$$\mathbb{P}^x(T_{D(0,r)}<T_{D(0,r+k)})\ge c/k. \qquad (2.55)$$

Proof. This is trivial if $k=1,2$ so we may assume that $k\ge 3$ and that $x\in D(0,r)^c$. By a rotation of the coordinate system, assume $x=(|x|,0)$. Let $Y_k=X_k\cdot(0,1)$, i.e., Y_k is the second component of the random vector X_k. Since the covariance matrix of X is the identity, this is also true after a rotation, so Y_k is symmetric, mean 0, and the variance of Y_1 is 1. Let S_1 be the line segment connecting $(r-1,-k^{1+\beta/2})$ with $(r-1,k^{1+\beta/2})$ and S_2 the line segment connecting $(r+k-1,-k^{1+\beta/2})$ to $(r+k-1,k^{1+\beta/2})$. We have $S_1\subset D(0,r)$ because

$$(r-1)^2+(k^{1+\beta/2})^2=r^2-2r+1+k^{2+\beta}\le r^2$$

by our assumption on k. Similarly $S_2\subset D(0,r+k)$ because

$$(r+k-1)^2+(k^{1+\beta/2})^2=(r+k)^2-2(r+k)+1+k^{2+\beta}\le(r+k)^2.$$

Let L_i be the line containing S_i, $i = 1, 2$, and Q the rectangle whose left and right sides are S_1 and S_2, resp.

If X hits L_1 before L_2 and does not exit Q before hitting L_1, then $T_{D(0,r)} < T_{D(0,r+k)}$. If $T_{L_1} \wedge T_{L_2}$ is less than $k^{2+\beta/2}$ and Y does not move more than $k^{1+\beta/2}$ is time $k^{2+\beta/2}$, then X will not hit exit Q through the top or bottom sides of Q. Therefore

$$\begin{aligned}\mathbb{P}^x(T_{D(0,r)} < T_{D(0,r+k)}) &\geq \mathbb{P}^x(T_{L_1} < T_{L_2}) - \mathbb{P}^x(T_{L_1} \wedge T_{L_2} \geq k^{2+\beta/2}) \\ &\quad - \mathbb{P}^x(\max_{j \leq k^{2+\beta/2}} |Y_j| \geq k^{1+\beta/2}) \\ &= I_1 - I_2 - I_3. \end{aligned} \tag{2.56}$$

By the gambler's ruin estimate (10, (4.2)), we have

$$I_1 \geq c \frac{(r+k-1) - |x|}{(r+k-1) - (r-1)} \geq \frac{c}{k}. \tag{2.57}$$

It follows from the one dimensional case of (9.10) that for some $\rho < 1$ and all sufficiently large k

$$I_2 \leq \mathbb{P}^x(T_{L_1} \wedge T_{L_2} \geq k^{2+\beta/2}) \leq \rho^{k^{\beta/4}} = o(1/k). \tag{2.58}$$

For I_3 we truncate the one-dimensional random walk Y_j at level $k^{1+\beta/4}$ and use Bernstein's inequality. If we let $\xi_j = Y_j - Y_{j-1}$, $\xi'_j = \xi_j 1_{(|\xi_j| \leq k^{1+\beta/4})}$ and $Y'_j = \sum_{i \leq j} \xi'_i$, then

$$\mathbb{P}^x(Y_j \neq Y'_j \text{ for some } j \leq k^{2+\beta/2}) \leq \frac{k^{2+\beta}}{(k^{1+\beta/4})^{3+2\beta}} = o(1/k). \tag{2.59}$$

By Bernstein's inequality ((3))

$$\mathbb{P}^x(\max_{j \leq k^{2+\beta/2}} |Y'_j| \geq k^{1+\beta/2}) \leq \exp\Big(- \frac{k^{2+\beta}}{2k^{2+\beta/2} + \frac{2}{3} k^{1+\beta/2} k^{1+\beta/4}} \Big). \tag{2.60}$$

This is also $o(1/k)$.

Step 2. Let

$$J_k = \max_{x \in A} \sum_{y \in U_k} G_A(x, y).$$

By the strong Markov property, we see that the maximum is taken when $x \in U_k$. Also, by the Markov property at the fixed time m,

$$J_k \leq m + \sup_{x \in U_k} \mathbb{E}^x \Big[\sum_{y \in U_k} G_A(X_m, y) \Big]. \tag{2.61}$$

We have

$$\begin{aligned}\mathbb{E}^x \Big[\sum_{y \in U_k} G_A(X_m, y); X_m \notin D(0, r+k-3) \Big] \\ \leq J_k \mathbb{P}^x(X_m \notin D(0, r+k-3)) \end{aligned} \tag{2.62}$$

and using (2.55)

$$\begin{aligned}
&\mathbb{E}^x\Big[\sum_{y\in U_k} G_A(X_m,y); X_m\in D(0,r+k-3)\Big] \\
&\qquad\qquad \le \mathbb{E}^x[\mathbb{E}^{X_m}[J_k; T_{U_k}<T_{D(0,r)}]; X_m\in D(0,r+k-3)] \\
&\qquad\qquad \le J_k(1-c/k)\mathbb{P}^x(X_m\in D(0,r+k-3)). \qquad (2.63)
\end{aligned}$$

Using (2.32), there exists m and $\varepsilon>0$ such that for all $x\in U_k$

$$\mathbb{P}^x(X_m\in D(0,r+k-3))\ge\varepsilon. \qquad (2.64)$$

Using (2.64) and combining (2.62) and (2.63) we see that for all $x\in U_k$

$$\mathbb{E}^x\Big[\sum_{y\in U_k} G_A(X_m,y)\Big]\le J_k(1-c'/k)$$

for some $0<c'<1$. Then by (2.61)

$$J_k\le m+J_k(1-c'/k), \qquad (2.65)$$

and solving for J_k yields (2.52).
□

Using (2.21) and (2.50) we can obtain that for some $\delta>0$

$$\begin{aligned}
&\mathbb{P}^x\left(T_{D(0,r)}<T_{D(0,R)^c}\,;\,T_{D(0,r)}=T_{\partial D(0,r-s)_s}\right) \qquad (2.66)\\
&=\frac{\log(R/|x|)+O(r^{-\delta})}{\log(R/r)}+O(s^{-1-2\beta}).
\end{aligned}$$

We now prove a bound for the Green's function in the exterior of the disk $D(0,n)$.

Lemma 2.5.

$$G_{D(0,n)^c}(x,y)\le c\log(|x|\wedge|y|),\qquad x,y\in D(0,n)^c. \qquad (2.67)$$

Proof of Lemma 2.5: Since $G_{D(0,n)^c}(x,y)=\mathbb{P}^x\left(T_y<T_{D(0,n)}\right)G_{D(0,n)^c}(y,y)$, and using the symmetry of $G_{D(0,n)^c}(x,y)$, it suffices to show that

$$G_{D(0,n)^c}(x,x)\le c\log(|x|),\qquad x\in D(0,n)^c. \qquad (2.68)$$

Let

$$\begin{aligned}
&U_1=0,\\
&V_i=\min\{k>U_i : |X_k|<n \text{ or } |X_k|\ge|x|^8\},\qquad i=1,2,\ldots,\\
&U_{i+1}=\min\{k>V_i : X_k=x\},\qquad i=1,2,\ldots
\end{aligned}$$

Then using the strong Markov property and (2.19)

$$G_{D(0,n)^c}(x,x) = \mathbb{E}^x\left\{\sum_{k<T_{D(0,n)}} 1_{\{x\}}(X_k)\right\} \tag{2.69}$$
$$\leq \sum_{i=1}^{\infty} \mathbb{E}^x\Big[\sum_{U_i\leq k<V_i} 1_{\{x\}}(X_k); U_i < T_{D(0,n)}\Big]$$
$$\leq \sum_{i=1}^{\infty} \mathbb{E}^x[G_{D(0,|x|^8)}(X_{U_i},x); U_i < T_{D(0,n)}]$$
$$\leq c_2 \log(|x|^8) \sum_{i=1}^{\infty} \mathbb{P}^x(U_i < T_{D(0,n)}).$$

We have

$$\mathbb{P}^x(U_{i+1} < T_{D(0,n)}) \leq \mathbb{E}^x[\mathbb{P}^{X_{U_i}}(T_{D(0,|x|^8)^c} < T_{D(0,n)}); U_i < T_{D(0,n)}].$$

By (2.20),

$$\mathbb{P}^{X_{U_i}}(T_{D(0,|x|^8)^c} < T_{D(0,n)}) = \mathbb{P}^x(T_{D(0,|x|^8)^c} < T_{D(0,n)}) \tag{2.70}$$
$$\leq \frac{\log(|x|/n) + O(n^{-1/4})}{\log(|x|^8/n)}$$
$$= \frac{\log|x| - \log n + O(n^{-1/4})}{8\log|x| - \log n}$$
$$\leq \frac{\log|x| + 1}{7\log|x|} \leq \frac{2}{7}.$$

Therefore

$$\mathbb{P}^x(U_{i+1} < T_{D(0,n)}) \leq \tfrac{2}{7}\mathbb{P}^x(U_i < T_{D(0,n)}).$$

By induction $\mathbb{P}^x(U_i < T_{D(0,n)}) \leq (\frac{2}{7})^i$, hence $\sum_i \mathbb{P}^x(U_i < T_{D(0,n)}) < c_4$. Together with (2.69), this proves (2.68).
□

Lemma 2.6.

$$\sup_{x\in D(0,n+s)^c} \mathbb{P}^x(T_{D(0,n+s)} \neq T_{\partial D(0,n)_s}) \leq cn^2 \log(n)\, s^{-3-2\beta} + cn^{-1-\beta}. \tag{2.71}$$

Proof of Lemma 2.6: Using (2.67)

$$\begin{aligned}
&\sup_{x\in D(0,n+s)^c} \mathbb{P}^x(T_{D(0,n+s)}\neq T_{\partial D(0,n)_s}) \qquad (2.72)\\
&= \sup_{x\in D(0,n+s)^c} \sum_{\substack{y\in D(0,n+s)^c\\ w\in D(0,n)}} G_{D(0,n)^c}(x,y)p_1(y,w)\\
&\leq c \sum_{\substack{y\in D(0,n+s)^c\\ w\in D(0,n)}} \log(|y|)\, p_1(y,w) \leq c \sum_{y\in D(0,n+s)^c} \log(|y|)\mathbb{P}(|X_1|\geq |y|-n)\\
&\leq c \sum_{y\in D(0,n+s)^c} \log(|y|)(|y|-n)^{-3-2\beta}\\
&\leq c\log(n) \sum_{n+s\leq |y|\leq 2n} (|y|-n)^{-3-2\beta} + c\sum_{2n<|y|} \log(|y|)(|y|-n)^{-3-2\beta}\\
&\leq cn^2\log(n)\, s^{-3-2\beta} + c\log(n)\, n^{-1-2\beta}.
\end{aligned}$$

□

3 Harnack inequalities

We next present some Harnack inequalities tailored to our needs. We continue to assume that Condition A and (1.2) hold.

Lemma 3.1 (Interior Harnack Inequality)**.** *Let $e^n \leq r = R/n^3$. Uniformly for $x, x' \in D(0,r)$ and $y \in \partial D(0,R)_{n^4}$*

$$H_{D(0,R)^c}(x,y) = \left(1+O(n^{-3})\right) H_{D(0,R)^c}(x',y). \qquad (3.1)$$

Furthermore, uniformly in $x \in \partial D(0,r)_{n^4}$ and $y \in \partial D(0,R)_{n^4}$,

$$\begin{aligned}
&\mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} < T_{D(0,r/n^3)}) \qquad (3.2)\\
&= \left(1+O(n^{-3})\right) \mathbb{P}^x(T_{D(0,R)^c} < T_{D(0,r/n^3)}) H_{D(0,R)^c}(x,y).
\end{aligned}$$

Proof of Lemma 3.1: It suffices to prove (3.1) with $x' = 0$. For any $y \in \partial D(0,R)_{n^4}$ we have the last exit decomposition

$$\begin{aligned}
H_{D(0,R)^c}(x,y) &= \sum_{z\in D(0,R)-D(0,3R/4)} G_{D(0,R)}(x,z)p_1(z,y)\\
&+ \sum_{z\in D(0,3R/4)-D(0,R/2)} G_{D(0,R)}(x,z)p_1(z,y)\\
&+ \sum_{z\in D(0,R/2)} G_{D(0,R)}(x,z)p_1(z,y). \qquad (3.3)
\end{aligned}$$

Let us first show that uniformly in $x \in D(0,r)$ and $z \in D(0,3R/4) - D(0,R/2)$

$$G_{D(0,R)}(x,z) = \left(1+O(n^{-3})\right) G_{D(0,R)}(0,z). \qquad (3.4)$$

To this end, note that by (2.3), uniformly in $x \in D(0,r)$ and $y \in D(0,R/2)^c$

$$\begin{aligned} a(y-x) &= \frac{2}{\pi}\log|x-y| + k + O(|x-y|^{-1}) \\ &= \frac{2}{\pi}\log|y| + k + O(n^{-3}) \\ &= a(y) + O(n^{-3}). \end{aligned} \tag{3.5}$$

Hence, by (2.8), and using the fact that $H_{D(0,R)^c}(z,\cdot)$ is a probability, we see that uniformly in $x \in D(0,r)$ and $z \in D(0,R) - D(0,R/2)$

$$\begin{aligned} &G_{D(0,R)}(x,z) = G_{D(0,R)}(z,x) \\ &= \left\{ \sum_{y \in D(0,R)^c} H_{D(0,R)^c}(z,y)a(y-x) \right\} - a(z-x) \\ &= \left\{ \sum_{y \in D(0,R)^c} H_{D(0,R)^c}(z,y)a(y) \right\} - a(z) + O(n^{-3}) \\ &= G_{D(0,R)}(z,0) + O(n^{-3}). \end{aligned} \tag{3.6}$$

By (2.18), $G_{D(0,R)}(z,0) \geq c > 0$ uniformly for $z \in D(0,3R/4) - D(0,R/2)$, which completes the proof of (3.4).

We next show that uniformly in $x \in D(0,r)$ and $z \in D(0,R) - D(0,3R/4)$

$$G_{D(0,R)}(x,z) = \left(1 + O(n^{-3})\right) G_{D(0,R)}(0,z). \tag{3.7}$$

Thus, let $z \in D(0,R) - D(0,3R/4)$, and use the strong Markov property to see that

$$G_{D(0,R)}(z,x) = \mathbb{E}^z \left(G_{D(0,R)}(X_{T_{D(0,3R/4)}}, x) \,;\, T_{D(0,3R/4)} < T_{D(0,R)} \right) \tag{3.8}$$

$$\begin{aligned} &= \mathbb{E}^z \left(G_{D(0,R)}(X_{T_{D(0,3R/4)}}, x) \,;\, T_{D(0,3R/4)} < T_{D(0,R)} \,;\, |X_{T_{D(0,3R/4)}}| > R/2 \right) \\ &+ \mathbb{E}^z \left(G_{D(0,R)}(X_{T_{D(0,3R/4)}}, x) \,;\, T_{D(0,3R/4)} < T_{D(0,R)} \,;\, |X_{T_{D(0,3R/4)}}| \leq R/2 \right) \end{aligned}$$

By (2.19) and (2.72) we can bound the last term by

$$c(\log R)\, \mathbb{P}^z \left(|X_{T_{D(0,3R/4)}}| \leq R/2 \right) \leq c(\log R)^2 R^{-1-2\beta}. \tag{3.9}$$

Thus we can write (3.8) as

$$\begin{aligned} &\mathbb{E}^z \left(G_{D(0,R)}(X_{T_{D(0,3R/4)}}, x) \,;\, T_{D(0,3R/4)} < T_{D(0,R)} \,;\, |X_{T_{D(0,3R/4)}}| > R/2 \right) \\ &\qquad = G_{D(0,R)}(z,x) + O(R^{-1-\beta}). \end{aligned} \tag{3.10}$$

Applying (3.4) to the first line of (3.10) and comparing the result with (3.10) for $x = 0$ we see that uniformly in $x \in D(0,r)$ and $z \in D(0,R) - D(0,3R/4)$

$$G_{D(0,R)}(x,z) = \left(1 + O(n^{-3})\right) G_{D(0,R)}(0,z) + O(R^{-1-\beta}). \tag{3.11}$$

Since, by (2.38), $G_{D(0,R)}(0,z) \geq c/R$, we obtain (3.7).

If X has finite range then the last sum in (3.3) is zero and the proof of (3.1) is complete. Otherwise, assume that (1.3) holds. We begin by showing that

$$\sum_{z\in D(0,R/2)} G_{D(0,R)}(x,z)p_1(z,y) = O(R^{-3-\beta}). \tag{3.12}$$

By (2.19) we have $G_{D(0,R)}(x,z) = O(\log R)$ so that the sum in (3.12) is bounded by $O(\log R)$ times the probability of a jump of size greater than $|y| - R/2 \geq R/2$, which gives (3.12).

To prove (3.1) it now suffices to show that uniformly in $y \in \partial D(0,R)_{n^4}$

$$R^{-3-\beta} \leq cn^{-3} H_{D(0,R)^c}(0,y). \tag{3.13}$$

Using once more the fact that $G_{D(0,R)}(0,z) \geq c/R$ by (2.38),

$$\begin{aligned} H_{D(0,R)^c}(0,y) &= \sum_{z\in D(0,R)} G_{D(0,R)}(0,z)p_1(z,y) \\ &\geq C\left(\sum_{z\in D(0,R)} p_1(y,z)\right) R^{-1}. \end{aligned} \tag{3.14}$$

(3.13) then follows from (1.3) and our assumption that $R \geq e^n$, completing the proof of (3.1).

Turning to (3.2), we have

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} < T_{D(0,r/n^3)}) \\ &= H_{D(0,R)^c}(x,y) - \mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} > T_{D(0,r/n^3)}). \end{aligned} \tag{3.15}$$

By the strong Markov property at $T_{D(0,r/n^3)}$

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} > T_{D(0,r/n^3)}) \\ &= \mathbb{E}^x(H_{D(0,R)^c}(X_{T_{D(0,r/n^3)}}, y);\, T_{D(0,R)^c} > T_{D(0,r/n^3)}). \end{aligned} \tag{3.16}$$

By (3.1), uniformly in $w \in D(0,r/n^3)$,

$$H_{D(0,n)^c}(w,y) = \left(1 + O(n^{-3})\right) H_{D(0,n)^c}(x,y)\,.$$

Substituting back into (3.16) we have

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} > T_{D(0,r/n^3)}) \\ &= \left(1 + O(n^{-3})\right) \mathbb{P}^x(T_{D(0,R)^c} > T_{D(0,r/n^3)}) H_{D(0,R)^c}(x,y). \end{aligned}$$

Combining this with (3.15) we obtain

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,R)^c}} = y\,,\, T_{D(0,R)^c} < T_{D(0,r/n^3)}) \\ &= \left(\mathbb{P}^x(T_{D(0,R)^c} < T_{D(0,r/n^3)}) + O(n^{-3})\right) H_{D(0,R)^c}(x,y). \end{aligned} \tag{3.17}$$

Since, by (2.20)

$$\inf_{x\in\partial D(0,r)_{n^4}} \mathbb{P}^x(T_{D(0,R)^c} < T_{D(0,r/n^3)}) \geq 1/4, \tag{3.18}$$

we obtain (3.2) which completes the proof of Lemma 3.1.
□

Using the notation n, r, R of the last Theorem, in preparation for the proof of the exterior Harnack inequality we now establish a uniform lower bound for the Greens function in the exterior of a disk:

$$G_{D(0,r+n^4)^c}(x,y) \geq c > 0, \qquad x \in \partial D(0,R)_{n^4},\, y \in D(0,R)^c. \tag{3.19}$$

Pick some $x_1 \in \partial D(0,R)$ and proceeding clockwise choose points $x_1, \ldots, x_{36} \in \partial D(0,R)$ which divide $\partial D(0,R)$ into 36 approximately equal arcs. The distance between any two adjacent such points is $\sim 2R\sin(5^\circ) \approx .17R$. It then follows from (2.21) that for any $j = 1, \ldots, 36$

$$\begin{aligned} &\inf_{x \in T_{D(x_j,R/5)}} \mathbb{P}^x(T_{D(x_{j+1},R/5)} < T_{D(0,r+n^4)}) \\ &\geq \inf_{x \in T_{D(x_{j+1},2R/5)}} \mathbb{P}^x(T_{D(x_{j+1},R/5)} < T_{D(x_{j+1},R/2)^c}) \geq c_1 > 0 \end{aligned} \tag{3.20}$$

for some $c_1 > 0$ independent of n, r, R for n large and where we set $x_{37} = x_1$. Hence, a simple argument using the strong Markov property shows that

$$\inf_{j,k} \inf_{x \in T_{D(x_j,R/5)}} \mathbb{P}^x(T_{D(x_k,R/5)} < T_{D(0,r+n^4)}) \geq c_2 =: c_1^{36} \tag{3.21}$$

Furthermore, it follows from (2.16) that for any $j = 1, \ldots, 36$

$$\begin{aligned} &\inf_{x,x' \in T_{D(x_j,R/5)}} \mathbb{P}^x(T_{x'} < T_{D(0,r+n^4)}) \\ &\geq \inf_{x \in T_{D(x',2R/5)}} \mathbb{P}^x(T_{x'} < T_{D(x',R/2)^c}) \geq c_3/\log R \end{aligned} \tag{3.22}$$

for some $c_3 > 0$ independent of n, r, R for n large. Since $\partial D(0,R)_{R/100} \subseteq \cup_{j=1}^{36} D(x_j, R/5)$, combining (3.21) and (3.22) we see that

$$\inf_{x,x' \in \partial D(0,R)_{R/100}} \mathbb{P}^x(T_{x'} < T_{D(0,r+n^4)}) \geq c_4/\log R. \tag{3.23}$$

It then follows from (2.13) that

$$\begin{aligned} &\inf_{x,x' \in \partial D(0,R)_{R/100}} G_{D(0,r+n^4)^c}(x,x') \\ &= \inf_{x,x' \in \partial D(0,R)_{R/100}} \mathbb{P}^x(T_{x'} < T_{D(0,r+n^4)}) G_{D(0,r+n^4)^c}(x',x') \\ &\geq (c_4/\log R)\, G_{D(x',R/2)}(x',x') \geq c_5 > 0 \end{aligned} \tag{3.24}$$

for some $c_5 > 0$ independent of n, r, R for n large.
Using the strong Markov property, (3.24), and (2.72) we see that

$$\begin{aligned} &\inf_{z \in D(0,1.01R)^c,\, x \in \partial D(0,R)_{R/100}} G_{D(0,r+n^4)^c}(z,x) \\ &\geq \mathbb{E}^z[G_{D(0,r+n^4)^c}(X_{T_{D(0,1.01R)}}, x); X_{T_{D(0,1.01R)}} \in \partial D(0,R)_{R/100}] \geq c > 0. \end{aligned} \tag{3.25}$$

Together with (3.24) this completes the proof of (3.19).

We next observe that uniformly in $x \in \partial D(0,R)_{n^4}$, $z \in D(0,2r) - D(0,5r/4)$, by (3.19) and (2.20),

$$\begin{aligned} &G_{D(0,r+n^4)^c}(x,z) = G_{D(0,r+n^4)^c}(z,x) \\ &= \mathbb{E}^z\left\{G_{D(0,r+n^4)^c}(X_{T_{D(0,R)^c}},x)\,;\,T_{D(0,R)^c} < T_{D(0,r+n^4)}\right\} \\ &\geq c\mathbb{P}^z\left\{T_{D(0,R)^c} < T_{D(0,r+n^4)}\right\} \\ &\geq c/\log n. \end{aligned} \tag{3.26}$$

Our final observation is that for any $\epsilon > 0$, uniformly in $x \in \partial D(0,R)_{n^4}$, $z \in D(0,2r) - D(0,r+(1+\epsilon)n^4)$,

$$G_{D(0,r+n^4)^c}(x,z) \geq c(R\log R)^{-1}. \tag{3.27}$$

To see this, we first use the strong Markov property together with (3.23) to see that uniformly in $x, x' \in \partial D(0,R)_{n^4}$ and $z \in D(0,2r) - D(0,r+(1+\epsilon)n^4)$,

$$\begin{aligned} G_{D(0,r+n^4)^c}(x,z) &\geq \mathbb{P}^x(T_{x'} < T_{D(0,r+n^4)})G_{D(0,r+n^4)^c}(x',z) \\ &\geq cG_{D(0,r+n^4)^c}(x',z)/\log R. \end{aligned} \tag{3.28}$$

In view of (2.38), if $x' \in \partial D(0,R)_{n^4}$ is chosen as close as possible to the ray from the the origin which passes through z

$$G_{D(0,r+n^4)^c}(x',z) \geq G_{D(x',|x'|-(r+n^4))}(x',z) \geq cR^{-1}. \tag{3.29}$$

Combining the last two displays proves (3.27).

Lemma 3.2 (Exterior Harnack Inequality)**.** *Let $e^n \leq r = R/n^3$. Uniformly for $x, x' \in \partial D(0,R)_{n^4}$ and $y \in \partial D(0,r)_{n^4}$*

$$H_{D(0,r+n^4)}(x,y) = \left(1 + O(n^{-3}\log n)\right) H_{D(0,r+n^4)}(x',y). \tag{3.30}$$

Furthermore, uniformly in $x \in \partial D(0,R)_{n^4}$ and $y \in \partial D(0,r)_{n^4}$,

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,r+n^4)}} = y\,;\,T_{D(0,r+n^4)} < T_{D(0,n^3R)^c}) \\ &= \Big(1 + O(n^{-3}\log n)\Big)\mathbb{P}^x(T_{D(0,r+n^4)} < T_{D(0,n^3R)^c})H_{D(0,r+n^4)}(x,y), \end{aligned} \tag{3.31}$$

and uniformly in $x, x' \in \partial D(0,R)_{n^4}$ and $y \in \partial D(0,r)_{n^4}$,

$$\begin{aligned} &\mathbb{P}^x(X_{T_{D(0,r+n^4)}} = y\,;\,T_{D(0,r+n^4)} < T_{D(0,n^3R)^c}) \\ &= \Big(1 + O(n^{-3}\log n)\Big)\mathbb{P}^{x'}(X_{T_{D(0,r+n^4)}} = y\,;\,T_{D(0,r+n^4)} < T_{D(0,n^3R)^c}). \end{aligned} \tag{3.32}$$

Proof of Lemma 3.2: For any $x \in \partial D(0,R)_{n^4}$ and $y \in \partial D(0,r)_{n^4}$ we have the last exit decomposition

$$\begin{aligned} H_{D(0,r+n^4)}(x,y) &= \sum_{z\in D(0,5r/4)-D(0,r+n^4)} G_{D(0,r+n^4)^c}(x,z)p_1(z,y) \\ &+ \sum_{z\in D(0,2r)-D(0,5r/4)} G_{D(0,r+n^4)^c}(x,z)p_1(z,y) \\ &+ \sum_{z\in D^c(0,2r)} G_{D(0,r+n^4)^c}(x,z)p_1(z,y). \end{aligned} \tag{3.33}$$

Let us first show that uniformly in $x,x'\in\partial D(0,R)_{n^4}$ and $z\in D(0,2r)-D(0,5r/4)$

$$G_{D(0,r+n^4)^c}(x,z)=\big(1+O(n^{-3}\log n)\big)\,G_{D(0,r+n^4)^c}(x',z). \tag{3.34}$$

To this end, note that by (2.3), uniformly in $x,x'\in\partial D(0,R)_{n^4}$ and $y\in D(0,2r)$

$$\begin{aligned} a(x-y) &= \frac{2}{\pi}\log|x-y|+k+O(|x-y|^{-1}) \\ &= \frac{2}{\pi}\log R+k+O(n^{-3}) \\ &= a(x'-y)+O(n^{-3}), \end{aligned} \tag{3.35}$$

and with $N\geq n^3R$ the same result applies with $y\in D(0,N)^c$. Hence, by (2.8) applied to the finite set $A(r+n^4,N)=:D(0,N)-D(0,r+n^4)$, and using the fact that $H_{A(r+n^4,N)^c}(z,\cdot)$ is a probability, we see that uniformly in $x,x'\in\partial D(0,R)_{n^4}$ and $z\in D(0,r+n^4)^c$

$$\begin{aligned} & G_{A(r+n^4,N)}(x,z)=G_{A(r+n^4,N)}(z,x) \\ &=\left\{\sum_{y\in D(0,r+n^4)\cup D(0,N)^c} H_{A(r+n^4,N)^c}(z,y)a(y-x)\right\}-a(z-x) \\ &=\left\{\sum_{y\in D(0,r+n^4)\cup D(0,N)^c} H_{A(r+n^4,N)^c}(z,y)a(y-x')\right\} \\ &\qquad\qquad -a(z-x')+O(n^{-3}) \\ &=G_{A(r+n^4,N)}(z,x')+O(n^{-3}). \end{aligned} \tag{3.36}$$

Since this is uniform in $N\geq n^3R$, using (2.67) we can apply the dominated convergence theorem as $N\to\infty$ to see that uniformly in $x,x'\in\partial D(0,R)_{n^4}$ and $z\in D(0,2r)$

$$G_{D(0,r+n^4)^c}(x,z)=G_{D(0,r+n^4)^c}(z,x')+O(n^{-3}). \tag{3.37}$$

Applying (3.26) now establishes (3.34).

We show next that uniformly in $x,x'\in\partial D(0,R)_{n^4}$ and $z\in D(0,5r/4)-D(0,r+n^4)$

$$G_{D(0,r+n^4)^c}(x,z)=\big(1+O(n^{-3}\log n)\big)\,G_{D(0,r+n^4)^c}(x',z)+O(r^{-1-2\beta}\log r). \tag{3.38}$$

To see this we use the strong Markov property together with (3.34) and (2.67) to see that

$$\begin{aligned} & G_{D(0,r+n^4)^c}(x,z)=G_{D(0,r+n^4)^c}(z,x) \\ &=\mathbb{E}^z\left\{G_{D(0,r+n^4)^c}(X_{D(0,5r/4)^c},x)\,;\,T_{D(0,5r/4)^c}<T_{D(0,r+n^4)}\right\}, \end{aligned} \tag{3.39}$$

and this in turn is bounded by

$$\begin{aligned} & \mathbb{E}^z\left\{G_{D(0,r+n^4)^c}(X_{D(0,5r/4)^c},x)\,;\,T_{D(0,5r/4)^c}<T_{D(0,r+n^4)}\,,\,|X_{D(0,5r/4)^c}|\leq 2r\right\} \\ &+\mathbb{E}^z\left\{G_{D(0,r+n^4)^c}(X_{D(0,5r/4)^c},x)\,;\,T_{D(0,5r/4)^c}<T_{D(0,r+n^2)}\,,\,|X_{D(0,5r/4)^c}|>2r\right\} \\ &\leq\big(1+O(n^{-3}\log n)\big)\,\mathbb{E}^z\left\{G_{D(0,r+n^2)^c}(X_{D(0,5r/4)^c},x')\,;\,T_{D(0,5r/4)^c}<T_{D(0,r+n^4)}\right\} \\ &\qquad +c\log(R)\,\mathbb{P}^z\left\{|X_{D(0,5r/4)^c}|>2r\right\}. \end{aligned}$$

Using (3.39) we see that

$$\mathbb{E}^z \left\{ G_{D(0,r+n^2)^c}(X_{D(0,5r/4)^c}, x') \,;\, T_{D(0,5r/4)^c} < T_{D(0,r+n^4)} \right\} = G_{D(0,r+n^4)^c}(x', z).$$

As in (2.45), by using (2.19) and (2.1) we have

$$\begin{aligned} &\mathbb{P}^z \left\{ |X_{D(0,5r/4)^c}| > 2r \right\} \qquad (3.40) \\ &= \sum_{\substack{|y|<5r/4 \\ 2r<|w|}} G_{D(0,5r/4)}(z,y) p_1(y,w) \\ &\leq c \log r \sum_{|y|<5r/4} \mathbb{P}(|X_1| \geq 3r/4) \\ &\leq c \log r \sum_{|y|<5r/4} \frac{1}{|r|^{3+2\beta}} \leq c r^{-1-2\beta} \log r. \end{aligned}$$

This establishes (3.38).

We next show that

$$\sum_{z \in D^c(0,2r)} G_{D(0,r+n^4)^c}(x,z) p_1(z,y) = O(r^{-3-\beta}). \qquad (3.41)$$

It follows from (2.67) that $G_{D(0,r+n)^c}(x,z) = O(\log R)$, so that the sum in (3.12) is bounded by $O(\log R)$ times the probability of a jump of size greater than $|z| - r - n \geq r - n \geq r/2$, which gives (3.41).

To prove (3.30) it thus suffices to show that uniformly in $x \in \partial D(0,R)_{n^4}$ and $y \in \partial D(0,r)_{n^4}$

$$r^{-1-2\beta} \log r \leq c n^{-3} H_{D(0,r+n^4)}(x,y). \qquad (3.42)$$

Using the last exit decomposition together with (3.27) we obtain

$$\begin{aligned} H_{D(0,r+n^4)}(x,y) &= \sum_{z \in D(0,r+n^4)^c} G_{D(0,r+n^4)^c}(x,z) p_1(z,y) \qquad (3.43) \\ &\geq c(R \log R)^{-1} \sum_{r+(1+\epsilon)n^4 \leq |z| \leq 2r} p_1(z,y) \end{aligned}$$

for any $\epsilon > 0$. Note that the annulus $\{z \,|\, r+(1+\epsilon)n^4 \leq |z| \leq 2r\}$ contains the disc $D(v, 2(1+\epsilon)n^4)$ where $v = (r + 3(1+\epsilon)n^4)y/|y|$, and we have $2(1+\epsilon)n^4 \leq |y - v| \leq 3(1+\epsilon)n^4$. Thus

$$\begin{aligned} \sum_{r+(1+\epsilon)n^4 \leq |z| \leq 2r} p_1(z,y) &\geq \sum_{z \in D(v,2(1+\epsilon)n^4)} p_1(z,y) \qquad (3.44) \\ &= \sum_{z \in D(v,2(1+\epsilon)n^4)} p_1(z-v, y-v) \\ &= \sum_{z \in D(0,2(1+\epsilon)n^4)} p_1(z, y-v). \end{aligned}$$

Hence by (1.3) and our assumption that $r \geq e^n$

$$H_{D(0,r+n^4)}(x,y) \geq c(R \log R)^{-1} e^{-(1+\epsilon)^{1/4}\beta n} \geq c r^{-1-\epsilon-(1+\epsilon)^{1/4}\beta}, \qquad (3.45)$$

and thus (3.42), and hence (3.30), follows by taking ϵ small.

The rest of Lemma 3.2 follows as in the proof of Lemma 3.1.

□

4 Local time estimates and upper bounds

The following simple lemma, the analog of (11, Lemma 2.1), will be used repeatedly.

Lemma 4.1. *For* $|x_0| < R$,

$$\mathbb{E}^{x_0}(L^0_{T_{D(0,R)^c}}) = G_{D(0,R)}(x_0, 0). \tag{4.1}$$

For all $z \geq 1$

$$\mathbb{P}^{x_0}(L^0_{T_{D(0,R)^c}} \geq z G_{D(0,R)}(0,0)) \leq c\sqrt{z} e^{-z} \tag{4.2}$$

for some $c < \infty$ *independent of* x_0, z, R.

Let $x_0 \neq 0$. *Let* $0 < \varphi \leq 1$ *and set* $\lambda = \varphi / G_{D(0,R)}(0,0)$. *Then*

$$\begin{aligned} &\mathbb{E}^{x_0}\left(e^{-\lambda L^0_{T_{D(0,R)^c}}}\right) \qquad (4.3) \\ &= 1 - \frac{\log(\frac{R}{|x_0|}) + O(|x_0|^{-1/4})}{\log(R)} \frac{\varphi}{1+\varphi}\left(1 + O(\frac{1}{\log(|x_0|)})\right) \end{aligned}$$

Proof of Lemma 4.1: Since

$$L^0_{T_{D(0,R)^c}} = \sum_{i < T_{D(0,R)^c}} 1_{\{X_i = 0\}}, \tag{4.4}$$

(4.1) follows from (2.4). Then we have by the strong Markov property that

$$\begin{aligned} \mathbb{E}^{x_0}(L^0_{T_{D(0,R)^c}})^k &= k!\mathbb{E}^{x_0}\left(\sum_{0 \leq j_1 \leq \cdots \leq j_k \leq T_{D(0,R)^c}} \prod_{i=1}^k 1_{\{X_{j_i}=0\}}\right) \\ &= k!\mathbb{E}^{x_0}\left(\sum_{0 \leq j_1 \leq \cdots \leq j_{k-1} \leq T_{D(0,R)^c}} \prod_{i=1}^{k-1} 1_{\{X_{j_i}=0\}} G_{D(0,R)}(0,0)\right) \\ &= k\mathbb{E}^{x_0}(L^0_{T_{D(0,R)^c}})^{k-1} G_{D(0,R)}(0,0)., \end{aligned}$$

By induction on k,

$$\mathbb{E}^{x_0}(L^0_{T_{D(0,R)^c}})^k = k! G_{D(0,R)}(x_0, 0)(G_{D(0,R)}(0,0))^{k-1}. \tag{4.5}$$

To prove (4.2), use (4.5), (2.17) and Chebyshev to obtain

$$\mathbb{P}^{x_0}(L^0_{T_{D(0,R)^c}} \geq z G_{D(0,R)}(0,0)) \leq \frac{k!}{z^k} \tag{4.6}$$

then take $k = [z]$ and use Stirling's formula.

For (4.3), note that, conditional on hitting 0, $L^0_{T_{D(0,R)^c}}$ is a geometric random variable with mean $G_{D(0,R)}(0,0)$. Hence,

$$\begin{aligned} &\mathbb{E}^{x_0}\left(e^{-\lambda L^0_{T_{D(0,R)^c}}}\right) \\ &= 1 - \mathbb{P}^{x_0}\left(T_0 < T_{D(0,R)^c}\right) \\ &\qquad + \mathbb{P}^{x_0}\left(T_0 < T_{D(0,R)^c}\right)\left(\frac{1}{(e^\lambda - 1)G_{D(0,R)}(0,0) + 1}\right). \qquad (4.7) \end{aligned}$$

Since by (2.13)

$$\frac{1}{G_{D(0,R)}(0,0)} = O(1/\log(R)) \tag{4.8}$$

we have

$$(e^{\lambda}-1)G_{D(0,R)}(0,0)+1 = 1+\varphi+O\Big(\frac{1}{\log(R)}\Big) \tag{4.9}$$

and (4.3) then follows from (4.7) and (2.16).
□

We next provide the required upper bounds in Theorem 1.2. Namely, we will show that for any $a \in (0,2]$

$$\limsup_{m\to\infty} \frac{\log\left|\left\{x \in D(0,m):\ L^x_{T_{D(0,m)^c}} \geq (2a/\pi)(\log m)^2\right\}\right|}{\log m} \leq 2-a \qquad a.s. \tag{4.10}$$

To see this fix $\gamma > 0$ and note that by (4.2) and (2.13), for some $0 < \delta < \gamma$, all $x \in D(0,m)$ and all large enough m

$$\mathbb{P}^0\left(\frac{L^x_{T_{D(x,2m)^c}}}{(\log m)^2} \geq 2a/\pi\right) \leq m^{-a+\delta} \tag{4.11}$$

Therefore

$$\begin{aligned} &\mathbb{P}^0\left(\left|\left\{x \in D(0,m):\ \frac{L^x_{T_{D(0,m)^c}}}{(\log m)^2} \geq 2a/\pi\right\}\right| \geq m^{2-a+\gamma}\right) \\ &\leq m^{-(2-a)-\gamma}\mathbb{E}^0\left(\left|\left\{x \in D(0,m):\ \frac{L^x_{T_{D(0,m)^c}}}{(\log m)^2} \geq 2a/\pi\right\}\right|\right) \\ &= m^{-(2-a)-\gamma}\sum_{x\in D(0,m)}\mathbb{P}^0\left(\frac{L^x_{T_{D(0,m)^c}}}{(\log m)^2} \geq 2a/\pi\right) \\ &\leq m^{-(2-a)-\gamma}\sum_{x\in D(0,m)}\mathbb{P}^0\left(\frac{L^x_{T_{D(x,2m)^c}}}{(\log m)^2} \geq 2a/\pi\right) \\ &\leq m^{-(\gamma-\delta)}. \end{aligned} \tag{4.12}$$

Now apply our result to $m = m_n = e^n$ to see by Borel-Cantelli that for some $N(\omega) < \infty$ a.s. we have that for all $n \geq N(\omega)$

$$\left|\left\{x \in D(0,e^n):\ \frac{L^x_{T_{D(0,e^n)^c}}}{(\log e^n)^2} \geq 2a/\pi\right\}\right| \leq e^{(2-a+\gamma)n}. \tag{4.13}$$

Then if $e^n \leq m \leq e^{n+1}$

$$\begin{aligned}
&\left|\left\{x \in D(0,m) : \frac{L^x_{T_{D(0,m)^c}}}{(\log m)^2} \geq 2a/\pi\right\}\right| \qquad (4.14)\\
&\leq \left|\left\{x \in D(0,e^{n+1}) : \frac{L^x_{T_{D(0,e^{n+1})^c}}}{(\log e^n)^2} \geq 2a/\pi\right\}\right|\\
&= \left|\left\{x \in D(0,e^{n+1}) : \frac{L^x_{T_{D(0,e^{n+1})^c}}}{(\log e^{n+1})^2} \geq 2a(1+1/n)^{-2}/\pi\right\}\right|\\
&\leq e^{(2-a(1+1/n)^{-2}+\gamma)(n+1)} \leq m^{(2-a(1+1/n)^{-2}+\gamma)(n+1)/n}.
\end{aligned}$$

(4.10) now follows by first letting $n \to \infty$ and then letting $\gamma \to 0$.
□

5 Lower bounds for probabilities

Fixing $a < 2$, we prove in this section

$$\liminf_{m\to\infty} \frac{\log\left|\left\{x \in D(0,m) : L^x_{T_{D(0,m)^c}} \geq (2a/\pi)(\log m)^2\right\}\right|}{\log m} \geq 2-a \quad a.s. \qquad (5.1)$$

In view of (4.10), we will obtain Theorem 1.2 .

We start by constructing a subset of the set appearing in (5.1), the probability of which is easier to bound below. To this end fix n, and let $r_{n,k} = e^n n^{3(n-k)}$, $k = 0,\dots,n$. In particular, $r_{n,n} = e^n$ and $r_{n,0} = e^n n^{3n}$. Set $K_n = 16 r_{n,0} = 16 e^n n^{3n}$.

Let $U_n = [2r_{n,0}, 3r_{n,0}]^2 \subseteq D(0,K_n)$. For $x \in U_n$, consider the x-bands $\partial D(x,r_{n,k})_{n^4}$; $k = 0,\dots,n$. We use the abbreviation $r'_{n,k} = r_{n,k} + n^4$. For $x \in U_n$ we will say that the path **does not skip x-bands** if

(1) $T_{D(x,r'_{n,0})} < T_{D(0,K_n)^c}$ and $T_{D(x,r'_{n,0})} = T_{\partial D(x,r_{n,0})_{n^4}}$.

(2) For any $t < T_{D(0,K_n)^c}$ such that $X_t \in \partial D(x,r_{n,k})_{n^4}$ we have:

(2′) if $k = 0$ then

$$\left(T_{D(x,r'_{n,1})} \wedge T_{D(0,K_n)^c}\right) \circ \theta_t = \left(T_{\partial D(x,r_{n,1})_{n^4}} \wedge T_{D(0,K_n)^c}\right) \circ \theta_t,$$

(2′′) if $k = 1,\dots,n-1$ then

$$\left(T_{D(x,r'_{n,k+1})} \wedge T_{D(x,r_{n,k-1})^c}\right) \circ \theta_t = \left(T_{\partial D(x,r_{n,k+1})_{n^4}} \wedge T_{\partial D(x,r_{n,k-1})_{n^4}}\right) \circ \theta_t,$$

(2′′′) if $k = n$ then

$$\left(T_{D(x,r_{n,n-1})^c}\right) \circ \theta_t = \left(T_{\partial D(x,r_{n,n-1})_{n^4}}\right) \circ \theta_t.$$

For $x \in D(0, K_n)$, let $N^x_{n,k}$ denote the number of excursions from $D(x, r_{n,k-1})^c$ to $D(x, r'_{n,k})$ until time $T_{D(0,K_n)^c}$. Set $\mathcal{N}_k = 3ak^2 \log k$, and $k_0 = 4 \vee \inf\{k \,|\, \mathcal{N}_k \geq 2k\}$. We will say that a point $x \in U_n$ is **n-successful** if the path does not skip x-bands, $N^x_{n,k} = 1$, $\forall k = 1, \ldots, k_0 - 1$ and

$$\mathcal{N}_k - k \leq N^x_{n,k} \leq \mathcal{N}_k + k \qquad \forall k = k_0, \ldots, n. \tag{5.2}$$

Let $\{Y(n,x)\,;\, x \in U_n\}$ be the collection of random variables defined by

$$Y(n,x) = 1 \quad \text{if } x \text{ is } n\text{-successful}$$

and $Y(n,x) = 0$ otherwise. Set $\bar{q}_{n,x} = \mathbb{P}(Y(n,x) = 1) = \mathbb{E}(Y(n,x))$.

The next lemma relates the notion of n-successful and local times. As usual we write $\log_2 n$ for $\log \log n$.

Lemma 5.1. *Let*

$$\mathcal{S}_n = \{x \in U_n \,|\, x \text{ is } n\text{-successful}\}.$$

Then for some $N(\omega) < \infty$ a.s., for all $n \geq N(\omega)$ and all $x \in \mathcal{S}_n$

$$\frac{L^x_{T_{D(0,K_n)^c}}}{(\log K_n)^2} \geq 2a/\pi - 2/\log_2 n.$$

Proof of Lemma 5.1: Recall that if x is n-successful then $N^x_{n,n} \geq \mathcal{N}_n - n = a(3n^2 \log n) - n$. Let $L^{x,j}$ denote the number of visits to x during the j^{th} excursion from $\partial D(x, r_{n,n})_{n^4}$ to $D(x, r_{n,n-1})^c$. Then for any $0 < \lambda < \infty$

$$\begin{aligned} P_x &:= \mathbb{P}\left(L^x_{T_{D(0,K_n)^c}} \leq (2a/\pi - 2/\log_2 n)(\log K_n)^2\,,\, x \in \mathcal{S}_n\right) \\ &\leq \mathbb{P}\left(\sum_{j=0}^{\mathcal{N}_n - n} L^{x,j} \leq (2a/\pi - 1/\log_2 n)(3n \log n)^2\right) \\ &\leq \exp\left(\lambda(2a/\pi - 1/\log_2 n)(3n\log n)^2\right) E\left(e^{-\lambda \sum_{j=0}^{\mathcal{N}_n - n} L^{x,j}}\right). \end{aligned} \tag{5.3}$$

If τ denotes the first time that the $(\mathcal{N}_n - n)^{th}$ excursion from $D(x, r_{n,n-1})^c$ reaches $\partial D(x, r_{n,n})_{n^4}$ then by the strong Markov property

$$\begin{aligned} &\mathbb{E}\left(e^{-\lambda \sum_{j=0}^{\mathcal{N}_n - n} L^{x,j}}\right) \\ &= \mathbb{E}\left(e^{-\lambda \sum_{j=0}^{\mathcal{N}_n - n - 1} L^{x,j}} \mathbb{E}^{X_\tau}\left(e^{-\lambda L^x_{T_{D(x, r_{n,n-1})^c}}}\right)\right). \end{aligned} \tag{5.4}$$

Set $\lambda = \phi / G_{D(x, r_{n,n-1})}(x,x)$. By (4.3), with $r = r_{n,n} = e^n$, $R = r_{n,n-1} = n^3 e^n$, for any $0 < \phi \leq 1$ and large n

$$\sup_{y \in \partial D(x, r_{n,n})_{n^4}} \mathbb{E}^y\left(e^{-\lambda L^x_{T_{D(x, r_{n,n-1})^c}}}\right) \leq \exp\left(-\frac{(1 - 1/2\log n)\,\phi}{1 + \phi} 3(\log n)/n\right). \tag{5.5}$$

Hence by induction

$$\mathbb{E}\left(e^{-\lambda \sum_{j=0}^{\mathcal{N}_n - n} L^{x,j}}\right) \leq \exp\left(-\frac{(1 - 1/\log n)\phi}{1 + \phi} 9an(\log n)^2\right). \tag{5.6}$$

Then with this choice of λ, noting that $G_{D(x,r_{n,n-1})}(x,x) \sim \frac{2}{\pi}n$ by (2.13), we have

$$P_x \le \inf_{\phi>0} \exp\left(\left\{ \phi(1 - 1/2\log_2 n) - \frac{(1-1/\log n)\phi}{1+\phi} \right\} 9an(\log n)^2 \right). \tag{5.7}$$

A straightforward computation shows that

$$\inf_{\phi>0} \left(\phi\alpha - \frac{\phi}{1+\phi}\beta \right) = -\left(\sqrt{\beta} - \sqrt{\alpha} \right)^2 \tag{5.8}$$

which is achieved for $\phi = \sqrt{\beta}/\sqrt{\alpha} - 1$. Using this in (5.7) we find that

$$P_x \le \exp\left(-cn(\log n/\log_2 n)^2 \right). \tag{5.9}$$

Note that $|U_n| \le e^{cn\log n}$. Summing over all $x \in U_n$ and then over n and applying Borel-Cantelli will then complete the proof of Lemma 5.1.

□

The next lemma, which provides estimates for the first and second moments of $Y(n,x)$, will be proved in the following sections. Recall $\bar{q}_{n,x} = \mathbb{P}(x \text{ is } n\text{-successful})$. Let $Q_n = \inf_{x\in U_n} \bar{q}_{n,x}$.

Lemma 5.2. *There exists $\delta_n \to 0$ such that for all $n \ge 1$,*

$$Q_n \ge K_n^{-(a+\delta_n)}, \tag{5.10}$$

and

$$Q_n \ge c\bar{q}_{n,x} \tag{5.11}$$

for some $c > 0$ and all n and $x \in U_n$.

There exists $C < \infty$ and $\delta'_n \to 0$ such that for all n, $x \neq y$ and $l(x,y) = \min\{m : D(x,r_{n,m}) \cap D(y,r_{n,m}) = \emptyset\} \le n$

$$\mathbb{E}(Y(n,x)Y(n,y)) \le CQ_n^2 (l(x,y)!)^{3a+\delta'_{l(x,y)}}. \tag{5.12}$$

Remark. Immediately following this remark we give a relatively quick proof of our main results, Theorem 1.2 and Theorem 1.1. It may be helpful at this point to give a short heuristic overview of the proof of Lemma 5.2. (5.10) and (5.11) are proven in Section 6. There we use estimates (2.49) and (2.66) on the probability of excursions between bands at level $k-1$ and k to show that the probability of making between $\mathcal{N}_k - k$ and $\mathcal{N}_k + k$ such excursions is about k^{-3a}, so that the the probability of making that many excursions for each $k = 1,\dots,n$ is about $(n!)^{-3a} \approx K_n^{-(a+\delta_n)}$. The lower bound (5.12) is proven in Sections 7 and 8. Since we only require a lower bound it suffices to bound the probability that we have the right number of excursions at all levels around x and at levels $k = l(x,y)+3,\dots,n$ around y. We choose $l(x,y)$ so that none of the bands at levels $k = l(x,y)+3,\dots,n$ around y intersect any of the bands around x. If this implied that the excursion counts around x and y were independent, the heuristic mentioned for obtaining (5.10) and (5.11) would give (5.12). It is here that the Harnack inequalities of Section 3 come in. The only way that excursions around y can have an effect on excursion counts around x is by influencing the initial and final points of the x-excursions. Our Harnack inequalities show that such an effect is negligible. The reader might find it useful to compare the proof given here with the one in (11).

Proof of Theorem 1.2. In view of (4.10) we need only consider the lower bound. To prove (5.1) we will show that for any $\delta > 0$ we can find $p_0 > 0$ and $N_0 < \infty$ such that

$$\mathbb{P}^0 \left(\sum_{x \in U_n} 1_{\{Y(n,x)=1\}} \geq K_n^{2-a-\delta} \right) \geq p_0 \tag{5.13}$$

for all $n \geq N_0$. Lemma 5.1 will then imply that for some $p_1 > 0$ and $N_1 < \infty$

$$\mathbb{P}^0 \left(\left| \left\{ x \in D(0, K_n) : \; L^x_{T_{D(0,K_n)^c}} \geq (2a/\pi - 2/\log_2 n)(\log K_n)^2 \right\} \right| \geq K_n^{2-a-\delta} \right) \geq p_1. \tag{5.14}$$

for all $n \geq N_1$. As in the proof of (4.10) and readjusting $\delta > 0$ we can find $p_2 > 0$ and $N_2 < \infty$ such that

$$\mathbb{P}^0 \left(\left| \left\{ x \in D(0, n) : \; L^x_{T_{D(0,n)^c}} \geq (2a/\pi)(\log n)^2 \right\} \right| \geq n^{2-a-\delta} \right) \geq p_2 \tag{5.15}$$

for all $n \geq N_2$. Then by Lemma 9.4, with a further readjustment of $\delta > 0$ we will have that

$$\mathbb{P}^0 \left(\left| \left\{ x \in \mathbb{Z}^2 : \; L^x_n \geq (a/2\pi)(\log n)^2 \right\} \right| \geq n^{1-a/2-\delta} \right) \geq p_3 \tag{5.16}$$

for some $p_3 > 0$ and all $n \geq N_3$ with $N_3 < \infty$. This estimate leads to (5.1) as in the proof of Theorem 5.1 of (4).

Recall the Paley-Zygmund inequality (see (7, page 8)): for any $W \in L^2(\Omega)$ and $0 < \lambda < 1$

$$\mathbb{P}(W \geq \lambda \mathbb{E}(W)) \geq (1-\lambda)^2 \frac{(\mathbb{E}(W))^2}{\mathbb{E}(W^2)}. \tag{5.17}$$

We will apply this with $W = W_n = \sum_{x \in U_n} 1_{\{Y(n,x)=1\}}$. We see by (5.10) of Lemma 5.2 that for some sequence $\delta_n \to 0$

$$\mathbb{E} \left(\sum_{x \in U_n} 1_{\{Y(n,x)=1\}} \right) = \sum_{x \in U_n} \bar{q}_{n,x} \geq K_n^{2-a-\delta_n}. \tag{5.18}$$

Thus to complete the proof of (5.13) it suffices to show

$$\mathbb{E} \left(\left\{ \sum_{x \in U_n} 1_{\{Y(n,x)=1\}} \right\}^2 \right) \leq c \left\{ \mathbb{E} \left(\sum_{x \in U_n} 1_{\{Y(n,x)=1\}} \right) \right\}^2 \tag{5.19}$$

for some $c < \infty$ all n sufficiently large. Furthermore, using (5.18) it suffices to show that

$$\mathbb{E} \left(\sum_{\substack{x,y \in U_n \\ x \neq y}} 1_{\{Y(n,x)=1\}} 1_{\{Y(n,y)=1\}} \right) \leq c \left\{ \mathbb{E} \left(\sum_{x \in U_n} 1_{\{Y(n,x)=1\}} \right) \right\}^2 \tag{5.20}$$

We let C_m denote generic finite constants that are independent of n. The definition of $l(x,y) \geq 1$ implies that $|x - y| \leq 2r_{n,l(x,y)-1}$. Recall that because $n \geq l$, there are at most $C_0 r^2_{n,l-1} \leq$

$C_0 K_n^2/n^{6(l-1)} \leq C_0 K_n^2 l^6 (l!)^{-6}$ points y in the ball of radius $2r_{n,l-1}$ centered at x. Thus it follows from Lemma 5.2 that

$$
\begin{aligned}
&\sum_{\substack{x,y\in U_n\\ 2r_{n,n}\leq|x-y|\leq 2r_{n,0}}} \mathbb{E}\left(Y(n,x)Y(n,y)\right) \qquad (5.21)\\
&\leq C_1 \sum_{\substack{x,y\in U_n\\ 2r_{n,n}\leq|x-y|\leq 2r_{n,0}}} Q_n^2 (l(x,y)!)^{3a+\delta'_{l(x,y)}}\\
&\leq C_2 Q_n^2 \sum_{x\in U_n}\sum_{j=1}^{n}\sum_{\{y\,|\,l(x,y)=j\}} (j!)^{3a+\delta'_j}\\
&\leq C_3 Q_n^2 K_n^2 \sum_{j=1}^{n} K_n^2 j^6 (j!)^{-6} (j!)^{3a+\delta'_j}\\
&\leq C_4 (K_n^2 Q_n)^2 \sum_{j=1}^{n} j^6 (j!)^{-3(2-a)+\delta'_j}\\
&\leq C_5 (K_n^2 Q_n)^2 \leq C_6 \left\{\mathbb{E}\left(\sum_{x\in U_n} Y(n,x)\right)\right\}^2
\end{aligned}
$$

where we used the fact which follows from the definitions that

$$
K_n^2 Q_n \leq c \sum_{x\in U_n} \bar{q}_{n,x} = c\mathbb{E}\left(\sum_{x\in U_n} Y(n,x)\right). \qquad (5.22)
$$

Because $Y(n,x) \leq 1$ and $\mathbb{E}Y(n,x) = \bar{q}_{n,y} \leq cQ_n$, we have

$$
\begin{aligned}
&\sum_{\substack{x,y\in U_n\\ |x-y|\leq 2r_{n,n}}} \mathbb{E}\left(Y(n,x)Y(n,y)\right) \leq \sum_{\substack{x,y\in U_n\\ |x-y|\leq 2r_{n,n}}} \mathbb{E}\left(Y(n,y)\right) \qquad (5.23)\\
&\leq C_7 \sum_{x\,;\,|x|\leq 2r_{n,n}} K_n^2 Q_n \leq C_8\, e^{2n} K_n^2 Q_n.
\end{aligned}
$$

By (5.10)

$$
K_n^2 Q_n \geq K_n^{2-a-\delta_n} \geq ce^{2n}.
$$

This and (5.22) show that the right hand side of (5.23) is bounded by

$$
C_9 \left\{\mathbb{E}\left(\sum_{x\in U_n} Y(n,x)\right)\right\}^2. \qquad (5.24)
$$

We know that if $x, y \in U_n$, then $|x-y| \leq 2r_{n,0}$. Thus combining (5.21), (5.23), and (5.24) completes the proof of (5.20) and hence of (5.1).
□

Proof of Theorem 1.1. The lower bound is an immediate consequence of Theorem 1.2. The upper bound is a consequence of (4.2) as follows; cf. the proof of (4, (2.8)). Let $\delta > 0$. By (2.13) and (4.2)

$$\begin{aligned}
&\mathbb{P}^0(\sup_{x\in D(0,R)} L^x_{T_{D(0,R)^c}} > \frac{4}{\pi}(1+\delta)\log^2 R) \qquad (5.25)\\
\leq\; & \sum_{x\in D(0,R)} \mathbb{P}^0(L^x_{T_{D(x,2R)^c}} > \frac{4}{\pi}(1+\delta)\log^2 R)\\
\leq\; & cR^2((1+2\delta)\log R)^{1/2}e^{-2(1+\delta/2)\log R} \leq cR^{-\delta/4}
\end{aligned}$$

for R large. By Borel-Cantelli there exists $M_0(\omega)$ such that if $m \geq M_0$, then

$$L^*_{T_{D(0,2^m)^c}} \leq \frac{4}{\pi}(1+\delta)\log^2(2^m).$$

If $m \geq M_0$, $2^m \leq n \leq 2^{m+1}$, and m is large,

$$L^*_{T_{D(0,n)^c}} \leq L^*_{T_{D(0,2^{m+1})^c}} \leq \frac{4}{\pi}(1+\delta)\log^2(2^{m+1}) \leq \frac{4}{\pi}(1+2\delta)\log^2 n.$$

Since δ is arbitrary, this and Lemma 9.4 prove the upper bound.
□

6 First moment estimates

Proof of (5.10) and (5.11): For $x \in U_n$ we begin by getting bounds on the probability that $T_{D(x,r'_{n,0})} < T_{D(0,K_n)^c}$ and $T_{D(x,r'_{n,0})} = T_{\partial D(x,r_{n,0})_{n^4}}$. Since

$$\begin{aligned}
&\mathbb{P}\left(T_{D(x,r'_{n,0})} < T_{D(0,K_n)^c}\,;\, T_{D(x,r'_{n,0})} = T_{\partial D(x,r_{n,0})_{n^4}}\right) \qquad (6.1)\\
&\geq \mathbb{P}\left(T_{D(x,r'_{n,0})} < T_{D(x,\frac{1}{2}K_n)^c}\,;\, T_{D(x,r'_{n,0})} = T_{\partial D(x,r_{n,0})_{n^4}}\right)
\end{aligned}$$

we see from (2.66) that uniformly in n and $x \in U_n$

$$\mathbb{P}\left(T_{D(x,r'_{n,0})} < T_{D(0,K_n)^c}\,;\, T_{D(x,r'_{n,0})} = T_{\partial D(x,r_{n,0})_{n^4}}\right) \geq c \qquad (6.2)$$

for some $c > 0$. And since for $x \in U_n$ and $y \in \partial D(x, r_{n,0})_{n^4}$

$$\begin{aligned}
&\mathbb{P}^y\left(T_{D(x,r'_{n,1})} < T_{D(x,\frac{1}{2}K_n)^c}\,;\, T_{D(x,r'_{n,1})} = T_{\partial D(x,r_{n,1})_{n^4}}\right) \qquad (6.3)\\
&\quad\leq \mathbb{P}^y\left(T_{D(x,r'_{n,1})} < T_{D(0,K_n)^c}\,;\, T_{D(x,r'_{n,1})} = T_{\partial D(x,r_{n,1})_{n^4}}\right)\\
&\leq \mathbb{P}^y\left(T_{D(x,r'_{n,1})} < T_{D(x,2K_n)^c}\,;\, T_{D(x,r'_{n,1})} = T_{\partial D(x,r_{n,1})_{n^4}}\right)
\end{aligned}$$

we see from (2.66) that uniformly in n, $x \in U_n$ and $y \in \partial D(x, r_{n,0})_{n^4}$

$$c/\log n \leq \mathbb{P}^y\left(T_{D(x,r'_{n,1})} < T_{D(0,K_n)^c}\,;\, T_{D(x,r'_{n,1})} = T_{\partial D(x,r_{n,1})_{n^4}}\right) \leq c'/\log n. \qquad (6.4)$$

Similarly, since for $x \in U_n$ and $y \in \partial D(x, r_{n,0})_{n^4}$

$$\mathbb{P}^y\left(T_{D(0,K_n)^c} < T_{D(x,r'_{n,1})}\right) \geq \mathbb{P}^y\left(T_{D(x,2K_n)^c} < T_{D(x,r'_{n,1})}\right) \qquad (6.5)$$

we see from (2.49) that uniformly in n, $x \in U_n$ and $y \in \partial D(x, r_{n,0})_{n^4}$

$$\mathbb{P}^y \left(T_{D(0,K_n)^c} < T_{D(x,r'_{n,1})} \right) \geq c > 0. \tag{6.6}$$

These bounds will be used for excursions at the 'top' levels. To bound excursions at 'intermediate' levels we note that using (2.49), we have uniformly for $x \in \partial D(0, r_{n,l})_{n^4}$, with $1 \leq l \leq n-1$

$$\begin{aligned} &\mathbb{P}^x \left(T_{D(0,r_{n,l-1})^c} < T_{D(0,r'_{n,l+1})} \, ; \, T_{D(0,r_{n,l-1})^c} = T_{\partial D(0,r_{n,l-1})_{n^4}} \right) \\ &= 1/2 + O(n^{-4-4\beta}), \end{aligned} \tag{6.7}$$

and using (2.66), we have uniformly for $x \in \partial D(0, r_{n,l})_{n^4}$, with $1 \leq l \leq n-1$

$$\begin{aligned} &\mathbb{P}^x \left(T_{D(0,r'_{n,l+1})} < T_{D(0,r_{n,l-1})^c} \, ; \, T_{D(0,r'_{n,l+1})} = T_{\partial D(0,r_{n,l+1})_{n^4}} \right) \\ &= 1/2 + O(n^{-4-4\beta}). \end{aligned} \tag{6.8}$$

For excursions at the 'bottom' level, let us note, using an analysis similar to that of (2.43), that uniformly in $z \in D(0, r_{n,n})_{n^4}$

$$\mathbb{P}^z \left(T_{D(0,r_{n,n-1})^c} = T_{\partial D(0,r_{n,n-1})_{n^4}} \right) = 1 + O(n^{-4-4\beta}). \tag{6.9}$$

Let $\bar{m} = (m_2, m_3, \ldots, m_n)$ and set $|\bar{m}| = 2\sum_{j=2}^n m_j + 1$. Let $\mathcal{H}_n(\bar{m})$, be the collection of maps, ('histories'),

$$\varphi : \{0, 1, \ldots, |\bar{m}|\} \mapsto \{0, 1, \ldots, n\}$$

such that $\varphi(0) = 1$, $\varphi(j+1) = \varphi(j) \pm 1$, $|\bar{m}| = \inf\{j \, ; \, \varphi(j) = 0\}$ and the number of upcrossings from $\ell - 1$ to ℓ

$$u(\ell) =: |\{(j, j+1) \, | \, (\varphi(j), \varphi(j+1)) = (\ell - 1, \ell)\}| = m_\ell.$$

Note that we cannot have any upcrossings from ℓ to $\ell + 1$ until we have first had an upcrossing from $\ell - 1$ to ℓ. Hence the number of ways to partition the $u(\ell+1)$ upcrossings from ℓ to $\ell + 1$ among and after the $u(\ell)$ upcrossings from $\ell - 1$ to ℓ is the same as the number of ways to partition $u(\ell + 1)$ indistinguishable objects into $u(\ell)$ parts, which is

$$\binom{u(\ell+1) + u(\ell) - 1}{u(\ell) - 1}. \tag{6.10}$$

Since $u(\ell) = m_\ell$ and the mapping φ is completely determined once we know the relative order of all its upcrossings

$$|\mathcal{H}_n(\bar{m})| = \prod_{\ell=2}^{n-1} \binom{m_{\ell+1} + m_\ell - 1}{m_\ell - 1}. \tag{6.11}$$

What we do next is estimate the probabilities of all possible orderings of visits to $\{\partial D(x, r_{n,j})_{n^4} : j = 0, \ldots, n\}$. Let $\Omega_{x,n}$ denote the set of random walk paths which do not skip x-bands until completion of the first excursion from $\partial D(x, r_{n,1})_{n^4}$ to $\partial D(x, r_{n,0})_{n^4}$. To each random walk path $\omega \in \Omega_{x,n}$ we assign a 'history' $h(\omega)$ as follows. Let $\tau(0)$ be the time of the first visit to $\partial D(x, r_{n,1})_{n^4}$, and define $\tau(1), \tau(2), \ldots$ to be the successive hitting times of different elements of

$$\{\partial D(x, r_{n,0})_{n^4}, \ldots, \partial D(x, r_{n,n})_{n^4}\}$$

until the first downcrossing from $\partial D(x, r_{n,1})_{n^4}$ to $\partial D(x, r_{n,0})_{n^4}$. Setting $\Phi(y) = k$ if $y \in \partial D(x, r_{n,k})_{n^4}$, let $h(\omega)(j) = \Phi(\omega(\tau(j)))$. Let $h_{|_k}$ be the restriction of h to $\{0, \ldots, k\}$. We claim that uniformly for any $\varphi \in \mathcal{H}_n(\bar{m})$ and $z \in \partial D(x, r_{n,1})_{n^4}$

$$\mathbb{P}^z \left\{ h_{|_{|\bar{m}|}} = \varphi \,;\, \Omega_{x,n} \right\} = \left(\frac{1}{2}\right)^{|\bar{m}|-m_n} \left\{1 + O(n^{-4-4\beta})\right\}^{|\bar{m}|}. \tag{6.12}$$

To see this, simply use the strong Markov property successively at the times

$$\tau(0), \tau(1), \ldots, \tau(|\bar{m}| - 1)$$

and then use (6.7)-(6.9).

Writing $m \overset{k}{\sim} \mathcal{N}_k$ if $m = 1$ for $k < k_0$ and $|m - \mathcal{N}_k| \le k$ for $k \ge k_0$ we see that uniformly in $m_n \overset{n}{\sim} \mathcal{N}_n$ we have that $\left\{1 + O(n^{-4-4\beta})\right\}^{|\bar{m}|} = 1 + O(n^{-1-3\beta})$. Combining this with (6.11) and (6.12) we see that uniformly in $z \in \partial D(x, r_{n,1})_{n^4}$

$$\begin{aligned}
&\sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \mathbb{P}^z \left\{ h_{|_{|\bar{m}|}} \in \mathcal{H}_n(\bar{m}) \,;\, \Omega_{x,n} \right\} \qquad (6.13)\\
&= (1 + O(n^{-1-3\beta})) \sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \left(\frac{1}{2}\right)^{|\bar{m}|-m_n} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1} + m_\ell - 1}{m_\ell - 1} \\
&= (1 + O(n^{-1-3\beta})) \frac{1}{4} \sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1} + m_\ell - 1}{m_\ell - 1} \left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell}.
\end{aligned}$$

Here we used the fact that $m_2 = 1$ so that $|\bar{m}| - m_n = 2 + \sum_{\ell=2}^{n-1}(m_{\ell+1} + m_\ell)$.

Lemma 6.1. *For some $C = C(a) < \infty$ and all $k \ge 2$, $|m - \mathcal{N}_{k+1}| \le k + 1$, $|\ell + 1 - \mathcal{N}_k| \le k$,*

$$\frac{C^{-1} k^{-3a-1}}{\sqrt{\log k}} \le \binom{m + \ell}{\ell} \left(\frac{1}{2}\right)^{m+\ell+1} \le \frac{C k^{-3a-1}}{\sqrt{\log k}}. \tag{6.14}$$

Proof of Lemma 6.1: It suffices to consider $k \gg 1$ in which case the binomial coefficient in (6.14) is well approximated by Stirling's formula

$$m! = \sqrt{2\pi} m^m e^{-m} \sqrt{m}(1 + o(1)) \,.$$

With $\mathcal{N}_k = 3ak^2 \log k$ it follows that for some $C_1 < \infty$ and all k large enough, if $|m - \mathcal{N}_{k+1}| \le 2k$, $|\ell - \mathcal{N}_k| \le 2k$ then

$$\left|\frac{m}{\ell} - 1 - \frac{2}{k}\right| \le \frac{C_1}{k \log k} \,. \tag{6.15}$$

Hereafter, we use the notation $f \sim g$ if f/g is bounded and bounded away from zero as $k \to \infty$, uniformly in $\{m : |m - \mathcal{N}_{k+1}| \le 2k\}$ and $\{\ell : |\ell - \mathcal{N}_k| \le 2k\}$. We then have by the preceding observations that

$$\binom{m+\ell}{\ell}\left(\frac{1}{2}\right)^{m+\ell+1} \sim \frac{(m+\ell)^{m+\ell}}{\sqrt{\ell}\,\ell^\ell m^m}\left(\frac{1}{2}\right)^{m+\ell} \sim \frac{\exp(-\ell I(\frac{m}{\ell}))}{\sqrt{k^2\log k}}\,, \tag{6.16}$$

where

$$I(\lambda) = -(1+\lambda)\log(1+\lambda) + \lambda\log\lambda + \lambda\log 2 + \log 2\,.$$

The function $I(\lambda)$ and its first order derivative vanishes at 1, with the second derivative $I_{\lambda\lambda}(1) = 1/2$. Thus, by a Taylor expansion to second order of $I(\lambda)$ at 1, the estimate (6.15) results with

$$|I(\frac{m}{\ell}) - \frac{1}{k^2}| \le \frac{C_2}{k^2\log k} \tag{6.17}$$

for some $C_2 < \infty$, all k large enough and m, ℓ in the range considered here. Since $|\ell - 3ak^2\log k| \le 2k$, combining (6.16) and (6.17) we establish (6.14).
□

Using the last Lemma we have that

$$\begin{aligned}\sum_{\substack{m_2,\ldots,m_n\\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \frac{C^{-1}\ell^{-3a-1}}{\sqrt{\log\ell}} &\le \sum_{\substack{m_2,\ldots,m_n\\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1}\left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell} \\ &\le \sum_{\substack{m_2,\ldots,m_n\\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \frac{C\ell^{-3a-1}}{\sqrt{\log\ell}}.\end{aligned} \tag{6.18}$$

Using the fact that $|\{m_\ell \,|\, m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell\}| = 2\ell+1$, this shows that for some $C_1 < \infty$,

$$\begin{aligned}n\prod_{\ell=2}^{n-1} \frac{C_1^{-1}\ell^{-3a}}{\sqrt{\log\ell}} &\le \sum_{\substack{m_2,\ldots,m_n\\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1}\left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell} \\ &\le n\prod_{\ell=2}^{n-1} \frac{C_1\ell^{-3a}}{\sqrt{\log\ell}}.\end{aligned} \tag{6.19}$$

Since for any $c < \infty$, for some $\zeta_n, \zeta'_n \to 0$

$$nc^n \prod_{\ell=2}^{n-1} \log\ell = n^{n\zeta_n} = (n!)^{\zeta'_n} \tag{6.20}$$

we see that for some $\delta_{1,n}, \delta_{2,n} \to 0$

$$\sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1} \left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell} = (n!)^{-3a-\delta_{1,n}} = r_{n,0}^{-a-\delta_{2,n}}. \tag{6.21}$$

(6.2)-(6.6) and (6.13) show that for some $0 < c, c' < \infty$

$$\begin{aligned} &\frac{c}{\log n} \sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1} \left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell} \\ &\leq Q_n = \inf_{x\in U_n} \mathbb{P}(x \text{ is } n\text{-successful}) \leq \sup_{x\in U_n} \mathbb{P}(x \text{ is } n\text{-successful}) \\ &\leq \frac{c'}{\log n} \sum_{\substack{m_2,\ldots,m_n \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1} \left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell}. \end{aligned} \tag{6.22}$$

Together with (6.21) this gives (5.10) and (5.11).
□

In the remainder of this section we prove two lemmas needed to complete the proof of Lemma 5.2.

Let $\Omega_{x,n,i,m}^{i-1,\ldots,j}$ denote the set of random walk paths which do not skip x-bands on excursions between levels $k = i-1, i, \ldots, j$ until completion of the first m excursions from $D(x, r'_{n,i})$ to $D(x, r_{n,i-1})^c$ and let $N^x_{n,i,m,k}$ denote the number of excursions from $D(x, r_{n,k-1})^c$ to $D(x, r'_{n,k})$ until completion of the first m excursions from $D(x, r'_{n,i})$ to $D(x, r_{n,i-1})^c$.

Lemma 6.2. *We can find $C < \infty$ and $\delta_{3,l} \to 0$ such that for all n and $1 \leq l < n$,*

$$\begin{aligned} &\sum_{\substack{m_l,\ldots,m_n \\ m_k \overset{k}{\sim} \mathcal{N}_k}} \mathbb{P}\left(N^x_{n,k} = m_k,\ k = l+1,\ldots,n\,;\, \Omega_{x,n,l,m_l}^{l-1,\ldots,n} \,|\, N^x_{n,l} = m_l\right) \\ &\qquad \leq C\, Q_n\ (l!)^{3a+\delta_{3,l}}. \end{aligned} \tag{6.23}$$

Proof of Lemma 6.2: The analysis of this section shows that uniformly in $m_k \overset{k}{\sim} \mathcal{N}_k,\ k = l, l+1, \ldots, n$ and $z \in \partial D(x, r_{n,l})_{n^4}$

$$\begin{aligned} &\mathbb{P}^z\left(N^x_{n,l,m_l,k} = m_k,\ k = l+1,\ldots,n\,;\, \Omega_{x,n,l,m_l}^{l-1,\ldots,n}\right) \\ &= (1+O(n^{-2\beta})) \prod_{k=l}^{n-1} \binom{m_{k+1}+m_k-1}{m_k-1} \left(\frac{1}{2}\right)^{m_{k+1}+m_k}. \end{aligned} \tag{6.24}$$

Our analysis also shows that for some $\delta_{3,l} \to 0$

$$\inf_{\substack{m_l \\ m_l \overset{l}{\sim} \mathcal{N}_l}} \left(\sum_{\substack{m_2,\ldots,m_{l-1} \\ m_j \overset{j}{\sim} \mathcal{N}_j}} \prod_{k=2}^{l-1} \binom{m_{k+1}+m_k-1}{m_k-1} \left(\frac{1}{2}\right)^{m_{k+1}+m_k} \right) \geq ((l-1)!)^{-3a-\delta_{3,l}}, \tag{6.25}$$

and since

$$\sum_{\substack{m_2,\ldots,m_n \\ m_j \overset{j}{\sim} \mathcal{N}_j}} \prod_{\ell=2}^{n-1} \binom{m_{\ell+1}+m_\ell-1}{m_\ell-1} \left(\frac{1}{2}\right)^{m_{\ell+1}+m_\ell} \tag{6.26}$$
$$\geq \sum_{\substack{m_l,\ldots,m_n \\ m_j \overset{j}{\sim} \mathcal{N}_j}} \prod_{k=l}^{n-1} \binom{m_{k+1}+m_k-1}{m_k-1} \left(\frac{1}{2}\right)^{m_{k+1}+m_k}$$
$$\inf_{\substack{m_l \\ m_\ell \overset{\ell}{\sim} \mathcal{N}_\ell}} \left(\sum_{\substack{m_2,\ldots,m_{l-1} \\ m_j \overset{j}{\sim} \mathcal{N}_j}} \prod_{k=2}^{l-1} \binom{m_{k+1}+m_k-1}{m_k-1} \left(\frac{1}{2}\right)^{m_{k+1}+m_k} \right),$$

where we used the fact that for $C(i,j), D(i,j)$ non-negative, we have

$$\sum_{i,j,k} C(i,j)D(j,k) = \sum_j \sum_i C(i,j) \sum_k D(j,k) \geq \left(\sum_{i,j} C(i,j) \right) \inf_j \sum_k D(j,k),$$

we see from (6.24) and (6.22) that uniformly in $z \in \partial D(x, r_{n,l})_{n^4}$

$$\sum_{\substack{m_l,\ldots,m_n \\ m_k \overset{k}{\sim} \mathcal{N}_k}} \mathbb{P}^z \left(N^x_{n,l,m_l,k} = m_k,\ k = l+1,\ldots,n\,;\, \Omega^{l-1,\ldots,n}_{x,n,l,m_l} \right) \leq C \log n\, Q_n\ (l!)^{3a+\delta_{3,l}}. \tag{6.27}$$

As in (6.4) we have that uniformly in n and $x \in U_n$

$$\mathbb{P}\left(T_{D(x,r'_{n,l})} < T_{D(0,K_n)^c}\,;\, T_{D(x,r'_{n,l})} = T_{\partial D(x,r_{n,l})_{n^4}} \right) \leq c'/l \log n \tag{6.28}$$

so that by readjusting $\delta_{3,l}$

$$\sum_{\substack{m_l,\ldots,m_n \\ m_k \overset{k}{\sim} \mathcal{N}_k}} \mathbb{P} \left(N^x_{n,l,m_l,k} = m_k,\ k = l+1,\ldots,n\,;\, \Omega^{l-1,\ldots,n}_{x,n,l,m_l} \right) \leq C\, Q_n\ (l!)^{3a+\delta_{3,l}}, \tag{6.29}$$

and (6.23) follows.
□

Lemma 6.3. *For some $C < \infty$ and $\delta_{3,l} \to 0$*

$$\sum_{\substack{m_2,\ldots,m_l \\ m_k \overset{k}{\sim} \mathcal{N}_k}} \mathbb{P} \left(N^x_{n,k} = m_k,\ k = 2,\ldots,l\,;\, \Omega^{1,\ldots,l}_{x,n,1,1} \right) \leq C\, (l!)^{-3a+\delta_{3,l}}. \tag{6.30}$$

Proof of Lemma 6.3: As before, uniformly in $m_k \overset{k}{\sim} \mathcal{N}_k,\ k = 2,3,\ldots,l$ and $z \in \partial D(x, r_{n,1})_{n^4}$

$$\mathbb{P}^z \left(N^x_{n,1,1,k} = m_k,\ k = 2,\ldots,l\,;\, \Omega^{1,\ldots,l}_{x,n,1,1} \right) \tag{6.31}$$
$$= (1 + O(n^{-2\beta})) \prod_{k=2}^{l-1} \binom{m_{k+1}+m_k-1}{m_k-1} \left(\frac{1}{2}\right)^{m_{k+1}+m_k}.$$

Using (6.14) as before, we obtain (6.30).
□

7 Second moment estimates

We begin by defining the σ-algebra $\mathcal{G}^x_{n,l}$ of excursions from $D(x, r_{n,l-1})^c$ to $D(x, r'_{n,l})$. To this end, fix $x \in \mathbb{Z}^2$, let $\overline{\tau}_0 = 0$ and for $i = 1, 2, \dots$ define

$$\begin{aligned} \tau_i &= \inf\{k \geq \overline{\tau}_{i-1} : \; X_k \in D(x, r'_{n,l})\}, \\ \overline{\tau}_i &= \inf\{k \geq \tau_i : \; X_k \in D(x, r_{n,l-1})^c\}. \end{aligned}$$

Then $\mathcal{G}^x_{n,l}$ is the σ-algebra generated by the excursions $\{e^{(j)}, j = 1, \dots\}$, where $e^{(j)} = \{X_k : \overline{\tau}_{j-1} \leq k \leq \tau_j\}$ is the j-th excursion from $D(x, r_{n,l-1})^c$ to $D(x, r'_{n,l})$ (so for $j = 1$ we do begin at $t = 0$).

The following Lemma is proved in the next section. Recall that for any σ-algebra $\mathcal{G}$ and event $B \in \mathcal{G}$, we have $\mathbb{P}(A \cap B \,|\, \mathcal{G}) = \mathbb{P}(A \,|\, \mathcal{G}) 1_{\{B\}}$.

Lemma 7.1 (Decoupling Lemma). *Let*

$$\Gamma^y_{n,l} = \{N^y_{n,i} = m_i; i = l+1, \dots, n\} \cap \Omega^{l-1,\cdots,n}_{x,n,l,m_l}.$$

Then, uniformly over all $l \leq n$, $m_l \overset{l}{\sim} \mathcal{N}_l$, $\{m_i : i = l, \dots, n\}$, $y \in U_n$,

$$\begin{aligned} &\mathbb{P}(\Gamma^y_{n,l}\,,\, N^y_{n,l} = m_l \,|\, \mathcal{G}^y_{n,l}) \\ &= (1 + O(n^{-1/2}))\mathbb{P}(\Gamma^y_{n,l} \,|\, N^y_{n,l} = m_l) 1_{\{N^y_{n,l} = m_l\}} \end{aligned} \tag{7.1}$$

Remark 1. The intuition behind the Decoupling Lemma is that what happens 'deep inside' $D(y, r'_{n,l})$, e.g., $\Gamma^y_{n,l}$, is 'almost' independent of what happens outside $D(y, r'_{n,l})$, i.e., $\mathcal{G}^y_{n,l}$.

Proof of (5.12): Recall that $\mathcal{N}_k = 3ak^2 \log k$ and that we write $m \overset{k}{\sim} \mathcal{N}_k$ if $m = 1$ for $k < k_0$ and $|m - \mathcal{N}_k| \leq k$ for $k \geq k_0$. Relying upon the first moment estimates and Lemma 7.1, we next prove the second moment estimates (5.12). Take $x, y \in U_n$ with $l(x, y) = l - 1$. Thus $2r_{n,l-1} + 2 \leq |x - y| < 2r_{n,l-2} + 2$ for some $2 \leq l \leq n$. Since $r_{n,l-3} - r_{n,l-2} \gg 2r_{n,l-1}$, it is easy to see that $\partial D(y, r_{n,l-1})_{n^4} \cap \partial D(x, r_{n,k})_{n^4} = \emptyset$ for all $k \neq l - 2$. Replacing hereafter l by $l \wedge (n-3)$, it follows that for $k \neq l-1, l-2$, the events $\{N^x_{n,k} \overset{k}{\sim} \mathcal{N}_k\}$ are measurable with respect to the σ-algebra $\mathcal{G}^y_{n,l}$.

We write

$$\Gamma^y_{n,l}(m_l, \dots, m_n) = \{N^y_{n,i} = m_i; i = l+1, \dots, n\} \cap \Omega^{l-1,\cdots,n}_{x,n,l,m_l}$$

to emphasize the dependence on $m_l, \dots, m_n$. With $J_l := \{l+1, \dots, n\}$ set

$$\widetilde{\Gamma}^y_n(J_l, m_l) = \bigcup_{m_k \overset{k}{\sim} \mathcal{N}_k\,;\, k \in J_l} \Gamma^y_{n,l}(m_l, \dots, m_n)$$

Similarly, with $M_{l-3} := \{2, \dots, l-3\}$ set

$$\widetilde{\Gamma}^x_n(M_{l-3}) = \{N^x_{n,k} \overset{k}{\sim} \mathcal{N}_k,\ k \in M_{l-3}\} \cap \Omega^{1,\dots,l-3}_{x,n,1,1},$$

and with $I_l := \{2, \dots, l-3, l, \dots, n\}$ set

$$\bar{\Gamma}^x_n(I_l) = \bigcup_{m'_l \overset{l}{\sim} \mathcal{N}_l} \widetilde{\Gamma}^x_n(J_l, m'_l) \cap \{N^x_{n,l} = m'_l\} \cap \widetilde{\Gamma}^x_n(M_{l-3}).$$

Using the previous paragraph we can check that $\bar{\Gamma}^x_n(I_l) \in \mathcal{G}^y_{n,l}$. Note that

$$\begin{aligned} &\{x, y \text{ are } n\text{-successful}\} \\ &\subseteq \bigcup_{m_l \overset{l}{\sim} \mathcal{N}_l} \left\{ \bar{\Gamma}^x_n(I_l) \bigcap \widetilde{\Gamma}^y_n(J_l, m_l) \bigcap \{N^y_{n,l} = m_l\} \right\}. \end{aligned} \tag{7.2}$$

Applying (7.1), we have that for some universal constant $C_3 < \infty$,

$$\begin{aligned} &\mathbb{P}\left(x \text{ and } y \text{ are } n\text{-successful}\right) \\ &\leq \sum_{m_l \overset{l}{\sim} n_l} \mathbb{E}\left[\mathbb{P}(\widetilde{\Gamma}^y_n(J_l, m_l)\,, N^y_{n,l} = m_l \,\middle|\, \mathcal{G}^y_{n,l})\,; \bar{\Gamma}^x_n(I_l) \right] \\ &\leq C_3 \mathbb{P}(\bar{\Gamma}^x_n(I_l), N^y_{n,l} = m_l) \sum_{m_l \overset{l}{\sim} n_l} \mathbb{P}(\widetilde{\Gamma}^y_n(J_l, m_l) \,|\, N^y_{n,l} = m_l) \\ &\leq C_3 \mathbb{P}(\bar{\Gamma}^x_n(I_l)) \sum_{m_l \overset{l}{\sim} n_l} \mathbb{P}(\widetilde{\Gamma}^y_n(J_l, m_l) \,|\, N^y_{n,l} = m_l). \end{aligned} \tag{7.3}$$

Using (6.23), for some universal constant $C_5 < \infty$,

$$\sum_{m_l \overset{l}{\sim} n_l} \mathbb{P}(\widetilde{\Gamma}^y_n(J_l, m_l) \,|\, N^y_{n,l} = m_l) \leq C_5\, Q_n\ (l!)^{3a + \delta_{3,l}}. \tag{7.4}$$

Noting that $\widetilde{\Gamma}^x_n(M_{l-3}) \in \mathcal{G}^x_{n,l}$, (7.1) then shows that

$$\begin{aligned} &\mathbb{P}\left(\bar{\Gamma}^x_n(I_l)\right) \\ &\leq \sum_{m_l \overset{l}{\sim} \mathcal{N}_l} \mathbb{E}\left[\mathbb{P}(\widetilde{\Gamma}^x_n(J_l, m_l),\ N^x_{n,l} = m_l \,\middle|\, \mathcal{G}^x_{n,l})\,; \widetilde{\Gamma}^x_n(M_{l-3}) \right] \\ &\leq C_6 \mathbb{P}\left(\widetilde{\Gamma}^x_n(M_{l-3}), N^x_{n,l} = m_l \right) \sum_{m_l \overset{l}{\sim} \mathcal{N}_l} \mathbb{P}(\widetilde{\Gamma}^x_n(J_l, m_l) \,|\, N^x_{n,l} = m_l) \\ &\leq C_6 \mathbb{P}\left(\widetilde{\Gamma}^x_n(M_{l-3}) \right) \sum_{m_l \overset{l}{\sim} \mathcal{N}_l} \mathbb{P}(\widetilde{\Gamma}^x_n(J_l, m_l) \,|\, N^x_{n,l} = m_l). \end{aligned} \tag{7.5}$$

Using (6.30) and (7.4) we get that, for some $\delta_{4,l} \to 0$

$$\mathbb{P}\left(\bar{\Gamma}^x_n(I_l)\right) \leq C_7 l^{15}\, (l!)^{\delta_{4,l}}\, Q_n. \tag{7.6}$$

Putting (7.3), (7.4) and (7.6) together and adjusting C and δ'_{l-1} proves (5.12) for $l(x, y) = l - 1$.

□

8 Approximate decoupling

The goal of this section is to prove the Decoupling Lemma, Lemma 7.1. Since what happens 'deep inside' $D(y, r'_{n,l})$, e.g., $\Gamma^y_{n,l}$, depends on what happens outside $D(y, r'_{n,l})$, i.e., on $\mathcal{G}^y_{n,l}$, only through the initial and end points of the excursions from $D(x, r'_{n,l})$ to $D(x, r_{n,l-1})^c$, we begin by studying the dependence on these initial and end points.

Consider a random path beginning at $z \in \partial D(0, r_{n,l})_{n^4}$. We will show that for l large, a certain σ-algebra of excursions of the path from $D(0, r'_{n,l+1})$ to $D(0, r_{n,l})^c$ prior to $T_{D(0,r_{n,l-1})^c}$, is almost independent of the choice of initial point $z \in \partial D(0, r_{n,l})_{n^4}$ and final point $w \in \partial D(0, r_{n,l-1})_{n^4}$. Let $\tau_0 = 0$ and for $i = 0, 1, \dots$ define

$$
\begin{aligned}
\tau_{2i+1} &= \inf\{k \geq \tau_{2i} :\ X_k \in D(0, r'_{n,l+1}) \cup D(0, r_{n,l-1})^c\} \\
\tau_{2i+2} &= \inf\{k \geq \tau_{2i+1} :\ X_k \in D(0, r_{n,l})^c\}\,.
\end{aligned}
$$

Abbreviating $\bar\tau = T_{D(0,r_{n,l-1})^c}$ note that $\bar\tau = \tau_{2I+1}$ for some (unique) non-negative integer I. As usual, $\mathcal{F}_j$ will denote the $\sigma-$algebra generated by $\{X_l,\ l = 0, 1, \dots, j\}$, and for any stopping time τ, $\mathcal{F}_\tau$ will denote the collection of events A such that $A \cap \{\tau = j\} \in \mathcal{F}_j$ for all j.

Let $\mathcal{H}_{n,l}$ denote the σ-algebra generated by the excursions of the path from $D(0, r'_{n,l+1})$ to $D(0, r_{n,l})^c$, prior to $T_{D(0,r_{n,l-1})^c}$. Then $\mathcal{H}_{n,l}$ is the σ-algebra generated by the excursions $\{v^{(j)}, j = 1, \dots, I\}$, where $v^{(j)} = \{X_k : \tau_{2j-1} \leq k \leq \tau_{2j}\}$ is the j-th excursion from $D(0, r'_{n,l+1})$ to $D(0, r_{n,l})^c$.

Lemma 8.1. *Uniformly in l, n, $z, z' \in \partial D(0, r_{n,l})_{n^4}$, $w \in \partial D(0, r_{n,l-1})_{n^4}$, and $B_n \in \mathcal{H}_{n,l}$,*

$$
\begin{aligned}
&\mathbb{P}^z(B_n \cap \Omega^{l-1,l,l+1}_{0,n,l,1} \,|\, X_{T_{D(0,r_{n,l-1})^c}} = w) \\
&= (1 + O(n^{-3}))\mathbb{P}^z(B_n \cap \Omega^{l-1,l,l+1}_{0,n,l,1}),
\end{aligned}
\tag{8.1}
$$

and

$$
\mathbb{P}^z(B_n \cap \Omega^{l-1,l,l+1}_{0,n,l,1}) = (1 + O(n^{-3}))\mathbb{P}^{z'}(B_n \cap \Omega^{l-1,l,l+1}_{0,n,l,1})\,. \tag{8.2}
$$

Proof of Lemma 8.1: Fixing $z \in \partial D(0, r_{n,l})_{n^4}$ it suffices to consider $B_n \in \mathcal{H}_{n,l}$ for which $\mathbb{P}^z(B_n) > 0$. Fix such a set B_n and a point $w \in \partial D(0, r_{n,l-1})_{n^4}$. Using the notation introduced right before the statement of our Lemma, for any $i \geq 1$, we can write

$$
\begin{aligned}
&\{B_n \cap \Omega^{l-1,l,l+1}_{0,n,l,1}, I = i\} \\
&= \{B_{n,i} \cap A_i, \tau_{2i} < \bar\tau\} \cap (\{I = 0\,, X_{\bar\tau} \in \partial D(0, r_{n,l-1})_{n^4}\} \circ \theta_{\tau_{2i}})
\end{aligned}
$$

for some $B_{n,i} \in \mathcal{F}_{\tau_{2i}}$, where

$$
A_i = \left\{X_{\tau_{2j-1}} \in \partial D(0, r_{n,l+1})_{n^4}\,, X_{\tau_{2j}} \in \partial D(0, r_{n,l})_{n^4}\,, \forall\, j \leq i\right\} \in \mathcal{F}_{\tau_{2i}}
$$

so by the strong Markov property at τ_{2i},

$$
\begin{aligned}
&\mathbb{E}^z[X_{\bar\tau} = w; B_n \cap \Omega^{l-1,l,l+1}_{0,n,l-1,1}, I = i] \\
&= \mathbb{E}^z\left[\mathbb{E}^{X_{\tau_{2i}}}(X_{\bar\tau} = w,\ I = 0); B_{n,i} \cap A_i, \tau_{2i} < \bar\tau\right]\,,
\end{aligned}
$$

and

$$\mathbb{P}^z\left(B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}, I=i\right)$$
$$= \mathbb{E}^z\left[\mathbb{E}^{X_{\tau_{2i}}}(I=0\,, X_{\bar{\tau}} \in \partial D(0, r_{n,l-1})_{n^4}); B_{n,i} \cap A_i, \tau_{2i} < \bar{\tau}\right] \,.$$

Consequently, for all $i \geq 1$,

$$\begin{aligned} &\mathbb{E}^z[X_{\bar{\tau}} = w; B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}, I=i] \\ &\geq \mathbb{P}^z\left(B_n \cap \Omega_{0,l,1}^{l-1,l,l+1}, I=i\right) \\ &\qquad\qquad \times \inf_{x \in \partial D(0,r_{n,l})_{n^4}} \frac{\mathbb{E}^x\left(X_{\bar{\tau}} = w;\ I=0\right)}{\mathbb{E}^x\left(I=0\,, X_{\bar{\tau}} \in \partial D(0, r_{n,l-1})_{n^4}\right)} \,. \end{aligned} \tag{8.3}$$

Necessarily $\mathbb{P}^z(B_n | I=0) \in \{0,1\}$ and is independent of z for any $B_n \in \mathcal{H}_{n,l}$, implying that (8.3) applies for $i=0$ as well. By (3.2), (3.1), (2.20) and (2.49) there exists $c < \infty$ such that for any $z, x \in \partial D(0, r_{n,l})_{n^4}$ and $w \in \partial D(0, r_{n,l-1})_{n^4}$,

$$\frac{\mathbb{E}^x\left(X_{\bar{\tau}} = w;\ I=0\right)}{\mathbb{E}^x\left(I=0\,, X_{\bar{\tau}} \in \partial D(0, r_{n,l-1})_{n^4}\right)} \geq (1 - cn^{-3}) H_{D(0,r_{n,l-1})^c}(z,w)\,.$$

Hence, summing (8.3) over $I = 0, 1, \ldots$, we get that

$$\begin{aligned} &\mathbb{E}^z\left[X_{\bar{\tau}} = w, B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}\right] \\ &\qquad\qquad \geq (1 - cn^{-3}) \mathbb{P}^z(B_n \cap \Omega_{0,n,l,1}^{l,l+1}) H_{D(0,r_{n,l-1})^c}(z,w)\,. \end{aligned} \tag{8.4}$$

A similar argument shows that

$$\begin{aligned} &\mathbb{E}^z\left[X_{\bar{\tau}} = w, B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}\right] \\ &\qquad\qquad \leq (1 + cn^{-3}) \mathbb{P}^z\left(B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}\right) H_{D(0,r_{n,l-1})^c}(z,w)\,, \end{aligned} \tag{8.5}$$

and we thus obtain (8.1).

By the strong Markov property at τ_1, for any $z \in \partial D(0, r_{n,l})_{n^4}$,

$$\begin{aligned} &\mathbb{P}^z(B_n \cap \Omega_{0,n,l,1}^{l,l+1}) = \mathbb{P}^z(B_n \cap \Omega_{0,l,1}^{l-1,l,l+1}, I=0) \\ &\quad + \sum_{x \in \partial D(0,r_{n,l+1})_{n^4}} H_{D(0,r'_{l+1}) \cup D(0,r_{n,l-1})^c}(z,x) \mathbb{P}^x\left(B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}\right) \end{aligned}$$

The term involving $\{B_n \cap \Omega_{0,n,l,1}^{l-1,l,l+1}, I=0\}$ is dealt with by (2.21) and (8.2) follows by (3.32). $\square$

Building upon Lemma 8.1 we quantify the independence between the σ-algebra $\mathcal{G}_l^x$ of excursions from $D(x, r_{n,l-1})^c$ to $D(x, r'_{n,l})$ and the σ-algebra $\mathcal{H}_{n,l}^x(m)$ of excursions from $D(x, r'_{n,l+1})$ to $D(x, r_{n,l})^c$ during the first m excursions from $D(x, r'_{n,l})$ to $D(x, r_{n,l-1})^c$. To this end, fix $x \in \mathbb{Z}^2$, let $\bar{\tau}_0 = 0$ and for $i = 1, 2, \ldots$ define

$$\begin{aligned} \tau_i &= \inf\{k \geq \bar{\tau}_{i-1} :\ X_k \in D(x, r'_{n,l})\}\,, \\ \bar{\tau}_i &= \inf\{k \geq \tau_i :\ X_k \in D(x, r_{n,l-1})^c\}. \end{aligned}$$

Then $\mathcal{G}_l^x$ is the σ-algebra generated by the excursions $\{e^{(j)}, j = 1, \dots\}$, where $e^{(j)} = \{X_k : \overline{\tau}_{j-1} \leq k \leq \tau_j\}$ is the j-th excursion from $D(x, r_{l-1})^c$ to $D(x, r'_{n,l})$ (so for $j = 1$ we do begin at $t = 0$).

We denote by $\mathcal{H}_{n,l}^x(m)$ the σ-algebra generated by all excursions from $D(x, r'_{n,l+1})$ to $D(x, r_{n,l})^c$ from time τ_1 until time $\overline{\tau}_m$. In more detail, for each $j = 1, 2, \dots, m$ let $\overline{\zeta}_{j,0} = \tau_j$ and for $i = 1, \dots$ define

$$
\begin{aligned}
\zeta_{j,i} &= \inf\{k \geq \overline{\zeta}_{j,i-1} : \; X_k \in D(x, r'_{n,l+1})\}, \\
\overline{\zeta}_{j,i} &= \inf\{k \geq \zeta_{j,i} : \; X_k \in D(x, r_{n,l})^c\}.
\end{aligned}
$$

Let $v_{j,i} = \{X_k : \zeta_{j,i} \leq k \leq \overline{\zeta}_{j,i}\}$ and $Z^j = \sup\{i \geq 0 : \overline{\zeta}_{j,i} < \overline{\tau}_j\}$. Then, $\mathcal{H}_{n,l}^x(m)$ is the σ-algebra generated by the intersection of the σ-algebras $\mathcal{H}_{n,l,j}^x = \sigma(v_{j,i}, i = 1, \dots, Z^j)$ of the excursions between times τ_j and $\overline{\tau}_j$, for $j = 1, \dots, m$.

Lemma 8.2. *There exists $C < \infty$ such that uniformly over all $m \leq (n \log n)^2$, l, $x \in \mathbb{Z}^2$ and $y_0, y_1 \in \mathbb{Z}^2 \setminus D(y, r'_{n,l})$, and $H \in \mathcal{H}_{n,l}^x(m)$,*

$$
\begin{aligned}
(1 - Cmn^{-3})\mathbb{P}^{y_1}(H \cap \Omega_{x,n,l,m}^{l-1,l,l+1}) &\leq \mathbb{P}^{y_0}(H \cap \Omega_{x,n,l,m}^{l-1,l,l+1} \,|\, \mathcal{G}_l^x) \\
&\leq (1 + Cmn^{-3})\mathbb{P}^{y_1}(H \cap \Omega_{x,n,l,m}^{l-1,l,l+1}).
\end{aligned}
\tag{8.6}
$$

Proof of Lemma 8.2: Applying the Monotone Class Theorem to the algebra of their finite disjoint unions, it suffices to prove (8.6) for the generators of the σ-algebra $\mathcal{H}_{n,l}^x(m)$ of the form $H = H_1 \cap H_2 \cap \cdots \cap H_m$, with $H_j \in \mathcal{H}_{n,l,j}^x$ for $j = 1, \dots, m$. Conditioned upon $\mathcal{G}_l^x$ the events H_j are independent. Further, each H_j then has the conditional law of an event B_j in the σ-algebra $\mathcal{H}_{n,l}$ of Lemma 8.1, for some random $z_j = X_{\tau_j} - x \in \partial D(0, r_{n,l})_{n^4}$ and $w_j = X_{\overline{\tau}_j} - x \in \partial D(0, r_{n,l-1})_{n^4}$, both measurable on $\mathcal{G}_l^x$. By our conditions, the uniform estimates (8.1) and (8.2) yield that for any fixed $z' \in \partial D(0, r_{n,l})_{n^4}$,

$$
\begin{aligned}
&\mathbb{P}^{y_0}(H \cap \Omega_{x,n,l,m}^{l-1,l,l+1} \,|\, \mathcal{G}_l^x) \\
&= \mathbb{P}^{y_0}(\cap_{j=1}^m (H_j \cap \Omega_{x,n,l,1}^{l-1,l,l+1}) \,|\, \mathcal{G}_l^x) \\
&= \prod_{j=1}^m \mathbb{P}^{z_j}(B_j \cap \Omega_{x,n,l,1}^{l-1,l,l+1} \,|\, X_{T_{D(0,r_l)^c}} = w_j) \\
&= \prod_{j=1}^m (1 + O(n^{-3}))\mathbb{P}^{z_j}(B_j \cap \Omega_{0,n,l,1}^{l-1,l,l+1}) \\
&= (1 + O(n^{-3}))^m \prod_{j=1}^m \mathbb{P}^{z'}(B_j \cap \Omega_{0,n,l,1}^{l-1,l,l+1}).
\end{aligned}
\tag{8.7}
$$

Since $m \leq (n \log n)^2$ and the right-hand side of (8.7) neither depends on $y_0 \in \mathbb{Z}^2$ nor on the extra information in $\mathcal{G}_l^x$, we get (8.6).
□

Corollary 8.3. *Let* $\Gamma^y_{n,l} = \{N^y_{n,i} = m_i; i = l+1,\dots,n\} \cap \Omega^{l-1,\cdots,n}_{y,n,l,m_l}$. *Then, uniformly over all* $n \geq l$, $m_l \overset{l}{\sim} \mathcal{N}_l$, $\{m_i : i = l, l+2, \dots, n\}$, $y \in U_n$ *and* $x_0, x_1 \in \mathbb{Z}^2 \setminus D(y, r'_{n,l})$,

$$\begin{aligned}&\mathbb{P}^{x_0}(\Gamma^y_{n,l}\,,\, N^y_{n,l} = m_l \,|\, \mathcal{G}^y_l) \\ &= (1 + O(n^{-1}\log n))\mathbb{P}^{x_1}(\Gamma^y_{n,l} \,|\, N^y_{n,l} = m_l)\mathbf{1}_{\{N^y_{n,l}=m_l\}}\end{aligned} \tag{8.8}$$

Proof of Corollary 8.3: For $j = 1, 2, \dots$ and $i = l+1, \dots, n$, let Z^j_i denote the number of excursions from $D(y, r_{n,i-1})^c$ to $D(y, r'_{n,i})$ by the random walk during the time interval $[\tau_j, \overline{\tau}_j]$. Clearly, the event

$$H = \{\sum_{j=1}^{m_l} Z^j_i = m_i : i = l+1, \dots, n\} \cap \Omega^{l+1,\dots,n}_{y,n,l+1,m_{l+1}}$$

belongs to the σ-algebra $\mathcal{H}^y_{n,l}(m_l)$ of Lemma 8.2. It is easy to verify that starting at any $x_0 \notin D(y, r'_{n,l})$, when the event $\{N^y_{n,l} = m_l\} \in \mathcal{G}^y_l$ occurs, it implies that $N^y_{n,i} = \sum_{j=1}^{m_l} Z^j_i$ for $i = l+1, \dots, n$. Thus,

$$\mathbb{P}^{x_0}(\Gamma^y_{n,l} \,|\mathcal{G}^y_l)\mathbf{1}_{\{N^y_{n,l}=m_l\}} = \mathbb{P}^{x_0}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l} \,|\mathcal{G}^y_l)\mathbf{1}_{\{N^y_{n,l}=m_l\}}\,. \tag{8.9}$$

With $m_l/(n^2 \log n)$ bounded above, by (8.6) we have, uniformly in $y \in \mathbb{Z}^2$ and $x_0, x_1 \in \mathbb{Z}^2 \setminus D(y, r'_{n,l})$,

$$\mathbb{P}^{x_0}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l} \,|\mathcal{G}^y_l) = (1 + O(n^{-1}\log n))\mathbb{P}^{x_1}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l})\,. \tag{8.10}$$

Hence,

$$\begin{aligned}&\mathbb{P}^{x_0}(\Gamma^y_{n,l} \,|\mathcal{G}^y_l)\mathbf{1}_{\{N^y_{n,l}=m_l\}} \\ &= (1 + O(n^{-1}\log n))\mathbb{P}^{x_1}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l})\mathbf{1}_{\{N^y_{n,l}=m_l\}}\,.\end{aligned} \tag{8.11}$$

Setting $x_0 = x_1$ and taking expectations with respect to $\mathbb{P}^{x_0}$, one has

$$\mathbb{P}^{x_1}(\Gamma^y_{n,l} \,|N^y_{n,l} = m_l) = (1 + O(n^{-1}\log n))\mathbb{P}^{x_1}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l}). \tag{8.12}$$

Hence,

$$\begin{aligned}&\mathbb{P}^{x_1}(\Gamma^y_{n,l} \,|N^y_{n,l} = m_l)\mathbf{1}_{\{N^y_{n,l}=m_l\}} \\ &= (1 + O(n^{-1}\log n))\mathbb{P}^{x_1}(H \cap \Omega^{l-1,l,l+1}_{y,n,l,m_l})\mathbf{1}_{\{N^y_{n,l}=m_l\}} \\ &= (1 + O(n^{-1}\log n))\mathbb{P}^{x_0}(\Gamma^y_{n,l} \,|\mathcal{G}^y_{n,l})\mathbf{1}_{\{N^y_{n,l}=m_l\}}\end{aligned} \tag{8.13}$$

where we used (8.11) for the last equality. Using that $\{N^y_{n,l} = m_l\} \in \mathcal{G}^y_{n,l}$, this is (8.8).
□

9 Appendix

Let $q_0(x) = \mathbf{1}_{\{0\}}(x)$ and for $n \geq 1$ let

$$q_n(x) = \frac{1}{2\pi n} e^{-|x|^2/2n}.$$

Proposition 9.1. *Suppose X_n is a strongly aperiodic symmetric random walk in $\mathbb{Z}^2$ with the covariance matrix of X_1 equal to the identity and with $3+2\beta$ moments. Then there exists c_1 such that*

$$\sup_{x\in\mathbb{Z}^2} |p_n(x) - q_n(x)| \leq cn^{-\frac{3}{2}-\beta}, \qquad n \geq 1.$$

Proof. Let ϕ be the characteristic function for X_1. Since X_1 is symmetric, the third moments are zero, that is, if $X_1 = (X_1^{(1)}, X_1^{(2)})$, $i_1, i_2 \geq 0$, $i_1 + i_2 = 3$, then $\mathbb{E}[(X_1^{(1)})^{i_1}(X_1^{(2)})^{i_2}] = 0$. So by a Taylor expansion,

$$\phi(\alpha/\sqrt{n}) = 1 - \frac{|\alpha|^2}{n} + E_1(\alpha, n),$$

where

$$|E_1(\alpha, n)| \leq c_2(|\alpha|/\sqrt{n})^{3+2\beta},$$

provided $\alpha \in [-\pi, \pi]^2$. Similarly

$$e^{-|\alpha|^2/n} = 1 - \frac{|\alpha|^2}{n} + E_2(\alpha, n),$$

where the error term $E_2(\alpha, n)$ has the same bound. We now follow Proposition 3.1 of (2), using the above estimate for $E_i(\alpha, n)$, $i = 1, 2$, in place of the one in that paper.
□

Proposition 9.2. *Let X_n be as above and*

$$a(x) = \sum_{n=0}^{\infty} [p_n(0) - p_n(x)].$$

Then for $x \neq 0$, $a(x) \geq 0$, and

$$a(x) = \frac{2}{\pi} \log |x| + k + o(1/|x|), \tag{9.1}$$

where k is a constant depending on p_1 but not x.

Proof. By (13), p. 76,

$$p_n(x) = c \int_C e^{-ix\cdot u} \varphi(u)^n \, du,$$

where φ is the characteristic function of X_1 and C is the cube of side length 2π centered at the origin. Then

$$a(x) = \int_C \frac{1 - e^{-iu\cdot x}}{1 - \varphi(u)} \, du. \tag{9.2}$$

Since X_1 is symmetric, φ is real. Since $a(x)$ is real, we take the real parts of both sides of (9.2) to obtain

$$a(x) = \int_C \frac{1 - \cos(u \cdot x)}{1 - \varphi(u)} \, du \geq 0.$$

To prove (9.1) we write

$$\begin{aligned} a(x) &= \sum_n [p_n(0) - q_n(0)] + \sum_n [q_n(0) - q_n(x)] \\ &\quad + \sum_n [q_n(x) - p_n(x)] \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{9.3}$$

Since $n^{-\frac{3}{2}-\beta}$ is summable, I_1 is a constant not depending on x. I_2 has the form given in the right hand side of (9.1); see the proof of Theorem 1.6.2 in (8). So it remains to show $I_3 = o(1/|x|)$.
We write

$$\begin{aligned} I_3 &= \sum_{n \leq N} p_n(x) + \sum_{n \leq N} q_n(x) + \sum_{n > N} [p_n(x) - q_n(x)] \\ &:= I_4 + I_5 + I_6, \end{aligned} \tag{9.4}$$

where we choose N to be the largest integer less than $|x|^2 / \log^2 |x|$.
Note

$$|I_4| \leq \mathbb{P}(\max_{n \leq N} |X_n| \geq |x|).$$

We estimate this using truncation and Bernstein's inequality. Let $\xi_i = X_i - X_{i-1}$, define $\xi_i' = \xi_i 1_{(|\xi_i| \leq N^{\frac{1}{2}-\frac{\beta}{4}})}$, and $X_n' = \sum_{i \leq n} \xi_i'$. We have

$$\begin{aligned} \mathbb{P}(X_n \neq X_n' \text{ for some } n \leq N) &\leq \mathbb{P}(\xi_n \neq \xi_n' \text{ for some } n \leq N) \\ &\leq N \max_{n \leq N} \mathbb{P}(\xi_n \neq \xi_n') \\ &\leq N c_1 N^{(\frac{1}{2}-\frac{\beta}{4})(3+2\beta)} \leq c_1 N^{-\frac{1}{2}-\frac{\beta}{8}}. \end{aligned}$$

With our choice of N we see that

$$\mathbb{P}(X_n \neq X_n' \text{ for some } n \leq N) = o(1/|x|). \tag{9.5}$$

By Bernstein's inequality ((3))

$$\begin{aligned} \mathbb{P}(\max_{n \leq N} |X_n'| \geq |x|) &\leq 2 \exp\Big(- \frac{|x|^2}{2c_2 N + \frac{2}{3}|x| N^{\frac{1}{2}-\frac{\beta}{4}}} \Big) \\ &\leq 2 e^{-c_3 \log^2 |x|} = o(1/|x|). \end{aligned} \tag{9.6}$$

Combining (9.5) and (9.6) yields the required bound on $|I_4|$.
We can show $I_5 = o(1/|x|)$ by straightforward estimates. Finally, by Proposition 9.1,

$$|I_6| \leq \sum_{n > N} c_4 n^{-\frac{3}{2}-\beta} = O(N^{-\frac{1}{2}-\beta}) = o(1/|x|).$$

Summing the estimates for I_4, I_5, and I_6 shows $I_3 = o(1/|x|)$ and completes the proof.
□

The following result holds for all mean zero finite variance random walks in any dimension d. To keep the notation uniform we use $D(0, n)$ to denote the ball (if $d \geq 3$) or disc (if $d = 2$) of radius n centered at the origin. When $d = 1$ we let $D(0, n) = (-n, n)$.

Lemma 9.3. *For some $c < \infty$*

$$\mathbb{E}^x(T_{D(0,n)^c}) \leq cn^2, \qquad x \in D(0, n), \qquad n \geq 1. \tag{9.7}$$

Proof. Let $T = \min\{j : |X_j - X_0| > 2n\}$. By the invariance principle

$$\begin{aligned} \mathbb{P}^x(T > c_1 n^2) &= \mathbb{P}^x(\sup_{j \leq c_1 n^2} |X_j - X_0| \leq 2n) \\ &= \mathbb{P}^0(\sup_{j \leq c_1 n^2} |X_j - X_0| \leq 2n) \\ &\leq \rho < 1 \end{aligned} \tag{9.8}$$

for all x if we take $c_1 = 1$ and n is large enough. Taking c_1 larger if necessary, we get the inequality for all n. Then letting θ_j be the usual shift operators and using the strong Markov property

$$\begin{aligned} \mathbb{P}^x(T > c_1(k+1)n^2) &\leq \mathbb{P}^x(T \circ \theta_{c_1 kn^2} > c_1 n^2, T > c_1 kn^2) \\ &= \mathbb{E}^x\left[\mathbb{P}^{X_{c_1 kn^2}}(T > c_1 n^2); T > c_1 kn^2\right] \\ &\leq \rho \mathbb{P}^x(T > c_1 kn^2). \end{aligned} \tag{9.9}$$

Using induction

$$\mathbb{P}^x(T > c_1 kn^2) \leq \rho^k, \tag{9.10}$$

and our result follows easily.
□

Equation (6) of (9) does the simple random walk case of the following.

Lemma 9.4. *We have*

$$\lim_{n\to\infty} \frac{\log T_{D(0,n)^c}}{\log n} = 2, \qquad \mathbb{P}^0\text{-a.s.}$$

Proof of Lemma 9.4: Let $\varepsilon > 0$. By Chebyshev and Lemma 9.3,

$$\mathbb{P}^0(T_{D(0,n)^c} > n^{2+\varepsilon}) \leq \frac{\mathbb{E}^0 T_{D(0,n)^c}}{n^{2+\varepsilon}} \leq cn^{-\varepsilon}.$$

So by Borel-Cantelli there exists $M_0(\omega)$ such that if $m \geq M_0$, then $T_{D(0,2^m)^c} \leq (2^m)^{2+\varepsilon}$. If $m \geq M_0$ and $2^m \leq n \leq 2^{m+1}$, then

$$T_{D(0,n)^c} \leq T_{D(0,2^{m+1})^c} \leq (2^{m+1})^{2+\varepsilon} \leq 2^{2+\varepsilon} n^{2+\varepsilon},$$

which, since ε is arbitrary, proves the upper bound.

By Kolmogorov's inequality applied to each component of the random walk,

$$\mathbb{P}^0(T_{D(0,n)^c} < n^{2-\varepsilon}) = \mathbb{P}^0(\sup_{k\leq n^{2-\varepsilon}} |X_k| > n) \leq c\frac{\mathbb{E}^0|X_{n^{2-\varepsilon}}|^2}{n^2} \leq cn^{-\varepsilon}.$$

So by Borel-Cantelli there exists $M_1(\omega)$ such that if $m \geq M_1$, then $T_{D(0,2^m)^c} \geq (2^m)^{2-\varepsilon}$. If $m \geq M_1$ and $2^m \leq n \leq 2^{m+1}$, then

$$T_{D(0,n)^c} \geq T_{D(0,2^m)^c} \geq (2^m)^{2-\varepsilon} \geq 2^{\varepsilon-2} n^{2-\varepsilon},$$

which proves the lower bound.
□

Acknowledgments We would like to thank Greg Lawler for making available to us some unpublished notes; these were very useful in preparing Sections 2 and 9.

References

[1] R. Bass and M. Kassmann, *Harnack inequalities for non-local operators of variable order*, Trans. Amer. Math. Soc. **357** (2005) 837–850. MR2095633

[2] R. Bass and D. Khoshnevisan, *Local times on curves and uniform invariance principles*, Probab. Th. rel. Fields **92** (1992), 465–492. MR1169015

[3] G. Bennett, Probability inequalities for the sums of independent random variables, J. Amer. Stat. Assoc. 57 (1962) 33–45.

[4] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni, *Thick points for planar Brownian motion and the Erdős-Taylor conjecture on random walk*, Acta Math. **186** (2001), 239–270. MR1846031

[5] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni, *Cover time for Brownian motion and random walks in two dimensions*, Ann. Math. **160** (2004), 433–467. MR2123929

[6] A. Dembo, Y. Peres, J. Rosen and O. Zeitouni, *Late points for random walks in two dimensions*, Ann. Probab. **34** (2006) 219–263. MR2206347

[7] J.-P. Kahane, *Some random series of functions: Second Edition*, Cambridge University Press, (1985). MR0833073

[8] G. Lawler, *Intersections of random walks.* Birkhäuser, Boston (1991). MR1117680

[9] G. Lawler, *On the covering time of a disc by a random walk in two dimensions*, In Seminar in Stochastic Processes 1992, 189-208. Birkhauser, (1993). MR1278083

[10] G. Lawler and T. Polaski, *Harnack inequalities and differences for random walks with infinite range*, J. Theor. Probab. **6** (1993), 781–802. MR1245395

[11] J. Rosen, *A random walk proof of the Erdős-Taylor conjecture*, Periodica Mathematica Hungarica, 50, (2005), 223-245. MR2162811

[12] P. Révész, *Random Walk in Random and Non-Random Environments*, World Scientific, Singapore (1990).

[13] F. Spitzer, *Principles of Random Walk*, Van Nostrand, Princeton, New Jersey (1964). MR0171290