DOI: 10.1214/EJP.v10-291

Electronic Journal of Probability

Vol. 10 (2005), Paper no. 40, pages 1359-1380.

Journal URL
http://www.math.washington.edu/~ejpecp/

The exact asymptotic of the time to collision

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Abstract

In this note we consider the time of the collision τ for n independent copies of Markov processes $X_t^1, \dots, X_t^n$, each starting from x_i, where $x_1 < \dots < x_n$. We show that for the continuous time random walk $\mathbb{P}_{\boldsymbol{x}}(\tau > t) = t^{-n(n-1)/4}(Ch(\boldsymbol{x}) + o(1))$, where C is known and $h(\boldsymbol{x})$ is the Vandermonde determinant. From the proof one can see that the result also holds for X_t being the Brownian motion or the Poisson process. An application to skew standard Young tableaux is given.

Keywords: continuous time random walk, Brownian motion, collision time, skew Young tableaux, tandem queue.

AMS 2000 Subject Classification: Primary: 60J27, 60J65. Secondary: 05E10.

Submitted to EJP on May 5, 2005. Final version accepted 18 October, 2005.

[1]This work was partially supported by KBN Grant No 2 P03A 020 23 (2002–2004).

1 Introduction and the results

In this note X_t is either a standard Brownian motion (SBM) W_t or the standard symmetric continuous time random walk (CTRW). Recall that a compound Poisson process with intensity parameter $\lambda = 1$ and jump distribution $\frac{1}{2}\delta_{-1} + \frac{1}{2}\delta_1$ is called CTRW; see also Asmussen [2, page 99], with his $\mu = \delta = \frac{1}{2}$. We consider a sequence $X_t^1, \ldots, X_t^n$ of independent copies of X_t, each starting from $X_0^i = x_i$. We assume that $\boldsymbol{x} \in W = \{\boldsymbol{y} \in \mathbb{R}^n : y_1 < y_2 < \ldots < y_n\}$. Two processes X_t^i and X_t^j collide at $\tau_{ij} = \min\{t > 0 : \; X_t^i = X_t^j\}$. The time of the collision is $\tau = \min_{1 \le i < j \le n} \tau_{ij}$.

Let $\boldsymbol{x} = (x_1, \ldots, x_n)$ and let

$$h(\boldsymbol{x}) = \det\left[\left(x_i^{j-1}\right)_{i,j=1}^n\right]$$

be the Vandermonde determinant. Our aim is the proof of the following theorem. The Brownian part was first given by Grabiner [6], see also a new proof of Doumerc and O'Connell [4] which uses the representation of collision time obtained there, and an elementary proof of Puchała [11].

Theorem 1.1 *For $t \to \infty$*

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) \sim C h(\boldsymbol{x}) t^{-n(n-1)/4}, \tag{1.1}$$

where

$$C = \frac{(2\pi)^{-n/2}}{\prod_{j=1}^{n-1} j!} \int_W e^{-\frac{|\boldsymbol{y}|^2}{2}} h(\boldsymbol{y})\, d\boldsymbol{y}. \tag{1.2}$$

We also remark that from the theorem proof and Proposition 6.1 from Doumerc and O'Connell [4] we may immediately conclude that the theorem also holds for X_t being the Poisson process with unit intensity. Following Mehta [10, page 354], we may rewrite the constant C in (1.2) in the following form:

$$C \;=\; \frac{1}{\prod_{j=1}^n j!} \prod_{j=1}^n \frac{\Gamma\left(1 + \frac{j}{2}\right)}{\Gamma\left(1 + \frac{1}{2}\right)} \,. \tag{1.3}$$

The proof is based on the following recent result by Doumerc and O'Connell [4], which expresses $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ in terms of Pfaffians. Let $\boldsymbol{P} = (p_{ij})_{i,j=1}^n$, where $p_{ij} = p_{ij}(t) = \mathbb{P}_{x_i,x_j}(\tau_{ij} > t)$ for $i \le j$ and $p_{ij} = -p_{ji}$. Then

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \begin{cases} \mathrm{Pf}(\boldsymbol{P}) & \text{if } n \text{ is even,} \\ \sum_{l=1}^n (-1)^{l+1} \mathrm{Pf}(\boldsymbol{P}_{(l)}) & \text{if } n \text{ is odd,} \end{cases} \tag{1.4}$$

where $\boldsymbol{P}_{(l)} = (p_{ij})_{i,j \ne l}$. By Pf we denote the Pfaffian. To recall this notion, let for n even $P_2(n)$ be the set of partitions of $\{1, \ldots, n\}$ into $\frac{n}{2}$ pairs and $c(\pi)$ is the number of crossings. For a given skew–symmetric matrix $\boldsymbol{A} = (a_{ij})_{i,j=1}^n$ we define the Pfaffian

$$\mathrm{Pf}(\boldsymbol{A}) = \sum_{\pi \in P_2} (-1)^{c(\pi)} \prod_{\{i<j\} \in \pi} a_{ij} \,.$$

In particular we have the formula:

$$\mathrm{Pf}(\boldsymbol{A}) = \sqrt{\det(\boldsymbol{A})}.$$

The paper is organized as follows. In Section 2 we give some preliminary ideas and we state without proofs two key lemmas and a proposition. We work out special cases for the Brownian motion and the continuous time random walk in Section 3. An application to Young tableaux is given in Section 4. There is also mentioned some relationship of Theorem 1.1 with Markovian tandem queues. Proofs are given in Section 5.

2 Preliminaries

We need some technical facts. Suppose that $\boldsymbol{x} \in \mathbb{R}^n$ and $n \in 2\mathbb{N}$. If

$$A_k(x_i) = \sum_{l=0}^{k} a_{k,2l+1} x_i^{2l+1}$$

are odd polynomials of degree $2k+1$, and $\boldsymbol{Q} = (q_{ij}(t))_{i,j=1}^n$, where

$$q_{ij}(t) = \sum_{k=0}^{\infty} t^{-k} A_k(x_j - x_i), \tag{2.5}$$

then $\mathrm{Pf}(\boldsymbol{Q})$ can be written in the form

$$\mathrm{Pf}(\boldsymbol{Q}) = \sum_{k=0}^{\infty} t^{-k} W_k(\boldsymbol{x}) \tag{2.6}$$

for some polynomials $W_k(\boldsymbol{x})$. A simple argument shows that $W_k(\boldsymbol{x})$ is a polynomial of degree $2k + \frac{n}{2}$. Furthermore $\mathrm{Pf}(\boldsymbol{Q})$ is a skew–symmetric polynomial of variable $\boldsymbol{x}$ (that is $\mathrm{Pf}(\boldsymbol{Q}(\sigma\boldsymbol{x})) = \mathrm{sign}(\sigma)\mathrm{Pf}(\boldsymbol{Q}(\boldsymbol{x}))$). Hence we conclude that all polynomials W_k must be skew symmetric polynomials too. We will also use the generalized Vandermonde determinant

$$h_{\boldsymbol{l}}(\boldsymbol{x}) = \det[(x_i^{l_j})_{i,j=1}^n],$$

where $\boldsymbol{l} = (l_1, \ldots, l_n)$. The special case is the Vandermonde determinant when $\boldsymbol{l} = (0, 1, \ldots, n-1)$. Since generalized Vandermonde determinants creates basis for skew polynomials (see Macdonald [8, page 24]) we can write W_k as a linear combination of generalized Vandermonde determinants.

The following lemmas will be useful in calculating asymptotics, which proofs will be demonstrated in Section 5. Notice that in both the lemmas we suppose that n is even, because only for this case Pfaffian is defined.

Lemma 2.1 *If n is even, then*

$$W_{n(n-2)/4}(\boldsymbol{x}) = C_{\text{even}}(n)h(\boldsymbol{x}),$$

where

$$C_{\text{even}}(n) = \det\left[\left(a_{i+j,2j+2i+1}\binom{2i+2j+1}{2i}\right)_{i,j=0}^{n/2-1}\right]. \tag{2.7}$$

Lemma 2.2 *If n is even, then*

$$W_{\frac{n^2}{4}}(\boldsymbol{x}) = \sum_{|\boldsymbol{u}|\le n(n+1)/2} C_{\boldsymbol{u}} h_{\boldsymbol{u}}(\boldsymbol{x}).$$

In particular

$$\begin{aligned} C_{(1,2,\ldots,n)} &= (-1)^{(n/2)}\det\left[\left(a_{i+j-1,2i+2j-1}\binom{2i+2j-1}{2i}\right)_{i,j=1}^{\frac{n}{2}}\right] \\ &= C_{\text{odd}}(n)\,. \end{aligned} \tag{2.8}$$

The next proposition will be the key to calculate asymptotics.

Proposition 2.3 *If $n \in 2\mathbb{N}$ then*

$$\lim_{t\to\infty} \text{Pf}(\boldsymbol{Q})t^{n(n-2)/4} = C_{\text{even}}(n)h(\boldsymbol{x})\,.$$

If $n \in 2\mathbb{N}+1$ then

$$\lim_{t\to\infty}\sum_{l=1}^{n}(-1)^{l+1}\text{Pf}(\boldsymbol{Q}_{(l)})t^{(n-1)^2/4} = C_{\text{odd}}(n-1)h(\boldsymbol{x})\,.$$

Notice that constants $C_{\text{even}}(n)$ and $C_{\text{odd}}(n)$ depend only on coefficients $a_{k,2k+1}$. The above lemmas and the proposition will be proved in Section 5.

3 Special cases

We find here details of expansions (2.5) for two special cases, from which with the use of Proposition 2.3 we may conclude the result of Theorem 1.1.

3.1 Brownian motion

To calculate $p_{ij}(t)$ we will use the reflection principle:

$$p_{ij}(t) = \mathbb{P}_{(x_i,x_j)}(\tau > t) = \mathbb{P}_0(-(x_j - x_i) < B_{2t} \le x_j - x_i).$$

Lemma 3.1 *For $t > 0$*

$$\mathbb{P}_0(-x < B_{2t} \le x) = \frac{1}{\sqrt{2\pi 2t}}\psi_{2t}(x) = \frac{1}{\sqrt{2\pi 2t}}\sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}(2t)^{-k},$$

where

$$a_{2k+1} = \frac{(-1)^k}{(2k+1)2^{k-1}k!}. \tag{3.9}$$

Proof. We have

$$\begin{aligned}\psi_{2t}(x) &= 2\int_0^x e^{-y^2/(2\cdot 2t)}\,dy = 2\int_0^x \sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\left(\frac{y^2}{2\cdot 2t}\right)^k\,dy \\ &= \sum_{k=0}^{\infty} a_{2k+1}x^{2k+1}(2t)^{-k}.\end{aligned}$$

□

Setting $q_{ij}(t) = \psi_{2t}(x_j - x_i)$, the assumptions of Proposition 2.3 are satisfied with

$$\begin{aligned} a_{k,l} &= 0 && \text{if } l \neq 2k+1, \\ a_{k,l} &= a_{2k+1} && \text{if } l = 2k+1. \end{aligned}$$

We have

$$\mathrm{Pf}(\boldsymbol{P}) = \mathrm{Pf}\left(\frac{1}{\sqrt{2\pi 2t}}\boldsymbol{Q}\right) = \left(\frac{1}{\sqrt{2\pi 2t}}\right)^{n/2}\mathrm{Pf}(\boldsymbol{Q}) .$$

Using formula (1.4) and Proposition 2.3 we obtain:
for n even:

$$\lim_{t\to\infty}(2\pi 2t)^{n/4}\mathbb{P}_{\boldsymbol{x}}(\tau > t)(2t)^{n(n-2)/4} = C_{\text{even}}(n)h(\boldsymbol{x}),$$

and for n odd:

$$\lim_{t\to\infty}(2\pi 2t)^{(n-1)/4}\mathbb{P}_{\mathbf{x}}(\tau > t)(2t)^{(n-1)^2/4} = C_{\text{odd}}(n-1)h(\boldsymbol{x}) ,$$

where the constants are defined in (2.7) and (2.8) respectively.

We can rewrite above

$$\lim_{t\to\infty}\mathbb{P}_{\boldsymbol{x}}(\tau > t)t^{n(n-1)/4} = (2\pi)^{-n/4}2^{-n(n-1)/4}C_{\text{even}}(n)h(\boldsymbol{x}),$$

and for n odd:

$$\lim_{t\to\infty}\mathbb{P}_{\mathbf{x}}(\tau > t)t^{n(n-1)/4} = (2\pi)^{-(n-1)/4}2^{-n(n-1)/4}C_{\text{odd}}(n-1)h(\boldsymbol{x}) ,$$

In Section 5.4 we find another expressions for $C_{\text{even}}(n)$ and $C_{\text{odd}}(n)$, which gives an alternative proof of Grabiner theorem (see Grabiner [6], Dourmerc and O'Connell [4], or Puchała [11]).

3.2 CTRW

Let S_t be the standard symmetric CTRW. Following Asmussen [2, page 99] (with his $\mu = \delta = 1/2$)

$$\mathbb{P}_0(S_t = r) = e^{-t} I_r(t),$$

where $I_r(t)$ is the modified Bessel function of order r. As in the Brownian case to calculate p_{ij} we will use the reflection principle.

$$p_{ij}(t) = \mathbb{P}_{(x_i,x_j)}(\tau > t) = \mathbb{P}_0(-(x_j - x_i) < S_{2t} \leq x_j - x_i) \ .$$

We need the asymptotic of

$$\mathbb{P}_0(-x < S_t \leq x) = 2\sum_{i=1}^{x-1} \mathbb{P}_0(S_t = i) + \mathbb{P}_0(S_t = 0) + \mathbb{P}_0(S_t = x) \ .$$

For this we define

$$\beta_t\left(r\right) = \sum_{k=0}^{\infty} \frac{\left(-1\right)^k \left(r,k\right)}{\left(2t\right)^k},$$

where

$$\left(r,k\right) = \frac{\left(4r^2 - 1^2\right)\left(4r^2 - 3^2\right)\ldots\left(4r^2 - \left(2k-1\right)^2\right)}{2^{2k} k!} \ .$$

Recall that a function f_t has the asymptotic expansion $\sum_{k=0}^{\infty} c_k t^{-k}$ at ∞ if $f_t - \sum_{k=0}^{n} c_k t^{-k} = o(t^{-n})$ for all $n = 0, 1 \ldots$. In this case we write $f_t \simeq \sum_{k=0}^{\infty} c_k t^{-k}$. For details and basic facts on the asymptotic expansions we refer to Knopp [7]. By Watson [13, page 203] we have

$$\sqrt{2\pi t}\,\mathbb{P}_0\left(S_t = r\right) \simeq \beta_t(r). \tag{3.10}$$

Thus

$$\mathbb{P}(-x < S_t \leq x) \simeq \frac{1}{\sqrt{2\pi t}}\left(2\sum_{i=1}^{x-1} \beta_t(i) + \beta_t(0) + \beta_t(x)\right)$$

as $t \to \infty$. Define

$$\begin{aligned} \varphi_t(x) &= 2\sum_{i=1}^{x} \beta_t(i) + \beta_t(0) - \beta_t(x) \\ &= \sum_{k=0}^{\infty} A_k^{\varphi}(x) t^{-k} \ . \end{aligned}$$

Setting $q_{ij}(t) = \varphi_{2t}(x_j - x_i)$, the assumptions of Proposition 2.3 are satisfied with

$$A_k(x) = A_k^{\varphi}(x) \ .$$

Hence we have (remember about doubling t)

$$\sqrt{2\pi 2t}p_{ij}(t) \simeq \sum_{k=0}^{\infty} A_k^{\varphi}(x_j - x_i)(2t)^{-k}. \tag{3.11}$$

Thus

$$\mathrm{Pf}(\boldsymbol{P}) \simeq \mathrm{Pf}\left(\frac{1}{\sqrt{2\pi 2t}}\boldsymbol{Q}\right) = \mathrm{Pf}\left(\frac{1}{\sqrt{2\pi 2t}}\sum_{k=0}^{\infty} A_k^{\varphi}(x_j - x_i)(2t)^{-k}\right) .$$

Lemma 3.2 (see [14])

$$\sum_{i=1}^{n} i^p = n^p + \sum_{k=0}^{p} \frac{B_k p!}{k!\,(p-k+1)!} n^{p-k+1} ,$$

where B_k are Bernoulli numbers.

In the next lemma we study polynomials $A_k^{\varphi}(x)$.

Lemma 3.3 *$A_k^{\varphi}(x)$ is an odd polynomial (that is with even coefficient vanishing) of order $2k+1$ with the leading coefficient a_{2k+1} defined in (3.9). That is*

$$A_k^{\varphi}(x) = \sum_{i=1}^{k} a_{k,2i+1} x^{2i+1},$$

where $a_{k,2k+1} = a_{2k+1}$.

Proof. We have that

$$2\sum_{i=1}^{x-1} i^{2m} + x^{2m}$$

is an odd polynomial of order $2m+1$ with the leading coefficient $\frac{2}{2m+1}$. This is because $B_0 = 1$, $B_1 = -\frac{1}{2}$ and $B_{2l+1} = 0$ for $l \geq 1$ and

$$\begin{aligned} 2\sum_{i=1}^{x-1} i^{2m} + x^{2m} &= 2\sum_{i=1}^{x} i^{2m} - 2x^{2m} + x^{2m} \\ &= 2x^{2m} + 2\sum_{k=0}^{2m} \frac{B_k\,(2m)!}{k!\,(2m-k+1)!} x^{2m-k+1} - x^{2m} . \end{aligned}$$

Hence the above equals to

$$\frac{2}{2m+1} x^{2m+1} + 2\sum_{k=2}^{2m} \frac{B_k\,(2m)!}{k!\,(2m-k+1)!} x^{2m-k+1} .$$

□

Using now Proposition 2.3 we get: for $n \in 2\mathbb{Z}$

$$\lim_{t\to\infty}(2\pi 2t)^{n/4}\mathbb{P}_{\boldsymbol{x}}(\tau > t)(2t)^{n(n-2)/4} = C_{\text{even}}(n)h(\boldsymbol{x})$$

and for $n \in 2\mathbb{N}+1$ we have

$$\lim_{t\to\infty}(2\pi 2t)^{(n-1)/4}\mathbb{P}_{\boldsymbol{x}}(\tau > t)(2t)^{(n-1)^2/4} = C_{\text{odd}}(n-1)h(\boldsymbol{x}) \ ,$$

where the constants are defined in (2.7) and (2.8) respectively. Therefore this case is identical to the Brownian case, which completes the proof of Theorem 1.1. We must notice that above considerations are valid for Poisson process N_t, this is because difference of two independent Poisson processes with intensity 1 is a CTRW with intensity 2.

$$N_t^1 - N_t^2 =_d S_{2t},$$

so all calculations are identical.

4 Applications

Since the result of Theorem 1.1 is also valid in the case of independent Poisson processes, we can apply it to obtain an aymptotics for Young tableaux, which generalizes some earlier results of Regev[12].

Thus let $X_t = (X_t^1, \ldots, X_t^n)$ be vector of independent Poisson processes with intensity 1 starting from $\boldsymbol{x} \in W$. Let σ_m denote the time of the m-th transition of X_t, and let $T = \min\{m > 0 : X_{\sigma_m} \notin W\}$. Observe that

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = \mathbb{P}(T > N_t) \ , \tag{4.12}$$

where $N_t = \max\{m : \sigma_m \leq t\}$ is a Poisson process with intensity n independent of T.

For integer partitions λ and μ with $\mu \leq \lambda$, let $f_{\lambda/\mu}$ denote the number of skew standard tableaux with shape λ/μ. Set $\delta = (n-1, n-2, \ldots, 1, 0)$. We denote by $\widetilde{\lambda}_1$ the hight of the Young diagram defined by partition λ (the number of boxes in the first row of the conjugate diagram), see for definitions Fulton[5]. Put

$$\bar{\phi}(k) = n^{-k} \sum_{\lambda \vdash k, \widetilde{\lambda}_1 \leq n} f_{\lambda/\mu} \ . \tag{4.13}$$

The key observation relating our theorem with Young tableaux is that

$$\bar{\phi}(k) = \mathbb{P}(T > k),$$

which together with (4.13) links the exit time theory with Young tableaux. The following corollary extends the asymptotics obtained by Regev[12] from Young tableaux to skew Young tableaux.

Corollary 4.1 *As $k \to \infty$*

$$\sum_{\lambda \vdash k, \widetilde{\lambda}_1 \le n} f_{\lambda/\mu} \sim n^k \left(\frac{k}{n}\right)^{-n(n-1)/4} \frac{h(\mu+\delta)}{h(\delta)} \frac{(2\pi)^{-n/2}}{n!} \int_{\mathbf{R}^n} e^{-|y|^2/2} |h(y)| dy \ .$$

Proof. Let N_t be a Poisson variable with intensity nt. We first show that for each $a > 0$, and $g(t) \sim ct^{-b}$, where $c > 0$ and $0 < b < 1/2$ we have

$$\lim_{t\to\infty} t^a \mathbb{P}\left(\left|\frac{N_t - nt}{t}\right| > ct^{-b}\right) = 0 \ . \tag{4.14}$$

For the proof, without loss of generality, we may assume $n = 1$. The Fenchel-Legendre transform for random variable $X - 1$, where X is Poisson distributed with mean 1 is

$$\Lambda^*(x) = \begin{cases} (1+x)\log(1+x) - x & x > -1, \\ \infty & x \le -1 \ . \end{cases}$$

see Dembo and Zeitouni [3, page 35]. Note that $\Lambda^*(x) = \frac{x^2}{2} + o(x^2)$ for $x \to 0$. Following Dembo and Zeitouni [3, page 27] we have the inequality: for all t nonnegative integer

$$\mathbb{P}\left(\left|\frac{N_t - t}{t}\right| > ct^{-b}\right) \le 2\exp(-t \inf_{x\in F} \Lambda^*(x)), \tag{4.15}$$

where $F = (-\infty, -g(t)] \cup [g(t), \infty)$, which the inequality can be extended to all $t \ge 0$. Since

$$\inf_{x\in F} \Lambda^*(x) = (1+g(t))\log(1+g(t)) - g(t) = \frac{g^2(t)}{2} + o(g^2(t))) \ ,$$

we have

$$t \inf_{x\in F} \Lambda^*(x) = t\left(\frac{g^2(t)}{2} + o(g^2(t))\right) \sim \frac{ct^{1-2b}}{2}$$

as $t \to \infty$. Thus from (4.15) the proof of (4.14) follows.

Observe now that

$$\begin{aligned} \mathbb{P}_{\boldsymbol{x}}(\tau > t) &= \mathbb{P}_{\boldsymbol{x}}(\tau > t, |N_t - nt| < k - nt) \\ &\quad + \mathbb{P}_{\boldsymbol{x}}(\tau > t, |N_t - nt| \ge k - nt) \\ &\ge \bar{\phi}(k)\mathbb{P}(|N_t - nt| < k - nt) \end{aligned} \tag{4.16}$$

and similarly

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) \le \bar{\phi}(k) + \mathbb{P}(|N_t - nt| > nt - k) \ . \tag{4.17}$$

We relate t and k above by $t(k) = \frac{k - k^{3/4}}{n}$ and then

$$\begin{aligned} &\bar{\phi}(k)\left(\frac{k}{n}\right)^{n(n-1)/4} \\ &\ge \ \mathbb{P}_{\boldsymbol{x}}(\tau > t(k))\left(\frac{k}{n}\right)^{n(n-1)/4} - \mathbb{P}(|N_{t(k)} - nt(k)| > k - nt(k))\left(\frac{k}{n}\right)^{n(n-1)/4} \ . \end{aligned}$$

By (4.14)

$$\mathbb{P}\left(\left|\frac{N_{t(k)}-nt(k)}{t(k)}\right| > \frac{nk^{3/4}}{k-k^{3/4}}\right)\left(\frac{k}{n}\right)^{n(n-1)/4} \to 0\ ,$$

as $k\to\infty$. We also have by Theorem 1.1

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t(k))\left(\frac{k}{n}\right)^{n(n-1)/4} = \mathbb{P}_{\boldsymbol{x}}(\tau > t)t^{n(n-1)/4}\left(\frac{k}{k-k^{3/4}}\right)^{n(n-1)/4} \to Ch(\boldsymbol{x}),$$

where C is from (1.2). Hence by (4.16) we have

$$Ch(\boldsymbol{x}) \le \liminf_{k\to\infty} \bar{\phi}(k)\left(\frac{k}{n}\right)^{n(n-1)/4}.$$

In the similar way, using (4.17) with $t(k) = (k+k^{3/4})/n$, we prove

$$\limsup_{k\to\infty} \bar{\phi}(k)\left(\frac{k}{n}\right)^{n(n-1)/4} \le Ch(\boldsymbol{x}).$$

which completes the proof of the corollary.

□

¿From the corollary we have that for $k\to\infty$

$$\sum_{\lambda\vdash k,\widetilde{\lambda}_1\le n} f_{\lambda/\mu} \sim \frac{h(\mu+\delta)}{h(\delta)} \sum_{\lambda\vdash k,\widetilde{\lambda}_1\le n} f_\lambda.$$

Note that following Regev [12, (F.4.5.1)]

$$\begin{aligned}\sum_{\lambda\vdash k,\widetilde{\lambda}_1\le n} f_\lambda &= S_n^{(1)}(k)\\ &\sim \frac{\sqrt{n}^{n(n-1)/2}}{n!}\Gamma\left(\frac{3}{2}\right)^{-n}\prod_{j=1}^{n}\Gamma\left(1+\frac{1}{2}j\right)\left(\frac{1}{\sqrt{k}}\right)^{n(n-1)/2} n^k \qquad (4.18)\end{aligned}$$

Now by (1.2) and (1.3) the right hand side of (4.18) equals

$$\begin{aligned}S_n^{(1)}(k) &\sim \sqrt{n}^{n(n-1)/2}\frac{1}{n!}\Gamma\left(\frac{3}{2}\right)^{-n}\prod_{j=1}^{n}\Gamma\left(1+\frac{1}{2}j\right)\left(\frac{1}{\sqrt{k}}\right)^{n(n-1)/2} n^k\\ &= n^k\left(\frac{k}{n}\right)^{-n(n-1)/4}\frac{1}{n!}\prod_{j=1}^{n}\frac{\Gamma(1+\frac{j}{2})}{\Gamma(1+\frac{1}{2})}\\ &= n^k\left(\frac{k}{n}\right)^{-n(n-1)/4}\frac{(2\pi)^{-n/2}}{n!}\int_{R^n} e^{-\frac{|\boldsymbol{y}|^2}{2}}|h(\boldsymbol{y})|d\boldsymbol{y}.\end{aligned}$$

The exit time result of the type like in Theorem 1.1 has also an application to the series of queues M/M$\to \ldots \to$M/1 with $n-1$ stations, with service rate μ_i on the i-th station and arrival rate μ_0. Correspondingly we consider independent Poisson processes $X_t^1, \ldots, X_t^n$ with intensity μ_{i-1} respectively and τ is the collision time. We observe that for $0 \le t \le \tau$, the queue size at the i-th station is $Q_i(t) = X_t^{i-1} - X_t^i$, if there is at $t = 0$, $q_i = x_{i-1} - x_i > 0$ jobs at i-th station. Thus in terms of the theory of queues, τ is the moment for the first time a station is empty. Define $\gamma = (\prod_{j=0}^{n-1} \mu_j)^{1/n}$ and $\beta_1 = \mu_0/\gamma, \beta_j = \mu_0 \cdots \mu_{j-1}/\gamma^j$. Let $\alpha = \sum_{j=0}^{n-1} \mu_j/n$. Massey [9] showed that, if $\beta_j < 1$

$$\mathbb{P}_{\boldsymbol{x}}(\tau > t) = O\left(\frac{\exp(-(n)(\alpha-\gamma)t)}{t\sqrt{t^{n-1}}}\right) .$$

No exact asymptotic is known. However for the case when $\mu_0 = \mu_1 = \ldots = \mu_{n-1}$, that is not fulfilling conditions of Massey [9], our Theorem 1.1 shows the right asymptotics. The exact asymptotics for $\mathbb{P}_{\boldsymbol{x}}(\tau > t)$ in the case of independent but not necessarily identically distributed process $X_t^1, \ldots, X_t^n$ is an open problem.

5 Proofs

In this section we show details of proofs. We use the following vector notations. By $\boldsymbol{l} = (l_1, \ldots, l_n)$, $\boldsymbol{k} = (k_1, \ldots, k_n)$, $\boldsymbol{s} = (s_1, \ldots, s_n)$ or $\boldsymbol{s} - \boldsymbol{l} = (s_1 - l_1, \ldots, s_n - l_n)$ we denote vectors from $\mathbb{Z}_+^n$, where n is the number of particles. Let $\mathbf{0} = (0, \ldots, 0) \in \mathbb{Z}_+^n$ and $\mathbf{1} = (1, \ldots, 1)$. In this section $\boldsymbol{m} = (m_1, \ldots, m_n) = 2\boldsymbol{k} + \mathbf{1}$. By $\boldsymbol{l} \le \boldsymbol{s}$ we mean that $l_i \le s_i$ for $i = 1, \ldots, n$. We also write

$$\sum_{\mathbf{0}\le \boldsymbol{l} \le \boldsymbol{m}} = \sum_{l_1=0}^{m_1} \cdots \sum_{l_n=0}^{m_n}, \qquad |\boldsymbol{k}| = k_1 + \ldots + k_n$$

By σ we denote a permutation of $(1, \ldots, n)$ and S_n is the family of all permutations. For $\boldsymbol{l} \in \mathbb{Z}_+^n$ we define $\sigma(\boldsymbol{l}) = (l_{\sigma(1)}, \ldots, l_{\sigma(n)})$.

5.1 Proof of Lemma 2.1

The proof is partitioned into lemmas.

Lemma 5.1 *For all $\boldsymbol{m} = (m_1, \ldots, m_n) \in \mathbb{Z}_+^n$ we have*

$$\sum_{\sigma \in S_n} \det\left[((x_i - x_j)^{m_{\sigma(i)}})_{i,j=1}^n\right] = \sum_{\mathbf{0}\le \boldsymbol{l} \le \boldsymbol{m}} \binom{m_1}{l_1} \cdots \binom{m_n}{l_n} h_{\boldsymbol{l}}(-\boldsymbol{x})\, h_{\boldsymbol{m}-\boldsymbol{l}}(\boldsymbol{x}) . \tag{5.19}$$

Proof. Using Newton coefficients we write the LHS of (5.19)

$$\sum_{\sigma \in S_n} \det\left[\left(\sum_{l_{\sigma(i)}=0}^{m_{\sigma(i)}} \binom{m_{\sigma(i)}}{l_{\sigma(i)}} x_i^{m_{\sigma(i)}-l_{\sigma(i)}} (-x_j)^{l_{\sigma(i)}}\right)_{i,j=1}^n\right] .$$

Next using elementary properties of determinants the above is

$$\sum_{\sigma\in S_n}\sum_{l_{\sigma(1)},\dots,l_{\sigma(n)}=0}^{m_{\sigma(1)},\dots,m_{\sigma(n)}}\binom{m_{\sigma(1)}}{l_{\sigma(1)}}\dots\binom{m_{\sigma(n)}}{l_{\sigma(n)}}x_1^{m_{\sigma(1)}-l_{\sigma(1)}}\dots x_n^{m_{\sigma(n)}-l_{\sigma(n)}}\det\left[\left((-x_i)^{l_{\sigma(j)}}\right)_{i,j=1}^n\right],$$

which can be written as

$$\sum_{\sigma\in S_n}\sum_{l_1,\dots,l_n=0}^{m_1,\dots,m_n}\binom{m_1}{l_1}\dots\binom{m_n}{l_n}x_1^{m_{\sigma(1)}-l_{\sigma(1)}}\dots x_n^{m_{\sigma(n)}-l_{\sigma(n)}}\mathrm{sgn}(\sigma)h_{\boldsymbol{l}}(-\boldsymbol{x})$$

and finally

$$\sum_{l_1,\dots,l_n=0}^{m_1,\dots,m_n}\binom{m_1}{l_1}\dots\binom{m_n}{l_n}h_{\boldsymbol{l}}(-\boldsymbol{x})\,h_{\boldsymbol{m}-\boldsymbol{l}}(\boldsymbol{x})\,. \tag{5.20}$$

□

Lemma 5.2 *Suppose that n is even and consider $\boldsymbol{m}=2\boldsymbol{k}+\boldsymbol{1}$ for which there exist $\boldsymbol{l},\boldsymbol{m}-\boldsymbol{l}\in\mathbb{Z}_+^n$ having different elements respectively. Then the minimal $\boldsymbol{k}$ is such that $|\boldsymbol{k}|=n(n-2)/2$.*

Proof. Notice that $|\boldsymbol{m}|=2|\boldsymbol{k}|+n$. Take $\boldsymbol{l}=(0,1,\dots,n-1)$ and $\boldsymbol{m}-\boldsymbol{l}$ a permutation of $\boldsymbol{l}$ which maps even numbers of $\boldsymbol{l}$ to odd numbers respectively. Then $\boldsymbol{m}$ has odd elements. Such a permutation exists when n is even. Now $|\boldsymbol{l}|=|\boldsymbol{m}-\boldsymbol{l}|=n(n-1)/2$ so $\boldsymbol{m}=n(n-1)$ and hence $|\boldsymbol{k}|=(|\boldsymbol{m}|-n)/2=n(n-2)/2$.

□

Lemma 5.3 *Let n be an even number. Suppose that for $\boldsymbol{x}\in\mathbb{R}^n$ we have $q_{ij}(t)=\sum_{k=0}^{\infty}A_k(x_j-x_i)t^{-k}$. Then*

$$\begin{aligned}\det\left[(q_{ij}(t))_{i,j=1}^n\right] &= \sum_{v=v_0}^{\infty}t^{-v}\sum_{\substack{|\boldsymbol{k}|=v\\ k_1\le\dots\le k_n}}J_{\boldsymbol{k}}\sum_{\sigma\in S_n}\det\left[\left(A_{k_{\sigma(i)}}(x_i-x_j)\right)_{i,j=1}^n\right],\\ &= \sum_{v=v_0}^{\infty}H_v(\boldsymbol{x}),\end{aligned} \tag{5.21}$$

where

$$H_v(\boldsymbol{x})=\sum_{s_1,\dots,s_n}^{2k_1+1,\dots,2k_n+1}\prod_{j=1}^n a_{k_j,s_j}\sum_{l_1,\dots,l_n}^{s_1,\dots,s_n}\prod_{i=1}^n\binom{s_i}{l_i}h_{\boldsymbol{l}}(\boldsymbol{x})\,h_{\boldsymbol{s}-\boldsymbol{l}}(-\boldsymbol{x})\,.$$

and

$$\begin{aligned}v_0 &= \frac{n(n-2)}{2},\\ J_{\boldsymbol{k}} &= \frac{1}{(\#\{k_i=1\})!(\#\{k_i=2\})!\dots(\#\{k_i=\nu\})!}.\end{aligned}$$

Proof. We write

$$\det\left[(q_{ij}(t))_{i,j=1}^{n}\right] = \sum_{v=0}^{\infty}\sum_{|\boldsymbol{k}|=v} t^{-\nu}\det\left[(A_{k_i}(x_i-x_j))_{i,j=1}^{n}\right]$$
$$= \sum_{v=0}^{\infty} t^{-v}\sum_{\substack{|\boldsymbol{k}|=v\\ k_1\le\cdots\le k_n}} J_{\boldsymbol{k}}\sum_{\sigma\in S_n}\det\left[\left(A_{k_{\sigma(i)}}(x_i-x_j)\right)_{i,j=1}^{n}\right]. \quad (5.22)$$

We now show that the first v_0-1 coefficients vanish. Then the inside sum in (5.22), for $|\boldsymbol{k}|=v$ and $k_1\le\ldots\le k_n$, with the use of Lemma 5.1, can be transformed as follows (note that s_i's are odd)

$$\sum_{\sigma\in S_n}\det\left[\left(A_{k_{\sigma(i)}}(x_i-x_j)\right)_{i,j=1}^{n}\right]$$
$$= \sum_{l_1,\ldots,l_n}^{2k_1+1,\ldots,2k_n+1}\prod_{j=1}^{n} a_{k_j,l_j}\sum_{\sigma\in S_n}\det\left[(x_i-x_j)^{l_i}\right]$$
$$= \sum_{s_1,\ldots,s_n}^{2k_1+1,\ldots,2k_n+1}\prod_{j=1}^{n} a_{k_j,s_j}\sum_{l_1,\ldots,l_n}^{s_1,\ldots,s_n}\prod_{i=1}^{n}\binom{s_i}{l_i} h_{\boldsymbol{l}}(\boldsymbol{x})\, h_{\boldsymbol{s}-\boldsymbol{l}}(-\boldsymbol{x}).$$

We now analyze the sum

$$\sum_{l_1,\ldots,l_n}^{s_1,\ldots,s_n}\prod_{i=1}^{n}\binom{s_i}{l_i} h_{\boldsymbol{l}}(\boldsymbol{x})\, h_{\boldsymbol{s}-\boldsymbol{l}}(-\boldsymbol{x}) .$$

For $h_{\boldsymbol{l}}(\boldsymbol{x})h_{\boldsymbol{s}-\boldsymbol{l}}(-\boldsymbol{x})\ne 0$, both the sequences $\boldsymbol{l},\boldsymbol{s}-\boldsymbol{l}\in\mathbb{Z}_+^n$ must have different elements respectively. By Lemma 5.2, the minimal possible case is when $\boldsymbol{l}$ and $\boldsymbol{s}-\boldsymbol{l}$ are permutations of $\{0,1,\ldots,n-1\}$. This corresponds to $v_0=|\boldsymbol{k}|=n(n-2)/2$.

□

Lemma 5.4 *Let $\boldsymbol{m}=2\boldsymbol{k}+\boldsymbol{1}$. Then*

$$\sum_{\substack{k_1\le\cdots\le k_n\\ |\boldsymbol{k}|=n(n-2)/2}} J_{\boldsymbol{k}}\sum_{\boldsymbol{0}\le\boldsymbol{l}\le\boldsymbol{m}}\prod_{i=1}^{n} a_{m_i}\binom{m_i}{l_i} h_{\boldsymbol{l}}(\boldsymbol{x})\, h_{\boldsymbol{m}-\boldsymbol{l}}(-\boldsymbol{x})$$
$$= \left(h(\boldsymbol{x})\det\left[\left(a_{2i+2j+1}\binom{2i+2j+1}{2i}\right)_{i,j=0}^{\frac{n}{2}-1}\right]\right)^2.$$

Proof. Let

$$g(\boldsymbol{l},\boldsymbol{s}) = \prod_{i=1}^{n} a_{l_i+s_i}\binom{l_i+s_i}{l_i} h_{\boldsymbol{l}}(\boldsymbol{x})\, h_{\boldsymbol{s}}(-\boldsymbol{x}).$$

Recall that since $\boldsymbol{k}$ has components $k_1 \leq \ldots \leq k_n$, then for components of $\boldsymbol{m}$ we have $m_1 \leq \ldots \leq m_n$ and they are odd. Moreover we have $|\boldsymbol{m}| = 2|\boldsymbol{k}| + n$. For $|\boldsymbol{k}| = v_0$, the only admissible splits are of the following form. Let S_n^{eo} be the set of all permutations σ of $(1, 2, \ldots, n)$ such that $\sigma(i)$ is odd if and only if i is even. We may identify this family with $S_{n/2} \times S_{n/2}$. We define for $\boldsymbol{s} \in \mathbb{Z}_+^n$, $\sigma(\boldsymbol{s}) = (s_{\sigma(1)}, \ldots, s_{\sigma(n)})$. Let $\boldsymbol{l} = \boldsymbol{l}^* = (0, 1, \ldots, n-1)$. Then $\{(\boldsymbol{l}^*, \sigma(\boldsymbol{l}^*) \ : \ \sigma \in S_n^{\mathrm{eo}}\}$ has the property that components of $\boldsymbol{s} = \boldsymbol{l}^* + \sigma(\boldsymbol{l}^*)$ are odd. However the components of $\boldsymbol{l}^* + \sigma(\boldsymbol{l}^*)$ are not always nondecreasing components, and therefore we must introduce another permutation $\sigma^{'}$, defined for a given $\boldsymbol{s} \in \mathbb{Z}_+$, which makes the components of $\sigma^{'}(\boldsymbol{s})$ nondecreasing. Let $\sigma^{'}$ be defined by $\boldsymbol{l}^* + \sigma(\boldsymbol{l}^*)$. Then the set of all admissible entries is

$$\{(\sigma^{'}(\boldsymbol{l}^*), \sigma^{'}(\sigma(\boldsymbol{l}^*))\}.$$

Fortunately, if $\sigma^{''}$ is defined by $\boldsymbol{l} + \boldsymbol{s}$, then

$$g(\boldsymbol{l}, \boldsymbol{s}) = g(\sigma^{''}(\boldsymbol{l}), \sigma^{''}(\boldsymbol{s})).$$

¿From these considerations we see that

$$\begin{aligned}
&\sum_{\substack{k_1 \leq \cdots \leq k_n \\ |\boldsymbol{k}| = n(n-2)/2}} J_{\boldsymbol{k}} \sum_{l_1, \ldots, l_n}^{m_1, \ldots, m_n} \prod_{i=1}^{n} a_{m_i} \binom{m_i}{l_i} h_{\boldsymbol{l}}(\boldsymbol{x})\, h_{\boldsymbol{m}-\boldsymbol{l}}(-\boldsymbol{x}) \\
&= \sum_{\sigma \in S_n^{\mathrm{eo}}} \prod_{i=1}^{n} a_{l_i + l_{\sigma(i)}} \binom{l_i + l_{\sigma(i)}}{l_i} h(\boldsymbol{x})\, h(\sigma(\boldsymbol{x}))\, (-1)^{n(n-1)/2} \\
&= h^2(\boldsymbol{x})\, (-1)^{n(n-1)/2} \sum_{\sigma \in S_n^{\mathrm{eo}}} \prod_{i=1}^{n} a_{l_i + l_{\sigma(i)}} \binom{l_i + l_{\sigma(i)}}{l_i} \mathrm{sign}(\sigma).
\end{aligned}$$

Now $\sigma \in S_n^{\mathrm{eo}}$ is identified with $(\eta, \xi) \in S_{n/2} \times S_{n/2}$. Notice that $\mathrm{sign}(\sigma) = (-1)^{n/2} \mathrm{sign}(\eta) \mathrm{sign}(\xi)$, where $(-1)^{n/2}$ is responsible for $n/2$ transpositions from odds to evens, and that $(-1)^{n/2} = (-1)^{n(n-1)/2}$. Hence the above can be rewritten in the form:

$$\begin{aligned}
&h^2(\boldsymbol{x}) \sum_{\eta \in S_{n/2}} \mathrm{sign}(\eta) \prod_{i=1}^{n/2} a_{l_{2i-1} + l_{2\eta(i)}} \binom{l_{2i-1} + l_{2\eta(i)}}{l_{2i-1}} \times \\
&\times \sum_{\xi \in S_{n/2}} \mathrm{sign}(\eta) \prod_{i=1}^{n/2} a_{l_{2i} + l_{2\xi(i)-1}} \binom{l_{2i} + l_{2\xi(i)-1}}{l_{2i}}.
\end{aligned}$$

Using standard properties of determinants we write above as:

$$\begin{aligned}
&h^2(\boldsymbol{x}) \det \left[\left(a_{2i+2j-3} \binom{2i+2j-3}{2i-2} \right)_{i,j=1}^{n/2} \right] \det \left[\left(a_{2i+2j-3} \binom{2i+2j-3}{2i-1} \right)_{i,j=1}^{n/2} \right] \\
= &\left(h(\boldsymbol{x}) \det \left[\left(a_{2i+2j+1} \binom{2i+2j+1}{2i} \right)_{i,j=0}^{\frac{n}{2}-1} \right] \right)^2
\end{aligned}$$

$\square$

The proof of lemma 2.1 is completed.

5.2 Proof of Lemma 2.2

We want to calculate constant $C_{\text{odd}}(n)$ defined in (2.8). Recall that this is the one which stands at $h_{(1,2,\dots,n)}(\boldsymbol{x})$ in polynomial $W_{n^2/4}(\boldsymbol{x})$. Since Pfaffian of a matrix is the square root of the corresponding determinant, it suffices to look for the constant in the asymptotic expansion of the determinant standing at $h_{(0,\dots,n-1)}(\boldsymbol{x})h_{(1,\dots,n)}(\boldsymbol{x})$ at $t^{-n(n-1)/2}$ and divide it by the already known constant $C_{even}(n)$. The following argument provides the proof of the uniqueness for this procedure, that is $h_{0,\dots,n-1}h_{1,\dots,n}$ cannot be represented as a combination of other Vandermondes. This means that if

$$h_{(0,\dots,n-1)}(\boldsymbol{x})h_{(1,\dots,n)}(\boldsymbol{x}) = \sum b_{\boldsymbol{u}}h_{\boldsymbol{u}}(\boldsymbol{x})b_{\boldsymbol{v}}h_{\boldsymbol{v}}(\boldsymbol{x}) \ ,$$

where the summation runs over $|\boldsymbol{u}| + |\boldsymbol{v}| \leq n^2$ without pairs $(\boldsymbol{u} = (0, 1, \dots, n-1), \boldsymbol{v} = (1, 2, \dots, n))$ or $(\boldsymbol{u} = (1, 2, \dots, n-1), \boldsymbol{v} = (0, 1, \dots, n-1))$, then

$$x_1 \dots x_n = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} b_{\boldsymbol{\mu}}s_{\boldsymbol{\mu}}(\boldsymbol{x})b_{\boldsymbol{\nu}}s_{\boldsymbol{\nu}}(\boldsymbol{x}) \ ,$$

where $\boldsymbol{\mu} = \boldsymbol{u}-(1, \dots, n)+\mathbf{1}$, $\boldsymbol{\nu} = \boldsymbol{v}-(1, \dots, n)+\mathbf{1}$ and $s_{\boldsymbol{\mu}}(\boldsymbol{x}), s_{\boldsymbol{\nu}}(\boldsymbol{x})$ are Schur polynomials. Now using the Littlewood-Richardson rule we may write above as

$$s_{(1,\dots,1)}(\boldsymbol{x}) = x_1 \dots x_n = \sum_{\boldsymbol{\mu},\boldsymbol{\nu}} b_{\boldsymbol{\mu}}b_{\boldsymbol{\nu}} \sum_{\boldsymbol{\lambda}} b^{\boldsymbol{\lambda}}_{\boldsymbol{\mu}\boldsymbol{\nu}}s_{\boldsymbol{\lambda}}(\boldsymbol{x}) \ .$$

Since Schur polynomials form a basis and $\boldsymbol{\mu}, \boldsymbol{\nu}$ is not a subtableaux of $(1, \dots, 1)$ the above equality cannot be satisfied.

The method of calculating is similar to the one presented before in Section 2.1. Since $v_0 = \frac{n(n-1)}{2}$, by Lemma 5.3 the coefficient standing with t^{-v_0} is

$$\sum_{|\boldsymbol{k}|=\frac{n(n-1)}{2}} J_{\boldsymbol{k}} \sum_{s_1,\dots,s_n}^{2k_1+1,\dots 2k_n+1} \prod_{i=1}^{n} a_{k_i,s_i} \sum_{l_1,\dots,l_n}^{s_1,\dots,s_n} \prod_{i=1}^{n} \binom{s_i}{l_i} h_{\boldsymbol{l}}\left(\boldsymbol{x}\right) h_{\boldsymbol{s}-\boldsymbol{l}}\left(-\boldsymbol{x}\right) \ .$$

Since we are only interested in $h_{\boldsymbol{l}}(\boldsymbol{x})h_{\boldsymbol{s}-\boldsymbol{l}}(\boldsymbol{x})$, where $\boldsymbol{l}$ is a permutation of $(0, 1, \dots, n-1)$ and $\boldsymbol{s} - \boldsymbol{l}$ is a permutation of $(1, 2, \dots, n)$, for which

$$|\boldsymbol{l}| + |\boldsymbol{s} - \boldsymbol{l}| = |\boldsymbol{s}| = \frac{n\left(n-1\right)}{2} + \frac{n\left(n+1\right)}{2} = n^2 \ ,$$

we also have

$$|2\boldsymbol{k} + \mathbf{1}| = 2\left|\boldsymbol{k}\right| + n = n^2 - n + n = n^2 \ .$$

Therefore s_i cannot be less than $2\boldsymbol{k}+\mathbf{1}$, and so we have

$$\sum_{|\boldsymbol{k}|=\frac{n(n-1)}{2}} J_{\boldsymbol{k}} \prod_{i=1}^{n} a_{k_i,2k_i+1} \sum_{l_1,\ldots,l_n}^{2k_1+1,\ldots,2k_n+1} \prod_{i=1}^{n} \binom{2k_i+1}{l_i} h_{\boldsymbol{l}}\left(\boldsymbol{x}\right) h_{(2\boldsymbol{k}+\mathbf{1}-\boldsymbol{l})}\left(-\boldsymbol{x}\right) \ .$$

Using the same argumentation as in the proof of Lemma 2.1 we can assume that $\boldsymbol{l}=(0,1,\ldots,n-1)$ and $2\boldsymbol{k}+\mathbf{1}-\boldsymbol{l}$ is a permutation of $(1,2,\ldots,n)$, which places even numbers into even places. We denote the resulting set of permutations by S_n^{ee}. Thus we have

$$\begin{aligned}&\sum_{\sigma\in S_n^{\mathrm{ee}}} \prod_{i=1}^{n} a_{i-1+\sigma(i)} \binom{i-1+\sigma(i)}{i-1} h_{(0,1,\ldots,n-1)}\left(\boldsymbol{x}\right) h_{\sigma(1,2,\ldots,n)}\left(-\boldsymbol{x}\right)\\ =\;& h\left(\boldsymbol{x}\right) h_{(1,2,\ldots,n)}\left(\boldsymbol{x}\right)(-1)^{\frac{n(n+1)}{2}} \sum_{\sigma\in S_n^{\mathrm{ee}}} \prod_{i=1}^{n} a_{i-1+\sigma(i)} \binom{i-1+\sigma(i)}{i-1} \mathrm{sign}(\sigma) \ .\end{aligned}$$

As in the proof of Lemma 5.3 we identify S_n^{ee} with the product of permutations $S_{n/2}\times S_{n/2}$ of even numbers and permutations of odd numbers respectively. Hence the above expression can be written as

$$\begin{aligned}&h\left(\boldsymbol{x}\right) h_{(1,2,\ldots,n)}\left(\boldsymbol{x}\right)(-1)^{\frac{n(n+1)}{2}} \sum_{\eta,\xi\in S_{n/2}} \prod_{i\in\{2,4,\ldots,n\}} a_{i-1+2\eta\left(\frac{i}{2}\right)} \binom{i-1+2\eta\left(\frac{i}{2}\right)}{i-1} \mathrm{sign}(\eta)\times\\ &\times \prod_{i\in\{1,3,\ldots,n-1\}} a_{i-1+2\xi\left(\frac{i+1}{2}\right)-1} \binom{i-1+2\xi\left(\frac{i+1}{2}\right)-1}{i-1} \mathrm{sign}(\xi) \ ,\end{aligned}$$

which equals

$$\begin{aligned}&h\left(\boldsymbol{x}\right) h_{(1,2,\ldots,n)}\left(\boldsymbol{x}\right)(-1)^{\frac{n(n+1)}{2}} \sum_{\eta,\xi\in S_{n/2}} \prod_{i=1}^{n/2} a_{2i-1+2\eta(i)} \binom{2i-1+2\eta(i)}{2i-1}\times\\ &\times \prod_{i=1}^{n/2} a_{2i-2+2\xi(i)-1} \binom{2i-2+2\xi(i)-1}{2i-2} \mathrm{sign}(\xi) \ .\end{aligned}$$

We now recognize in the expression above the product of two determinants, so we rewrite it in the form

$$\begin{aligned}&h\left(\boldsymbol{x}\right) h_{(1,2,\ldots,n)}\left(\boldsymbol{x}\right)(-1)^{\frac{n(n+1)}{2}} \det\left[a_{2i-1+2j}\binom{2i-1+2j}{2i-1}\right] \det\left[a_{2i-3+2j}\binom{2i-3+2j}{2i-2}\right]\\ =\;& C_{\mathrm{even}}(n) h\left(\boldsymbol{x}\right) h_{(1,2,\ldots,n)}\left(\boldsymbol{x}\right)(-1)^{\frac{n(n+1)}{2}} \det\left[\left(a_{2i+2j-1}\binom{2i+2j-1}{2i-1}\right)_{i,j=1}^{\frac{n}{2}}\right] \ .\end{aligned}$$

Thus we conclude that

$$C_{\mathrm{odd}}(n)=(-1)^{\frac{n}{2}} \det\left[\left(a_{2i+2j-1}\binom{2i+2j-1}{2i}\right)_{i,j=1}^{\frac{n}{2}}\right] \ .$$

5.3 Proof of Proposition 2.3

Suppose first n is even. Since polynomials W_k are linear combinations of generalized Vandermonde determinants and the minimal degree of Vandermonde determinant is $n(n-1)/2$ we have

$$\mathrm{Pf}(\boldsymbol{Q}) = \sum_{k=n(n-2)/4}^{\infty} t^{-k} W_k(\boldsymbol{x}) \ ,$$

which together with Lemma 2.1 yield

$$\lim_{t\to\infty} \mathrm{Pf}(\boldsymbol{Q}) t^{n(n-2)/4} = C_{\mathrm{even}}(n) h(\boldsymbol{x}) \ .$$

Suppose now that n is odd. Recall the notation $\boldsymbol{Q}_{(l)} = (p_{ij})_{i,j\neq l}$. We need then to work out the sum:

$$\begin{aligned}
\sum_{l=1}^{n}(-1)^{l+1}\mathrm{Pf}(\boldsymbol{Q}_{(l)}) &= \sum_{l=1}^{n}(-1)^{l+1}\sum_{k=0}^{\infty} t^{-k} W_k(\boldsymbol{x}_{(l)}) \\
&= \sum_{k=0}^{\infty} t^{-k} \sum_{l=1}^{n}(-1)^{l+1} W_k(\boldsymbol{x}_{(l)}) \\
&= \sum_{k=0}^{\infty} t^{-k} \sum_{l=1}^{n}(-1)^{l+1} \sum_{|\boldsymbol{u}|\leq 2k+\frac{n-1}{2}} c(k;\boldsymbol{u}) h_{\boldsymbol{u}}(\boldsymbol{x}_{(l)}) \\
&= \sum_{k=0}^{\infty} t^{-k} \sum_{|\boldsymbol{u}|\leq 2k+\frac{n-1}{2}} \sum_{l=1}^{n}(-1)^{l+1} c(k;\boldsymbol{u}) h_{\boldsymbol{u}}(\boldsymbol{x}_{(l)}) \\
&= \sum_{k=0}^{\infty} t^{-k} \sum_{|\boldsymbol{u}|\leq 2k+\frac{n-1}{2}} c(k;\boldsymbol{u}) h_{(0,\boldsymbol{u})}(\boldsymbol{x}) \ .
\end{aligned}$$

We now make the following observations. If there is a zero entry in $\boldsymbol{u}$, then $h_{(0,\boldsymbol{u})}(\boldsymbol{x}) = 0$. Similarly $\boldsymbol{u}$ cannot have two entries the same. Thus the first non-vanishing element is for $\boldsymbol{u} = (1, 2, \ldots, n-1)$, which yields the minimal exponent $(n-1)^2/4$. The constant standing at $t^{(n-1)^2/4} h_{(1,2,\ldots,n-1)}$ in the asymptotic expansion of Pfaffian (remember that of matrix of size $n-1$) is called $C_{\mathrm{odd}}(n-1)$ which is calculated in Lemma (2.2). Hence we have

$$\lim_{t\to\infty} \left(\sum_{l=1}^{n} (-1)^{l+1} \mathrm{Pf}(\boldsymbol{Q}_{(l)}) \right) t^{(n-1)^2/4} = C_{\mathrm{odd}}(n-1) h(\boldsymbol{x}), \tag{5.23}$$

5.4 Calculating constants

Let n be even. We will work out alternative expressions for constant (2.7):

$$
\begin{aligned}
C_{\text{even}}(n) &= \det\left[\left(a_{2i+2j+1}\binom{2i+2j+1}{2i}\right)_{i,j=0}^{\frac{n}{2}-1}\right] \\
&= \det\left[\left(\frac{(-1)^{i+j}}{(i+j)!2^{i+j-1}\,(2i+2j+1)}\frac{(2i+2j+1)!}{(2i)!\,(2j+1)!}\right)_{i,j=0}^{\frac{n}{2}-1}\right] \\
&= \prod_{i=0}^{\frac{n}{2}-1}\frac{1}{(2i)!\,(2i+1)!2^{i-1}2^{i}}\det\left[\left(\frac{(2i+2j)!}{(i+j)!}\right)_{i,j=0}^{\frac{n}{2}-1}\right] \\
&= 2^{-\frac{n(n-4)}{4}}\prod_{i=0}^{n-1}\frac{1}{i!}\det\left[\left(\frac{(2i+2j)!}{(i+j)!}\right)_{i,j=0}^{\frac{n}{2}-1}\right]
\end{aligned}
$$

We now consider the determinant in the product above (with the substitution $K=\frac{n}{2}-1$). Since

$$
\frac{(2i+2j)!}{(i+j)!}=\frac{(2i)!}{i!}2^{i}\prod_{k=0}^{i-1}(2j+1+2k)
$$

we have

$$
\det\left[\left(\frac{(2i+2j)!}{(i+j)!}\right)_{i,j=0}^{K}\right]=\prod_{i=0}^{K}\left(\frac{(2i)!2^{i}}{i!}\right)\det\left[\left(\prod_{k=0}^{i-1}(2j+1+2k)\right)_{j,i=0}^{K}\right].
$$

Using

$$
\begin{aligned}
\det\left[\left(\prod_{k=0}^{i-1}(2j+1+2k)\right)_{j,i=0}^{K}\right] &= h\left((2j+1)_{j=0}^{K}\right) \\
&= \prod_{i<j}(2j+1-2i-1)=\prod_{i=1}^{K}(2i)!!
\end{aligned}
$$

we write

$$
\begin{aligned}
\det\left[\left(\frac{(2i+2j)!}{(i+j)!}\right)_{i,j=0}^{K}\right] &= \prod_{i=0}^{K}\left(\frac{(2i)!2^{i}\,(2i)!!}{i!}\right) \\
&= \prod_{i=0}^{K}\left((2i)!2^{i}2^{i}\right)=2^{K(K+1)}\prod_{i=0}^{K}(2i)!\,.
\end{aligned}
$$

Thus finally

$$
\begin{aligned}
C_{\text{even}}(n) &= \det\left[\left(a_{2i+2j+1}\binom{2i+2j+1}{2i}\right)_{i,j=0}^{\frac{n}{2}-1}\right] \\
&= 2^{-\frac{n(n-4)}{4}} \prod_{i=0}^{n-1} \frac{1}{i!} 2^{\left(\frac{n}{2}-1\right)\frac{n}{2}} \prod_{i=0}^{\frac{n}{2}-1} (2i)! \\
&= 2^{\frac{n}{2}} \prod_{i=0}^{n-1} \frac{1}{i!} \prod_{i=0}^{\frac{n}{2}-1} (2i)!\,.
\end{aligned}
$$

Consider now the constant defined in (2.8). Similarly as before

$$
\begin{aligned}
C_{\text{odd}}(n) &= (-1)^{\frac{n}{2}} \det\left[\left(a_{2i+2j-1}\binom{2i+2j-1}{2i}\right)_{i,j=1}^{\frac{n}{2}}\right] \\
&= (-1)^{\frac{n}{2}} \det\left[\left(\frac{(-1)^{i+j-1}}{(i+j-1)!2^{i+j-2}\,(2i+2j-1)} \frac{(2i+2j-1)!}{(2i)!\,(2j-1)!}\right)_{i,j=1}^{\frac{n}{2}}\right] \\
&= \prod_{i=1}^{\frac{n}{2}} \frac{1}{(2i)!\,(2i-1)!2^{i-2}2^{i}} \det\left[\left(\frac{(2i+2j-2)!}{(i+j-1)!}\right)_{i,j=1}^{\frac{n}{2}}\right] \\
&= 2^{\frac{4n}{4}-\frac{n}{2}\left(\frac{n}{2}+1\right)} \prod_{i=1}^{n} \frac{1}{i!} \det\left[\left(\frac{(2i+2j-2)!}{(i+j-1)!}\right)_{i,j=1}^{\frac{n}{2}}\right] \\
&= 2^{-\frac{n(n-2)}{4}} \prod_{i=1}^{n} \frac{1}{i!} \det\left[\left(\frac{(2i+2j-2)!}{(i+j-1)!}\right)_{i,j=1}^{\frac{n}{2}}\right]\,.
\end{aligned}
$$

The determinant in the product above is (with $K = \frac{n}{2}$)

$$
\det\left[\left(\frac{(2i+2j-2)!}{(i+j-1)!}\right)_{i,j=1}^{K}\right] = \prod_{i=1}^{K}\left(\frac{(2i)!2^{i-1}}{i!}\right) \det\left[\left(\prod_{k=1}^{i-1}(2j-1+2k)\right)_{j,i=1}^{K}\right]\,.
$$

Notice that

$$
\begin{aligned}
\det\left[\left(\prod_{k=1}^{i-1}(2j-1+2k)\right)_{j,i=1}^{K}\right] &= h\left((2j-1)_{j=1}^{K}\right) \\
&= \prod_{i=1}^{K-1}(2i)!!
\end{aligned}
$$

and hence

$$\det\left[\left(\frac{(2i+2j)!}{(i+j)!}\right)_{i,j=1}^{K}\right] = \prod_{i=1}^{K}\left(\frac{(2i)!2^{i-1}}{i!}\right)\prod_{i=1}^{K-1}(2i)!!$$
$$= \prod_{i=0}^{K}\left((2i)!2^{i-1}\right)\prod_{i=1}^{K-1}2^{i}\frac{1}{K!} = \frac{2^{K(K-1)}}{K!}\prod_{i=0}^{K}(2i)!\,.$$

Thus

$$C_{\text{odd}}(n) = (-1)^{\frac{n}{2}}\det\left[\left(a_{2i+2j+1}\binom{2i+2j-1}{2i}\right)_{i,j=1}^{\frac{n}{2}}\right]$$
$$= 2^{-\frac{n(n-2)}{4}}\prod_{i=1}^{n}\frac{1}{i!}2^{\frac{n(n-2)}{4}}\frac{1}{\left(\frac{n}{2}\right)!}\prod_{i=0}^{\frac{n}{2}}(2i)! = \frac{1}{\left(\frac{n}{2}\right)!}\prod_{i=0}^{n}\frac{1}{i!}\prod_{i=0}^{\frac{n}{2}}(2i)!\,.$$

We now demonstrate that our constants are consistent with the ones in Grabiner theorem. Following Mehta [10, p. 354],

$$E\left[h(Y)\,1_W(Y)\right] = (2\pi)^{-n/2}\int_W e^{-\frac{|\boldsymbol{y}|^2}{2}}h(\boldsymbol{y})\,d\boldsymbol{y}$$
$$= \frac{1}{n!}\prod_{j=1}^{n}\frac{\Gamma\left(1+\frac{j}{2}\right)}{\Gamma\left(1+\frac{1}{2}\right)}\,,$$

where Y is the vector of i.i.d. standard random variables. Suppose $n \in 2\mathbb{N}$. Since

$$\Gamma\left(\frac{1}{2}+j\right) = \frac{(2j-1)!!}{2^j}\sqrt{\pi}$$

(see e.g. Abramowitz and Stegun [1, formula 6.1.12]), we have

$$\frac{1}{n!}\prod_{j=1}^{n}\frac{\Gamma\left(1+\frac{j}{2}\right)}{\Gamma\left(1+\frac{1}{2}\right)} = \frac{1}{n!}\left(\frac{1}{2}\sqrt{\pi}\right)^{-n}2^{-\frac{n}{2}\left(\frac{n}{2}+1\right)/2}\left(\sqrt{\pi}\right)^{\frac{n}{2}}\prod_{j=1}^{\frac{n}{2}}(2j-1)!!\prod_{j=1}^{\frac{n}{2}}\left(2^j2^{-j}j!\right)$$
$$= \frac{1}{n!}(2\pi)^{-\frac{n}{4}}\,2^{-\frac{n(n-8)}{8}}2^{-\frac{n}{2}\left(\frac{n}{2}+1\right)/2}\prod_{j=1}^{\frac{n}{2}}(2j-1)!!\prod_{j=1}^{\frac{n}{2}}(2j)!!$$
$$= \frac{1}{n!}(2\pi)^{-\frac{n}{4}}\,2^{-\frac{n(n-3)}{4}}\prod_{j=1}^{\frac{n}{2}}(2j)! = (2\pi)^{-\frac{n}{4}}\,2^{-\frac{n(n-3)}{4}}\prod_{j=1}^{\frac{n}{2}-1}(2j)!\,.$$

Suppose now n is odd. Then

$$\begin{aligned} E\left[h\left(y\right)1_W\left(Y\right)\right] &= \frac{1}{n!}\left(2\pi\right)^{-\frac{n-1}{4}}2^{-\frac{(n-1)(n-1-3)}{4}}\prod_{j=1}^{\frac{n-1}{2}}\left(2j\right)!\frac{\Gamma\left(1+\frac{n}{2}\right)}{\Gamma\left(1+\frac{1}{2}\right)} \\ &= \frac{1}{n!}\left(2\pi\right)^{-\frac{n-1}{4}}2^{-\frac{(n-1)(n-4)}{4}}\prod_{j=1}^{\frac{n-1}{2}}\left(2j\right)!\frac{n!!}{2^{\frac{n+1}{2}}}2 \\ &= \left(2\pi\right)^{-\frac{n-1}{4}}2^{-\frac{n^2-3n+2}{4}}\prod_{j=1}^{\frac{n-1}{2}}\left(2j\right)!\frac{n!!}{n!}\ . \end{aligned}$$

Since

$$\frac{n!!}{n!}=\frac{1}{(n-1)!!}=\frac{1}{2^{\frac{n-1}{2}}\left(\frac{n-1}{2}\right)!},$$

the above equals

$$E\left[h\left(Y\right)1_W\left(Y\right)\right]=\left(2\pi\right)^{-\frac{n-1}{4}}2^{-\frac{n(n-1)}{4}}\prod_{j=1}^{\frac{n-1}{2}}\left(2j\right)!\frac{1}{\left(\frac{n-1}{2}\right)!}\ .$$

We conclude the considerations in this subsection.

Lemma 5.5 *If n is even, then*

$$E\left[h\left(Y\right)1_W\left(Y\right)\right]=\left(2\pi\right)^{-\frac{n}{4}}2^{-\frac{n(n-3)}{4}}\prod_{j=1}^{\frac{n}{2}-1}\left(2j\right)!$$

and if n is odd, then

$$E\left[h\left(Y\right)1_W\left(Y\right)\right]=\left(2\pi\right)^{-\frac{n-1}{4}}2^{-\frac{n(n-1)}{4}}\prod_{j=1}^{\frac{n-1}{2}}\left(2j\right)!\frac{1}{\left(\frac{n-1}{2}\right)!}\ .$$

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