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The Time for a Critical Nearest Particle System to reach Equilibrium starting with a large Gap

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Abstract: We consider the time for a critical nearest particle system, starting in equilibrium subject to possessing a large gap, to achieve equilibrium.

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1 Introduction

In this paper we consider critical Nearest Particle Systems. A Nearest Particle System (NPS) is a spin system on $\{0,1\}^{\mathbb{Z}}$. For $x \in \mathbb{Z}$ and $\eta \in \{0,1\}^{\mathbb{Z}}$, the flip rate at site x for configuration η is given by

$$c(x,\eta) = \begin{cases} 1 & \text{if} \quad \eta(x) = 1, \\ f(l_\eta(x), r_\eta(x)) & \text{if} \quad \eta(x) = 0, \end{cases}$$

where f is a real valued nonnegative function defined on ordered pairs (l, r), where each coordinate is a strictly positive integer or infinity: $l_\eta(x) = x - \sup\{y < x : \eta(y) = 1\}$ and $r_\eta(x) = \inf\{y > x : \eta(y) = 1\} - x$ (either or both possibly ∞).

Of particular interest are the so-called *reversible* NPSs. These are systems where $f(l,r)$ is of the form $\frac{\beta(l)\beta(r)}{\beta(l+r)}$, $f(l,\infty) = f(\infty,l) = \beta(l)$, $f(\infty,\infty) = 0$, where β is a real valued nonnegative function on the strictly positive integers. This class of particle systems was introduced by [13]. A NPS is reversible in the classical sense only if $f(.,.)$ is of this form (see [7]). These processes are of mathematical interest partly because there is an array of reversible Markov chain techniques with which to analyze them. This paper considers reversible NPSs.
We will also require the condition

$$\frac{\beta(n)}{\beta(n+1)} \downarrow 1 \quad \text{as} \quad n \to \infty. \qquad (*)$$

The convergence of the quotient to one is equivalent to the reversible NPS being *Feller*, a naturally desirable property. The assumption of monotonic convergence down to one ensures that the process is *attractive*. This makes the process much more mathematically tractable; see [7] for a complete treatment of NPSs as well as of attractiveness.

In this paper we will use the adjective *infinite* to describe a reversible NPS on $\{0,1\}^{\mathbb{Z}}, (\eta_t : t \geq 0)$ such that a.s. for all t, $\sum_{x\leq 0} \eta_t(x) = \sum_{x\geq 0} \eta_t(x) = \infty$. A reversible NPS $(\eta_t : t \geq 0)$ such that a.s. for all t, $\sum_{x\leq 0} \eta_t(x) = \infty$ and $\sum_{x\geq 0} \eta_t(x) < \infty$ or such that a.s. for all t, $\sum_{x\geq 0} \eta_t(x) = \infty$ and $\sum_{x\leq 0} \eta_t(x) < \infty$ will be called *semi-infinite*. A *right sided* reversible NPS is a semi-infinite reversible NPS for which for all time t, a.s. , $\sum_{x<0} \eta_t(x) = \infty$ and $\sum_{x>0} \eta_t(x) < \infty$. For such processes we denote the position of the rightmost particle at time t ($\sup\{x : \eta_t(x) = 1\}$) by r_t. Similarly a *left sided* reversible NPS is a semi-infinite NPS for which a.s. for all time $t, \sum_{x>0} \eta_t(x) = \infty$ and $\sum_{x<0} \eta_t(x) < \infty$. For such processes we denote the position of the leftmost particle at time t, ($\inf\{x : \eta_t(x) = 1\}$), by l_t. Finally a *finite* reversible NPS on $\{0,1\}^{\mathbb{Z}}(\eta_t : t \geq 0)$ is one for which, a.s., for all $t \geq 0, \sum_x \eta_t(x) < \infty$. In similar fashion, we will speak of finite, infinite , semi-infinite, right sided and left sided configurations $\eta \in \{0,1\}^{\mathbb{Z}}$ so that for instance a reversible NPS $(\eta_t : t \geq 0)$ on $\{0,1\}^{\mathbb{Z}}$ is finite if and only if a.s. for every $t \geq 0, \eta_t$ is finite.

In this article we will treat reversible NPSs corresponding to functions β such that $\sum_{n=1}^{\infty} \beta(n) < \infty$ (in fact we will assume rather more, see $(***)$ below). In this case, as is easily seen, a reversible NPS $(\eta_t : t \geq 0)$ is finite, infinite, right sided or left sided if and only if the initial configuration, η_0, is a.s. finite, infinite, right sided or left sided respectively.

One has two notions of survival for a reversible NPS, $\eta_{\cdot}$. For finite systems one says that $\eta_{\cdot}$ survives if $P^{\eta_0}(\eta_t \neq \underline{0}\ \forall\ t \geq 0) > 0$, where $\underline{0}$ is the trap state of all 0's (similarly $\underline{1}$ will denote the configuration consisting of all 1's) and $P^{\eta}(.)$ is the probability measure for a reversible NPS starting from state η. For infinite systems, one says that $\eta_{\cdot}$ survives if there exists a non-trivial equilibrium measure μ. That is a measure μ on $\{0,1\}^{\mathbb{Z}}$ so that for all continuous functions f defined on this space and all positive t

$$\int f(\eta) d\mu = \int P_t f(\eta) d\mu,$$

where $(P_t)_{t \geq 0}$ is the semigroup for the reversible NPS. [4] proves that for all reversible NPS with β satisfying $(*)$ and such that $\sum_{n=1}^{\infty} \beta(n) < \infty$, the finite processes survive if and only if $\sum_{n=1}^{\infty} \beta(n) > 1$. In the infinite case with $\beta(.)$ satisfying $(*)$ there is survival if and only if either $\sum_{n=1}^{\infty} \beta(n) > 1$ or $\sum_{n=1}^{\infty} \beta(n) = 1$ and $\sum_{n=1}^{\infty} n\beta(n) < \infty$ (which certainly hold under our further assumption $(***)$ below). Thus while conditions on $\beta(.)$ for survival of finite and infinite reversible NPSs do not coincide, the cases where

$$\sum_{n=1}^{\infty} \beta(n) = 1, \qquad (**)$$

are critical for both. In this paper we will restrict attention to a class of critical reversible NPSs, i.e., reversible NPSs with β satisfying $(*)$ and $(**)$.

If the conditions $(*)$ and $(**)$ hold and the condition

$$n\left(\frac{\beta(n)}{\beta(n+1)} - 1\right) \to k \in [1500, \infty), \qquad (***)$$

is also satisfied, then a non-trivial equilibrium measure exists and is equal to the renewal measure on $\{0,1\}^{\mathbb{Z}}$ that corresponds to the probability measure $\beta(\cdot)$ on the integers. Subsequently we will denote this renewal measure by $Ren(\beta)$. It is known (see e.g. [7]) that $Ren(\beta)$ is the *upper equilibrium* for β-NPSs in the sense that it is greater than any other equilibrium in the natural partial order on measures on $\{0,1\}^{\mathbb{Z}}$. The strong condition $(***)$ is similar to that imposed in [8] and is certainly not optimal. However we have sought to avoid adding to the technical aspects of the paper, since we cannot reduce the bound 1500 to a "realistic" bound. [10] discusses the case

$$n^{\alpha}\left(\frac{\beta(n)}{\beta(n+1)} - 1\right) \to v \in (0, \infty) \text{ for } 0 < \alpha < 1$$

where the scaling is different.

Henceforth, $\beta(\cdot)$ will denote a fixed positive function on the positive integers satisfying $(*)$, $(**)$ and $(***)$. A $\beta-$NPS will be a critical reversible NPS whose flip rates correspond to this fixed function β. In particular, the process will be reversible, Feller, attractive and critical. In addition $Ren(\beta)$ will denote the renewal measure on $\{0,1\}^{\mathbb{Z}}$ associated with $\beta(\cdot)$.

We will assume that the β-NPS $(\eta_t : t \geq 0)$, is generated by a given Harris system, which will also generate auxiliary comparison processes (see [1] for a general treatment of Harris constructions). For this, we suppose that we are given for each $x \in \mathbb{Z}$ two Poisson processes D_x and B_x, independent of each other and independent over $x \in \mathbb{Z}$, with D_x of rate 1 and B_x of rate $M = \frac{(\beta(1))^2}{\beta(2)}$, the maximum flip rate from spin value 1 to value 0. The spin value (or simply spin) at site x can flip at time t only if there is a jump in either D_x or in B_x at time t, i.e., $t \in D_x$ or $t \in B_x$. The process D_x corresponds to flips of 1's at site x (or deaths of particles) and is simple: if $t \in D_x$ then $\eta_t(x) = 0$, irrespective of its value immediately preceding time t. The Poisson process B_x corresponds to flips of 0's to 1's at site x (or births at site x). For this process, associated to the i'th point $t_i \in B_x$ is a random variable $U_{x,i}$ that is uniform on [0, 1]. At time $t = t_i \in B_x$, we have $\eta_t(x) = 1$ if either $\eta_{t-}(x) = 1$ or if $U_{x,i} \leq \frac{c(x,\eta_{t-})}{M}$, where η_{t-} is the limiting configuration immediately before time t. The uniform random variables $U_{x,i}$ are independent as x and i vary and also are independent of the Poisson processes $\{B_x, D_x\}_{\mathbb{Z}}$. If t is the i'th point of B_x we also denote $U_{x,i}$ by $U^{x,t}$. We may on occasion assume that additional independent Poisson processes belong to the system.

In this paper, a gap (for a configuration η) is an interval on which the configuration is zero. We say that configuration η has a gap of size R (or an R gap) in interval V, if there exists an interval I of length R contained in V on which η is zero.

The aim of this paper is to consider how quickly a β-NPS, starting from a configuration η_0 distributed as $Ren(\beta)$ conditioned upon having a large gap in some large interval V, converges to the upper equilibrium measure, $Ren(\beta)$. This question is somewhat vague. We propose three more or less equivalent formulations of the question. The first addresses the time for the gap to disappear, the second gives a coupling notion of equilibrium and the third a distributional convergence result. In all cases we consider a family of β-NPSs $(\eta_t^N : t \geq 0)$ with N a positive integer so that for $a < b$ fixed,

A) η_0^N is identically zero on (aN, bN),

B) The distribution of the restriction of η_0^N to interval $(-\infty, [aN]]$, $\eta_0^N|_{(\infty,[aN]]}$, is the restriction of Ren(β) conditioned on $\eta_0^N([aN]) = 1$ (here [ ] denotes the integer part :[x] is the greatest integer less than or equal to x),

C) The distribution of the restriction of η_0^N to interval $[[(bN+1)-], \infty)$, $\eta_0^N|_{[[(bN+1)-],\infty)}$ is Ren(β) conditioned on $\eta_0^N([(bN+1)-]) = 1$ and is independent of $\eta_0^N|_{(-\infty,[aN]]}$. (Here $[x-]$ denotes the greatest integer strictly less than x.)

The interval (aN, bN) is said to be the initial gap. In order to talk of "the gap" at time t for $\eta_\cdot^N$, we wish to define processes $r_\cdot^N$ and $\ell_\cdot^N$ and stopping time σ^N (w.r.t. the natural filtration of the Harris system and η_0) that satisfy the following conditions:

(I) $\forall\, t < \sigma^N, r_t^N < \ell_t^N, \quad \eta_t^N(r_t^N) = \eta_t^N(\ell_t^N) = 1$, and $\eta_t(x) = 0 \qquad \forall\, x \in (r_t^N, \ell_t^N)$;

(II) $\forall\, t \geq \sigma^N, r_t^N = \ell_t^N$.

In particular, we wish σ^N to denote the time at which the gap vanishes. This is slightly delicate because when a birth occurs within the gap, one needs to determine whether or not that birth should be interpreted as a mere shrinking of the gap from one side or the other or the vanishing time when the two edges meet in the middle. Furthermore there are also times where a birth (or attempted birth) outside the gap should be interpreted as one edge overtaking the other and which accordingly should be regarded as the vanishing time. In both instances, our definition considers whether or not the birth would have occurred in a related semi-infinite process as follows:
(1) $r_0^N = [aN], \ell_0^N = [(bN+1)-], \quad t_0^N = 0$;
(2) $t_1^N = \inf\{t > t_0^N : \eta_{t_-}(x) \neq \eta_t^N(x)$ some $x \in [r_{t_0^N}, \ell_{t_0^N}],\ t \in B^x, U^{x,t} \leq \beta(x - r_0^N)/M$ for some $x > \ell_0^N$ or $t \in B^x, U^{x,t} \leq \beta(\ell_0^N - x)/M$ for some $x < r_0^N\}$;
(3) for $t \in (t_0^N, t_1^N)$, $r_t^N = r_{t_0^N}^N$ and $\ell_t^N = \ell_{t_0^N}^N$;
(4)
(a) if $\eta_{t_1^N}^N(r_{t_0^N}^N) \neq 1$, then

$$\begin{aligned} r_{t_1^N}^N &= \sup\{x < r_{t_0^N}^N : \eta_{t_1^N-}^N(x) = 1\} \text{ and} \\ \ell_{t_1^N}^N &= \ell_{t_0^N}^N, \end{aligned}$$

(b) if $\eta_{t_1^N}^N(\ell_{t_0^N}^N) \neq 1$, then

$$\begin{aligned} r_{t_1^N}^N &= r_{t_0^N}^N \text{ and} \\ \ell_{t_1^N}^N &= \inf\{x > \ell_{t_0^N}^N : \eta_{t_1^N-}^N(x) = 1\}, \end{aligned}$$

(c) if $\eta_{t_1^N}^N(x) = 1$ some $x \in \left[\frac{\ell_{t_0^N}^N + r_{t_0^N}^N}{2}, \ell_{t_0^N}^N\right)$, then

(i) if $U^{x,t_1^N} \leq \beta(x - r_{t_0^N}^N)/M$, then

$$\begin{aligned} r_s^N &= \ell_{t_0^N}^N \text{ and } \ell_s^N = \ell_{t_0^N}^N\ \forall\, s \geq t_1^N, \\ \sigma^N &= t_1^N. \end{aligned}$$

(ii) otherwise

$$r_{t_1^N} = r_{t_0^N} \text{ and } \ell_{t_1^N} = x,$$

(d) if $\eta^N_{t_1^N}(x) = 1$ some $x \in \left(r^N_{t_0^N}, \frac{\ell^N_{t_0^N}+r^N_{t_0^N}}{2}\right)$, then

(i) if $U^{x,t_1^N} \leq \beta(\ell^N_{t_0^N} - x)/M$, then

$$\begin{aligned} r^N_s &= r^N_{t_0^N} \text{ and } \ell^N_s = r^N_{t_0^N} \ \forall\ s \geq t_1^N, \\ \sigma^N &= t_1^N. \end{aligned}$$

(ii) otherwise

$$\ell_{t_1^N} = \ell_{t_0^N} \text{ and } r_{t_1^N} = x;$$

(e) If $t \in B^x, U^{x,t} \leq \beta(x - r_0^N)/M$ for some $x > \ell_0^N$, then

$$\begin{aligned} r^N_s &= \ell^N_{t_0^N} \text{ and } \ell^N_s = \ell^N_{t_0^N} \ \forall\ s \geq t_1^N, \\ \sigma^N &= t_1^N; \end{aligned}$$

(f) if $t \in B^x, U^{x,t} \leq \beta(\ell_0^N - x)/M$ for some $x < r_0^N$, then

$$\begin{aligned} r^N_s &= r^N_{t_0^N} \text{ and } \ell^N_s = r^N_{t_0^N} \ \forall\ s \geq t_1^N, \\ \sigma^N &= t_1^N. \end{aligned}$$

The above inductive construction is repeated until such time as (4)(c)(i), (4)(d)(i), (4)(e) or (4)(f) happen, at which point the full construction is achieved.

We will henceforth regard the "gap" at time t as being the interval (r_t^N, ℓ_t^N).

We now consider the evolution of $r^N_\cdot$ (and by reflection of $\ell^N_\cdot$). For $r_t^N + x$ in interval (r_t^N, ℓ_t^N), the flip rate at time t is

$$\frac{\beta(x)\beta(\ell_t^N - r_t^N - x)}{\beta(\ell_t^N - r_t^N)} \;=\; \beta(x) + \beta(x)\left(\frac{\beta(\ell_t^N - r_t^N - x)}{\beta(\ell_t^N - r_t^N)} - 1\right).$$

For x fixed and $(\ell_t^N - r_t^N)$ of order N this is (using assumption (***))

$$\beta(x) + \beta(x)\left(\frac{kx}{\ell_t^N - r_t^N}\right) \;+\; o(\frac{1}{N}).$$

We consider these three terms in turn. The first term, $\beta(x)$, is the flip rate if the spins to the right of $r^N_\cdot$ were set to zero (or equally the flip rate that obtains if the 1's to the right of $r^N_\cdot$ are ignored). The second term is small (of order $\frac{1}{N}$) but non-negligible, and will be regarded as an "extra fliprate" while the third it will eventually turn out is just a nuisance factor.

Consider the process $(\eta_t^{R,N} : t \geq 0)$ where $\eta_t^{R,N}(x) \ = \ I_{x \leq r_t^N} \eta_t^N(x)$ is approximately a rightsided β-NPS (at least if $\ell_t^N - r_t^N$ is large). It is easy to verify that for a right sided β-NPS, $(\eta_t : t \geq 0)$, if the distribution of

$$\Xi_0(x) \ = \ \eta_0(x + r_0) \text{ for } x \ \leq \ 0$$

(recall $r_0 \ = \ \sup\{x : \eta_0(x) \ = \ 1\}$) is $\mathrm{Ren}(\beta)$ restricted to $\{0,1\}^{(-\infty,0]}$ and conditioned to have a 1 at the origin then for all $t \geq 0$

$$\Xi_t(x) \ = \ \eta_t(x + r_t) \text{ for } x \ \leq \ 0$$

shares this distribution. We say, with a small abuse of terminology, that a right sided β-NPS $(\eta_t : t \geq 0)$ is in equilibrium if Ξ_0, as defined above, has the above distribution. By considering $Ren(\beta)$ restricted to $[0, \infty)$ and conditioned to have 1 at the origin, this notion of equilibrium extends to left sided β-NPSs.

We say that a right sided configuration η_0, at time 0, is supported by $(-\infty, x]$ if its rightmost particle is at site x. We will use the following notation throughout the article: $Ren^{(x,y]}(\beta)$ will represent the measure $Ren(\beta)$ restricted to $\{0,1\}^{(x,y]}$ and conditioned on there being a 1 at site y but with no conditioning on the open boundary point. If the term $(x, y]$ is replaced by $[x, y]$, then $Ren(\beta)$ is renewal measure on $\{0,1\}^{[x,y]}$ conditioned to have a 1 at sites x and y. Equally $Ren^{(x,y)}(\beta)$ is simply the unconditioned restriction of $Ren(\beta)$ to $\{0,1\}^{(x,y)}$. Thus, for example, if η_0 is an equilibrium right sided configuration supported on $(-\infty, x]$, then its distribution is $Ren^{(-\infty,x]}(\beta)$. Note that $Ren^{(0,n]}(\beta)$ is not the same measure as $Ren^{[1,n]}(\beta)$.

In much of the following we establish or quote results for right sided $\beta - NPSs$. By symmetry the results also apply in an obvious fashion to left sided $\beta-$NPSs and will be used in this way without comment.

In [11] it was shown:

Theorem 1 *Let $(\eta_t^R : t \geq 0)$ be a right sided $\beta-NPS$ in equilibrium supported by $(-\infty, 0]$ at time 0. Denote the position of its rightmost occupied site at time t by r_t. Then as N tends to infinity $(\frac{r_{N^2 t}}{N} : t \geq 0)$ tends in distribution to a positive constant σ times a standard Brownian motion.*

For details of the invariance principle for the rightmost particle process $(r_t : t \geq 0)$ the reader is referred to [11]. Here and throughout the paper convergence in distribution of a sequence of processes defined on an interval $[0, T]$ (respectively $[0, \infty)$) is meant in the sense of Skorohod convergence on the space $D[0, T]$ (respectively $D[0, \infty)$).

In studying $\eta_\cdot^{R,N}(x) \ = \ I_{x \leq r_\cdot^N} \eta_\cdot^N(x)$ up to time σ^N, we will be considering what can be regarded as a rightsided β-NPS with extra flip rate to the right of $r_\cdot^N$, which are of order $\frac{1}{N}$, $\beta(y)\frac{ky}{\ell_\cdot^N - r_\cdot^N}$ y units to the right of $r_\cdot^N$. These "extra" jumps will over a time interval of order N^2 (suggested by the invariance principle above) contribute an effect of order $N^2 \times \frac{1}{N}$ and so cannot be ignored. In fact if we consider the (normalized) process

$$(\frac{\ell^N_{N^2t} - r^N_{N^2t}}{N} : t \geq 0),$$

it will (it turns out) tend in distribution to $(X_t : t \geq 0)$ the diffusion on $[0, \infty)$, starting at value $(b-a)$ for which 0 is a trap and so that

$$dX_t = \sqrt{2}\sigma dW_t - \frac{2}{X_t}\nu dt$$

for $X_t \neq 0$, where $W_{.}$ is a standard Brownian motion, σ the constant fixed by Theorem 1 and ν is equal to $\sum_n kn\beta(n)c(n)$ for constant k given by $(***)$ and the positive constants $c(\cdot)$ defined and discussed in Section 5.

We define for any $d \in (1/2, 1)$ fixed and positive integer N,

$$\tau^N = \inf\{t > 0 : \text{ for } \eta^N_t \text{ there is no } \quad N^d \text{ gap } \text{ in } [-N^2, N^2]\}.$$

The choice of d is not important beyond the fact that it must be strictly below 1 and "not too small" so that for a $\beta-$NPS $\eta_{.}$ in $Ren(\beta)$ equilibrium, the occurrence of a N^d gap in interval $[-N^2, N^2]$ during time interval $[0, \lambda N^2]$ is an event of small probability as N becomes large with λ held fixed. See the remark after the statement of Theorem 2.

Theorem 2 *Let σ^N, τ^N be as previously defined. For any fixed $d \in (1/2, 1)$ as N tends to infinity,*

$$\frac{\tau^N}{N^2}, \qquad \frac{\sigma^N}{N^2} \quad \xrightarrow{D} \quad \tau$$

where τ is the hitting time of zero for the diffusion $X_{.}$ as above which starts from value $b - a$.

Remark: In general the stopping times τ^N, σ^N should have little to do with one another. A priori one could have a birth near the center of the gap (as defined above) well before an interval of length of order N disappears. However the condition $(***)$ makes such an occurrence highly unlikely. Equally it could be the case that σ^N occurs well after the gap has been reduced in size to order N^d. In point of fact both stopping times turn out to be very close (to scale N^2) to stopping times $\tau^{N,\epsilon} = \inf\{t \geq 0 : \ell^N_t - r^N_t \leq N\epsilon\}$ for ϵ small.

We can also consider $\eta^N_{.}$ to have achieved equilibrium in the following coupling way : define $(\tilde{\eta}^N_t : t \geq 0)$ to be the β-NPS run with the same Harris system as $(\eta^N_t : t \geq 0)$ so that $\tilde{\eta}^N_0$ is distributed as Ren(β) conditioned on $\tilde{\eta}^N_0(x) = \eta^N_0(x)\ \forall\ x \in (-\infty, [aN]] \cup [[bN + 1-]\infty)$. Let $\tilde{\tau}^N = \inf\{t : \eta^N_t = \tilde{\eta}^N_t\}$ then

Theorem 3 *For $\tau^N, \tilde{\tau}^N$ defined as above,*

$$\frac{|\tilde{\tau}^N - \tau^N|}{N^2} \xrightarrow{pr} 0.$$

Note that the distribution of $\tilde{\eta}_t^N$ is stochastically above Ren(β) $\forall t$ and stochastically below a β-NPS starting from all 1's. Since this latter process tends to Ren(β) in distribution as t tends to infinity we have that $\tilde{\eta}_t^N \xrightarrow{D}$ Ren(β) as $t \to \infty$ uniformly in N.

Another approach to this question is to fix a cylinder function f and consider $E[f(\eta_t^N)]$ for $E[\cdot]$ the expectation operator. For this, we need to introduce a Markov process $(X_t^1, X_t^2), t \geq 0$ on the set $\{(x,y) \in \mathbb{R}^2 : x \leq y\}$ for which all states $(x,x), x \in \mathbb{R}$ are traps and so that for $X_t^1 < X_t^2$,

$$\begin{aligned} dX_t^1 &= \sigma dW_t^1 + \tfrac{\nu dt}{X_t^2 - X_t^1}, \\ dX_t^2 &= \sigma dW_t^2 - \tfrac{\nu dt}{X_t^2 - X_t^1}, \end{aligned}$$

for $W_\cdot^i$ independent Brownian motions and σ and ν strictly positive constants. Also, for measure μ on $\{0,1\}^{\mathbb{Z}}$ and measurable function f, $< \mu, f >$ will represent $\int f(\eta) d\mu$.

Theorem 4 *Let f be a fixed cylinder function on $\{0,1\}^{\mathbb{Z}}$ and for every positive integer N let $\eta_\cdot^N$ be a $\beta-$NPS with η_0^N satisfying condition A,B and C of page 4. For fixed $t \geq 0$, as N tends to infinity*

$$E[f(\eta_{N^2 t}^N)] \to \lambda_t < Ren(\beta), f > + (1-\lambda_t) f(\underline{0})$$

where $\underline{0}$ denotes the configuration of all zeros,

$$\lambda_t = P(0 \notin (X_t^1, X_t^2),\ \tau > t) \ + \ P(\tau \leq t)$$

for process $(X_\cdot^1, X_\cdot^2)$ described above with $X_0^1 = a < b = X_0^2$ and $\tau = \inf\{t \geq 0 : X_t^1 = X_t^2\}$.

Remark: We use τ to denote both a stopping time in Theorem 3 and a hitting time in Theorem 2. However, as is easily seen the two τ's have the same distribution.

The main part of this paper consists in dealing with the evolution of right (and so left) sided β-NPSs, $\eta_\cdot^R$, with a small extra flip rate to the right of the rightmost particle. There are two issues to be addressed. The first is to quantify the effect of the extra jumps on the evolution of the rightmost particle. The second is to show that the extra jumps do not affect "too much" the distribution of

$$\Xi_t(x) \ = \ \eta_t^R(x + r_t) \text{ for } x \ \leq \ 0.$$

That is we wish to show that for relevant times t the distribution of the above configuration is approximately $Ren^{(-\infty,0]}(\beta)$.

The paper is organized as follows: In Section 2 we assemble some simple mixing results for β-NPSs and introduce some finite state comparison Markov chains. In Section 3 we introduce, for positive integer n, a comparison Markov chain $\eta^{F,n}_{\cdot}$ on state space $\{0,1\}^{(-n,0]}$ to "track" the (non-Markov) process

$$(\eta^R_t(r_t - x) I_{x \in (-n,0]} : t \geq 0)$$

for $\eta^R_{\cdot}$ a rightsided β-NPS having rightmost occupied site process $R_{\cdot}$ and with η^R_0 distributed as $Ren^{(-\infty,0]}(\beta)$. The importance of this comparison chain is that (as will be shown next in Section Four) the two processes can be coupled so that

$$\eta^R_t(r_t + x) = \eta^{F,n}_t(x) \text{ for } 0 \leq t \leq n^4, \ -n/2 < x \leq 0$$

with high probability. Given the good mixing properties of the chain $\eta^{F,n}_{\cdot}$, this amounts to showing that "with high probability $\eta^R_{\cdot}$ has good mixing properties". Section 5 begins to investigate the effect of extra jumps to the right of $r_{\cdot}$ for $r_{\cdot}$ the rightmost occupied site of a right sided β-NPS and shows the existence of constants $c(v)$ such that (in an averaging sense) the effect of an extra jump of v to the right by $r_{\cdot}$ on the longterm evolution of $r_{\cdot}$ is essentially a shift by $c(v)$. Section 6 applies these results and considers the effect on the position of the rightmost occupied site for a right sided β-NPS with an additional (low) rate of flips to the right of the rightmost particle. The key idea is that due to mixing and the fact that the time between "extra" jumps is typically large, the overall effect of the "extra" jumps is essentially the sum over the effects associated with the "individual" jumps. Section 7 considers the distribution of a right sided β-NPS starting from distribution $Ren^{(-\infty,0]}(\beta)$ with an additional flip rate to the right of the rightmost occupied site of the process. If this total extra rate is of order $\frac{1}{N}$, then it is shown that on a time interval $[0, N^{1+\alpha}]$ for $0 < \alpha < \frac{1}{2}$, that the distribution of the process is not very different from $Ren^{(-\infty,0]}(\beta)$. A convergence in distribution result is shown for $(\frac{r_{N^2 t}}{N} : t \geq 0)$ where $r_{\cdot}$ again denotes the position of the rightmost occupied site of the process. Section 8 extends this approach to give a limit law for the process $((\frac{r^N_{N^2 t}}{N}, \frac{\ell^N_{N^2 t}}{N}) : t \geq 0)$. Finally in the final section the technical details are supplied to obtain Theorems 2-4 from this convergence in law

Simplifying assumptions: in order to reduce notation in the following we assume

$$\sigma = 1, \qquad a = 0, \qquad b = 1;$$

it will immediately be seen that the arguments given do not require special values of a, b or σ for their validity, so while the assumption that $\sigma = 1$ represents a further restriction on the class of β considered directly, the results given will be valid for all β satisfying conditions $(*)$, $(**)$ and $(***)$.

Some conventions: in this article we will use c, C and K and other letters to denote constants which may vary from line to line (or even from one side of an equation to the other).

For a typical positive integer n and positive c, the number n^c will not be an integer, however when an integer is demanded (such as in discussing a distance or position for $\mathbb{Z}$) we will take n^c to represent the integer part of n^c.

We will make extensive use of the indicator function I_A. If A is an event then I_A is the random variable equal to 1 on the event A and equal to 0 on the complement of A. If A is some logical condition, then I_A will be one if A holds and 0 otherwise. $I_{x \in A}$ may also be used to specify a configuration in $\{0,1\}^{\mathbb{Z}}$.

We will abuse notation and take the supremum of the empty set to be $-\infty$ and the infimum to be ∞.

For a cadlag process $G_{\cdot}$ indexed by positive continuous time, G_{t-} will represent the left hand limit of G_s as s tends up to t.

We will on occasion employ the usual o and O terminology: a quantity $g(n)$, indexed by a variable n, is said to be $O(f(n))$ as n tends to infinity if there exists a finite K so that for n sufficiently large $|g(n)| \leq Kf(n)$; if K may be taken as small as desired $g(n)$ is said to be $o(f(n))$.

We will use the notation $E[\ ]$ to denote expectation. If we are dealing with a Markov process and ν is a probability measure on the appropriate state space, $E^\nu[\ ]$ and $P^\nu(\ )$ will denote respectively, the expectation and the probability for the process starting from initial distribution ν. If ν is a point mass at η then $E^\eta[\ ]$ and $P^\eta(\ )$ will be used instead of $E^{\delta_\eta}[\ ]$ and $P^{\delta_\eta}(\ )$.

When arguing for asymptotic results in a variable n, it will be tacitly taken that n is sufficiently large to justify asymptotic relations, e.g., it may be taken that n is sufficiently large to justify $\beta(l) \geq \frac{1}{n^{3/2}}, \forall l \leq \frac{1}{n^{3/2(k+1)}}$.

We will often use shifts by $x \in \mathbb{Z}$, $\theta_x \circ$, of configurations where for $\eta \in \{0,1\}^{\mathbb{Z}}$, $(\theta_x \circ \eta)(y) = \eta(y-x)$ for $y \in \mathbb{Z}$.

For a function f defined on a set Ω, $||f||_\infty$ shall as usual denote $\sup_{x \in \Omega} |f(x)|$.

2 Background results for semi-infinite β-NPSs

In this section we assemble some results concerning semi-infinite β-NPSs in equilibrium and "regeneration" or mixing results. These are taken from [8] as well as [9].

Lemma 5 *Let $(\eta_t : t \geq 0)$ be a right sided β-NPS with rightmost occupied site at time t equal to r_t. If η_0 is distributed according to $Ren^{(-\infty,0]}(\beta)$, we have*

$$P(\sup_{s \leq t} |r_s| \geq t^{1/2} log^2(t)) \leq \frac{C}{t^{(k-3)/2}}$$

where k is the constant fixed in condition $(***)$.

PROOF. This is Lemma 1.2 (i) in [8].

$\diamondsuit$

The next lemma follows quickly from Lemma 4.3 of [8]

Lemma 6 *Let $\eta^R_\cdot$, be a right sided β-NPS with η^R_0 distributed as $Ren^{(-\infty,0]}(\beta)$, while β-NPS $\eta_\cdot$ is such that $\eta_0 \equiv \eta^R_0$ on $(-\infty, 0]$. With probability tending to one as n tends to infinity*

$$\eta_s \equiv \eta^R_s \text{ on } (-\infty, -n^{3/2}] \;\; \forall \;\; 0 \leq s \leq n^{5/2}.$$

We now recall some finite state Markov chains used in [8]. Given an interval I we let $Z^I_\cdot$ be the Markov chain on $\{0,1\}^I$ with 1's fixed at the endpoints of I but otherwise having flip rates of a β-NPS on the interior of I. We assume (unless otherwise stated) that $Z^I_\cdot$ will be generated by the same Harris system of Poisson processes as a given β-NPS, $\eta_\cdot$. If the interval I is equal to $[-n, n]$ we also denote the chain as $Z^n_\cdot$. If the chain starts from all 1's it is denoted by $Z^{I,1}_\cdot$. From the attractiveness property we have, for a β-NPS $\eta_\cdot$ starting with initial configuration equal to Z^I_0 on the interior of I, uniformly over intervals I, t and η_0 that on I, $\eta_t \;\leq Z^I_t \;\leq\; Z^{I,1}_t$ in the natural partial order. Given an interval I of length $2n$, we denote by $Y^I_\cdot$ the Markov chain on $\{0,1\}^I$ with 1's fixed at the endpoints of I, 0's fixed within $n^{1/3}$ of the endpoints and other sites having flip rates corresponding to a β-NPS. Again $Y^I_\cdot$ is taken to be derived from the same Harris system as a relevant β-NPS $\eta_\cdot$. $X^I_\cdot$ is the Markov chain derived from $Y^I_\cdot$ for which the configuration that is zero on all interior sites of I is forbidden.

A configuration η is said to be *bad* on an interval I of length $2n$ if the process $Y^I_\cdot$ with $Y^I_0 = \eta$ on I (except, of course within $n^{1/3}$ of the endpoints of I where it is zero) has probability at least $n^{-k/6}$ of hitting the (forbidden for $X^I_\cdot$) configuration that is all 0's on the interior of I for k the constant in condition $(***)$.

Proposition 7 *Let $(\eta^R_t : t \geq 0)$ be a right sided β-NPS in equilibrium supported by $(-\infty, 0]$ at time 0 and let r_t be the position of rightmost particle at time t. Then*

(i) *the probability that for some $0 \leq t \leq T$, there is a gap of size $n^{1/3}$ in interval $(r_t - S, r_t]$ is bounded above by $K(T+1)(S+1)n^{-(k-1)/3}$,*

(ii) *the probability that for some $0 \leq t \leq T$, $\theta_{r_t - x} \circ \eta_t|_{(-n,n)}$ is bad for $n \leq x \leq S$ is bounded above by $K(T+1)(S+1)n \cdot n^{-k/6}$ where $\theta_{r_t-x} \circ \eta_t|_{(-n,n)}$ denotes the restriction to interval $(-n, n)$ of the shift of configuration η_t by $r_t - x$.*

for some K not depending on n, S, T and k the constant of condition $(***)$.

PROOF. This is essentially Lemma 2.2 in [8].
The following result is shown in the same (basic) fashion.

Proposition 8 *For* $(\eta_t : \ t \geq 0)$ *a* β*-NPS with* η_0 *(and therefore* η_t *for all* t*) distributed as* $Ren(\beta)$ *and for* $0 \leq T, S < \infty$,

(i) *The probability that for some* $0 \leq t \leq T$*, there is a gap of size* $n^{1/3}$ *within spatial interval* $[-S, S]$ *is bounded by* $K(T+1)(S+1)n^{-(k-1)/3}$,

(ii) *The probability that for some* $0 \leq t \leq T$, $\theta_x \circ \eta_t|_{(-n,n)}$ *is bad for some* $|x| \leq S$ *is bounded by* $K(T+1)(S+1)n \cdot n^{-k/6}$

for k *the constant of* $(***)$ *and for some* K *not depending on* n, S *or* T.

Remark: The proof of these two results result simply relies on bounds for $Ren(\beta)$ or $Ren^{(-\infty,0]}(\beta)$ and (crude) upper bounds on the relevant total flip rate. As such the conclusions also apply to the processes $\eta^{F,n}_{\cdot}$ to be defined in the next section.

Lemma 9 *Let* n *be a positive integer. For* X^n_0 *arbitrary in* $\{0,1\}^{[-n,n]} \backslash \underline{0}$*, If* $X^n_{\cdot}$ *and* $Z^{n,1}_{\cdot}$ *are derived from the same Harris systems, then for some constant* C *uniformly in* n *and* X^n_0*, outside of an event of probability* $Cn^2/n^{k/3}$,

$$Z^{n,1}_{n^4}(x) = X^n_{n^4}(x) \ \ \forall x \in [-4n/5, 4n/5].$$

Proof. This is Corollary 2.1 in [8].

$\diamondsuit$

Lemma 10 *Let* n *be a positive integer. Consider* $Z^n_{\cdot}$ *and* $Y^n_{\cdot}$ *run with the same Harris system. Suppose* Z^n_0 *and* Y^n_0 *are derived from the restriction to* $(-n, n)$ *of a renewal process* γ *on* $\mathbb{Z}$ *(or a renewal process* γ *conditioned to have* $\gamma(x) = 1$ *for some fixed* x *with* $|x| \geq n$*) so that*

(i) $Z^n_0 \equiv \gamma$ *on* $(-n, n)$ *and*

(ii) $Y^n_0 \equiv \gamma$ *on* $(-n + n^{1/3}, n - n^{1/3})$,

then there exists a finite constant c *not depending on* n *such that*

$$P(\exists t \leq n^8 \text{so that } \ Z^n_t(x) \neq \ Y^n_t(x) \text{ for some } x \in (-4n/5, 4n/5)) \leq \frac{c}{n^{k/3-10}}.$$

Proof. This is Lemma 4.3 in [8].

$\diamondsuit$

3 Introduction of a finite state comparison Markov chain

As in [8] the main element in our attack is the use of spectral gap estimates applied to finite state comparison Markov chains. This method has its roots in [5] and [2] and was specifically applied to particle systems in [12]. We are ultimately interested in analyzing semi-infinite β-NPSs and seeing how quickly they return to equilibrium from a "reasonable" initial configuration. To this end we introduce a comparison finite state Markov chain which will be "close" to the non-Markov process obtained by looking at the n sites to the left of the rightmost particle of a right sided β-NPS in equilibrium.

For n a positive integer we define a chain $\eta^{F,n}_{\cdot}$, on $\Omega^n = \{\eta \in \{0,1\}^{(-n,0]} : \eta(0) = 1\}$ via the jump rates $q(\eta,\xi) = q_1(\eta,\xi) + q_2(\eta,\xi) + q_3(\eta,\xi)$ where the $q_i(,)$ are defined as follows. For a configuration $\eta \in \Omega^n$, let $j = j(\eta) = \inf\{-n < x \le 0 : \eta(x) = 1\}$. For $-n < x < 0$, let $\eta^x \in \Omega^n$ satisfy $\eta^x(y) = \eta(y)$ if and only if $x \neq y$.

(i) For $j \le x < 0$

$$q_1(\eta,\eta^x) = \begin{cases} 1 & \text{if } \eta(x) = 1, \\ \beta(l_\eta(x), r_\eta(x)) & \text{if } \eta(x) = 0, \end{cases}$$

and $q_1(\eta,\xi) = 0$ for other configurations ξ.

(ii) For $-n < x < j$, we have

$$q_1(\eta,\eta^x) = \frac{\bar\beta(x+n)\beta(j-x)}{\bar\beta(j+n)}, \quad \text{where } \bar\beta(r) = \sum_{y \ge r} \beta(y).$$

(iii) For $1 \le \ell \le n$, let $\eta^{F,+\ell} \in \Omega^n$ be defined by:

$$\begin{aligned} \eta^{F,+\ell}(0) &= \ 1 \\ \eta^{F,+\ell}(x) &= \ 0 \text{ if } -\ell < x < 0 \\ \eta^{F,+\ell}(x) &= \ \eta(x+\ell) \text{ if } -n < x \le -\ell. \end{aligned}$$

$$q_2(\eta,\xi) = \begin{cases} \beta(\ell) & \text{if } \xi = \eta^{F,+\ell} \text{ some } \ell \in [1, n-1]; \\ \bar\beta(n) & \text{if } \xi = \eta^{F,+n}, \end{cases}$$

(and is zero if ξ is not of this form).

(iv) Let $v = \sup\{x < 0 : \eta(x) = 1\} \vee -n$. Then for $\xi \in \Omega^n$ such that for $-n-v < x \le 0$,

$$\xi(x) = \eta(x+v),$$

$q_3(\eta,\xi)$ equals the $Ren^{(-n,0]}(\beta)$ probability of ξ given $\xi(0),\ \xi(-1),\ \cdots \xi(-n-v+1)$, where, if $v = -n$ there is no conditioning. Again if ξ is not of this form $q_3(\eta,\xi)$ is zero.

We can think of the jumps of this chain as corresponding to q_1, q_2 or q_3 even though it is possible to find η and $\xi \in \Omega^n$ so that $q_i(\eta, \xi)$ is strictly positive for all three values of i.

We think of a jump corresponding to q_1 as being a flip, a jump from η to $\eta^{F,+l}$ corresponding to q_2 is described, somewhat counterintuitively, as a *positive shift by* l, (or a positive l-shift) while a jump of q_3 type is a *negative shift.* To understand the definitions the reader should keep in mind the goal: to provide a finite state Markov chain that will well approximate a right sided $\beta-$NPS as seen n sites to the left of its rightmost particle at site r_t. Jumps governed by function q_1 simply follow those of the $\beta-$NPS not involving a change of rightmost particle. For the $\beta-$NPS the flip rates for births occurring to the left of the leftmost occupied site of $(r_t - n, r_t]$ depend on the $\beta-$NPS outside this interval. Our comparison Markov chain takes them to be the expected rates for the $\beta-$NPS if it were distributed as Ren(β) conditioned on the configuration in interval $(r_t - n, r_t]$. The jumps that arise from function q_2 correspond to the occurrence of jumps to the right of the rightmost particle for the $\beta-$NPS and so in particular jumps of size n or greater will all result in the $\beta-$NPS seeing nothing but 0s in the $n-1$ sites to the left of the new rightmost occupied site. The negative shifts for Markov chain $\eta^{F,n}_{\cdot}$ correspond to the deaths of the rightmost particle in the $\beta-$NPS. It is for this reason that the counterintuitive adjectives *positive* and *negative* are applied to the various shifts. We note that for some $\eta \in \Omega^n$ the flip rates $q_2(\eta, \eta)$ or $q_3(\eta, \eta)$ may be strictly positive. Nonetheless for reasons given in the next section, though in these cases the state of the Markov chain may not change we may still regard a shift as having taken place. The chain $\eta^{F,n}_{\cdot}$ is readily seen to be an irreducible Markov chain on a finite state space and thus possessing of a unique equilibrium.

Lemma 11 *The Markov chain $\eta^{F,n}_{\cdot}$ on $\{0,1\}^{(-n,0]}$ defined by (i) to (iv) above is reversible with respect to $Ren^{(-n,0]}(\beta)$.*

Proof. One simply checks directly the detailed balance equations.

$\diamondsuit$

A key element in our approach is to use mixing properties for our comparison Markov chains. Many desirable mixing inequalities follow from a good bound on the spectral gap for the chain: the difference between the largest eigenvalue (which is zero) for the Markov chain operator (acting on the space of L^2 functions with respect to $Ren^{(-n,0]}(\beta)$) and the next largest eigenvalue.

The following is proven in [14] and follows in a straightforward manner from the approach to spectral gap estimation employed in [9].

Proposition 12 *Let $Gap(\eta^{F,n}_{\cdot})$ be the spectral gap of the chain $\eta^{F,n}_{\cdot}$. There exists a constant $c \in (0, \infty)$ such that for every positive integer n*

$$n^2 \ Gap(\eta^{F,n}_{\cdot}) > c.$$

4 The coupling

In this section we consider a natural coupling of finite comparison processes and right sided processes with the property that if both the two processes are in equilibrium and are initially "close" then they will remain so for a long time. Thus the mixing properties of the former process (implied by Proposition 12 yield mixing information for right sided β-NPSs. The chief result is Proposition 13 which is then applied to obtain various regeneration results such as Proposition 15.

We begin by detailing a Harris system construction for the Markov chain $\eta^{F,n}_{\cdot}$. We suppose given independent Poisson processes

D'_x for $-n < x \leq 0$ of rate 1,

B'_x for $-n < x < 0$ of rate M, plus independent U[0,1] random variables G^x_i associated with the i'th point of B'_x for $i = 1, 2 \cdots$,

V_x of rate $\beta(x)$ for $0 < x < n$ and V_n of rate $\bar{\beta}(n)$,

X of rate 1 plus independent U[0,1] random variables H_i associated with the i'th point of X for $i = 1, 2 \cdots$.

We use the processes D'_x, B'_x and random variables G^x_i to generate the flips for $\eta^{F,n}_{\cdot}$ in the same way as we use a Harris system to generate a $\beta-$NPS. The Poisson processes V_l are used to generate positive l shifts, while the process X generates the negative shifts with the associated random variables H_i being employed to determine which configuration the process "negative shifts" to (for each of the finite $\eta \in \Omega^n$, we partition $[0, 1]$ into a finite number of intervals $J_1, J_2, \cdots J_r$ (r depends on η) which are in 1-1 correspondence with the $\xi_1, \xi_2, \cdots \xi_r$ such that $q_3(\eta, \xi_j) > 0$ so that for each $1 \leq i \leq r,\ q_3(\eta, \xi_j)$ is equal to the length J_i. For $t \in X, \eta^{F,n}_{t-} = \eta$, a negative shift to ξ_i will occur if and only if the associated random variable H is in interval J_i). Thus even though $q_i(\eta^{F,n}_{t-}, \eta^{F,n}_t)$ may be simultaneously positive for more than one $i \in \{1, 2, 3\}$, we can rigorously talk of positive shifts, negative shifts and flips.

As previously remarked, for some configurations $\eta \in \Omega^n$, $q_2(\eta, \eta)$ and/or $q_3(\eta, \eta)$ are positive and so a positive or negative shift may well result in no change in the chain $\eta^{F,n}_{\cdot}$.

We now couple a right sided $\beta-$NPS, $(\eta^R_t : t \geq 0)$ for which at time t the position of the rightmost particle is r_t and a finite comparison Markov chain, $\eta^{F,n}_{\cdot}$ in the "obvious" way. We suppose that we are given the Harris system $\{B_x\}, \{D_x\}$ and $\{U_{x,i}\} - \infty < x < \infty$ generating $\eta^R_{\cdot}$. We take as our system above, generating $\eta^{F,n}_{\cdot}$,

D'_x for $-n < x \leq 0$: $t \in D'_x$ if and only if $t \in D_{r_{t-}+x}$

B'_x for $-n < x < 0$: $t \in B'_x$ if and only if $t \in B_{r_{t-}+x}$. The random variables G^x_i

associated with t which is the i'th point of B'_x, will equal the corresponding random variable $U_{x+r_{t-},j}$ associated to t considered as an element of $B_{x+r_{t-}}$.

A point t will belong to V_x for $0 < x < n$ if and only if $r_t - r_{t-} = x$; $t \in V_n$ if and only if $r_t - r_{t-} \geq n$,

A point t will be in X if and only if $r_t - r_{t-}$ is strictly negative and the random variables H_i will be independent of the Harris system $\{B_x\}, \{D_x\}, \{U_{x,i}\} -\infty < x < \infty$ and $(\eta_t^R : t \geq 0)$.
The coupling given above is called the *natural coupling*.

In context, for the process $\eta_\cdot^{F,n}$ on $\{0,1\}^{(-n,0]}$ one can define $r_0^{F,n} = 0$ and

$$r_t^{F,n} = \sum_{0 \leq s \leq t} \Delta_s^F$$

where

$$\begin{aligned}
\Delta_s^F &= \ell && \text{if there is a positive } \ell \text{ shift of } \eta_\cdot^{F,n} \text{ at time } s \text{(or} \\
& && \quad \text{equivalently if and only if } s \in V_\ell); \\
\Delta_s^F &= -\ell && \text{if there is a negative shift of } \eta_\cdot^{F,n} \text{ at time } s \text{ and} \\
& && \quad \sup\{j < 0 : \eta_{s-}^{F,n}(j) = 1\} \vee -n = -\ell; \\
\Delta_s^F &= 0 && \text{otherwise.}
\end{aligned}$$

Again, to motivate our choice of adjectives in describing shifts, note that positive shifts of $\eta_\cdot^{F,n}$ correspond to positive changes of $r_\cdot^{F,n}$, negative shifts to negative changes. The random variables $r_t^{F,n}$ for $t \geq 0$ are not measurable with respect to the natural filtration of $\eta_\cdot^{F,n}$ (indeed according to the above definitions, it is possible that at in an interval $r_\cdot^{F,n}$ may change while the process $\eta_\cdot^{F,n}$ remains constant) and so we assume that the filtration is suitably enlarged to accommodate $r_\cdot^{F,n}$. For each $t \geq 0$, we can treat $\eta_t^{F,n}$ to be defined on $(r_t^{F,n} - n, r_t^{F,n}]$ by taking the spin value at site $x + r_t^{F,n}$ to equal that at site x for the originally defined chain. With a little abuse of our notation, we will simultaneously regard the process $(\eta_t^{F,n} : t \geq 0)$ as being defined on $\{0,1\}^{(-n,0]}$ and on $\{0,1\}^{(r_t^F - n, r_t^F]}$.

Proposition 13 *Let $\eta_\cdot^R$ be a right sided β-NPS with η_0^R distributed as $Ren^{(-\infty,0]}(\beta)$ and let the site of the rightmost particle of η_t^R at time t be r_t. Let $\eta_\cdot^{F,n}$ be an equilibrium comparison finite Markov chain defined on $(r_t^{F,n} - n, r_t^{F,n}]$ at time t with $r_0^{F,n} = r_0 = 0$, naturally coupled with $\eta_\cdot^R$. Assume further $\eta_0^R = \eta_0^{F,n}$ on $[-\frac{2n}{3}, 0]$, then we have the following*

(i)

$$P(\exists t \leq n^{23/12} : \quad r_t \neq r_t^{F,n}) \leq C n^3 n^{-k/6},$$

(ii)

$$P(\exists t \leq n^{23/12} : \eta_t^R(x) \neq \eta_t^{F,n}(x) \textit{ for some } x \in [r_t^{F,n} - \frac{n}{2}, r_t^{F,n}]) \leq Cn^3 n^{-k/6},$$

(iii)

$$P(r^{F,n}_{n^{23/12}} \neq r_{n^{23/24}} \textit{ or } \eta^R_{n^{23/12}}(x) \neq \eta^{F,n}_{n^{23/12}}(x) \textit{ for some } x \in [r^{F,n}_{n^{23/12}} - \frac{2n}{3}, r^{F,n}_{n^{23/12}}])$$
$$\leq \; Cn^3 n^{-23k/288}.$$

Remark: For $\eta^R_\cdot$ and $\eta^{F,n}_\cdot$ naturally coupled, we say the coupling *breaks down* on time interval $[S,T]$, if there exists some t in this interval so that either $r_t \neq r_t^{F,n}$ or $\eta_t^R(x) \neq \eta_t^{F,n}(x)$ for some $x \in [r_t^{F,n} - \frac{n}{2}, r_t^{F,n}]$.

Proof. We wish to argue that outside of the null set where points occur simultaneously for distinct and therefore independent Poisson processes, the event

$$\{\exists t \leq n^{23/12} : \quad r_t \neq r_t^{F,n} \text{ or } \eta_t^R(x) \neq \eta_t^{F,n}(x) \text{ for some } x \in [r_t^{F,n} - \frac{n}{2}, r_t^{F,n}]\}$$

is contained in the event $A \cup B \cup C$ where
A is the event

$$\{\sup_{s \leq n^{23/12}} |r_s| \geq \frac{n}{12}\}.$$

By Lemma 5 and the fact that $k \geq 1500$, $P(A) \leq \frac{C}{n^{(k-3)/2}} < Cn^{10} n^{-k/3}$.

B is the event

$$\{\forall s \leq n^{23/12}, \text{ there is no } (n/15)^{1/3} \text{ gap for } \eta_s^R \text{ in } (r_s - n, r_s]\}^c \cup$$

$$\{\forall s \leq n^{23/12}, \text{ there is no } (n/15)^{1/3} \text{ gap for } \eta_s^{F,n} \text{ in } (r_s^{F,n} - n, r_s^{F,n}]\}^c.$$

By Proposition 7 (which can be applied to process $\eta^{F,n}_\cdot$ as well as to $\eta^R_\cdot$), $P(B) \leq Cn n^{23/12}/n^{(k-1)/3}$.

C is the event that for interval $I = [-\frac{39n}{60}, -\frac{31n}{60}] \subset [-\frac{2n}{3}, 0]$, the processes $Y^I_\cdot$ and $Z^I_\cdot$ run with the Harris system of $\eta^R_\cdot$ and such that on $(-\frac{39n}{60}, -\frac{31n}{60})$, $Z_0^I(x) = \eta_0^R(x)$ and on $(-\frac{39n}{60} + (n/15)^{1/3}, -\frac{31n}{60} - (n/15)^{1/3})$, $Y_0^I(x) = \eta_0^R(x)$, we have

$$\exists s \leq n^{23/12}, x \in J \text{ so that} \qquad Y_s^I(x) \; \neq \; Z_s^I(x),$$

where J is the central subinterval of I having length $(n/15)^{1/3}$. By Lemma 10, $P(C)$ is bounded by $Cn^{10}/n^{k/3}$.

We first examine the consequences of $A \cup B \cup C$ not occurring for process $\eta^R_\cdot$. By attractiveness, on event $A^c \cap \{\forall s \leq n^{23/12}, \text{ there is no } (n/15)^{1/3} \text{ gap for } \eta_s^R \text{ in } (r_s - n, r_s]\}$,

$$\forall s \leq n^{23/12}, x \in (-\frac{39n}{60}, -\frac{31n}{60}), Y_s^I(x) \; \leq \; \eta_s^R(x) \; \leq \; Z_s^I(x)$$

and so in particular on event $(A \cup B \cup C)^c$

$$\forall s \leq n^{23/12}, x \in J, \quad Y_s^I(x) \;=\; \eta_s^R(x) \;=\; Z_s^I(x).$$

Analysing the chain $\eta_\cdot^{F,n}$ for $A \cup B \cup C$ not occurring is slightly more difficult given the positive and negative shifts which potentially could violate "natural" attractiveness relations, however a little thought shows this not to be a problem on $A^c \cap \{\forall s \;\leq\; n^{23/12}$, there is no $(n/15)^{1/3}$ gap for $\eta_s^{F,n}$ in $(r_s^{F,n} - n, r_s^{F,n}]\}$ for times $t < \lambda = \inf\{s : r_s \neq r_s^{F,n}\}$ and we have that on event $(A \cup B \cup C)^c$

$$\forall s < n^{23/12} \wedge \lambda, x \in J, \qquad Y_s^I(x) \;=\; \eta_s^R(x) \;=\; Z_s^I(x)$$

and so under these conditions

$$\forall s < n^{23/12} \wedge \lambda, x \in J, \qquad \eta_s^R(x) \;=\; \eta_s^{F,n}(x).$$

Now observe that λ can only occur at a point in $D_{r_{\lambda-}}$ and so (on event $(A \cup B)^c$) if λ is less than or equal to $n^{23/12}$, $\eta_\cdot^R$ and $\eta_\cdot^{F,n}$ will be unchanged on interval J at this moment. Thus we have a.s. on event $(A \cup B \cup C)^c$ that

$$\forall s \quad \leq \quad n^{23/12} \wedge \lambda, x \in J,\ \eta_s^R(x) \;=\; \eta_s^{F,n}(x).$$

But further on $(A \cup B \cup C)^c$ these two configurations are never identically zero on J during this closed time interval. From the nearest particle nature of $\beta-$NPSs it follows that

$$\forall s \leq\ n^{23/12} \wedge \lambda, x \text{ in or to the right of } J,\ \eta_s^R(x) \;=\; \eta_s^{F,n}(x)$$

(where these values are defined). This gives a contradiction and completes the proof of parts (i) and (ii).

To prove part (iii), we know the rightmost particles will not move more than $\frac{n}{12}$ for both processes, outside of an event of suitably small probability, from the first part of our proof. So we only need to look at interval $[-\frac{5n}{6}, -\frac{n}{2}]$. Now we need to use the approach of [8], Section Two. First divide up $[-\frac{5n}{6}, -\frac{n}{2}]$ into $1 \leq m \leq Kn/N$ equal disjoint naturally ordered intervals of length $N = n^{23/48}$ (so that $N^4 = n^{23/12}$), I_i. Now choose disjoint intervals $J_i,\ 0 \leq i \leq m$ so that the left endpoint of I_i is the center of J_{i-1} and the right endpoint of I_m is the center of J_m.

Let us consider processes $Z_t^{I_i,1}$, the β-NPS on $\{0,1\}^{I_i}$ with 1's fixed at the endpoints of I_i and such that $Z_0^{I_i} \equiv 1$ on I_i and $Y_t^{I_i}$ the β-NPS on $\{0,1\}^{I_i}$ with 1's fixed at the endpoints of I_i and 0's fixed within $N^{1/3}$ of the endpoints and (subject to the above) $Y_0^{I_i} \equiv \eta_0^R$ on I_i. We also consider Y_t^{F,I_i} the β-NPS on $\{0,1\}^{I_i}$ with 1's fixed at the endpoints of I_i and 0's fixed within $N^{1/3}$ of the endpoints and (subject to the above) $Y_0^{F,I_i} \equiv \eta_0^{F,n}$ on I_i.

Then

(1) Outside of an event of probability $n^{\frac{23}{12}} n N^{-\frac{1}{3}(k-1)}$ (by Proposition 7), there is no $(N/2)^{\frac{1}{3}}$ gap for η^R_t or $\eta^{F,n}_t$ for $0 \leq t \leq n^{\frac{23}{24}}$, within n of $r_t = r^F_t$.

(2) Outside of an event of probability $\frac{C}{n^{(k-3)/2}}$ (by Lemma 5) $|r_s| \leq \frac{1}{12} n$.

(3) Outside of an event of probability $C(n+N)N^{-k/6}$ (by Proposition 7 and the definition of bad) $\forall i$, $Y^{I_i}_t, Y^{F,I_i}_t$ are not identically zero on the interior of I_i, $\forall t \leq n^{\frac{23}{12}}$. Similarly for the Y processes associated to the intervals J_i. So $Y^{I_i}_t = X^{I_i}_t \ \forall t \leq n^{23/12}$ and $Y^{I_i,F}_t = X^{I_i,F}_t$.

(4) Outside of an event of probability $C \frac{n}{N} N^2 N^{-k/3}$ (by Lemma 9) $\forall i$ and $\forall x$ in the central $\frac{4}{5}$ of I_i, $X^{I_i}_{n^{23/12}}(x) = X^{F,I_i}_{n^{23/12}}(x) \ = \ Z^i_{n^{23/12}}(x)$.

We have that outside of an event of probability $C(n^{10} n^{-k/3} + n^3 n^{-23k/288})$ (from (1) to (4) above),

$$Z^i_{n^{23/12}}(x) = \eta^R_{n^{23/12}}(x) = \eta^{F,n}_{n^{23/12}}(x) = Y^i_{n^{23/12}}(x)$$

$\forall$ i and $\forall$ x in the central $\frac{4}{5}$ of I_i for all $I_i \subset [-\frac{5n}{6}, -\frac{n}{2}]$.

Repeating this argument with J_i, we obtain that outside of an event of probability $C(n^{10} n^{-k/3} + n^3 n^{-23k/288}) \leq C n^3 n^{-23k/288}$

$$\eta^R_{n^{23/12}}(x) = \eta^{F,n}_{n^{23/12}}(x) \ \ \forall \ x \in [-\frac{5n}{6}, -\frac{n}{2}].$$

$\diamondsuit$

By using the above proposition repeatedly, we have

Corollary 14 *Let $\eta^R_\cdot$ and $\eta^{F,n}_\cdot$ be as in Proposition 13 and let b be a real number greater than $23/12$, then there exists finite C not depending on b or on N such that*

$$P(\eta^{F,n}_t(x) \neq \eta^R_t(x) \text{ for some } x \in [r^{F,n}_t - \frac{n}{2}, r^{F,n}_t] \text{ or } r^F_t \neq r^{F,n}_t \text{ for some } t \leq n^b)$$

$$\leq C n^{b-23/12} \frac{n^3}{n^{23k/288}}.$$

Arguing in the same way we achieve

Proposition 15 *Let $\eta^R_\cdot, \xi^R_\cdot$ be right sided β-NPSs generated by the same Harris system both in equilibrium, initially supported on $(-\infty, 0]$ and such that $\eta^R_0 = \xi^R_0$ on $(-n, 0]$. Then outside of an event of probability $C n^{32-23/12}/n^{23k/288}$, $\eta^R_{n^{23/12}} = \xi^R_{n^{23/12}}$ on $(-3n^{32}, \infty)$.*

This result is similar to "regeneration" results found in [8]; the difference is that here the coupling includes the rightmost particles. We also mention two coupling results which, though they do not make use of the the statement of Proposition 13, can be obtained by similar finite state Markov chain comparisons and so belong here.

Lemma 16 *Let δ be a fixed, strictly positive constant and $\eta^R_\cdot$ a right sided equilibrium β-NPS initially supported on $(-\infty, N\delta]$ where N is a positive integer. Let $\eta^{\underline{1}}_\cdot$ be the β-NPS for which all sites are initially occupied and run with the same Harris system as $\eta^R_\cdot$. As $N \to \infty$, with probability tending to one, η^R_N equals $\eta^{\underline{1}}_N$ on $(-4N^2, \frac{3N\delta}{4})$.*

Lemma 17 *Let $\epsilon \in (0, 1)$ be a fixed constant and f be a fixed increasing cylinder function bounded in absolute value by 1 and let $\xi_\cdot$ be a β-NPS such that $\xi_0(x) = I_{x \leq N\epsilon}$. There exists $C \in (0, \infty)$ not depending on N or ϵ so that for integers N sufficiently large*

$$E[f(\xi_t)] \quad \geq \quad < Ren(\beta), f > \quad - \quad C\epsilon,$$

uniformly over $0 \leq t \leq N^2\epsilon^3$.

(Here as before $< \mu, f >$ denotes $\int f d\mu$.)

5 The convergence of $E[r_t]$

In this section we consider the asymptotic behaviour of $E[r_t]$, for r_t the position of the rightmost particle at time t of a right sided β-NPS, $\xi^R_\cdot$, initially in equilibrium on $(-\infty, 0]$ conditioned on the occupied sites in $[-l, 0]$ at time 0 being precisely $-l$ and 0 for a positive integer l. In fact we show that a limiting value exists. The plan of attack is to use our finite state space Markov chains to show that $\frac{dE[r_t]}{dt}$ is small for large t. We also obtain some simple but useful bounds on the limit.

Lemma 18 *Let l and n be positive integers and let $\xi^R_\cdot$ be a right sided β-NPS with ξ^R_0 having distribution $Ren^{(-\infty,0]}(\beta)$ conditioned on the occupied sites in $[-l, 0]$ being precisely $-l$ and 0. Let r_t be the rightmost occupied site of ξ^R_t for $t \geq 0$. Let $\xi^{F,n}_\cdot$ be the finite state Markov chain naturally coupled with $\xi^R_\cdot$ and so that $\xi^{F,n}_0 = \xi^R_0$ on $(r_0 - n, r_0] = (-n, 0]$. Let the associated rightmost particle functional be $r^{F,n}_\cdot$ and consider $\xi^{F,n}_t$ to be defined on $(r^{F,n}_t - n, r^{F,n}_t]$. Then*

$$P(r^{F,n}_t \neq r_t \text{ or } \xi^{F,n}_t(x) \neq \xi^R_t(x) \text{ for some } x \in [r^{F,n}_t - \frac{n}{2}, r^{F,n}_t] \text{ and some } t \leq n^4)$$

$$\leq Cn^{25/12} l^{k+1} \frac{n^3}{n^{23k/288}}.$$

PROOF. Let $\eta^R_\cdot$ be a right sided process starting in equilibrium on $(-\infty, 0]$ and let r^η_t be the position of its rightmost particle at time t. Let $\eta^{F,n}_\cdot$ be the finite state Markov chain on $\{0,1\}^{(-n,0]}$ naturally coupled with $\eta^R_\cdot$ and starting with $\eta_0(x) = \eta^{F,n}_0(x)$ on $(-n, 0]$. Denote by $r^{F,n,\eta}_\cdot$ its associated rightmost particle functional. Then

$$P(r^F_t \neq r_t \text{ or } \xi^{F,n}_t(x) \neq \xi^R_t(x) \text{ for some } x \in [r^{F,n}_t - \frac{n}{2}, r^{F,n}_t] \text{ for } t \leq n^4)$$

$$= P(r^{F,n,\eta}_t \neq r^\eta_t \text{ or } \eta^{F,n}_t(x) \neq \eta^R_t(x) \text{ for some } x \in [r^{F,n,\eta}_t - \frac{n}{2}, r^{F,n,\eta}_t] \text{ for } t \leq n^4 | A_l)$$

where A_l is the event that the two rightmost occupied sites of η^R_0 are separated by distance l. By Corollary 14, the above probabilities are

$$\leq C \frac{n^{25/12}}{\beta(l)} \frac{n^3}{n^{23k/288}}.$$

The conclusion follows as $\beta(l) \geq \frac{1}{l^{k+1}}$ for l large.

$\diamondsuit$

For a right sided configuration ξ^R, we define $h(\xi^R)$ to be the distance between the rightmost particle and the second rightmost particle . So for a process $\xi^R_\cdot$,

$$h(\xi^R_t) = \ r_t \ - \ \sup\{x < r_t : \xi^R_t(x) = 1\}.$$

It is helpful to note that

$$\frac{d}{dt} E[r_t] = \sum_{\ell=1}^{\infty} \ell \beta(\ell) \ - \ E[h(\xi^R_t)],$$

is small for large t.

For $\xi^{F,n}$ a finite configuration on $\{0,1\}^{(-n,0]}$, we abuse notation by defining

$$h(\xi^{F,n}) = \inf\{-x > 0 \ : \xi^{F,n}_t(x) = 1 \text{ for } x < 0\} \wedge n.$$

Let $\xi^R_\cdot$ be as in Lemma 18 and denote by r_t the position of its rightmost particle at time t.

As previously stated, in this section, our goal is to bound above $|\sum i\beta(i) - E[h(\xi^R_{n^4})]|$ for right sided β-NPSs $\xi^R_\cdot$. First an elementary calculation gives

Lemma 19 *There exists a constant c not depending on n so that for all positive integers n,*

$$\frac{c}{n^{k-2}} \geq \sum_{i=n+1}^{\infty} (i-n)\beta(i) = |\sum_{i=1}^{\infty} i\beta(i) \ - \ < Ren^{(-n,0]}(\beta), h > |$$

Lemma 20 *Let $\xi^R_\cdot, \xi^{F,n}_\cdot$ be as in Lemma 18, then for $t \geq 0$ and finite C not depending on t or l,*

$$E[h^2(\xi^{F,n}_t)],\ E[h^2(\xi^R_t)] \leq \frac{C}{\beta(l)^{3/k}} \leq Cl^4,$$

$$E[h^4(\xi^{F,n}_t)],\ E[h^4(\xi^R_t)] \leq \frac{C}{\beta(l)^{6/k}} \leq Cl^7.$$

Proof. Given the similarity of proof, we only treat process $\xi^R_\cdot$ and power 2.

By Hölder's inequality

$$E[h^2(\eta^R_t)I_{h(\eta^R_0)=\ell}] \ \leq \ (E[h^{2k/3}(\eta^R_t)])^{3/k}(P(h(\eta^R_0) = \ell))^{1-3/k}.$$

for $(\eta^R_t : t \geq 0)$ a right sided β-NPS in equilibrium, initially supported on $(-\infty, 0]$, so

$$E[h^2(\xi^R_t)] \leq (E[h^{2k/3}(\eta^R_t)])^{3/k}(P(h(\eta^R_0) = \ell))^{1-3/k}/P(h(\eta^R_0) = \ell).$$
$$= (E[h^{2k/3}(\eta^R_t)])^{3/k}/(P(h(\eta^R_0) = \ell))^{3/k} \leq \frac{C}{\beta(l)^{3/k}}.$$

$\diamondsuit$

Recall that $Ren^{(-n,0]}(\beta)$ is the equilibrium distribution of $(\xi^{F,n}_t : t \geq 0)$. It is easily verified that

Lemma 21 *There exists a constant $\nu < \infty$ so that for every positive integer n and every configuration η in $\{0,1\}^{(-n,0]}$*

$$Ren^{(-n,0]}(\beta)(\{\eta\}) \ \geq \ e^{-\nu n}.$$

This and the spectral gap bound, Proposition 12, yield,

Proposition 22 *There exists finite positive constants c, K and C such that for all positive integers n and $\eta^{F,n}_0 \in \{0,1\}^{(-n,0]}$,*

$$(1)\ \forall A \subset \{0,1\}^{(-n,0]}, \qquad \left|\frac{P^{\eta^{F,n}_0}(\eta^{F,n}_{n^4/2} \in A)}{Ren^{(-n,0]}(\beta)(A)} - 1\right| \leq Ce^{-cn^2}.$$

$$(2)\ |P^{F,n}_{n^4}(h)(\eta^{F,n}_0) - < Ren^{(-n,0]}(\beta), h > | \leq Ce^{-cn^2}$$

where for function f

$$P^{F,n}_t(f)(\eta^{F,n}_0) = E^{\eta^{F,n}_0}[f(\eta^{F,n}_t)].$$

$$(3)\ |P^{F,n}_{n^4}(h^2)(\eta^{F,n}_0) \ - \ < Ren^{(-n,0]}(\beta), h^2 > | \ \leq \ Ce^{-cn^2}.$$

See [9] or, for a general account, [2]. We can now prove

Proposition 23 *Let l be a positive integer. For $\xi_\cdot^R$ a right sided β-NPS with ξ_0^R distributed as $Ren^{(-\infty,0]}(\beta)$ conditioned on $h(\xi_0^R)=\ell$, there exists a constant C (valid for all positive ℓ and $t\geq 0$) so that*

$$|\sum i\beta(i)-E[h(\xi_t^R)]|\leq C\ell^{7/4}\left(l^{k+1}t^{25/48}\frac{t^{3/4}}{t^{23k/1152}}\right)^{3/4}.$$

Proof. We consider without loss of generality t of the form $t=n^4$, n an integer. We need only consider n large. We introduce $\xi_\cdot^{F,n}$ on $\{0,1\}^{(-n,0]}$ naturally coupled with $\xi_\cdot^R$ so that ξ_0^R and $\xi_0^{F,n}$ agree on $(-n,0]$. Let A be the event that either $\theta_{r_{n^4}}\circ\xi_{n^4}^R$ and $\xi_{n^4}^{F,n}$ are not equal on $(-n/2,0]$ or that $h(\xi_{n^4}^{F,n})\geq n/2$. By Lemma 18 , $P(A)\leq 2Cl^{k+1}n^{25/12}\frac{n^3}{n^{23k/288}}$.

We have seen in Proposition 22, (3), that for c,C not depending on n,$|E[h(\xi_{n^4}^{F,n})]-<Ren^{(-n,0]}(\beta),h>|\leq Ce^{-cn^2}$ and so

$$|E[h(\xi_{n^4}^{F,n})]-\sum_l\beta(l)l|\leq Ce^{-cn^2}+|\sum_{l=1}^{\infty}\beta(l)l-<Ren^{(-n,0]}(\beta),h>|\ \leq\ C/n^{k-2}$$

by Lemma 19. So

$$|E[h(\xi_{n^4}^R)]-\sum_{l=1}^{\infty}\beta(l)l|\ =\ |E[h(\xi_{n^4}^R)]-E[h(\xi_{n^4}^{F,n})]+E[h(\xi_{n^4}^{F,n})]-\sum_{l=1}^{\infty}\beta(l)l|$$

$$\leq|E[h(\xi_{n^4}^R)]-E^F[h(\xi_{n^4}^{F,n})]|\ +\ C/n^{k-2}\leq\ E[I_Ah(\xi_{n^4}^R)]\ +\ E^F[I_Ah(\xi_{n^4}^{F,n})]\ +\ C/n^{k-2}$$

since on event A^c, $h(\xi_{n^4}^R)=h(\xi_{n^4}^{F,n})$.

By the Hölder inequality

$$E[I_Ah(\xi_{n^4}^R)]\leq(E[h^4(\xi_{n^4}^R)])^{1/4}(P(A))^{3/4}\ \leq\ Cl^{7/4}\left(l^{k+1}n^{25/12}\frac{n^3}{n^{23k/288}}\right)^{3/4}$$

by Lemmas 18 and 20. Similarly for $E[I_Ah(\xi_{n^4}^{F,n})]$ and the bound follows.

$\diamondsuit$

Corollary 24 *Let l be a positive integer and $\xi_\cdot^{l,R}$ be a right sided β-NPS so that $\xi_0^{l,R}(x)=\delta_l(x)$ for $x>0$ (where δ denotes Kronecker's delta function) and $\xi_0^{l,R}$ on $(-\infty,0]$ has distribution $Ren^{(-\infty,0]}(\beta)$. Let r_t^l be the rightmost point of the process at time t. Then $E[r_t^l]$ converges to some $c(l)>0$ as $t\to\infty$ and uniformly over $l\leq N^{3/(2(k+1)}$,*

$$|E[r_{N^{1/4}}^l]-c(l)|\leq N^{-1/8}.$$

for N sufficiently large.

Proof. Of course the distribution of $\xi_0^{l,R}$ is simply that of $\xi_\cdot^R$ in preceding propositions shifted l to the right. Therefore, for instance, Proposition 23 can and will be used in analyzing $\xi_\cdot^{l,R}$ without further comment.

For fixed l, as already noted, $\frac{d}{dt}(E[r_t^l])$ exists and equals $\sum_{i=1}^{\infty} i\beta(i) - E[h(\xi_t^l)]$.

But from Proposition 23, we know that

$$|\sum i\beta(i) - E[h(\xi_t^l)]| \leq C\ell^{7/4}\left(l^{k+1}t^{25/48}\frac{t^{3/4}}{t^{23k/1152}}\right)^{3/4} \leq C\ell^k t^{-5}$$

(as $k \geq 1500$). Since $\int_1^\infty \frac{1}{t^5}dt$ exists, $c(l) \ = \ \lim_{t\to\infty} E[r_t^l]$ exists.

Also $|c(l) - E[r^l_{N^{\frac{1}{4}}}]| \ = \ |\int_{N^{1/4}}^\infty \sum i\beta(i) - E[h(\xi_t^l)]dt| \ \leq \ C\ell^{7/4}\int_{N^{1/4}}^\infty \left(\frac{t^{3/4}t^{25/48}l^{k+1}}{t^{23k/1152}}\right)^{3/4} dt.$ So (as $k \geq 1500$) for $l \leq N^{3/2(k+1)}$

$$|E[r^l_{N^{1/4}}] - c(l)| \leq N^{-1/8}.$$

It remains to prove the strict positivity of the constants $c(l)$. We first treat the special case $l = 1$. We know that $\eta_\cdot^R$, a right sided β-NPS in equilibrium initially supported on $(-\infty, 0]$ with rightmost occupied site at positive time t, r_t, satisfies, by reversibility, $E[r_t] = 0$. Now if we condition on the event $h(\eta_0) = 1$, the distribution of η_0 is stochastically above $Ren^{(-\infty,0]}(\beta)$. Therefore attractiveness allows us to conclude that

$$E[r_t|h(\eta_0^R) = 1] \geq 0.$$

Thus by the standard translation invariance properties,

$$E[r_t^1] \ \geq \ 1 \quad \forall t \geq 0.$$

For the general case we consider $\eta_\cdot^R$ as above and introduce the event

$$B \ = \ \{r_t = 0 \ \forall \ 0 \leq t \leq 1\}.$$

It is elementary that conditioned on event B, η_1^R has distribution $Ren^{(-\infty,0]}(\beta)$ and hence for all $t \geq 1, E[r_t|B] \ = \ 0$. Thus $E[r_t|B^c] \ = \ 0$. We may use the same Harris system to generate the processes $\xi_\cdot^{l,R}$ and $\eta_\cdot^R$ and, furthermore couple $\xi_0^{l,R}$ and η_0^R to be equal on $(-\infty, 0]$ (and so that $\eta_t^R \leq \xi_t^{l,R} \ \forall t \geq 0$). Immediately we have $r_t^l \geq r_t \ \forall t \geq 0$. Let event B remain defined for process $\eta_\cdot^R$ and let C be the event that $\xi_1^{l,R}(1) = 1$. The important point is that event $B \cap C$ has strictly positive probability (albeit tending to zero as l tends to infinity), and that given either $B \cap C$ or $B \cap C^c$, the conditional distribution of η_1^R is equilibrium supported on $(-\infty, 0]$. Thus for $t \geq 1$

$$\begin{aligned} E[r_t^l] \ &= \ E[r_t^l I_{B^c}] \ + \ E[r_t^l I_{B\cap C}] \ + \ E[r_t^l I_{B\cap C^c}] \\ &\geq E[r_t I_{B^c}] \ + E[r_t^l I_{B\cap C}] \ + \ E[r_t I_{B\cap C^c}] \\ &= 0 + E[r_t^l I_{B\cap C}] + 0. \end{aligned}$$

But by attractiveness and the fact that $E[r_t^1] \geq 1$ we have for all $t \geq 1, E[r_t^l] \geq P(B \cap C) > 0$.

$\diamondsuit$

We will also need the following bound which is clearly far from optimal but adequate to our needs. The proof is sketched since the ideas are already in place.

Lemma 25 *For $\xi_\cdot^{l,R}$ and $r_\cdot^l$ as in Corollary 24, there exists a constant K, not depending on l, so that for all positive integers l, $c(l) \leq l + K$.*

Sketch of proof: Given $\xi_0^{l,R}$ as in Corollary 24, we introduce $\eta_0^{l,R}$ to equal $\xi_0^{l,R}$ on $(-\infty, 0]$ and on $[l, \infty)$ and on $[0, l]$ to have distribution $Ren^{[0,l]}(\beta)$ conditionally independent of $\xi_0^{l,R}$. By the attractiveness of the processes, it is sufficient to show that for appropriately chosen K

$$E[r_t^{\eta,l}] \;\leq\; l + K$$

for all t where $r_t^{\eta,l}$ is the rightmost particle of $\eta_t^{l,R}$ for $t \geq 0$. As in Proposition 23 we have

$$E[r_t^{\eta,l}] \;=\; l + \int_0^t (\sum_{n=1}^{\infty} \beta(n)n - E[h(\eta_t^{l,R})])dt.$$

It suffices therefore to obtain a good bound for $E[h(\eta_t^{l,R})]$. The key observation is that there exists a strictly positive constant g not depending on the positive l so that the $Ren^{(-\infty,0]}(\beta)$ probability that a configuration has a 1 at $-l$ exceeds g. Thus the distribution of $\eta_0^{l,R}$ has bounded Radon-Nykodym derivative with respect to $Ren^{(-\infty,l]}(\beta)$. Therefore arguing as in Proposition 23 we obtain the bound for $t \geq 1$,

$$|\sum i\beta(i) - E[h(\xi_t^{l,R})]| \leq C \left(t^{25/48} \frac{t^{3/4}}{t^{23k/1152}} \right)^{3/4}.$$

The result follows for $K = \int_1^\infty C \left(t^{25/48} \frac{t^{3/4}}{t^{23k/1152}} \right)^{3/4} dt \;+\; E[r_1^{\eta,l}]$.

$\diamondsuit$

We will also have need of the following "regeneration" result.

Proposition 26 *Given a positive integer N, a right sided β-NPS $(\eta_t^R : t \geq 0)$ in equilibrium and time $T \geq 0$, there is a configuration γ with distribution $Ren^{(-\infty,0]}(\beta)$, independent of $\sigma\{\eta_s^R, s \leq T\}$, so that for constant C (not depending on N or T), outside of an event of probability $C\frac{N^{33/16}}{N^{23k/(288)(16)}}$,*
$\gamma(x) = \eta_{T+N^{1/4}}^R(r_{T+N^{1/4}} + x) \; \forall x \in (-3N^2, 0]$.

Proof. We consider the finite comparison Markov chain $(\eta_t^{F,N^{1/16}} : t \geq T)$ coupled naturally with $\eta_\cdot^R$ so that at time T, $\theta_{R_T} \circ \eta_T^R$ is equal to $\eta_T^{F,N^{1/16}}$ on $(-N^{1/16}, 0]$ where, as before, $r_\cdot$ denotes the position of the rightmost particle of $\eta_\cdot^R$. Given Proposition

22, (1), we have that there exists random Y in $\{0,1\}^{(-N^{1/16},0]}$ which is
(i) independent of $\sigma\{\eta^R_s,\ s \leq T\ \}$
and
(ii) equal to $\eta^{F,N^{1/16}}_{T+N^{1/4}/2}$ (suitably shifted) outside of an event of probability $e^{-cN^{1/8}}$.
Choose configuration $\xi^R_{T+N^{1/4}/2}$, independent of $\sigma\{\eta^R_s\ s \leq T\}$ so that

a) $\xi^R_{T+N^{1/4}/2}$ is in equilibrium supported on $(-\infty, 0]$ and

b) $\xi^R_{T+N^{1/4}/2}$ is equal to Y on $(-N^{1/16}, 0]$.

We now generate $(\xi^R_s :\ s \geq T + N^{1/4}/2)$, starting from this configuration with the same Harris system as $\eta^R_\cdot$ and take γ to be $\xi_{T+N^{1/4}}$. The conclusion of the Proposition follows from Propositions 13 and 15 and Lemma 5.

$\diamondsuit$

6 A coupling result

We have established a good approximation for $E[r^l_{N^{1/4}}]$ in the previous section. Given the coupling results of Section 4, this can be used to gain information about $E[r^{F,N^{1/16}}_{N^{1/4}}]$ for a process starting in equilibrium conditioned on $h(\eta^{F,N^{1/16}}_0) = l$ (see Lemma 28 below). This is then used to consider $E[r^{F,N^{1/16}}_{N^{1+\alpha}}]$ for $\alpha \in (0, \frac{1}{2})$ and for a $\eta^{F,N^{1/16}}_\cdot$ process to which extra positive jumps are added at rate of order $\frac{1}{N}$. The essential idea is that most "extra" jumps will not occur within $N^{1/4}$ of other "extra" jumps and so when extra jumps occur the distribution of the process $\eta^{F,N^{1/16}}_\cdot$ will be close to $Ren^{(-N^{1/16},0]}(\beta)$ (extra jumps notwithstanding) and so an analogue of Corollary 24 may be applied. A problem to be addressed is the treatment of two extra jumps which occur close together. In dealing with this (Lemma 29) we do not aim for an optimal bound merely a sufficient one.

The section culminates in the result Proposition 30. This result is useful in proving Theorems 2-4 as it identifies the drift term.

Recall that for our process $(\eta^{F,n}_t : t \geq 0)$ on $\{0,1\}^{(-n,0]}$ we have defined

$$r^{F,n}_t = \sum_{0 \leq s \leq t} \Delta^F_s$$

in Section 4.

Lemma 27 *Let $\eta^R_\cdot$ be a right sided $\beta-$NPS with rightmost particle process $r_\cdot$ and for each positive integer n, let $\eta^{F,n}_\cdot$ be a finite comparison Markov chain with right most* *particle process* $r_\cdot^{F,n}$. *For fixed integer* $4 \leq \ell \leq (k-1)/20$ *and each positive integer* n, *let* $g(n^\ell) = E^{Ren^{(-\infty,0]}(\beta)}[(r_{n^\ell})^2]$ *and* $g'(n^\ell) = E^{Ren^{(-\infty,0]}(\beta)}[(r_{n^\ell})^4]$. *Then*

$$|E^{Ren^{(-n,0]}(\beta)}[(r_{n^\ell}^{F,n})^2] - g(n^\ell)| \leq \frac{1}{n},$$

and

$$|E^{Ren^{(-n,0]}(\beta)}[(r_{n^\ell}^{F,n})^4] - g'(n^\ell)| \leq \frac{1}{n}.$$

PROOF. We only concern ourselves with the first inequality; the second follows in a nearly identical fashion. By the definitions of the two renewal measures we can choose $\eta_0^{F,n}$ and η_0^R so that

(1) $\eta_0^{F,n}$ has the distribution $Ren^{(-n,0]}(\beta)$,
(2) η_0^R has the distribution $Ren^{(-\infty,0]}(\beta)$,
(3) $\eta_0^R|_{(-n,0]} = \eta_0^{F,n}$.

Let $\eta_\cdot^R$ and $\eta_\cdot^{F,n}$be naturally coupled as in Proposition 13. By Corollary 14 the probability that

$$\eta_t^{F,n}|_{(-n/2,0]} = \eta_t^R \circ \theta_{r_t}|_{(-n/2,0)} \neq \underline{0} \;\forall\; 0 \leq t \leq n^\ell,$$

is at least $1 - Cn^{\ell-23/12}\frac{n^3}{n^{23k/288}}$. On this event $r_{n^\ell}^{F,n} = r_{n^\ell}$. Denote the complement of this event by A, so

$$\begin{aligned}
& |E[(r_{n^\ell}^{F,n})^2] - g(n^\ell)| \\
= \;& |E[(r_{n^\ell}^{F,n})^2 I_A] - E[(r_{n^\ell})^2 I_A]| \\
\leq \;& |E[(r_{n^\ell}^{F,n})^2 I_A]| + |E[(r_{n^\ell})^2 I_A]|
\end{aligned}$$

and the proof of the lemma reduces to providing appropriate bound for these two terms.

$$\begin{aligned}
E[r_{n^\ell}^2 I_A] &\leq (E^{Ren(\beta)}[(r_{n^\ell})^4])^{1/2}(P(A))^{1/2} \\
&\leq Kn^\ell P(A)^{1/2},
\end{aligned}$$

by Proposition 3.2 of [8]. Equally we can easily bound $E[(r_{n^\ell}^F)^4]$ by $Kn^{4\ell}$ and so

$$(E[(r_{n^\ell}^F)^4])^{1/2} P(A)^{1/2} \leq Kn^{2\ell} P(A)^{1/2}$$

and the result follows. (Recall $k \geq 1500$.)

$\diamondsuit$

We now consider (for positive integer ℓ such that $\beta(\ell) \geq \frac{1}{N^{3/2}}$), the expectation of $r^{F,N^{1/16}}_{N^{1/4}}$ for $r^{F,N^{1/16}}_{\cdot}$ the rightmost particle process associated with a finite comparison Markov chain $\eta^{F,N^{1/16}}_{\cdot}$ initially distributed as $Ren^{(-N^{1/16},0]}(\beta)$ conditioned on having $h(\eta^{F,N^{1/16}}_0)$ equal to ℓ. By Corollary 24 for a right sided process, $\eta^R_{\cdot}$ starting in equilibrium with $r_0 = 0$ conditioned on $\{h(\eta^R_0) = l\}$, we know that for N large

$$|E[r_{N^{1/4}}] - (c(\ell) - \ell)| \leq \frac{1}{N^{1/8}}.$$

We expect a similar result here.

Lemma 28 *For $\eta^{F,N^{1/16}}_{\cdot}$ as in Lemma 27 and positive integer l so that $\beta(l) \geq \frac{1}{N^{3/2}}$,*

$$|E[r^{F,N^{1/16}}_{N^{1/4}}] - (c(\ell) - \ell)| \leq 2\frac{1}{N^{1/8}},$$

for N sufficiently large.

Proof. As with Lemma 18 we simply consider $E[(r^{F,N^{1/16}}_{N^{1/4}} - r_{N^{1/4}})]$ for $r_{\cdot}$ derived from a right sided process, $\eta^R_{\cdot}$, initially equal to $\eta^{F,N^{1/16}}_0$ on $(-N^{1/16}, 0]$ and distributed as $Ren(\beta)$ conditioned on $h(\eta^R_0) = \ell$.
Suppose $\eta^R_{\cdot}$ and $\eta^{F,N^{1/16}}_{\cdot}$ are naturally coupled as in Proposition 13. Let A be the event that the natural coupling breaks down. Let $\xi^R_{\cdot}$ be a right sided $\beta-NPS$ with ξ^R_0 distributed as $Ren^{(-\infty,0]}(\beta)$ and let $\xi^{R,N^{1/16}}_{\cdot}$ be a comparison Markov chain naturally coupled with $\xi^R_{\cdot}$ and satisfying $\xi^R_0 = \xi^{R,N^{1/16}}_0$ on $(-N^{1/16}, 0]$. Let the position of the rightmost particle of $\xi^R_{\cdot}$ be denoted by $r^{\xi}_{\cdot}$ and the rightmost particle functional of $\xi^{R,N^{1/16}}_{\cdot}$ be denoted by $r^{\xi,F,N^{1/16}}_{\cdot}$. We will abuse notation and denote also by A, the event corresponding to A, as previously defined in the proof of Lemma 27, but with $(\eta^R_{\cdot}, \eta^{F,N^{1/16}}_{\cdot})$ replaced by $(\xi^R_{\cdot}, \xi^{F,N^{1/16}}_{\cdot})$.

$$\begin{aligned}
|E[r^{F,N^{1/16}}_{N^{1/4}} - r_{N^{1/4}}]| &\leq |E[(r^{F,N^{1/16}}_{N^{1/4}} - r_{N^{1/4}})I_A]| + |E[(r^{F,N^{1/16}}_{N^{1/4}} - r_{N^{1/4}})I_{A^c}]| \\
&= |E[(r^{F,N^{1/16}}_{N^{1/4}} - r_{N^{1/4}})I_A]| \\
&\leq E[|r^{F,N^{1/16}}_{N^{1/4}}|I_A] + E[|r_{N^{1/4}}|I_A] \\
&\leq \frac{E[|r^{\xi,F,N^{1/16}}_{N^{1/4}}|I_A]}{\beta(\ell)} + \frac{E[|r^{\xi}_{N^{1/4}}|I_A]}{\beta(\ell)} \\
&\leq N^{3/2}E[|r^{\xi,F,N^{1/16}}_{N^{1/4}}|I_A] + N^{3/2}E[|r^{\xi}_{N^{1/4}}I_A|]
\end{aligned}$$

by the condition on l. But

$$\begin{aligned}
E[|r^{\xi,F,N^{1/16}}_{N^{1/4}}I_A|] &\leq (E[(r^{\xi,F,N^{1/16}}_{N^{1/4}})^2])^{1/2}P(A)^{1/2} \text{ by Cauchy Schwarz} \\
&\leq CN^{1/8}P(A)^{1/2} \text{ by the previous lemma.}
\end{aligned}$$

Similarly for $E[|r^{\xi}_{N^{1/4}} I_A|]$ and so the result follows by Proposition 13 and Corollary 24.

$\diamondsuit$

In our analysis we will have to deal with perturbations of the Markov chain $(\eta_t^{F,N^{1/16}} : t \in [0, N^{1+\alpha}])$ for a constant $0 < \alpha \leq 1/2$. We suppose that we are given a Harris system that generates a comparison Markov chain in equilibrium, $\eta_{\cdot}^{F,N^{1/16}}$. For two positive integer valued random variables X_1 and X_2 with support on $[1, N^{3/2(k+1)}]$ and a fixed time $s \in [0, N^{1/4}]$, we say $\xi_{\cdot}^{F,N^{1/16}}$ is the (X_1, X_2, s)-perturbation of $\eta_{\cdot}^{F,N^{1/16}}$ if

(i) $\xi_0^{F,N^{1/16}}$ is equal to $\eta_0^{F,N^{1/16}}$ after a positive shift by X_1,

(ii) for $t \in (0, s)$, $\xi_t^{F,N^{1/16}}$ evolves according to the rules for $\eta_{\cdot}^{F,N^{1/16}}$ applied to the Harris system given, as detailed at the start of Section 4

(iii) at fixed time s, $\xi_{\cdot}^{F,N^{1/16}}$ undergoes a positive shift by X_2,

(iv) for $t > s$, $\xi_t^{F,N^{1/16}}$ evolves according to the rules for $\eta_{\cdot}^{F,N^{1/16}}$ applied to the Harris system given as in Section 4.

For such a chain the rightmost functional $r_{\cdot}^{\xi,F,N^{1/16}}$ is defined in the same way as $r_{\cdot}^{F,N^{1/16}}$ with positive shifts X_1, X_2 contributing to the functional, so for instance $r_0^{\xi,F,N^{1/16}} = X_1$ and not 0.

We shall need the following crude bound

Lemma 29 *Let N be a positive integer, $s \in [0, N^{1/4}]$ and $\xi_{\cdot}^{F,N^{1/16}}$ be a (X_1, X_2, s)-perturbation of equilibrium finite comparison chain $\eta_{\cdot}^{F,N^{1/16}}$ where X_1, X_2 are i.i.d. positive integer valued random variables also independent of the Harris system generating $\eta_{\cdot}^{F,N^{1/16}}$ with*

$$\text{for } 1 \leq i \leq N^{3/2(k+1)} \quad P(X_1 = i) = \frac{\gamma(i)\beta(i)}{\sum_{j=1}^{N^{3/2(k+1)}} \gamma(j)\beta(j)}$$

where for all $1 \leq i \leq N^{3/2(k+1)}, 0 \leq \gamma(i) \leq c_1\gamma(1)i$.

Then $|E[r_{s+N^{1/4}}^{\xi,F,N^{1/16}}]| < KN^{1/8}$ for K depending on c_1 but not on N.

Proof. In this proof C will denote a constant depending on c_1 but not on N. This constant may change from line to line (or within a line).

$|E[r_{s+N^{1/4}}^{\xi,F,N^{1/16}}]|$ is less than the sum $|E[r_{s+N^{1/4}}^{\xi,F,N^{1/16}} - r_{s-}^{\xi,F,N^{1/16}}]| + |E[r_{s-}^{\xi,F,N^{1/16}} - r_0^{\xi,F,N^{1/16}}]| + E[X_1]$. Given condition $(***)$ on function $\beta(.)$ and the conditions on the function $\gamma(.)$ we immediately have $E[X_1] \leq \sum_{j=1}^{N^{3/2(k+1)}} c_1 j\beta(j)j/\beta(1) < \infty$, supposing as we may, that $c_1 > 1$. It remains to bound the first two terms. We bound the two separately.

Let V be the Radon-Nykodym derivative of the distribution of $\xi_0^{F,N^{1/16}}$ with respect to $Ren^{(-N^{1/16},0]}(\beta)$. Since $V(\xi) = c\gamma(i)$ (for $c = c(c_1)$) on $h(\xi) = i,\ i \leq N^{3/2(k+1)}; = 0$ for $h(\xi) > N^{3/2(k+1)}$ it is easily seen, using the conditions on $\gamma(.)$ that V possesses all moments less than $k-2$'th order and that the bounds do not depend on N. By Hölder's inequality

$$\begin{aligned}|E[r_{s-}^{\xi,F,N^{1/16}} - r_0^{\xi,F,N^{1/16}}]| &\leq E^{Ren^{(-N^{1/16},0]}(\beta)}[|V(\eta_0^{F,N^{1/16}})r_s^{F,N^{1/16}}|] \\ \leq \quad (E^{Ren^{(-N^{1/16},0]}(\beta)}[V^{3/2}])^{2/3}&(E^{Ren^{(-N^{1/16},0]}(\beta)}[|r_s^{F,N^{1/16}}|^3])^{1/3} \leq KN^{1/8}\end{aligned}$$

by Lemma 27.

In the same way let W be the Radon-Nykodym derivative of the distribution of $\xi_s^{F,N^{1/16}}$ with respect to $Ren^{(-N^{1/16},0]}(\beta)$.

$$\begin{aligned}|E[r_{s+N^{1/4}}^{\xi,F,N^{1/16}} - r_{s-}^{\xi,F,N^{1/16}}]| &\leq E[X_2] + |E^{Ren^{(-N^{1/16},0]}(\beta)}[W(\eta_0^{F,N^{1/16}})r_{N^{1/4}}^{F,N^{1/16}}]| \\ \leq \quad C \ + \ (E^{Ren^{(-N^{1/16},0]}(\beta)}&[W^{3/2}])^{2/3}(E^{Ren^{(-N^{1/16},0]}(\beta)}[|r_{N^{1/4}}^{F,N^{1/16}}|^3])^{1/3} \\ &\leq \ (C + (E^{Ren^{(-N^{1/16},0]}(\beta)}[W^{3/2}])^{2/3})N^{1/8}.\end{aligned}$$

So it remains to bound $E^{Ren^{(-N^{1/16},0]}(\beta)}[W^{3/2}]$.

Now let the Radon-Nikodym derivative of $\eta_{s-}^{F,N^{1/16}}$ with respect to measure $Ren^{(-N^{1/16},0]}(\beta)$ be V_s. For l an integer between 1 and $N^{1/16}$ and $\xi \in \Omega^{N^{1/16}}$ with $h(\xi) = l$ we have

$$W(\xi)Ren^{(-N^{1/16},0]}(\beta)(\{\xi\}) \ = \ P(\xi_s^{F,N^{1/16}} = \xi) = P(X_2 = l)\sum_{\eta \in A(\xi)} P(\xi_{s-}^{F,N^{1/16}} = \eta)$$

(where $A(\xi) = \{\eta : \eta(x) = \xi(x-l) \text{ for } 0 \leq \ -x \ < n-l\}$; note that the integer l is a function of configuration ξ and so no suffix l is required for $A(\xi)$)

$$\leq C\beta(l)l \sum_{\eta \in A(\xi)} V_s(\eta)Ren^{(-N^{1/16},0]}(\beta)(\{\eta\})$$

$$= CRen^{(-N^{1/16},0]}(\beta)(\{\xi\})lG(\xi),$$

where

$$G(\xi) \ = \ \frac{\sum_{\eta \in A(\xi)} V_s(\eta)Ren^{(-N^{1/16},0]}(\beta)(\{\eta\})}{\sum_{\eta \in A(\xi)} Ren^{(-N^{1/16},0]}(\beta)(\{\eta\})}.$$

Thus, we conclude, $W(\xi) \leq ClG(\xi) = Ch(\xi)G(\xi)$ and so

$$E[W^{3/2}] \leq C \sum_{\xi \in \Omega^{N^{1/16}}} Ren^{(-N^{1/16},0]}(\beta)(\{\xi\})h(\xi)^{3/2}(G(\xi))^{3/2}.$$

By Holder's inequality this is less than

$$\left(\sum_{\xi\in\Omega^{N^{1/16}}} Ren^{(-N^{1/16},0]}(\beta)(\{\xi\})h(\xi)^6\right)^{1/4}\left(\sum_{\xi\in\Omega^{N^{1/16}}} Ren^{(-N^{1/16},0]}(\beta)(\{\xi\})(G(\xi))^2\right)^{3/4}.$$

So, since all moments of h exist uniformly in $N^{1/16}$, to bound $E[W^{3/2}]$ it will suffice to bound

$$\sum_{\xi\in\Omega^{N^{1/16}}} Ren^{(-N^{1/16},0]}(\beta)(\{\xi\})(G(\xi))^2 = \sum_{l=1}^{[N^{3/2(k+1)}]}\beta(l)\sum_{\xi:h(\xi)=l} Ren^{(-N^{1/16},0]}(\beta)(\{A(\xi)\})G(\xi)^2.$$

But for each l in the summation range, the sets $A(\xi)$ form a partition of $\Omega^{N^{1/16}}$ as ξ ranges over configurations on which h equals l. Also by Jensen's inequality for any $\xi\in\Omega^{N^{1/16}}$,

$$Ren^{(-N^{1/16},0]}(\beta)(\{A(\xi)\})G(\xi)^2 \le \sum_{\eta\in A(\xi)}(V_s)^2(\eta)Ren^{(-N^{1/16},0]}(\beta)(\{\eta\}).$$

Therefore for each suitable l,
$\sum_{\xi:h(\xi)=l} Ren^{(-N^{1/16},0]}(\beta)(\{A(\xi)\})G(\xi)^2 \le < Ren^{(-N^{1/16},0]}(\beta),(V_s)^2 >$.

Hence

$$\sum_{l=1}^{[N^{3/2(k+1)}]}\beta(l)\sum_{\xi:h(\xi)=l} Ren^{(-N^{1/16},0]}(\beta)(\{A(\xi)\})G(\xi)^2 \le$$

$$\sum_{l=1}^{[N^{3/2(k+1)}]}\beta(l) < Ren^{(-N^{1/16},0]}(\beta),(V_s)^2 > < < Ren^{(-N^{1/16},0]}(\beta),(V_s)^2 > .$$

Since $< Ren^{(-N^{1/16},0]}(\beta),(V_s)^2 > \le < Ren^{(-N^{1/16},0]}(\beta),V^2 >$ and all moments of V less than $k-2$ exist, we are done.

$\diamondsuit$

We now consider a Markov chain on $\Omega^{N^{1/16}}$ which is a (slight) modification of the chain $\eta_\cdot^{F,N^{1/16}}$. We say that $\xi_\cdot^{F,N^{1/16}}$ is a (γ,N)- modification of $\eta_\cdot^{F,N^{1/16}}$ for
(i) N a positive integer,
(ii) $\gamma:\{1,2,3,\cdots,[N^{3/2(k+1)}]\}\to \mathbb{R}_+$,
if it is a chain with the same jump rates as $\eta_\cdot^{F,N^{1/16}}$ except that the positive shifts by l for $l\le[N^{3/2(k+1)}]$ occur at rate $\beta(l)(1+\frac{\gamma(l)}{N})$. In this section we will only consider γ so that for some $c_1\in(1,\infty)$, $\gamma(1)<c_1$,$\forall\ 2\le i\le N^{3/2(k+1)}, 0\le\gamma(i)\le c_1 i\gamma(1)$, so that the preceding lemma may be applied. We can define as before the associated rightmost particle functional $r_\cdot^{\gamma,N^{1/16}}$ For our modified finite comparison Markov chains, we consider the extra jumps to be produced by Poisson processes $V^l, 1\le l\le[N^{3/2(k+1)}]$

of rate $\gamma(l)\beta(l)/N$ separate from and independent of the Poisson processes generating a finite comparison Markov chain $\eta_{\cdot}^{F,N^{1/16}}$ so that a point in V^l engenders a positive l shift for $\xi_{\cdot}^{F,N^{1/16}}$. Thus we can speak of "extra" jumps unambiguously, as jumps corresponding to points in $\cup_l V^l$ and we can define A, the event that at most two of the extra jumps occur within time $N^{1/4}$ of each other during the time interval $[0, N^{1+\alpha}]$, no jumps occur within time $N^{1/4}$ of the endpoints of time interval $[0, N^{1+\alpha}]$. It is elementary that $P(A) \geq 1 - \frac{CN^{2\alpha}}{N^{3/2}} - C/N^{3/4}$. (The bound $C/N^{3/4}$ for the probability of the event that an extra point occurs in intervals $[0, N^{1/4}]$ or $[N^{1+\alpha} - N^{1/4}, N^{1+\alpha}]$ is clear given the rate of extra jumps. By elementary large deviations on Poisson processes we have for some finite c, C depending on c_1 but not on N, that with probability at least $1 - Ce^{-cN^{\alpha}}$, at most $2CN^{\alpha}$ extra jumps occur in time interval $[0, N^{1+\alpha}]$. By the strong Markov property, the number of extra jumps among the first $2CN^{\alpha}$ which have the property that another extra jump occurs in the $N^{1/4}$ time units following their own arrival, is exactly a Binomial with parameters $2CN^{\alpha}$ and $C'/N^{3/4}$. We can then apply elementary inequalities to this random variable to obtain the required bound. Then

Proposition 30 *Let $c_1 \in (1, \infty)$. Suppose that (γ, N) are such that, $\gamma(1) < c_1$ and $\forall\ 2 \leq i \leq N^{3/2(k+1)}, \gamma(i) \leq c_1 i \gamma(1)$.*

Let α be a constant in $(0, 1/2)$ and $\xi_{\cdot}^{F,N^{1/16}}$ be a (γ, N)- modification of $\eta_{\cdot}^{F,N^{1/16}}$ with $\xi_0^{F,N^{1/16}}$ distributed as $Ren^{(-N^{1/16},0]}(\beta)$ and let $r_{\cdot}^{\gamma,N^{1/16}}$ be the associated rightmost particle process. For A the event above,

$$|E[r_{N^{1+\alpha}}^{\gamma,N^{1/16}}|A] - N^{\alpha} \sum_{i=1}^{[N^{\frac{3}{2(k+1)}}]} c(i)\gamma(i)\beta(i)| \leq C \left(N^{3\alpha/4} + N^{\alpha - 1/8} + N^{1/8}\right)$$

for C depending on c_1 but not further on γ or on N and $c(i)$ equal to the constants of Corollary 24.

Remark: The time interval $[0, N^{1+\alpha}]$ and time $N^{1+\alpha}$ can be replaced by $[0, W]$ and W respectively for any W in the interval $[N^{1+\alpha}/2, 2N^{1+\alpha}]$ and the result will remain valid with $N^{\alpha}\sum_{i=1}^{[N^{\frac{3}{2(k+1)}}]} c(i)\gamma(i)\beta(i)$ replaced by $\frac{W}{N}\sum_{i=1}^{[N^{\frac{3}{2(k+1)}}]} c(i)\gamma(i)\beta(i)$.

Proof. We condition on the event A described above. Let the V be the (random) number of extra jumps and let the times of the extra jumps, that is jumps corresponding to times $t \in \cup_l V^l$ be

$$0 < N^{1/4} < t_1 < t_2 ... < t_V < N^{1+\alpha} - N^{1/4}.$$

V is a Poisson (cN^{α}) random variable for $c = \sum_{l=1}^{N^{3/2(k+1)}} \beta(l)\gamma(l)$, conditioned on A, an event of probability at least $1/2$ (for large N), so (as is easily seen from direct calculation using Stirling's formula) outside of an event of probability $e^{-hN^{\alpha/2}}$

$$\begin{array}{lll} \text{there are between} & cN^{\alpha} - N^{3\alpha/4} & \text{and} \\ & cN^{\alpha} + N^{3\alpha/4} & \text{extra jumps} \end{array}$$

for $h > 0$ depending on c_1 but not on N or α. We easily obtain the bound

$$E[|r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|I_{\{|V-cN^\alpha|\geq N^{3\alpha/4}\}}] \;\leq\; N^2N^{1+\alpha}e^{-hN^{\alpha/2}/2}.$$

Now suppose that the extra jump, occurring at t_i is a positive shift by x_i. Then the x_i are independent identically distributed random variables with law $P(x_i = j) = \gamma(j)\beta(j)/c$ where for $j \leq [N^{\frac{3}{2(k+1)}}]$, $0 \leq \gamma(j)/c \leq c_1 j/\beta(1)$, as the conditioning event A is independent of the sizes of the extra jumps. The value $E[r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|A]$ is the expectation (taken over t_i, x_i) of

$$\begin{aligned} & E[r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|\underline{t},\underline{x}] \\ = \;& \sum_{i=1}^{V} x_i + \int_0^{N^{1+\alpha}} \left(\sum_1^\infty \beta(i)(i\wedge N^{1/16}) - E[h(\xi^{F,N^{1/16}}_s)|\underline{t},\underline{x}]\right) ds. \end{aligned}$$

We suppose first that, in fact, none of the extra jump times t_i occur within time $N^{1/4}$ of another. We divide up $[0, N^{1+\alpha}]$ into intervals

$$\begin{aligned} I_i \; &= [t_i, t_i + N^{1/4}] \qquad & i = 1, \cdots V \\ \text{and } J_h & & h = 1, \cdots V + 1, \end{aligned}$$

the remaining ordered intervals, $J_1 = [0, t_1]$, $J_2 = [t_1 + N^{1/4}, t_2]$ and so on.
As our process starts with distribution $Ren^{(-N^{1/16},0]}(\beta)$, the integral $\int_{J_1} \left(\sum_1^\infty \beta(i)(i\wedge N^{1/16}) - E[h(\xi^{F,N^{1/16}}_s)|\underline{t},\underline{x}]\right) ds$ is just equal to the corresponding integral for a non-perturbed equilibrium process $\eta^{F,N^{1/16}}_{\cdot}$,

$$\int_{J_1} (\sum_1^\infty \beta(i)(i\wedge N^{1/16}) - E^{Ren^{(-N^{1/16},0]}(\beta)}[h(\eta^{F,N^{1/16}}_s)])ds = 0.$$

For $h > 1$, $J_h = [t_{h-1} + N^{1/4}, t_h)$ with t_{V+1} taken to be $N^{1+\alpha}$. For s in J_h we have by the independent increments property of Poisson processes,

$$E[h(\xi^{F,N^{1/16}}_s)|\underline{t},\underline{x}] = E[h(\xi^{F,N^{1/16}}_s)|t_j, x_j,\; j < h].$$

But Proposition 22 (2) yields

$$|E[h(\xi^{F,N^{1/16}}_t)|t_j, x_j, j < h] - \sum_i^\infty \beta(i)(i\wedge N^{1/16})| \leq e^{-CN^{1/8}}$$

for $t \in J_h$ and so

$$|\sum_{h=1}^{V+1} \int_{J_h} \left(\sum^\infty \beta(i)(i\wedge N^{1/16}) - E[h(\xi^{F,N^{1/16}}_s)|\underline{t},\underline{x}]\right) ds| \leq N^{1+\alpha}e^{-CN^{1/8}}.$$

We now consider intervals $I_j = [t_j, t_j + N^{1/4}]$. By Proposition 22 (1),

$$|P(\xi_{t_j}^{F,N^{1/16}} = \chi) - \frac{Ren^{(-N^{1/16},0]}(\beta)(\{\eta : \eta = \chi\})}{Ren^{(-N^{1/16},0]}(\beta)(\{\eta : h(\eta) = x_j\})}| \leq e^{-cN^{1/8}}.$$

Thus

$$\int_{I_j} (\Sigma\beta(i)(i \wedge N^{1/16}) - E[h(\xi_s^{F,N^{1/16}})|\underline{t}, \underline{x}])ds$$

$$= \int_0^{N^{1/4}} \left(\Sigma\beta(i)(i \wedge N^{1/16}) - E^{Ren^{(-N^{1/16},0]}(\beta)}[h(\eta_s^{F,N^{1/16}})|h(\eta_0^{F,N^{1/16}}) = x_j]\right) ds$$
$$+\mathrm{O}(e^{-cN^{1/8}} N^{1/4} N^{1/16}) \quad .$$

But by Lemma 28 this term is within $CN^{-1/8}$ of $c(x_j) - x_j$ and we have that

$$|E[r_{N^{1+\alpha}}^{\gamma,N^{1/16}}|\underline{t}, \underline{x}] - \Sigma c(x_j)| \leq K\ VN^{-1/8}$$

and so (recall x_i are i.i.d., independent of t_i), on event
$A \cap \{$ no two extra jumps are within $N^{1/4}$ of each other $\}$,

$$|E[r_{N^{1+\alpha}}^{\gamma,N^{1/16}}|\underline{t}] - V \sum_{i=1}^{N^{3/2(k+1)}} c(i)\gamma(i)\beta(i)| \leq\ K\ VN^{-1/8}. \qquad (+)$$

For the remaining part of event A where there is exactly one extra jump followed within $N^{1/4}$ by another, suppose that the extra jumps occur at $t_1 < t_2 < \cdots < t_V$ and that j is the unique integer between 1 and V such that $t_j < t_{j+1} < t_j + N^{1/4}$. Again we divide up the interval $[0, N^{1+\alpha}]$ into intervals:

$$\begin{array}{llll} & I_i & = [t_i, t_{i+N^{1/4}}] & i = 1, \cdots j-1 \\ & I_j & = [t_j, t_{j+1+N^{1/4}}] & \\ & I_i & = [t_{i+1}, t_{i+1+N^{1/4}}] & i = j+1, \cdots V-1 \\ \text{and} & J_h & & h = 1, \cdots V \end{array}$$

the remaining naturally ordered intervals, so that $J_h = [t_{h-1} + N^{1/4}, t_h]$ for $h \leq j$ (and taking $t_{-1} + N^{1/4}$ to be 0); $= [t_h + N^{1/4}, t_{h+1}]$ for $h > j$ (and taking t_{V+1} to be $N^{1+\alpha}$).

We have just as before that for $k \neq j$

$$\int_{I_k} (\Sigma\beta(i)(i \wedge N^{1/16}) - E[h(\xi_s^{F,N^{1/16}}|\underline{t}, \underline{x}])ds$$

$$= \int_0^{N^{1/4}} \Sigma\beta(i)(i \wedge N^{1/16}) - E^{Ren^{(-N^{1/16},0]}(\beta)}[h(\eta_s^{F,N^{1/16}})|h(\eta_0^{F,N^{1/16}}) = x_k]ds$$
$$+\mathrm{O}(e^{-cN^{1/8}} N^{1/4} N^{1/16})$$

and

$$|\sum_{h=1}^{V}\int_{J_h}\sum^{\infty}\beta(i)(i\wedge N^{1/16})-E[h(\xi_s^{F,N^{1/16}})|\underline{t},\underline{x}]ds|\leq N^{1+\alpha}e^{-cN^{1/8}}.$$

Thus, using Lemma 29 to bound the increment over I_j by $KN^{1/8}$, we obtain on event $A\cap\{$ two extra jumps are within $N^{1/4}$ of each other $\}$,

$$|E[r_{N^{1+\alpha}}^{\gamma,N^{1/16}}|\underline{t}]-V\sum_{i=1}^{N^{3/2(k+1)}}c(i)\gamma(i)\beta(i)|\leq\ K\ VN^{-1/8}+KN^{1/8}.$$

Using this and (+) and integrating over $\underline{t}$ (and also using our bound for the expectation on event $\{|V-cN^{1+\alpha}|>N^{3/4}\}$ we obtain the result.

$\diamondsuit$

7 Perturbed semi-infinite $\beta-$NPSs

In this section we consider classes of NPSs which are right sided (though obviously the results obtained will transfer to the analogous left sided processes) and are perturbations of β-NPSs in that the flip rates for sites to the left of or equal to the rightmost occupied site are those of a β-NPS. The key element in their analysis is Girsanov's Theorem which enables us to transfer various large deviations, regeneration and stability results from equilibrium right sided β-NPSs to the "perturbed" processes. We also prove a weak convergence result, the argument given being a simpler prototype for the weak convergence arguments given in proving Theorems 2-4. We state the following result not for its novelty or difficulty (it simply follows from Brownian embedding) but simply to illuminate the approach and to motivate technical results that follow.

Proposition 31 *Let α be fixed in $(0,1)$. For each positive integer N let $(W_i^N), i\geq 1$ and $(W_i^{N\prime}), i\geq 1$ be sequences of random variables so that for each T positive and fixed,*
(i) with probability tending to one $W_i^N = W_i^{N\prime}$ for $1\leq i\leq TN^{1-\alpha}$ as N tends to infinity,
(ii) $E[W_i^{N\prime}|\mathcal{F}_{i-1}]=0$, $E[(W_i^{N\prime})^2|\mathcal{F}_{i-1}]/N^{1+\alpha}\to 1$ uniformly in i as N tends to infinity, where $\mathcal{F}_0$ is the trivial σ-field and for $r\geq 1$, $\mathcal{F}_r=\sigma\{W_i^{N\prime},\ i\leq r\}$,
(iii) $\forall N\geq 1$, $E[(W_i^{N\prime})^4|\mathcal{F}_{i-1}]/N^{2(1+\alpha)}\leq K$ where K does not depend on N, then

$$(X_t^N:t\geq 0)\ =\ \left(\frac{\sum_{i=1}^{[tN^{1-\alpha}]}W_i^N}{N}:t\geq 0\right)$$

converges in distribution to standard Brownian motion.

Throughout this section $\gamma(.)$ will be a positive function on the positive integers such that

$$\gamma(1) < c_1, \forall i \geq 2, \gamma(i) \leq c_1 i \gamma(1),$$

for c_1 a positive constant. Let $\Gamma(c_1)$ denote the set of positive functions on the positive integers satisfying this constraint. A NPS, $(\xi_t^N : t \geq 0)$ is called a (γ, N) perturbed β-NPS if its flip rates $c(x, \xi^N)$ are

$$\begin{cases} 1 & \text{if } \xi^N(x) = 1 \\ \beta(\ell_{\xi^N}(x), r_{\xi^N}(x)) = \frac{\beta(\ell_{\xi^N}(x))\beta(r_{\xi^N}(x))}{\beta(\ell_{\xi^N}(x)+r_{\xi^N}(x))} & \text{if } \xi^N(x) = 0 \text{ and } \ell_{\xi^N}(x), r_{\xi^N}(x) < \infty \\ \beta^N(\ell_{\xi^N}(x), \infty) = \beta(\ell_{\xi^N}(x))(1 + \frac{\gamma(\ell_{\xi^N}(x))}{N}) & \text{if } \xi^N(x) = 0 \text{ and } \ell_{\xi^N}(x) < \infty \ r_{\xi^N}(x) = \infty \\ \beta^N(\infty, r_{\xi^N}(x)) = \beta(r_{\xi^N}(x))(1 + \frac{\gamma(r_{\xi^N}(x))}{N}) & \text{if } \xi^N(x) = 0 \text{ and } \ell_{\xi^N}(x) = \infty \ r_{\xi^N}(x) < \infty. \end{cases}$$

The similarity with the (γ, N) modifications of the finite state space processes is not accidental. Unless otherwise stated we will assume in the following that ξ_0^N is distributed as $Ren^{(-\infty,0]}(\beta)$. We denote the position of the rightmost particle of ξ_t^N by $r_t^{\xi,N}$. We prove the following proposition.

Proposition 32 *For (γ, N) perturbed β-NPS $(\xi_t^N : t \geq 0)$ as defined above starting from initial distribution $Ren^{(-\infty,0]}(\beta)$,*
$(\frac{r_{tN^2}^{\xi,N}}{N} : t \geq 0) \xrightarrow{D} (X_t : t \geq 0)$ *as* $N \to \infty$,
where
$X_0 = 0 \qquad dX_t = dW_t + v dt,$ *and* $v = \sum_{\ell=1}^{\infty} \beta(\ell)\gamma(\ell)c(\ell)$
for the constants $c(.)$ defined by Corollary 24 and $W_.$ is a standard Brownian motion.

A result of this flavour was proven for a process denoted as $\eta_t^{N'}$ in [8] but in this case the process in question was defined so that $\forall t \ \theta_{r_t^{N'}} \circ \eta_t^{N'}$ had distribution $Ren^{(-\infty,0]}(\beta)$. For us the fact that, as seen from the rightmost particle, ξ_t^N does not have distribution $Ren^{(-\infty,0]}(\beta)$ will constitute the major difficulty.

The above result is of intrinsic interest but also introduces the main approach of Section 8 and intermediate results (Propositions 36-38) shown in the course of proving Proposition 32 will also be needed in our proof of Theorem 2.

The main tool for us will be the Girsanov's formula.

Lemma 33 *Consider the space Ω of cadlag functions from $\mathbb{R}_+$ to right sided configurations ω in $\{0,1\}^{\mathbb{Z}}$, equipped with the usual Skorohod topology. Let Q be the law on Borel subsets of Ω for which the paths are a (γ, N) perturbed β-NPS, while P is the law for the β-NPS, both laws having $\omega(0)$ distributed as $Ren^{(-\infty,0]}(\beta)$. Fix $\alpha \in (0, 1]$.*

Let $N(r)(= N(r,\omega))$ for $\omega \in \Omega$ be the number of times that the rightmost site jumps r units to the right during the time interval $[0, TN^{1+\alpha}]$. Then Q and P a.s. $N(r)$ is defined for every positive integer r and

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}} = e^{-cTN^{\alpha}}\Pi_{r=1}^{\infty}\left(1+\frac{\gamma(r)}{N}\right)^{N(r)} \quad \textit{for } c = \sum_{r=1}^{\infty}\beta(r)\gamma(r),$$

where $\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}}$ denotes the Radon-Nikodym derivative of Q with respect to P on the sigma-field generated by the paths on the time interval $[0, TN^{1+\alpha}]$.

This follows directly from [6] page 320. This result is stated for finite state Markov chains but the extension to our case is minor.

Lemma 34 *Let Q and P be as in Lemma 33. For positive integer $\ell < k-1$, $\alpha \in [0,1]$, $T \geq 0$ fixed and all N sufficiently large, $E^P\left[\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}}\right)^{\ell}\right] \leq K_{\ell,T,c_1}$ for some finite K_{ℓ,T,c_1} depending on T, c_1 and ℓ but not on N or on the particular $\gamma \in \Gamma(c_1)$ underlying measure Q.*

PROOF. By the preceding lemma, $\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}}\right)^{\ell} = e^{-cT\ell N^{\alpha}}\Pi_{r=1}^{\infty}\left(\left(1+\frac{\gamma(r)}{N}\right)^{\ell}\right)^{N(r)}$ and under P the random variables $N(r)$ are independent Poisson random variables of mean $T\beta(r)N^{1+\alpha}$, thus

$$\begin{aligned} E^P\left[\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}}\right)^{\ell}\right] &= \exp(-cT\ell N^{\alpha})\Pi_{r=1}^{\infty}\exp\left(T\beta(r)N^{1+\alpha}[(1+\frac{\gamma(r)}{N})^{\ell}-1]\right) \\ &= \Pi_{r=1}^{\infty}\exp\left(T\beta(r)N^{1+\alpha}[(1+\frac{\gamma(r)}{N})^{\ell}-1-\frac{\gamma(r)}{N}\ell]\right) \end{aligned}$$

(for $\exp(x) = e^x$),

$$\begin{aligned} &\leq \Pi_{r=1}^{\infty}\exp\left(T\beta(r)\sum_{j=2}^{\ell}\binom{\ell}{j}\gamma(r)^j\frac{1}{N^{j-2}}\right) \\ &= \Pi_{j=2}^{\ell}\exp\left(\frac{T}{N^{j-2}}\binom{\ell}{j}\sum_{r=1}^{\infty}\beta(r)\gamma(r)^j\right) \qquad < K_{\ell,T,c_1} \end{aligned}$$

since $\gamma \in \Gamma(c_1)$ and $\sum_{r=1}^{\infty}\beta(r)r^j$ is summable for $j < k-1$.

$\diamondsuit$

Lemma 35 *Let Q and P be as in Lemma 33. For fixed $\alpha < \frac{1}{2}$, there exists a constant $K = K(\alpha, c_1, T)$, not depending on N or the particular $\gamma \in \Gamma(c_1)$, so that*

$$P\left(\left| \left. \frac{dQ}{dP} \right|_{\mathcal{F}_{TN^{1+\alpha}}} - 1 \right| \geq \frac{1}{N^{(1-\alpha)/3}} \right) \leq \frac{KN^{1+\alpha}N^{2/10}}{N^{k/10}},$$

$$Q\left(\left| \left. \frac{dQ}{dP} \right|_{\mathcal{F}_{TN^{1+\alpha}}} - 1 \right| \geq \frac{1}{N^{(1-\alpha)/3}} \right) \leq \frac{KN^{1+\alpha}N^{2/10}}{N^{k/10}}.$$

Proof. It is only necessary to establish a bound for large N. We treat explicitly the first inequality. We write $\left.\frac{dQ}{dP}\right|_{\mathcal{F}_{TN^{1+\alpha}}}$ as

$$\exp\left[- \sum_{r=1}^{\infty} \gamma(r)\beta(r)TN^{\alpha} + N(r)\log\left(1 + \frac{\gamma(r)}{N}\right) \right],$$

for $N(r)$ defined as in Lemma 33. Thus it is a question of dealing with

$$\sum_{r=1}^{\infty} \left(N(r)\log(1 + \frac{\gamma(r)}{N}) - \gamma(r)\beta(r)TN^{\alpha} \right) =$$

$$\sum_{r=1}^{[N^{1/10}]} \left(N(r)\log(1 + \frac{\gamma(r)}{N}) - \gamma(r)\beta(r)TN^{\alpha} \right) + \sum_{r=[N^{1/10}]+1}^{\infty} \left(N(r)\log(1 + \frac{\gamma(r)}{N}) - \gamma(r)\beta(r)TN^{\alpha} \right).$$

Now with P(or Q) probability at least $1 - \frac{TN^{1+\alpha}}{N^{(k-2)/10}}K$ (for K not depending on N or T; in the following K may be increased but it will always retain this property), there are no jumps of $r_{\cdot}$ by more than $[N^{1/10}]$ to the right in the interval of time $[0, TN^{1+\alpha}]$, in which case

$$\begin{aligned} \left| \sum_{r=[N^{1/10}]+1}^{\infty} \log(1 + \frac{\gamma(r)}{N})N(r) - \gamma(r)\beta(r)TN^{\alpha} \right| &= \sum_{r=[N^{1/10}]+1}^{\infty} \gamma(r)\beta(r)TN^{\alpha} \\ \leq TN^{\alpha}c_1 \sum_{r=[N^{1/10}]+1}^{\infty} r\beta(r) &\leq \frac{1}{4N^{(1-\alpha)/3}} \text{ for } N \text{ large },\end{aligned}$$

since, for r large, $r\beta(r) < \frac{1}{r^{12}}$. The other sum can be written

$$\sum_{r=1}^{[N^{1/10}]} (\log(1 + \tfrac{\gamma(r)}{N}) - \tfrac{\gamma(r)}{N})N(r)$$

$$+ \sum_{r=1}^{[N^{1/10}]} \tfrac{\gamma(r)}{N}(N(r) - \beta(r)TN^{1+\alpha}).$$

The first part is necessarily negative and, by a Taylor series argument, its absolute value is bounded by

$$\sum_{r=1}^{[N^{1/10}]} \frac{\gamma(r)^2}{2N^2} N(r) \leq \frac{\gamma(1)^2 c_1^2}{2N^2} \sum_{r=1}^{[N^{1/10}]} r^2 N(r) < \frac{\gamma(1)^2 c_1^2}{2N^{9/5}} \sum_{r=1}^{[N^{1/10}]} N(r)$$

since $\gamma \in \Gamma(c_1)$. Under P, $\sum_{r=1}^{[N^{1/10}]} N(r)$ is simply a Poisson random variable of parameter $TN^{1+\alpha} \sum_{r=1}^{[N^{1/10}]} \beta(r) < 2TN^{1+\alpha}$. For large N, $4TN^{1+\alpha} < \frac{N^{18/10}}{3\gamma(1)^2 c_1^2 N^{(1-\alpha)/3}}$ by the fact that $\alpha \in (0, \frac{1}{2})$. So, except on a set of exponentially small probability in N, $\sum_{r=1}^{[N^{1/10}]} N(r) \leq \frac{N^{18/10}}{3\gamma(1)^2 c_1^2 N^{(1-\alpha)/3}}$, which implies that off this small set $|\sum_{r=1}^{[N^{1/10}]} (\log(1 + \frac{\gamma(r)}{N}) - \frac{\gamma(r)}{N}) N(r)| \leq \frac{1}{4N^{(1-\alpha)/3}}$.

It remains to consider

$$U = \frac{1}{N} \sum_{r=1}^{[N^{1/10}]} \gamma(r)(N(r) - \beta(r)TN^{1+\alpha}) = \frac{1}{N} \sum_{1}^{[N^{1/10}]} \gamma(r) X_r$$

where X_r are independent centered Poisson r.v.s of variance $\beta(r)TN^{1+\alpha}$. Calculating as in Lemma 34 we find that for $c = \pm N^{(1-\alpha)/2}$

$$E[e^{cU}] \; = \; \exp(\sum_{r=1}^{[N^{1/10}]} \beta(r)TN^{1+\alpha}(e^{c\frac{\gamma(r)}{N}} - 1 - c\frac{\gamma(r)}{N})) \; \leq \; K.$$

So, by usual Tchebychev exponential bounds we find

$$P\left(\frac{1}{N}|\sum_{1}^{[N^{1/10}]} \gamma(r) X_r| \geq \frac{1}{4N^{(1-\alpha)/3}}\right)$$

$$\leq K \exp(-hN^{(1-\alpha)/6}),$$

where K, h do not depend on N. Thus we have that outside a set of probability $\frac{KN^{1+\alpha}N^{2/10}}{N^{k/10}}$ for large N,

$$\left|\sum_{r=1}^{\infty} \left(N(r)\log(1 + \frac{\gamma(r)}{N}) - \gamma(r)\beta(r)TN^{\alpha}\right)\right| \leq \frac{3}{4N^{(1-\alpha)/3}}$$

which will (for large N) imply that $|\frac{dQ}{dP}\Big|_{\mathcal{F}_{TN^{1+\alpha}}} - 1| \leq \frac{1}{N^{(1-\alpha)/3}}$. We can obtain a similar bound with probability Q if we, for instance, consider $\frac{1}{N}\sum_{r=1}^{[N^{1/10}]} \gamma(r)[N(r) - \beta(r)(1 + \frac{\gamma(r)}{N})TN^{1+\alpha}]$ instead of $\frac{1}{N}\sum_{r=1}^{[N^{1/10}]} \gamma(r)[N(r) - \beta(r)TN^{1+\alpha}]$.

$\diamondsuit$

From this result we can easily transfer results from equilibrium distribution processes to our perturbed processes.

We obtain the following from Lemma 5.

Proposition 36 *Let $\alpha \in (0,1/2)$ and $\xi^N_\cdot$ be a (γ, N) perturbed β-NPS for $\gamma \in \Gamma(c_1)$, initially distributed as $Ren^{(-\infty,0]}(\beta)$ with rightmost particle $r^{\xi,N}_\cdot$. The probability that, for $s \leq N^{1+\alpha} - N^{1/4}$ fixed,*

$$|r^{\xi,N}_{s+N^{1/4}} - r^{\xi,N}_s| \geq N^{1/8} \log^3 N$$

is bounded by $K/N^{k/24}$ for K depending on c_1 but not on N; the probability that

$$\sup_{s \in [0,N^{1+\alpha}]} \{|r^{\xi,N}_s - r^{\xi,N}_0|\} \geq N^{(1+\alpha)/2} \log^3 N$$

is less than $KN^{2/10+1+\alpha}/N^{k/10}$ where again K does not depend on N.

Proof. We prove the result on canonical path space with probabilities Q and P being as in Lemma 35. We simply use the inequality, valid for any event A in $\mathcal{F}_{N^{1+\alpha}}$,

$$Q(A) \;\leq\; 2E^Q[\frac{dP}{dQ}\Big|_{\mathcal{F}_{N^{1+\alpha}}} I_A] + Q(\{\frac{dP}{dQ}\Big|_{\mathcal{F}_{N^{1+\alpha}}} < \frac{1}{2}\}) \leq 2P(A) + Q(\{\frac{dP}{dQ}\Big|_{\mathcal{F}_{N^{1+\alpha}}} < \frac{1}{2}\}).$$

So for instance

$$Q(\{|r^{\xi,N}_{s+N^{1/4}} - r^{\xi,N}_s| \geq N^{1/8} \log^3 N\})$$
$$\leq 2P(\{|r_{s+N^{1/4}} - r_s| \geq N^{1/8} \log^3 N\}) \;+\; Q(\{\frac{dP}{dQ}\Big|_{\mathcal{F}_{N^{1+\alpha}}} < \frac{1}{2}\}),$$

where we have omitted the superscripts on r to emphasize that under probability P, the rightmost particle process is simply that of a rightsided β-NPS. By Lemma 5 the first term is bounded by $2K/N^{k/24}$, while by Lemma 35, the second probability is bounded by $\frac{K' N^{1+\alpha} N^{2/10}}{N^{k/10}}$. The first result follows (after increasing K) as the first bound is dominant. Similarly for the second part except that the second bound is now the dominant one.

$\diamondsuit$

Again employing the inequality $Q(A) \;\leq 2P(A) + Q(\{\frac{dP}{dQ}\Big|_{\mathcal{F}_{N^{1+\alpha}}} < \frac{1}{2}\})$, Proposition 8 becomes

Proposition 37 *For $\xi^N_\cdot$ and $r^{\xi,N}_\cdot$ as in Proposition 36, the probability that for some $t \in [0, N^{1+\alpha}]$, ξ^N_t has a $N^{1/3}$ gap in $[r^{\xi,N}_t - N^3, r^{\xi,N}_t]$ is bounded by $KN^{2/10+1+\alpha}/N^{k/10}$ for constant K not depending on N.*

We can also use the Radon-Nykodym derivative to get effective bounds on moments of $r_t^{\xi,N}$.

Proposition 38 *Let $\xi^N_\cdot$ and $r^{\xi,N}_\cdot$ be as in Proposition 36 and let A be an event of probability $1-o(1)$ as N becomes large. Then*

$$\frac{E\left[(r^{\xi,N}_{N^{1+\alpha}})^2|A\right]}{N^{1+\alpha}} = 1+o(1) \qquad \text{and} \qquad E\left[\left(r^{\xi,N}_{N^{1+\alpha}}\right)^4|A\right]/N^{2(1+\alpha)} \le K,$$

for K not depending on N. Equally for time $N^{1+\alpha}$ replaced with $N^{1+\alpha}-N^{1/4}$.

Proof.
As

$$E\left[\left(r^{\xi,N}_{N^{1+\alpha}}\right)^2|A\right] = \frac{E\left[\left(r^{\xi,N}_{N^{1+\alpha}}\right)^2 I_A\right]}{P(A)},$$

it is only necessary to analyse the number

$$E\left[\left(r^{\xi,N}_{N^{1+\alpha}}\right)^2 I_A\right].$$

This equals (and here again we omit the superscripts on r to emphasize that with respect to the probability P, the underlying process is simply a rightsided β-NPS)

$$E^P\left[(r_{N^{1+\alpha}})^2 \frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} I_A\right] = E^P\left[r^2_{N^{1+\alpha}} I_A\right] + E^P\left[r^2_{N^{1+\alpha}}\left(\frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} - 1\right) I_A\right].$$

By Proposition 3.2 of [8] the first term is equal to $N^{1+\alpha}(1+o(1))$ as N becomes large. The magnitude of the second term is bounded by

$$E^P\left[r^2_{N^{1+\alpha}}\frac{1}{N^{\frac{1-\alpha}{3}}}\right] + E^P\left[r^2_{N^{1+\alpha}}\left|\frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} - 1\right| I_{|\frac{dQ}{dP}|_{\mathcal{F}_{N^{1+\alpha}}}-1|>\frac{1}{N^{\frac{1-\alpha}{3}}}}\right]$$
$$\le 2N^{1+\alpha}\frac{1}{N^{\frac{1-\alpha}{3}}} + E^P\left[r^2_{N^{1+\alpha}}\left|\frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} - 1\right| I_{|\frac{dQ}{dP}|_{\mathcal{F}_{N^{1+\alpha}}}-1|>\frac{1}{N^{\frac{1-\alpha}{3}}}}\right]$$

for N large. The last term above, by Hölder's inequality is less than

$$\left(E^P\left[|r_{N^{1+\alpha}}|^{k/2}\right]\right)^{4/k}\left(E[\left|\frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} - 1\right|^{k/4}]\right)^{4/k}\left(P(\left|\frac{dQ}{dP}\Big|_{\mathcal{F}_{N^{1+\alpha}}} - 1\right| > \frac{1}{N^{\frac{1-\alpha}{3}}})\right)^{1-8/k}$$
$$\le C_k N^{1+\alpha}\left(\frac{N^{1+\alpha}N^{2/10}}{N^{k/10}}\right)^{1-8/k}$$

by Lemmas 34 and 35. Thus first part of the lemma follows. The second part is proven in the same manner. The inequalities with $N^{1+\alpha}$ replaced by $N^{1+\alpha}-N^{1/4}$ are shown in an entirely similar way.

$\diamondsuit$

A (γ, N) perturbation of a β-NPS, $\xi_{\cdot}^{N}$, can be coupled naturally with a (γ, N) modification $\xi_{\cdot}^{F,N^{1/16},N}$ of the Markov chain $\eta_{\cdot}^{F,N}$ as in Proposition 13 so that "extra" jumps occur simultaneously for both processes. In accordance with previous definitions we say that with the above coupling $(\xi_{\cdot}^{N}, \xi_{\cdot}^{F,N^{1/16},N})$ remain successfully coupled on time interval $[0, V]$ if $\forall t \in [0, V]$,

$$\theta_{r_t^{\xi,N}} \circ \xi_t^N ||_{[-N^{1/16}/2,0)} = \xi_t^{F,N^{1/16},N} ||_{[-N^{1/16}/2,0)} \neq \underline{0}$$

and (with the obvious definition), $r_t^{\xi,N} = r_t^{F,N^{1/16},N}$.

As our Radon-Nykodym derivative is really a derivative on families of Poisson processes, the Radon-Nykodym derivative given in Lemma 33 also serves as the derivative of the law of $(\xi_{\cdot}^{N}, \xi_{\cdot}^{F,N^{1/16},N})$ with respect to that of $(\eta_{\cdot}^{R}, \eta_{\cdot}^{F,N^{1/16}})$, where $\eta_{\cdot}^{R}$ is a right sided β-NPS staring from $Ren^{(-\infty,0]}(\beta)$ and $\eta_{\cdot}^{F,N^{1/16}}$ is the finite state comparison Markov chain naturally coupled with $\eta_{\cdot}^{R}$. We therefore obtain, using the argument of Proposition 36,

Proposition 39 *Let $\gamma \in \Gamma(c_1)$ and $\xi_{\cdot}^{N}$ be a (γ, N)-perturbed $\beta-$NPS, initially distributed as $Ren^{(-\infty,0]}(\beta)$, and naturally coupled with (γ, N) modification $\xi_{\cdot}^{F,N^{1/16},N}$ so that initially $\xi_{\cdot}^{N}$ and $\xi_{\cdot}^{F,N^{1/16},N}$ agree on $(-N^{1/16}, 0]$. The probability that the natural coupling between $\xi_{\cdot}^{N}, \xi_{\cdot}^{F,N^{1/16},N}$ does not break down on interval $[0, N^{1+\alpha}]$ is at least $1 - KN^{(1+\alpha)-23/(12.16)} \frac{N^{3/16}}{N^{23k/(16)(288)}}$.*

Arguing as before we obtain the following complement to Proposition 38.

Corollary 40 *For $\gamma, \xi_{\cdot}^{N}, r_{\cdot}^{\xi,N}$ as in Proposition 39, let A be the event that on time intervals $[0, N^{1/4}]$ and $[N^{1+\alpha} - N^{1/4}, N^{1+\alpha}]$, there are no extra jumps (equally put, on this time interval the process evolves as $\beta-$NPS) and that there are at most two "extra" jumps occurring within $N^{1/4}$ of each other. Then*

$$|E[r_s^{\xi,N}|A] - N^{\alpha} \sum_{i=1}^{N^{\frac{3}{2(k+1)}}} c(i)\gamma(i)\beta(i)| \leq C\left(N^{3\alpha/4} + N^{\alpha-1/8} + N^{1/8}\right)$$

for $s = N^{1+\alpha}$ or $N^{1+\alpha} - N^{1/4}$ where $c(i)$ $i = 1, 2, \cdots$ are the constants defined in Corollary 24.

Proof. We treat explicitly the case $s = N^{1+\alpha}$. Let the underlying probability measure be Q. We suppose that N is so large that $Q(A) \geq 1/2$ (see discussion preceding Proposition 30). We consider the comparison (γ, N) modified Markov chain, $\xi_{\cdot}^{F,N^{1/16}}$, which is naturally coupled with process $\xi_{\cdot}^{N}$. Denote by $r_{\cdot}^{\gamma,N^{1/16}}$ its rightmost particle process. Let the event that the coupling breaks down be C. We have, as argued previously,

$$|E^Q[r^{\xi,N}_{N^{1+\alpha}}|A] - E^Q[r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|A]| \leq E^Q[|r^{\xi,N}_{N^{1+\alpha}}|I_C|A] + E^Q[|r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|I_C|A].$$

To bound the second term of the righthand side we use the inequalities

$$\begin{aligned} E^Q[|r^{\xi,N}_{N^{1+\alpha}}|I_C|A] &\leq E^Q[|r^{\xi,N}_{N^{1+\alpha}}|I_C]/Q(A) \leq 2E^Q[|r^{\xi,N}_{N^{1+\alpha}}|I_C]) \\ &= 2E^P[|r^{\xi,N}_{N^{1+\alpha}}|I_C\frac{dQ}{dP}], \end{aligned}$$

for P the measure with respect to which $\xi^N_\cdot, \xi^{F,N^{1/16}}_\cdot$ are non perturbed processes.

By Hölder's inequality this is less than

$$= 2(E^P[|r^{\xi,N}_{N^{1+\alpha}}|^4])^{1/4}(E^P[(\frac{dQ}{dP})^4])^{1/4}(P(C))^{1/2}.$$

By Proposition 3.2 of [8] the first factor is bounded by $CN^{(1+\alpha)/2}$, the second is bounded by $K_{4,1,c_1}$ by Lemma 34, while $P(C)$ is bounded by $KN^{3/16}N^{-23k/(16\ 288)}$ by Proposition 13, thus by our extreme lower bound for k we have $E^Q[|r^{\xi,N}_{N^{1+\alpha}}|I_C|A] \leq 2E^P[|r^{\xi,N}_{N^{1+\alpha}}|I_C\frac{dQ}{dP}] < KN^{1/8}$. Similarly but also making use of Lemma 27 we have $E^Q[|r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|I_C|A] < KN^{1/8}$.

Thus we have (for large values of N)

$$|E^Q[r^{\xi,N}_{N^{1+\alpha}}|A] - E^Q[r^{\gamma,N^{1/16}}_{N^{1+\alpha}}|A]| \leq KN^{1/8}.$$

The result follows from Proposition 30.

$\diamondsuit$

Proposition 26 and Lemma 35 yield the following.

Proposition 41 *Let $T \geq 0$ and let $(\xi^N_t : t \geq 0)$ be a (γ, N) perturbed $\beta-$NPS, starting from initial distribution $Ren^{(-\infty,0]}(\beta)$ for $\gamma \in \Gamma(c_1)$. Let $r^{\xi,N}_\cdot$ be its rightmost particle process. Then for $N^{1/4} \leq V \leq N^{1+\alpha}$, conditioned on the event that there are no extra jumps on time intervals $[0, N^{1/4}]$ and $[V - N^{1/4}, V]$ and that at most two extra jumps occur within $N^{1/4}$ of each other, outside of an event of probability $C\frac{N^{33/16}}{N^{23k/(288)(16)}} + KN^{1+\alpha}N^{2/10}/N^{k/10}$,*

$$\xi^N_{V+N^{1/4}} = \Omega_{V+N^{1/4}} \text{ on } (r^{\xi,N}_V - 3N^2, \infty)$$

where $(\Omega_s : s \geq V + N^{1/4})$ is a (γ, N) perturbed $\beta-$NPS run with the same Harris system as $\xi^N_\cdot$ and $\theta_{r^{\xi,N}_{V+N^{1/4}}} \circ \Omega_{V+N^{1/4}}$ is a configuration with distribution $Ren^{(-\infty,0]}(\beta)$, independent of $\sigma\{\xi^N_s\ s \leq V\}$.

Furthermore, outside of an event of this probability, the rightmost particles of ξ^N_t and Ω_t will agree for $V + N^{1/4} \leq t \leq TN^2$ and for each t in this range Ω_t and ξ^N_t will agree on the interval $[r^{\xi,N}_t - 2N^2, r^{\xi,N}_t]$.

We are now ready to prove Proposition 32.

Proof of Proposition 32
We prove weak convergence on $[0,1]$; the general case is an automatic extension. By Proposition 36, it is enough to show convergence for the process $(X_t^N : t \geq 0)$ where

$$X_t^N \;=\; \sum_{j=1}^{[tN^{1-\alpha}]} \left(r^{\xi,N}_{jN^{1+\alpha}} - r^{\xi,N}_{(j-1)N^{1+\alpha}} \right) = \sum_{j=1}^{[tN^{1-\alpha}]} V_j.$$

We fix $1/2 > \alpha > 1/4$. We write

$$V_j = r^{\xi,N}_{jN^{1+\alpha}} - r^{\xi,N}_{(j-1)N^{1+\alpha}} \;=\; U_j + \omega_j$$

where

$$U_j = r^{\xi,N}_{jN^{1+\alpha}-N^{1/4}} - r^{\xi,N}_{(j-1)N^{1+\alpha}}, \text{ and } \omega_j \;=\; V_j - U_j.$$

As before we split up the Poisson process generating $\xi^N_{\cdot}$ into the "equilibrium" Poisson processes and an independent set of Poisson processes $S^x_{\cdot}$ generating at rate $\frac{\gamma(x)\beta(x)}{N}$ jumps x to the right of $r_t^{\xi,N}$ at time t. That is the $\{S^x_{\cdot}\}$ generate the extra jumps.

We have (see the discussion preceding Proposition 30) with probability bounded by $KN^{1-\alpha}(\frac{1}{N^{3/4}} + \frac{N^{2\alpha}}{N^{3/2}}) = o(1)$ that $\forall\ 0 \leq j \leq N^{1-\alpha}$ the event $A(j)$ occurs, where $A(j)$ is the event
(I) no extra points occur in time intervals $[(j-1)N^{1+\alpha}, (j-1)N^{1+\alpha} + N^{1/4}]$ and $[jN^{1+\alpha} - N^{1/4}, jN^{1+\alpha}]$
(II) at most two extra points are within $N^{1/4}$ of each other in $[(j-1)N^{1+\alpha}, jN^{1+\alpha}]$.

Given this we have, conditioned on $\cap_j A(j)$ by repeatedly applying Proposition 41 at times $V = jN^{1+\alpha} - N^{1/4}$, that outside of an event of probability $\left(\frac{KN^{33/16}N^{1-\alpha}}{N^{23k/(288)(16)}}\right)$ U_j are equal to U_j' where U_j' are i.i.d. random variables equal in distribution to $r^{N,\xi}_{N^{1+\alpha}-N^{1/4}}$ conditioned on the event that at most two extra points are within $N^{1/4}$ of each other in $[0, N^{1+\alpha} - N^{1/4}]$ and no extra jumps in $[0, N^{1/4}]$. By Proposition 36,

$$|\omega_j| \leq N^{1/8} \log^3 N \text{ for each } j$$

with probability tending to one as N tends to infinity. The result now follows from Proposition 31 with W_i^N equal to $U_j - E[U_j']$ and $W_i^{N\prime}$ equal to $U_j' - E[U_j']$, given Proposition 38 and Corollary 40.

$\diamondsuit$

Before concluding this section we state a result which belongs in this section but will be used in the next. We suppose given a filtration $\{\mathcal{F}_t : t \geq 0\}$ containing the natural filtration of the Harris system but with respect to which the underlying Poisson processes remain Poisson processes. We construct the process as follows: We take $\gamma_0 \in \Gamma(c_1)$ and ξ_0^N to be measurable with respect to $\mathcal{F}_0$ and distributed as $Ren^{(-\infty,0]}(\beta)$; on the interval $[N^{1/4}, N^{1+\alpha} - N^{1/4})$, $\xi_\cdot^N$ evolves under the given Harris system as a (γ_0, N) perturbed β-NPS. On the intervals $[0, N^{1/4}]$ and $[N^{1+\alpha} - N^{1/4}, N^{1+\alpha}]$, $\xi_\cdot^N$ evolves as a $\beta-$NPS. For $i = 1, 2, \cdots$, we take $\gamma_i \in \Gamma(c_1)$, measurable with respect to $\mathcal{F}_{iN^{1+\alpha}-N^{1/4}}$ and on the interval, $[iN^{1+\alpha}+N^{1/4}, (i+1)N^{1+\alpha}-N^{1/4})$, $\xi_\cdot^N$ evolves under the given Harris system as a (γ_i, N) perturbed β-NPS. On the intervals $[iN^{1+\alpha}, iN^{1+\alpha} + N^{1/4})$, $([i+1)N^{1+\alpha} - N^{1/4}, (i+1)N^{1+\alpha}]$ it evolves under the Harris system as a β-NPS. We call such a process a piecewise $(\Gamma(c_1), N)$ perturbed process. The following is a consequence of Proposition 41 and Lemma 6,

Proposition 42 *Fix finite positive T and c_1 and $1/2 > \alpha > 1/4$. For the process defined above, $\xi_\cdot^N$, let the rightmost occupied site at time t be denoted by $r_t^{\xi,N}$. There exist processes $(\eta_t^{N,i} \ : t \geq iN^{1+\alpha})$ for $i = 1, 2, \cdots TN^{1-\alpha}$ so that*
(i) $\theta_{r^{\xi,N}_{iN^{1+\alpha}}} \circ \eta^{N,i}_{iN^{1+\alpha}}$ has distribution $Ren^{(-\infty,0]}(\beta)$ independent of $\theta_{r^{\xi,N}_{jN^{1+\alpha}}} \circ \eta^{N,j}_{jN^{1+\alpha}}$ for $j < i$ and of $\mathcal{F}_{iN^{1+\alpha}-N^{1/4}}$,
(ii) on the time interval $[iN^{1+\alpha}+N^{1/4}, (i+1)N^{1+\alpha}-N^{1/4})$, $\eta_\cdot^{N,i}$ evolves under the given Harris system as a (γ_i, N) perturbed $\beta-$NPS; on the intervals $[iN^{1+\alpha}, iN^{1+\alpha} + N^{1/4}]$ $[(i+1)N^{1+\alpha} - N^{1/4}, (i+1)N^{1+\alpha}]$ it evolves as a $\beta-$NPS,
(iii) with probability tending to one as N tends to infinity (with T and c_1 fixed) for all $i \leq TN^{1-\alpha}$ and $t \in [iN^{1+\alpha}, (i+1)N^{1+\alpha}]$, $\eta_t^{N,i}(x) = \xi_t^N(x) \ \forall \ x \geq r_t^{\xi,N} - 2N^2$.

In the next section we will also use (γ, N)-perturbed β-NPS to refer to left sided β-NPSs and we will speak of piecewise $(\Gamma(c_1), N)$ perturbed left sided β-NPSs. The obvious definition will apply.

8 Convergence results

In this section we apply Proposition 42 to establish a weak convergence result for the joint process $((\frac{r^N_{N^2t}}{N}, \frac{\ell^N_{N^2t}}{N}) : t \geq 0)$. Recall from the discussion in the introduction that for $\eta_t^{L,N}(x) = \eta_t^N(x) I_{x \leq r_t^N}$ and $\eta_t^{R,N}(x) = \eta_t^N(x) I_{x \geq \ell_t^N}$, the two processes evolve "almost" as perturbed β-NPSs.

The principal difficulty to be confronted is that the "extra jump" rates for the two processes are dependent on previous evolutions and on each other. We address this problem by introducing processes $(r_\cdot^{N,+}, \ell_\cdot^{N,+})$ and $(r_\cdot^{N,-}, \ell_\cdot^{N,-})$ to which Proposition 42 may be directly applied and so that with large probability $r_t^{N,-} \leq r_t^N \leq r_t^{N,+}$ for $t \leq \tau^{N,\epsilon} = \inf\{s : \ell_s^N - r_s^N \leq \epsilon N\}$. We first give the definitions of these auxiliary processes, then we establish weak convergence results for the processes $((\frac{r^{N,+}_{N^2t}}{N}, \frac{\ell^{N,+}_{N^2t}}{N}) : t \geq 0)$ and $((\frac{r^{N,-}_{N^2t}}{N}, \frac{\ell^{N,-}_{N^2t}}{N}) : t \geq 0)$. We then show our weak convergence result for $((\frac{r^N_{N^2t}}{N}, \frac{\ell^N_{N^2t}}{N}) : t \geq 0)$ by establishing an (with large probability) ordering relation.

In showing weak convergence we will use the following analogue of Proposition 31, which again follows from Brownian embedding arguments which are left to the reader.

Proposition 43 *For each positive integer N, let (V_i^N, W_i^N), $i \geq 1$ and $(V_i^{N,\prime}, W_i^{N,\prime})$, $i \geq 1$ be sequences of pairs of random variables such that with respect to the natural filtration $\{\mathcal{F}_r^N\}_{r\geq 0}$ generated by $(V_i^{N,\prime}, W_i^{N,\prime}), i \geq 1$, for each $r \geq 1$,*
(i) the random variables $V_r^{N,\prime}$ and $W_r^{N,\prime}$ are conditionally independent given $\mathcal{F}_{r-1}$,
(ii) $E[V_r^{N,\prime}|\mathcal{F}_{r-1}] = E[W_r^{N,\prime}|\mathcal{F}_{r-1}] = 0$.
Further suppose that for each positive λ and $r \geq 1$,
(a) $E[(V_r^{N,\prime})^2|\mathcal{F}_{r-1}^N]/N^{4/3}$ and $E[(W_r^{N,\prime})^2|\mathcal{F}_{r-1}^N]/N^{4/3} \to 1$ as N tends to infinity uniformly in $1 \leq r \leq \lambda N^{2/3}$,
(b) for each $r, N \geq 1$, $E[(V_r^{N,\prime})^4|\mathcal{F}_{r-1}^N]/N^{2(4/3)} + E[(W_r^{N,\prime})^2|\mathcal{F}_{r-1}^N]/N^{2(4/3)} \leq K$ where K does not depend on N or r,
(c) $V_i^N = V_i^{N,\prime}, W_i^N = W_i^{N,\prime}$ for all $1 \leq i \leq \lambda N^{2/3}$ with probability tending to one as N tends to infinity. Then

$$(\underline{X}_t^N : t \geq 0) \ = \ (\frac{\sum_{i=1}^{[tN^{2/3}]}(V_i^N, W_i^N)}{N} : t \geq 0)$$

converges in distribution to standard two dimensional Brownian motion.

Recall that $\eta_\cdot^N$ has $\eta_0^N \equiv 0$ on $(0, N)$ and $\eta_0^N|_{(-\infty,0]}$ and $\eta_0^N|_{[N,\infty)}$ are independent and distributed respectively as $Ren^{(-\infty,0]}(\beta)$ and $Ren^{[N,\infty)}(\beta)$. We introduced processes $r_\cdot^N, \ell_\cdot^N$ in the introduction to be the endpoints of what we consider to be the gap.

Let λ be fixed but arbitrarily large. We now introduce semi-infinite comparison processes (to which Proposition 42 may be directly applied)

$$\begin{array}{l}(\eta_t^{L,N,\pm} : t \geq 0) \text{ with rightmost particles } (r_t^{N,\pm} : t \geq 0)\\ (\eta_t^{R,N,\pm} : t \geq 0) \text{ with leftmost particles } (\ell_t^{N,\pm} : t \geq 0)\end{array}$$

so that (as we will see in Proposition 47 and Corollary 48) with probability tending to one as $N \to \infty$ for $t \ < \ \tau^{N,\epsilon,\pm} = \inf\{t > 0 : \ell_t^{N,\pm} - r_t^{N,\pm} < N\epsilon\} \wedge \lambda N^2$,

$$\eta_t^{L,N,-} \leq \eta_t^{L,N} \leq \eta_t^{L,N,+} \text{ and } \eta_t^{R,N,-} \leq \eta_t^{R,N} \leq \eta_t^{R,N,+},$$

where $\eta_t^{L,N}(x) = \eta_t^N(x) I_{x \leq r_t^N}$ and $\eta_t^{R,N}(x) = \eta_t^N(x) I_{x \geq \ell_t^N}$. We note (and apologize for) the fact that, at variance with the conventions of Theorem 1, the suffix R will now denote leftsided processes (which are on the right of rightsided processes denoted with suffix L). Both $\eta_\cdot^{L,N,\pm}$ and $\eta_\cdot^{R,N,\pm}$ will be generated by the same Harris system as $\eta_\cdot^N$. All four processes will depend on ϵ for their time domain of meaningful definition but we suppress reference to this in the notation.

We define recursively $\eta_{\cdot}^{L,N,+},\ \eta_{\cdot}^{R,N,+}$ to be piecewise $(\Gamma(\frac{C}{\epsilon}),N)$ perturbed $\beta-$NPSs for suitable C (see Lemma 44 later for justification) as follows:

For $iN^{4/3}+N^{1/4}\leq t\leq (i+1)N^{4/3}-N^{1/4}$, with $iN^{4/3}-N^{1/4}<\tau^{N,\epsilon,+}$ (otherwise we need not bother with a definition),
$\eta_{\cdot}^{L,N,+}$ evolves under the given Harris system as a right sided (γ_{+}^{i},N) perturbed β-NPS where the function $\gamma_{+}^{i}(l)$ is zero for $l>N^{3/2(k+1)}$ and for $l\leq N^{3/2(k+1)}$ is given by $\gamma_{+}^{i}(l)=\left(\frac{\beta(L_i^+-N^{4/5}-\ell)}{\beta(L_i^+-N^{4/5})}-1\right)N$ where $L_i^+=\ell^{N,+}_{\ell N^{4/3}-N^{1/4}}-r^{N,+}_{iN^{4/3}-N^{1/4}}$. That is with flip rates $f_i^+(\cdot,\cdot)$ on time interval $[iN^{4/3}+N^{1/4},(i+1)N^{4/3}-N^{1/4}]$ given by the following:

$$
\begin{aligned}
& f_i^+(\ell,r) &=& \ \frac{\beta(\ell)\beta(r)}{\beta(r+\ell)} \text{ for } \ell,\ r \text{ finite },\\
\text{for } \ell\leq N^{3/2(k+1)}, \qquad & f_i^+(\ell,\infty) &=& \ \tfrac{\beta(\ell)\beta(L_i^+-N^{4/5}-\ell)}{\beta(L_i^+-N^{4/5})} \text{ for } L_i^+=\ell^{N,+}_{iN^{4/3}-N^{1/4}}-r^{N,+}_{iN^{4/3}-N^{1/4}}\\
& &=& \ \beta(\ell)\left(1+\left(\tfrac{\beta(L_i^+-N^{4/5}-\ell)}{\beta(L_i^+-N^{4/5})}-1\right)\right)=\beta(\ell)\left(1+\tfrac{\gamma_+^i(\ell)}{N}\right);\\
\text{for } \ell> N^{3/2(k+1)}, \qquad & f_i^+(\ell,\infty) &=& \ \beta(\ell).
\end{aligned}
$$

For the remaining times (in intervals of the form $[iN^{4/3},iN^{4/3}+N^{1/4})$ or $[iN^{4/3}-N^{1/4},iN^{4/3}])$, $\eta_{\cdot}^{L,N,+}$ evolves as a rightsided β-NPS. Similarly $\eta_{\cdot}^{R,N,+}$ evolves as a left sided (γ_{+}^{i},N) perturbed $\beta-$NPS with flip rates given by

$$f_i^+(\ell,r) = \frac{\beta(\ell)\beta(r)}{\beta(r+\ell)} \text{ for } \ell,\ r \text{ finite },$$

$$f_i^+(\infty,r)=\beta(r)\left(1+\frac{\gamma_+^i(r)}{N}\right).$$

We similarly define $\eta_{\cdot}^{L,N,-},\eta_{\cdot}^{R,N,-}$ so that they evolve as perturbed $(\Gamma(\frac{C}{\epsilon}),N)$ β-NPSs (see Lemma 44 below) with

$$\gamma_{-}^{i}(l)=\begin{cases}\left(\frac{\beta(L_i^-+N^{4/5}-\ell)}{\beta(L_i^-+N^{4/5})}-1\right)N & \text{if } \ell\leq N^{3/2(k+1)};\\ 0 & \text{otherwise,}\end{cases}$$

and $L_i^-=\ell^{N,-}_{iN^{4/3}-N^{1/4}}-r^{N,-}_{iN^{4/3}-N^{1/4}}$ for i so that $iN^{4/3}-N^{1/4}<\tau^{N,\epsilon,-}$.

In generating the "extra" jumps for these four processes we still use the given Harris system and auxiliary random variables and will not introduce any additional Poisson processes. This is to maintain closeness to the original process.

In order to use previous results we must verify that the $\gamma_{\pm}^{i}$ satisfy appropriate boundedness conditions.

Lemma 44 *Let $\gamma_{\pm}^{i}$ be as defined above. Uniformly over possible $\gamma_{\pm}^{i}$ we have on i such that $iN^{4/3}-N^{1/4}<\tau^{N,\epsilon,\pm}$,*

$$\sup_{\ell \leq N^{3/2(k+1)}} |k - \frac{\gamma_{\pm}^{i}(\ell) L_{i}^{\pm}}{\ell N}| \quad \rightarrow 0 \quad \textit{as } N \rightarrow \infty.$$

In particular there exists $C < \infty$ so that for all N sufficiently large and i such that $\tau^{N,\epsilon,\pm} < iN^{4/3} - N^{1/4}$, the $\gamma_{\pm}^{i}$ defined above belong to $\Gamma(\frac{C}{\epsilon})$.

Thus for the above perturbed systems the results of the previous section and of Lemma 29 and Proposition 30 are applicable.

Proposition 42 (and for part (2) Proposition 36) can be applied to the processes to show

Proposition 45 *For the joint process $(\eta_{\cdot}^{L,N,+}, \eta_{\cdot}^{R,N,+})$ we have that*
(1) there exist processes

$$(\eta_t^{N,j,L} \ : t \geq jN^{4/3}), \ (\eta_t^{N,j,R} \ : t \geq jN^{4/3}) \textit{ for } j = 1, 2, \cdots \lambda N^{2/3}$$

so that
(i)$\theta_{r^{N,+}_{jN^{4/3}}} \circ \eta_{jN^{4/3}}^{N,j,L}$ has distribution $Ren^{(-\infty,0]}(\beta)$ independent of

$$\theta_{\ell^{N,+}_{kN^{4/3}}} \circ \eta_{kN^{4/3}}^{N,k,R} \textit{ for } \quad k \leq j,$$

$$\theta_{r^{N,+}_{kN^{4/3}}} \circ \eta_{kN^{4/3}}^{N,k,R} \textit{ for } \quad k < j,$$

and of

$$\sigma\{\eta_s^{L,N,+}, \eta_s^{R,N,+} : s \leq jN^{4/3} - N^{1/4}\}.$$

$\theta_{\ell^{N,+}_{jN^{4/3}}} \circ \eta_{jN^{4/3}}^{N,j,R}$ has distribution $Ren^{[0,\infty)}(\beta)$ independent of

$$\theta_{\ell^{N,+}_{kN^{4/3}}} \circ \eta_{kN^{4/3}}^{N,k,R} \textit{ for } \quad k < j,$$

$$\theta_{r^{N,+}_{kN^{4/3}}} \circ \eta_{kN^{4/3}}^{N,k,L} \textit{ for } \quad k \leq j,$$

and of

$$\sigma\{\eta_s^{L,N,+}, \eta_s^{R,N,+} : s \leq jN^{4/3} - N^{1/4}\}.$$

(ii) on the time interval $[jN^{4/3} + N^{1/4}, (j+1)N^{4/3} - N^{1/4})$, $\eta_{\cdot}^{N,j,L}$, $\eta_{\cdot}^{N,j,R}$ evolve under the given Harris system as a (γ_j^+, N) perturbed $\beta-$NPS conditioned not to have two "extra" jumps within $N^{1/4}$ of each other; on the intervals $[jN^{4/3}, jN^{4/3} + N^{1/4})$, $[(j+1)N^{4/3} - N^{1/4}, (j+1)N^{4/3}]$ they evolve as a $\beta-$NPS,
(iii) with probability tending to one as N tends to infinity (with λ fixed) for all $j : jN^{4/3} - N^{1/4} < \lambda N^2 \wedge \tau^{N,\epsilon,+}$ and $t \in [jN^{4/3}, (j+1)N^{4/3}]$,

$$\eta_t^{N,j,L}(x) = \eta_t^{L,N,+}(x) \ \forall \ x \ \geq r_t^{N,+} - 2N^2$$

and

$$\eta_t^{N,j,R}(x) = \eta_t^{R,N,+}(x) \;\forall\; x \;\leq \ell_t^{N,+} + 2N^2.$$

(2) with probability tending to one as N *tends to infinity for all* j *so that* $jN^{4/3} - N^{1/4} < \tau^{N,\epsilon,+} \wedge \lambda N^2$

$$\sup_{jN^{4/3}-N^{1/4}<t<(j+1)N^{4/3}} |r_t^{N,+} - r_{jN^{4/3}-N^{1/4}}^{N,+}| < N^{4/5}/5.$$

Analogous results hold for $(\eta_\cdot^{L,N,-}, \eta_\cdot^{R,N,-})$.

Remark: We suppose fixed such a collection of processes $\{\eta_\cdot^{N,j,R}, \eta_\cdot^{N,j,L}\}$. Given part (1), part (2) is a simple consequence of Proposition 36.

Proposition 46 *The pairs of random processes*

$$(X_t^N, Y_t^N)_{t\in[0,\lambda]} = \left(\frac{r^{N,+}_{N^2t\wedge\tau^{N,\epsilon,+}}}{N}, \frac{\ell^{N,+}_{N^2t\wedge\tau^{N,\epsilon,+}}}{N}\right)_{t\in[0,\lambda]}$$

and

$$(U_t^N, V_t^N)_{t\in[0,\lambda]} = \left(\frac{r^{N,-}_{N^2t\wedge\tau^{N,\epsilon,-}}}{N}, \frac{\ell^{N,-}_{N^2t\wedge\tau^{N,\epsilon,-}}}{N}\right)_{t\in[0,\lambda]}$$

both converge in distribution to $(X^1_{t\wedge\tau^\epsilon}, X^2_{t\wedge\tau^\epsilon})$ *where* $X_0^1 = 0$, $X_0^2 = 1$, *and for* $t < \tau^\epsilon = \inf\{t : X_t^2 - X_t^1 = \epsilon\} \wedge \lambda$,
$dX_t^1 = dW_t^1 + \frac{cdt}{X_t^2 - X_t^1}$, $\qquad dX_t^2 = dW_t^2 - \frac{cdt}{X_t^2 - X_t^1}$,
for independent Brownian motions $W_\cdot^i$ *and where* $c = \sum_{\ell=1}^\infty \beta(\ell)k\ell c(\ell)$.

Proof. First by Proposition 45,(2) and Proposition 36 the processes

$$(X_t^N, Y_t^N)_{t\in[0,\lambda]}$$

and

$$\left(X_t^{N,'}, Y_t^{N,'}\right)_{t\in[0,\lambda]} = \left(\frac{r^{N,+}_{N^{4/3}[N^{2/3}t]\wedge T^+}}{N}, \frac{\ell^{N,+}_{N^{4/3}[N^{2/3}t]\wedge T^+}}{N}\right)_{t\in[0,\lambda]}$$

satisfy

$$\sup_{s\in[0,\lambda]} |X_t^N - X_t^{N,'}| + |Y_t^{N,'} - Y_t^N| \xrightarrow{pr} 0.$$

Here $T^+ = \inf\{jN^{4/3} : j \geq 0, jN^{4/3} \geq \tau^{N,\epsilon,+}\}$.

Secondly again by Proposition 45, (2), we have for

$$X_t^{N,''} = \frac{1}{N} \sum_{j=1}^{[N^{2/3}t]\wedge(T^+/N^{4/3})} \left(r^{N,+}_{(j+1)N^{4/3}-N^{1/4}} - r^{N,+}_{jN^{4/3}}\right)$$

$$Y_t^{N,''} = \frac{1}{N} \sum_{j=1}^{[N^{2/3}t]\wedge(T^+/N^{4/3})} \left(\ell^{N,+}_{(j+1)N^{4/3}-N^{1/4}} - \ell^{N,+}_{jN^{4/3}}\right)$$

that

$$\sup_{s\in[0,\lambda]} |X_t^{N,'} - X_t^{N,''}| + |Y_t^{N,'} - Y_t^{N,''}| \xrightarrow{pr} 0.$$

Furthermore, by Proposition 45,(1), we have with probability tending to one as $N \to \infty$, for all $t \in [0,\lambda]$

$$X_t^{N,''} = \frac{1}{N} \sum_{j=1}^{[N^{2/3}t]\wedge(T^+/N^{4/3})} \left(r^{N,j,L}_{(j+1)N^{4/3}-N^{1/4}} - r^{N,j,L}_{jN^{4/3}} \right)$$

$$Y_t^{N,''} = \frac{1}{N} \sum_{j=1}^{[N^{2/3}t]\wedge(T^+/N^{4/3})} \left(\ell^{N,j,R}_{(j+1)N^{4/3}-N^{1/4}} - \ell^{N,j,R}_{jN^{4/3}} \right),$$

where $r^{N,j,L}_{\cdot}$ (respectively $\ell^{N,j,R}_{\cdot}$) is the rightmost occupied site (respectively leftmost occupied site) of $\eta^{N,j,L}_{\cdot}$ (respectively $\eta^{N,j,R}_{\cdot}$) the processes fixed after Proposition 45.

Define, for $T^+ > (i-1)N^{4/3}$,

$$V_i^{N,\prime} = r^{N,i-1,L}_{iN^{4/3}-N^{1/4}} - r^{N,i-1,L}_{(i-1)N^{4/3}} - E[r^{N,i-1,L}_{iN^{4/3}-N^{1/4}} - r^{N,i-1,L}_{(i-1)N^{4/3}} | \mathcal{G}_{i-1}],$$
$$W_i^{N,\prime} = \ell^{N,i-1,L}_{iN^{4/3}-N^{1/4}} - \ell^{N,i-1,L}_{(i-1)N^{4/3}} - E[\ell^{N,i-1,L}_{iN^{4/3}-N^{1/4}} - \ell^{N,i-1,L}_{(i-1)N^{4/3}} | \mathcal{G}_{i-1}],$$

and $\mathcal{G}_i = \sigma\{W_j^{N,\prime}, V_j^{N,\prime} : j \le i\}$. For $T^+ \le (i-1)N^{4/3}$ we can arbitrarily define $V_i^{N,\prime}, W_i^{N,\prime}$ so that they are in conformity with the hypotheses of Proposition 43.

It is readily seen that $\{(X_{\cdot}^{N,''}, Y_{\cdot}^{N,''})\}_{N=1}^{\infty}$ is tight (and any limit is continuous) and that, by Corollary 40 and Lemma 44

$$\frac{1}{N} \sum_{j=1}^{[N^{2/3}t]\wedge(T^+/N^{4/3})} E[\ell^{N,i-1,L}_{iN^{4/3}-N^{1/4}} - \ell^{N,i-1,L}_{(i-1)N^{4/3}} | \mathcal{G}_{i-1}] - \int_0^{t\wedge(\tau^{N,\epsilon,+}/N^2)} \frac{cds}{X_s^{N,''} - Y_s^{N,''}}$$

converges in probability to zero for $c = \sum_{\ell=1}^{\infty} \beta(\ell) k\ell c(\ell)$. It therefore follows easily from Proposition 45, given Proposition 38 and Corollary 40 (where the bounds hold uniformly for $\gamma \in \Gamma(C/\epsilon)$) and Proposition 43 that $(X_{\cdot}^{N,''}, Y_{\cdot}^{N,''})$ converge in distribution to $(X^1_{t\wedge\tau^\epsilon}, X^2_{t\wedge\tau^\epsilon})$ for $X_0^1 = 0, \quad X_0^2 = 1$
$dX_t^1 = dW_t^1 + \frac{cdt}{X_t^2 - X_t^1}, \qquad dX_t^2 = dW_t^2 - \frac{cdt}{X_t^2 - X_t^1}$, for $c = \sum_{\ell=1}^{\infty} \beta(\ell) k\ell c(\ell)$
(details are standard and left to the reader). Thus the result holds for $(X_{\cdot}^N, Y_{\cdot}^N)$.

We have, using identical arguments, that the same distributional convergence result holds for

$$(U_t^N, V_t^N)_{t\in[0,\lambda]} = \left(\frac{r^{N,-}_{N^2 t\wedge\tau^{N,\epsilon,-}}}{N}, \frac{\ell^{N,-}_{N^2 t\wedge\tau^{N,\epsilon,-}}}{N} \right)_{t\in[0,\lambda]}.$$

$\diamondsuit$

Proposition 47 *Let $X^N_\cdot, Y^N_\cdot, U^N_\cdot$ and $V^N_\cdot$ be as in Proposition 46. As N tends to infinity $\sup_{0\leq t\leq\lambda}\{|X^N_t - U^N_t| + |Y^N_t - V^N_t|\}$ tends to zero in probability. Furthermore with probability tending to one as N tends to infinity,*

$$\forall\ 0 \leq t \leq \tau^{N,\epsilon,+} \qquad \eta^{L,N,+}_t \geq \eta^{L,N,-}_t, \qquad \eta^{R,N,+}_t \geq \eta^{R,N,-}_t$$

and

$$\forall\ 0 \leq t \leq \tau^{N,\epsilon,-} \qquad \eta^{L,N,+}_t \geq \eta^{L,N,-}_t \text{ and } \eta^{R,N,+}_t \geq \eta^{R,N,-}_t.$$

(In the last inequalities we replace ϵ by $\epsilon/2$ for the definition of processes $\eta^{R,N,+}_\cdot$ and $\eta^{L,N,+}_\cdot$ so that they are meaningfully defined up to time $\tau^{N,\epsilon,-}$ with high probability.)

Proof. We assume that the following three events all hold:
(1) $\forall j : jN^{4/3} - N^{1/4} < \tau^{N,\epsilon,+}$,

$$\sup_{t\in[jN^{4/3}-N^{1/4},(j+1)N^{4/3}]} |r^{N,+}_t - r^{N,+}_{jN^{4/3}-N^{1/4}}| < N^{4/5}/5;$$

(2) there does not exist $t < \tau^{N,\epsilon,+}$ such that $\eta^{L,N,+}_t$ has a $N^{1/3}$ gap in $[r^{N,+}_t - 2N^2, r^{N,+}_t]$,
(3) $\forall t < \tau^{N,\epsilon,\pm}, |r^{N,\pm}_t| \leq N^{3/2}$.

We also assume the analogous events hold for $\eta^{R,N,\pm}_\cdot$. That these events occur with probability tending to one as N ends to infinity is a consequence of Proposition 45, Proposition 36 and Proposition 37.

The problem to be confronted is that the perturbed β-NPSs are not attractive. Nonetheless, as we will see, it is "close" to being attractive.

Suppose that $jN^{4/3} - N^{1/4} < \tau^{N,\epsilon,+}$ and

$$\eta^{L,N,+}_s \ \geq\ \eta^{L,N,-}_s,\ \eta^{R,N,+}_s \ \geq\ \eta^{R,N,-}_s,\ \forall s \leq jN^{4/3}$$

then we have that $L^+_j \ \geq\ L^-_j$ and so during the time interval $[jN^{4/3}, (j+1)N^{4/3}]$ the flip rate ℓ to the right of the rightmost particle of $\eta^{L,N,+}_\cdot$,

$$f^+_j(\ell,\infty) \ \geq\ f^-_j(\ell,\infty) \ =\ \beta(\ell)(1 + \frac{\gamma^i_-(l)}{N}),$$

the flip rate of $\eta^{L,N,-}_\cdot$ at the site ℓ to the right of its rightmost particle. Both functions $f^\pm_j(\ell,\infty)$ are decreasing in ℓ as quotient $\frac{\beta(j)}{\beta(j+1)}$ is decreasing.

The relation

$$\eta^{L,N,+}_t \ \geq\ \eta^{L,N,-}_t$$

cannot be violated by a death at a site; furthermore by the attractiveness of the original (unperturbed) β-NPS it is immediate that this inequality cannot be violated by a birth at a site x to the left of $r^{N,-}_\cdot$ by the attractiveness of $\beta-$NPSs. Thus we are left with dealing with births to the right of $r^{N,-}_\cdot$. Since, as noted above, for

$\ell < \ell'$, $f_j^-(\ell',\infty) \;\leq\; f_j^+(\ell,\infty)$ we need only consider births at site $x \in \; (r_t^{N,-}, r_t^{N,+})$ where also $x \leq r_t^{N,-} + N^{3/2(k+1)}$. Now the flip rate for $\eta_.^{L,N,-}$ at such a site is equal to

$$\frac{\beta(x - r_t^{N,-})\beta(r_t^{N,-} + L_j^- - x)}{\beta(L_j^-)}$$

which will automatically be less than the flip rate at site x (if it is vacant) for process $\eta_.^{L,N,+}$ unless there is a gap of size $\epsilon N - N^{4/5}$ for this process in interval $[r_t^{N,-}, r_t^{N,+}]$.

It is readily seen that provided event (3) occurs, this event is contained in the event

$$\{\exists \text{ a } N\epsilon/2 \text{ gap for } \eta_.^{L,N,+} \text{ in } [r_t^{N,+} - N^{3/2}, r_t^{N,+}] \text{ for some } t \in [jN^{4/3}, (j+1)N^{4/3}]\}.$$

That is event (2) must fail (for N sufficiently large).

From this we see that if events (1)-(3) hold we must have the desired domination for all $t \in [jN^{4/3}, (j+1)N^{4/3}]$ for j such that $jN^{4/3} - N^{1/4} < \tau^{N,\epsilon,+}$. Similarly for $\eta_.^{R,N,+}$ and $\eta_.^{R,N,-}$. That is

$$U_t^N \leq X_t^N \qquad Y_t^N \leq V_t^N \qquad \forall t \leq \; \tau^{N,\epsilon,+}.$$

Since $(X_.^N, Y_.^N)$ and $(U_.^N, V_.^N)$ have the same limiting distribution, this is easily seen to imply

$$\sup_{0\leq t\leq\lambda} |U_t^N - X_t^N| + |V_t^N - Y_t^N| \xrightarrow{pr} 0 \qquad\qquad (++).$$

The relations for $t \leq \tau^{N,\epsilon,-}$ are handled similarly.

$\diamondsuit$

We are now in a position to deal with

$$((R_t, L_t) : t \geq 0) = \left((\frac{r_{N^2t}^N}{N}, \frac{\ell_{N^2t}^N}{N}) : t \geq 0 \right).$$

Corollary 48 *With probability tending to one as N tends to infinity*

$$\forall\; 0 \leq t \leq \tau^{N,\epsilon,+} \qquad\qquad \eta_t^{L,N,+} \geq \eta_t^{L,N} \geq \eta_t^{L,N,-},$$

and

$$\eta_t^{R,N,+} \geq \eta_t^{R,N} \geq \eta_t^{R,N,-}.$$

Proof. Again suppose that conditions (1)-(3) of the previous proposition hold. There are three slight differences to the previous case. Firstly the "extra" jumps of, for instance, $\eta_.^{L,N,+}$ are restricted to sites at most $N^{3/2(k+1)}$ to the right of the rightmost particle, while no such restriction applies to process $\eta_.^{L,N}$. However it will be clear that the probability of such jumps occurring will be small. Secondly the flip rates for process $\eta_\cdot^{L,N}$ on the time interval $[jN^{4/3},(j+1)N^{4/3}]$ will change in a way depending on $\eta_\cdot^{R,N}$ and so the processes must be considered together. Thirdly on time intervals of the form $[iN^{4/3}, iN^{4/3}+N^{1/4}]$ or $[(i+1)N^{4/3}-N^{1/4},(i+1)N^{4/3}]$ the processes $\eta_\cdot^{R,N}$ and $\eta_\cdot^{L,N}$ have extra jump rates while $\eta_\cdot^{R,N,\pm}$ and $\eta_\cdot^{L,N,\pm}$ do not. However, as with the discussion preceding Proposition 30, we have that the probability that this extra rate on such intervals produces extra jumps before time λN^2 is small and so we do not consider this possibility further.

We consider the event that the domination relations $\eta_t^{L,N,+} \geq \eta_t^{L,N}$, $\eta_t^{R,N,+} \geq \eta_t^{R,N}$ are first violated at time $t \in [jN^{4/3},(j+1)N^{4/3}]$ and that the inequality breaks down for the first pair of processes . We wish to show that if properties (1)-(3) hold, then an event of small probability is entailed. As argued previously this violation can only occur with a birth to the right of r_t^N for process $\eta_\cdot^{L,N}$ We can and do take t to belong to $[jN^{4/3}+N^{1/4},(j+1)N^{4/3}-N^{1/4}]$. There are now two cases to consider

a) $x \in (r_t^N, r_t^{N,+} \wedge r_t^N + N^{3/2(k+1)})$: this can be treated with the same argument that showed the relations $\eta_\cdot^{L,N,-} \leq \eta_\cdot^{L,N,+}$, provided (2) and (3) hold.

b) $x > r_t^N + N^{3/2(k+1)}$: in this case there may be no extra flip rate at site x for process $\eta_\cdot^{L,N,+}$, however the probability that $\eta_\cdot^{L,N}$ admits an extra jump of this size on $[jN^{4/3}, \tau^{N,\epsilon,+} \wedge (j+1)N^{4/3}]$ is readily seen to be dominated by $K\lambda N^{2/3}/(\epsilon N)(N^{3(k-3)/2(k+1)})$. Thus the chance that such a jump occurs for any relevant j while $\eta_\cdot^{L,N,+}$ dominates $\eta_\cdot^{L,N}$ is bounded by $KN/N^{3(k-3)/2(k+1)}$ which tends to zero as N tends to infinity and therefore may be neglected.

Similarly we have that the relation

$$\eta_t^{L,N,-} \leq \eta_t^{L,N} \text{ or } \eta_t^{R,N,-} \leq \eta_t^{R,N}$$

can only be violated by a birth at a site $x \in (r_t^{N,-}, r_t^N)$ or in $(l_t^N, l_t^{N,-})$. However given the (a-priori) lack of regularity of $\eta_\cdot^{L,N}$ the argument given for $(\eta_\cdot^{L,N,-},\ \eta_\cdot^{L,N,+})$ does not work with the pair $(\eta_\cdot^{L,N,-},\ \eta_\cdot^{L,N,})$. However we know by Proposition 46 and previous arguments of this proof) that, with probability tending to one as N tends to infinity before this inequality is violated

$$(A) \quad r_{t-}^{N,-} \leq r_{t-}^N \leq r_{t-}^{N,+}$$

and

$$(B) \quad r_{t-}^{N,+} - r_{t-}^{N,-} \leq \epsilon N/2.$$

Together (A) and (B) ensure that the flip rate for($\eta_\cdot^{L,N}$ vacant sites) x in $(r_t^{N,-}, r_t^N)$ for process $\eta_\cdot^{L,N}$ exceeds that for process $\eta_\cdot^{L,N,-}$.

Thus we are done.

$\diamondsuit$

Remark: To see that the desired domination relations hold asymptotically with probability one up to time $\tau^{N,\epsilon,-}$ we simply observe that by (++) with probability tending to one as N tends to infinity $\tau^{N,\epsilon/2,+} > \tau^{N,\epsilon,-}$. We have thus established a convergence in distribution result:

Corollary 49 *For each $\epsilon > 0$,*

$$\left((\frac{r^N_{N^2 t\wedge\tau^{N,\epsilon}}}{N}, \frac{\ell^N_{N^2 t\wedge\tau^{N,\epsilon}}}{N}) : t \geq 0 \right) \xrightarrow{D} \left((X^1_{t\wedge\tau^\epsilon}, X^2_{t\wedge\tau^\epsilon}) : t \geq 0 \right)$$

for $(X^1_\cdot,\ X^2_\cdot)$ as in Proposition 46. Also as N tends to infinity

$$\left\| \frac{\tau^{N,\epsilon}}{N^2} - \frac{\tau^{N,\epsilon,+}}{N^2} \right\| \ \rightarrow\ 0$$

and

$$\left\| \frac{\tau^{N,\epsilon}}{N^2} - \frac{\tau^{N,\epsilon,-}}{N^2} \right\| \ \rightarrow\ 0$$

in probability.

9 Proof of Theorems

Given Corollary 48, Theorems 2-4 announced in the introduction are very intuitive. The essential technical issues to be broached are

a) To show that for each $\delta > 0,\ \exists\epsilon > 0, N_0 < \infty$ so that $\forall N \geq N_0, P(|\frac{\tau^N}{N} - \frac{\tau^{N,\epsilon}}{N}| > \delta) \leq \delta$ (and similarly for σ^N and $\tilde{\tau}^N$),

b) To show for $t \leq \tau^N$ the distribution of η^N_t "reasonably" to the left of r^N_t is close to $Ren(\beta)$ (and similarly to the right of ℓ^N_t).

Corollary 49 is almost enough to prove Theorem 3, to supplement it we wish to record some "regeneration" results: We aim to show that for large t and "most" configurations, $P_t f(\eta)$ is close to $< Ren(\beta), f >$ for a fixed cylinder function f (here "most" means with respect to $Ren(\beta)$ or $Ren^{(-\infty,M]}(\beta)$ where M is large compared to $\sqrt{t}$). This phenomenon is related to the regeneration properties of equilibrium β-NPSs. Since the arguments are minor reformulations of work already done in [8] we will simply sketch the approach to the proofs.

Definition: A right sided configuration η_0 with rightmost particle at position r is said to be N^α *-rich* if for η'_0 chosen conditionally independently of η_0, given r_0 with distribution $Ren^{(-\infty,r_0-N^\alpha]}(\beta)$ and $\eta_\cdot, \eta'_\cdot$ generated with the same Harris system, the probability that r'_t, the rightmost particle of η'_t is not less than r_t for some t in $[0, N^2)$ or that $\eta'_\cdot$is not dominated by $\eta_\cdot$ on $[r_0 - 2N^2, \infty)$ over time interval $[N^\alpha, N^2]$ is at most $N^{-k/96}$. We similarly define N^α *-rich* for left sided processes.

The idea is that a rich configuration has good regeneration properties and that over interval $[N^\alpha, N^2]$, a process beginning from a rich configuration evolves similarly to a process beginning from a one sided equilibrium distribution.

Proposition 50 *For $(\eta_t^R : \ t \geq 0)$ a right sided β-NPS in equilibrium, there exists a finite K not depending on N or on λ so that*

$$P(\eta_s^R \textit{ is not } N^{1/4} - \textit{rich} \ \forall \ 0 \ \leq s \leq \lambda N^2) \ \leq \ \lambda K N^4 N^{-k/96}.$$

Sketch: For definiteness we consider right sided processes. Following Lemma 5 we can show that outside of an event of probability $N^{-k/16}$ for every $s \in [0, \lambda N^2]$ the conditional probability that in the next $N^{1/4}$ units of time r_t changes by $N^{1/4}/2$ is bounded by $N^2 N^{-k/16}$.

Similarly by Proposition 7 the probability that for some $s \in [0, \lambda N^2]$, η_s^R is bad for some $2N^{1/16}$ interval within $[r_s - 2N^2, r_s]$ or that the conditional probability that in the next $N^{1/4}$ time units there is a $N^{1/48}$ gap for η_t in $[r_s - 2N^2, r_s]$ is bounded by $\lambda K N^4 N^{-k/96}$. Given this we can use the regeneration argument of [8] or that of Proposition 13 to obtain the claimed result.

Given Lemma 33 and Proposition 45, we can easily show the following.

Corollary 51 *With probability tending to one as N tend to infinity, for all $t \leq \tau^{N,\epsilon,\pm} \wedge \lambda N^2$, the configurations $\eta_t^{L,N,\pm}, \eta_t^{R,N,\pm}$, as previously defined, are $N^{1/4}$-rich.*

The following is a consequence of the definition of N^α-rich, Proposition 50 and Schinazi's invariance principle.

Corollary 52 *Fix $1 > \epsilon > 0$ and cylinder function f. There exists constant C not depending on ϵ or N so that if η_0 is a right sided configuration with rightmost occupied site in $(\epsilon N, N)$ that is $N^{1/4}$-rich, then uniformly on $s \in [N^{1/2}, N^2 \epsilon]$ and such η_0,*

$$\left| E^{\eta_0}[f(\eta_s)] - < Ren(\beta), f > \right| \leq C\epsilon ||f||_\infty,$$

for N sufficiently large.

Using the regeneration results of [8] or of [9] we easily show the following.

Proposition 53 *Fix strictly positive t and ϵ with $\epsilon < t$. For $(\eta_s : s \geq 0)$ a β-NPS in $Ren(\beta)$ equilibrium (or stochastically greater than this) with natural filtration $\{\mathcal{F}_s\}_{s \geq 0}$ and f a fixed cylinder function, the event*

$$\{\exists s \leq N^2(t - \epsilon) : \left| E[f(\eta_{N^2 t}) | \mathcal{F}_s] - < Ren(\beta), f > \right| \geq \epsilon\}$$

has probability less than ϵ for N sufficiently large.

The following is the key result in addressing problem a) stated at the start of this section and so we supply the proof. We know that if an equilibrium right sided β-NPS has rightmost occupied site sufficiently to the left of the leftmost occupied site of an equilibrium left sided β-NPS, then the two will evolve almost independently even though they may be generated by the same Harris system. We wish to show that if initially the separation is small with respect to N, then the rightmost particle of the right sided β-NPS will become definitely bigger than the leftmost particle of the leftsided process in time that is small with respect to N^2.

Proposition 54 *Let $\epsilon > 0$ be fixed. Let $\eta^R_\cdot$ be a right sided β-NPS, initially in equilibrium with rightmost particle at the origin and let $\eta^L_\cdot$ be a left sided β-NPS also initially in equilibrium with leftmost particle at x for $0 \leq x \leq N\epsilon$. Suppose processes $(\eta^R_\cdot, \eta^L_\cdot)$ are generated by the same Harris system and $T^\epsilon = \inf\{s > 0 : r_s > l_s + N\epsilon/8\}$ where $r_\cdot$ is the rightmost particle of $\eta^R_\cdot$ and $l_\cdot$ is the leftmost particle of $\eta^L_\cdot$, then there exists a strictly positive c and a finite C (not depending on N, x or ϵ) so that for all N sufficiently large, uniformly in appropriate x,*

$$P(T^\epsilon > N^2\epsilon) \ \leq \ C\epsilon^c.$$

Proof. Let $\{\mathcal{F}_s\}_{s\geq 0}$ be the natural filtration of the Harris system but with $\mathcal{F}_s$ augmented so that η^L_0 and η^R_0 are measurable with respect to $\mathcal{F}_0$. Let

$$V = \inf\{s \geq 0 : h(\eta^L_s) > N^{1/6} \text{ or } h(\eta^R_s) > N^{1/6} \text{ or } \eta^L_s \text{ or } \eta^R_s \text{ are not } N^{1/4}-\text{rich }\}.$$

Then we have by Proposition 50 and elementary calculations that $P(V \leq N^2\epsilon) \leq KN^4N^{-k/96}$.

We define successively the stopping times τ_i for $\frac{log_2 \frac{1}{\epsilon}}{3} \geq i \ \geq \ 0$, by $\tau_0 = 0$, and

$$\tau_i \ = \ \inf\{t > \tau_{i-1} : r_t - l_t > N\epsilon/8 \text{ or } l_t - r_t \geq (2(l_{\tau_{i-1}} - r_{\tau_{i-1}}) \vee 2\epsilon N) \wedge (\epsilon^2N^2 2^{2i} + \tau_{i-1})\}.$$

Observe here that our definition of stopping times τ_i entails that

$$\tau_{\frac{log_2 \frac{1}{\epsilon}}{3}-1} \leq \sum_{i=1}^{\frac{log_2 \frac{1}{\epsilon}}{3}-1} \epsilon^2N^2 2^{2i} < N^2\epsilon - \epsilon^2N^2 2^{2\frac{log_2 \frac{1}{\epsilon}}{3}}.$$

Observe also that for

$$\tau_i < V, \ l_{\tau_i} - r_{\tau_i} \leq 2^i\epsilon N(1 + \frac{N^{1/6}}{\epsilon N})^i \leq 2 \ 2^i\epsilon N$$

for all $i \leq \frac{log_2 \frac{1}{\epsilon}}{3}$ if N is large. Thus for $j \leq \frac{log_2 \frac{1}{\epsilon}}{3}$, on $\{\tau_{j-1} < V\}$ we have (by the definition of rich and Schinazi's invariance principle) that, with $\delta N = (l_{\tau_{i-1}} - r_{\tau_{i-1}} \vee \epsilon N)/2$,

$$P(r_{\delta^2N^2+\tau_{i-1}} > 2N\delta + r_{\tau_{i-1}}, \inf_{s \in [\tau_{i-1}, \ \delta^2N^2+\tau_{i-1}]} r_s > -N\delta/10 + r_{\tau_{i-1}} | \mathcal{F}_{\tau_{i-1}}) \geq g > 0$$

and for the same g,

$$P(\ell_{\delta^2N^2+\tau_{i-1}} < -2N\delta + \ell_{\tau_{i-1}}, \sup_{s \in [\tau_{i-1}, \ \delta^2N^2+\tau_{i-1}]} \ell_s < N\delta/10 + \ell_{\tau_{i-1}} | \mathcal{F}_{\tau_{i-1}}) \geq g > 0.$$

Here for a stopping time $\mathcal{F}_T$ is the usual sigma field of information "up to time " T. Thus by the FKG inequality which applies to the above increasing (in the natural sense with respect to subsequent points of the Harris system) events

$$P(r_{\tau_i} - l_{\tau_i} \geq N\epsilon/8 | \mathcal{F}_{\tau_{i-1}}) > g^2$$

provided that N is large enough. Thus we deduce $P(T^\epsilon > N^2\epsilon) < P(V < N^2\epsilon) + g^{2\frac{1}{3}log_2(\frac{1}{\epsilon})}$.

The result follows.

$\diamondsuit$

Proof of Theorem 3

Recall that
$\tau^N = \inf\{t > 0 : \text{ there is no } N^d \text{ gap for } \eta_t^N \text{ in } [-N^2, N^2]\}$

$$\tau^{N,\varepsilon} = \inf\{t > 0 : \ell_t^N - r_t^N \leq N\epsilon\},$$

$$\tau^{N,\epsilon,-} = \inf\{t > 0 : \ell_t^{N,-} - r_t^{N,-} \leq N\epsilon\}.$$

By Proposition 8 applied to $\tilde{\eta}_\cdot^N$ (which is stochastically above the renewal equilibrium) with $S = N^2$ and $T = \lambda N^2$ (and Corollary 49) we have that $\tilde{\tau}^N \wedge \lambda N^2$ is at least both $\tau^N \wedge \lambda N^2$ and $\tau^{N,\varepsilon} \wedge \lambda N^2$ with probability tending to one as N tends to infinity. Also with probability tending to one as N tends to infinity, by Proposition 47,

$$\tau^{N,\epsilon,-} \ \leq \ \tau^{N,\varepsilon} \wedge \lambda N^2 \ \leq \ \tau^N \wedge \lambda N^2.$$

So (by the arbitrariness of λ and the fact that as $\lambda \to \infty, P(\tau^{N,\epsilon,-} < \lambda N^2)$ tends to one uniformly in N) to show the theorem it will suffice to show that for $\delta > 0$ fixed, there exists ϵ so that for all N sufficiently large, the probability that

$$\tilde{\tau}^N \ \geq \ \tau^{N,\epsilon,-} + \delta N^2$$

is less than δ.

Let A_1 be the event that either $\eta^{L,N,-}_{\tau^{N,\epsilon,-}}$ or $\eta^{R,N,-}_{\tau^{N,\epsilon,-}}$ are not $N^{1/4}$-rich. By Corollary 51, $P(A_1) \to 0$ as N tends to infinity.

Let $(\xi_t^L : \ t \geq \tau^{N,\epsilon,-})$, $(\xi_t^R : \ t \geq \tau^{N,\epsilon,-})$ be semi-infinite $\beta-$NPSs run with the given Harris system and so that

$$\theta_{r^{N,-}_{\tau^{N,\epsilon,-}} - N^{1/4}} \circ \xi^L_{\tau^{N,\epsilon,-}}$$

has distribution $Ren^{(-\infty,0]}(\beta)$,

$$\theta_{\ell^{N,-}_{\tau^{N,\epsilon,-}} + N^{1/4}} \circ \xi^R_{\tau^{N,\epsilon,-}}$$

has distribution $Ren^{[0,\infty)}(\beta)$ and these two distributions are mutually independent and independent of the Harris system. Let A_2 be the event that for some t with $\tau^{N,\epsilon,-} + N^2 \geq t \geq \tau^{N,\epsilon,-} + N^{1/4}$ and some $x \in [r^{N,-}_{\tau^{N,\epsilon,-}} - N^2, \infty)$, $\xi_t^L(x) > \eta_t^N(x)$ or for some t so that $\tau^{N,\epsilon,-} + N^2 \geq t \geq \tau^{N,\epsilon,-} + N^{1/4}$ and some $x \in (-\infty, \ell^{N,-}_{\tau^{N,\epsilon,-}} +$

$N^2]$, $\xi_t^R(x) > \eta_t^N(x)$. By the definition of $N^{1/4}$-rich and Proposition 47, $P(A_2)$ tends to zero as N tends to infinity.

Define for $t \geq \tau^{N,\epsilon,-}$, r_t^L to be the rightmost occupied site for ξ_t^L and l_t^R the leftmost occupied site of ξ_t^R. Let

$$S_\epsilon \;=\; \inf\{t > \tau^{N,\epsilon,-} + N^{1/4} : r_t^L > l_t^R + \epsilon N/9\}.$$

By Proposition 50 and Proposition 54 and Lemma 5, event A_3,

$$\{S_\epsilon > \tau^{N,\epsilon,-} + N^2\epsilon \text{ or } r_t^L > l_t^R + \epsilon N/8 + N^{1/3} \text{ or } \xi_{S_\epsilon}^R \text{ is not } N^{1/4}\text{−rich or } \xi_{S_\epsilon}^L \text{ is not } N^{1/4}\text{−rich } \}$$

has probability less than $C\epsilon^c$ for N large.

We define $(\eta^{S_\epsilon,\underline{1}} : t \geq S_\epsilon)$ to be the β-NPS run with the given Harris system starting, at time S_ϵ from all 1s. By the definition of $N^{1/4}$−rich and Lemma 16, the event

$$A_4 = A_3^c \cap \{\exists x \in [l_{S_\epsilon}^R + N\epsilon/20, l_{S_\epsilon}^R + 2N^2) : \xi_{S_\epsilon+N}^R(x) < \eta_{S_\epsilon+N}^{S_\epsilon,\underline{1}}(x) \text{ or}$$

$$\exists x \in [r_{S_\epsilon}^L - 2N^2, r_{S_\epsilon}^L - N\epsilon/20) \xi_{S_\epsilon+N}^L(x) < \eta_{S_\epsilon+N}^{S_\epsilon,\underline{1}}(x)\}$$

has probability tending to zero as N becomes large.

So, by attractiveness on $(A_1 \cup A_2 \cup A_3 \cup A_4)^c$, we have

$$\eta_{S_\epsilon+N}^N(x) \geq \eta_{S_\epsilon+N}^{S_\epsilon,\underline{1}}(x) \geq \tilde{\eta}_{S_\epsilon+N}^N(x) \;\forall\; x \in [r_{S_\epsilon}^L - 2N^2, l_{S_\epsilon}^R + 2N^2].$$

We have easily that event $A_5 = \{|r_{S_\epsilon}^L| + |r_{S_\epsilon}^L| \;\geq\; N^{3/2}/100\}$ has probability tending to zero as N tends to infinity.

Thus by Lemma 6 we have on the complement of the union of events $A_i, i = 1, 2, \cdots 5$ that $\eta_{S_\epsilon+N}^N = \tilde{\eta}_{S_\epsilon+N}^N$. That is

$$\limsup_{N\to\infty} P\left(\tilde{\tau}^N > \tau^{N,\epsilon,-} + \epsilon N^2 + \epsilon N\right) \;\leq\; C\epsilon^c.$$

We are done by the arbitrariness of $\epsilon > 0$.

$\diamondsuit$

Proof of Theorem 2 By definition, for N large enough $\tau^N \wedge \lambda N^2 \geq \tau^{N,\epsilon} \wedge \lambda N^2$ unless for some $s \leq \tau^{N,\epsilon} \wedge \lambda N^2$, $|r_s^N|$ or $|\ell_s^N|$ exceed $N^{3/2}$ (an event whose probability tends to zero as N becomes large by Corollary 49), equally it is clear that the event $\{\sigma^N < \tau^{N,\epsilon} \wedge \lambda N^2\}$ is contained in the union of events

$$\{\exists t \leq \lambda N^2, x \geq \frac{N\epsilon}{2} : t \in B_{r_{t-}^N + x},\; U^{x,t} \leq \beta(x)\}$$

$$\{\exists t \leq \lambda N^2, x \geq \frac{N\epsilon}{2} : t \in B_{\ell_{t-}^N - x},\; U^{x,t} \leq \beta(x)\}.$$

Thus $\{\sigma^N < \tau^{N,\epsilon} \wedge \lambda N^2\}$ has probability bounded by $2\lambda N^2 \sum_{x \geq N\epsilon} \beta(x)$. We conclude

$$P(\tau^N \wedge \sigma^N < \tau^{N,\epsilon} \wedge \lambda N^2) \rightarrow \ 0.$$

Furthermore, by Proposition 8, we have that $P(\tau^N > \tilde{\tau}^N)$ tends to zero as N tend to infinity, which implies, given Theorem 3, that

$$P(\tau^N > \tau^{N,\epsilon,-} + 2\epsilon N^2) \ \leq \ C\epsilon^c$$

for N large. For σ^N we simply observe that if processes $(\eta_s^{L,\prime}, \eta_s^{R,\prime} : s \geq \tau^{N,\epsilon,-})$ satisfy $\eta^{L,\prime}_{\tau^{N,\epsilon,-}} = \eta^{L,N}_{\tau^{N,\epsilon,-}}$ and evolve under the given Harris system as one sided β-NPSs, then (by Corollary 48) with probability tending to one as N tend to infinity, σ^N is less than or equal to the the minimum of λN^2 and the first time the rightmost particle of $\eta^{L,\prime}_{\cdot}$ exceeds the leftmost particle of $\eta^{R,\prime}_{\cdot}$. Thus, in the notation of the proof of Theorem 3, $P(\sigma^N < S_\epsilon)$ tends to zero as N tends to infinity and so by the proof of Theorem 3

$$\limsup_{N \rightarrow \infty} P(\sigma^N > \tau^{N,\epsilon,-} + 2\epsilon N^2) \ \leq \ C\epsilon^c.$$

Thus we have that

$$\limsup_{N \rightarrow \infty} P(\lambda N^2 \wedge (\tau^N \vee \sigma^N) > \tau^{N,\epsilon,-} + 2\epsilon N^2) \leq C\epsilon^c.$$

The result now follows from the arbitrariness of λ and the fact that, by Proposition 54 and Proposition 46 , the limit in distribution as $N \rightarrow \infty$ and then $\epsilon \rightarrow 0$ of $\tau^{N,\epsilon}/N^2 \wedge \lambda$ is $\tau \wedge \lambda$.

$\diamondsuit$

Proof of Theorem 4

We will use the (easily proven) facts that the distributions of the stopping times τ and τ^ϵ defined for the diffusions $(X^1_{\cdot}, X^2_{\cdot})$ have no atoms and that at time t, conditioned on $\{\tau > t\}$, the laws of X^i_t have no atoms without proof. We consider f, a cylinder function supported on $[-M, M]$. We assume without loss of generality that f is increasing with $f(\underline{0}) = 0$ and $f(\underline{1}) \leq 1$. We fix $\epsilon > 0$. Recall $\tau^{N,\epsilon,-} = \inf\{t : \ell^{N,-}_t - r^{N,-}_t \leq N\epsilon\} \wedge \lambda N^2$. We suppose that $\lambda > t$. We write

$$\begin{aligned} E[f(\eta^N_{N^2 t})] \ &= \ E[f(\eta^N_{N^2 t}) I_{\tau^{N,\epsilon,-} \leq N^2(t-3\epsilon)}] && (A) \\ &+ \ E[f(\eta^N_{N^2 t}) I_{\tau^{N,\epsilon,-} > N^2(t-3\epsilon)} I_{M \leq r^{N,-}_{N^2(t-3\epsilon)} - N\epsilon^{1/4}}] && (B) \\ &+ \ E[f(\eta^N_{N^2 t}) I_{\tau^{N,\epsilon,-} > N^2(t-3\epsilon)} I_{-M \geq \ell^{N,-}_{N^2(t-3\epsilon)} + N\epsilon^{1/4}}] && (C) \\ &+ \ E[f(\eta^N_{N^2 t}) I_{\tau^{N,\epsilon,-} > N^2(t-3\epsilon)} I_{r^{N,-}_{N^2(t-3\epsilon)} < M + N\epsilon^{1/4}, \ \ell^{N,-}_{N^2(t-3\epsilon)} > -M - N\epsilon^{1/4}}] \end{aligned}$$

Case (A): $E[f(\eta^N_{N^2 t}) I_{\tau^{N,\epsilon,-} \leq N^2(t-3\epsilon)}]$ is at least

$$P(\tau^{N,\epsilon,-} < N^2(t-3\epsilon)) \inf_{s<N^2(t-\epsilon)} \{E[f(\tilde{\eta}_t)|\mathcal{F}_s]\}$$

$$-P(\tau^{N,\epsilon,-} < N^2(t-3\epsilon),\ \tilde{\tau}^N \ > \ N^2(t-\epsilon)).$$

By Proposition 53 and Proposition 54, as well as the proof of Theorem 3 we have that (A) exceeds

$$(< Ren(\beta), f > -\epsilon)(P(\tau^{N,\epsilon,-} < N^2(t-3\epsilon)) - \epsilon) \ - \ 2C\epsilon^c$$

for N large.

Case (B): We assume without loss of generality that $N^2(t-3\epsilon)$ is of the form $jN^{4/3}$. By the increasing property of function f, Corollary 51, Corollary 48 and Lemmas 16 and 17, we have that $E[f(\eta^N_{N^2t})I_{\tau^{N,\epsilon,-}>N^2(t-3\epsilon)}I_{M\le r^{N,-}_{N^2(t-3\epsilon)}-N\epsilon^{1/4}}]$ exceeds

$$(< Ren(\beta), f > -C\epsilon)P(r^N_{N^2(t-3\epsilon)} \ge M + N\epsilon^{1/4}, \tau^{N,\epsilon,-} \le N^2(t-3\epsilon))$$

and (C) has a corresponding bound. We conclude that

$$\begin{aligned}
&\liminf_{N\to\infty} E[f(\eta^N_{N^2t})] \\
&\quad \ge \ (< Ren(\beta), f > -\epsilon)P(\tau^{N,\epsilon,-} < N^2(t-3\epsilon)) \\
&\quad + \ < Ren(\beta), f > P(r^{N,-}_{N^2(t-3\epsilon)} \ge M + N\epsilon^{1/4}, \tau^{N,\epsilon,-} \ge N^2(t-3\epsilon)) \\
&\quad + \ < Ren(\beta), f > P(\ell^{N,-}_{N^2(t-3\epsilon)} \le -(M + N\epsilon^{1/4}), \tau^{N,\epsilon,-} \le N^2(t-3\epsilon)) \\
&\quad - \ 2C\epsilon^c - 2C\epsilon.
\end{aligned}$$

Letting $\epsilon \to 0$ we conclude via Proposition 46 that

$$\liminf_{N\to\infty} E[f(\eta^N_{N^2t})] \quad \ge \quad \lambda_t < Ren(\beta), f > .$$

For the converse inequality we proceed with similar arguments: Recall $\tau^{N,\epsilon,+} = \inf\{t > 0 : \ell^{N,+}_t - r^{N,+}_t \le N\epsilon\} \wedge \lambda N^2$. Fix $\delta > 0$, $\delta << \epsilon^2$. By attractiveness and the increasing property of f,

$$\begin{aligned}
E[f(\eta^N_{N^2t})] \ &\le \ P(\tau^{N,\epsilon,+} \le N^2(t-\delta))P_{N^2\delta}f(\underline{1}) \\
&+ \ P(\tau^{N,\epsilon,+} > N^2(t-\delta),\ r^{N,+}_{N^2(t-\delta)} > -N\epsilon - M)P_{N^2\delta}f(\underline{1}) \\
&+ \ P(\tau^{N,\epsilon,+} > N^2(t-\delta),\ r^{N,+}_{N^2(t-\delta)} \le -N\epsilon - M,\ l^{N,+}_{N^2(t-\delta)} < N\epsilon + M)P_{N^2\delta}f(\underline{1}) \\
&+ \ P(|r^{N,+}_{N^2t} - r^{N,+}_{N^2(t-\delta)}| + |l^{N,+}_{N^2t} - l^{N,+}_{N^2(t-\delta)}| \ge N\epsilon \cup \{\tau^{N,\epsilon,+} \in [N^2(t-\delta), N^2t]\}) \\
&+ \ P(A(N,t))
\end{aligned}$$

where $A(N,t)$ is the event

$$\{\tau^{N,\epsilon,+} > N^2t\} \cap \{\exists s \le tN^2 : \eta^{\ell,N}_s \not\le \ \eta^{\ell,N,+}_s \text{ or } \eta^{R,N}_s \not\le \ \eta^{R,N,+}_s\}.$$

So

$$
\begin{aligned}
E[f(\eta^N_{N^2t})] &\leq P_{N^2\delta}f(\underline{1})[P(\tau^{N,\epsilon,+} \leq N^2(t-\delta)) \\
&+ P(\tau^{N,\epsilon,+} > N^2(t-\delta),\ r^{N,+}_{N^2(t-\delta)} > -N\epsilon - M) \\
&+ P(\tau^{N,\epsilon,+} > N^2(t-\delta),\ r^{N,+}_{N^2(t-\delta)} \leq -N\epsilon - M,\ l^{N,+}_{N^2(t-\delta)} < N\epsilon + M)] \\
&+ C(\epsilon,\delta)
\end{aligned}
$$

for all N where for fixed ϵ large (by Propositions 47 and 46), $C(\epsilon,\delta) \to 0$ as $\delta \to 0$.

For given ϵ, we choose δ so that $C(\epsilon,\delta) < \epsilon$; then we have $\limsup_{N\to\infty} E[f(\eta^N_{N^2t})] \leq \epsilon$

$$
+P_{N^2\delta}f(\underline{1})[P(\tau^{N,\epsilon,+} \leq N^2(t-\delta)) + P(\tau^{N,\epsilon,+} > N^2(t-\delta), r^{N,+}_{N^2(t-\delta)} > -N\epsilon - M)
$$

$$
+ P(\tau^{N,\epsilon,+} > N^2(t-\delta),\ r^{N,+}_{N^2(t-\delta)} \leq -N\epsilon - M,\ l^{N,+}_{N^2(t-\delta)} < N\epsilon + M)].
$$

Now letting ϵ tend to zero we obtain $\limsup_{N\to\infty} E[f(\eta^N_{N^2t})] \leq \lambda_t < Ren(\beta), f >$. We are done.

$\diamondsuit$

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