DOI: 10.1214/ECP.v9-1103

Elect. Comm. in Probab. **9** (2004)14–21

ELECTRONIC COMMUNICATIONS in PROBABILITY

INVARIANCE PRINCIPLES FOR RANKED EXCURSION LENGTHS AND HEIGHTS

ENDRE CSÁKI[1]
A. Rényi Institute of Mathematics, Hungarian Academy of Sciences, Reáltanoda u. 13–15, P.O.B. 127, Budapest, H-1364 Hungary
email: `csaki@renyi.hu`

YUEYUN HU
Laboratoire de Probabilités et Modèles Aléatoires (CNRS UMR–7599), Université Paris VI, 4 Place Jussieu, F–75252 Paris cedex 05, France
email: `hu@ccr.jussieu.fr`

Submitted 9 January 2004, *accepted in final form* 2 February 2004

AMS 2000 Subject classification: 60F17, 60F05, 60F15
Keywords: excursion lengths, excursion heights, invariance principle

Abstract
In this note we prove strong invariance principles between ranked excursion lengths and heights of a simple random walk and those of a standard Brownian motion. Some consequences concerning limiting distributions and strong limit theorems will also be presented.

1 Introduction

Let $X_1, X_2, \dots$ be independent random variables with distribution

$$P(X_i = +1) = P(X_i = -1) = \frac{1}{2}.$$

Put $S_0 = 0$, $S_i = X_1 + \dots X_i$, $i = 1, 2, \dots$. Then the sequence $\{S_i\}_{i=0}^{\infty}$ is called a simple symmetric random walk on the line. Consider the return times defined by $\rho_0 = 0$,

$$\rho_i = \min\{k > \rho_{i-1} : S_k = 0\}, \quad i = 1, 2, \dots$$

Further, let

$$\xi(n) = \#\{k : 0 < k \le n, S_k = 0\}$$

be the local time of the random walk at zero, i.e. the number of returns to the origin up to time n.
The parts

$$(S_{\rho_{i-1}}, \dots, S_{\rho_i - 1}), \quad i = 1, 2, \dots$$

[1]RESEARCH SUPPORTED BY THE HUNGARIAN NATIONAL FOUNDATION FOR SCIENTIFIC RESEARCH, GRANTS T 037886 AND T 043037

between consecutive returns are called excursions. Consider the lengths

$$\tau_i = \rho_i - \rho_{i-1},$$

and heights

$$\mu_i = \max_{\rho_{i-1} \le k \le \rho_i - 1} |S_k|,$$

$$\mu_i^+ = \max_{\rho_{i-1} \le k \le \rho_i - 1} S_k$$

of i-th excursion.
Clearly, the random walk does not change sign within an excursion. We may call the excursion positive (negative) if the random walk assumes positive (negative) values within this excursion. If the i-th excursion is negative, then $\mu_i^+ = 0$.
In this paper we consider the ranked lengths and heights of excursions up to time n. In general, however the (fixed) time n need not be an excursion endpoint, and we include the length and height of this last, possibly incomplete, excursion as well. Consider the sequences

$$L^{(1)}(n) \ge L^{(2)}(n) \ge \dots,$$

$$M^{(1)}(n) \ge M^{(2)}(n) \ge \dots,$$

and

$$M_+^{(1)}(n) \ge M_+^{(2)}(n) \ge \dots,$$

where $L^{(j)}(n)$ is the j-th largest in the sequence

$$(\tau_1, \tau_2, \dots, \tau_{\xi(n)}, n - \rho_{\xi(n)}),$$

$M^{(j)}(n)$ is the j-th largest in the sequence

$$(\mu_1, \mu_2, \dots, \mu_{\xi(n)}, \max_{\rho_{\xi(n)} \le k \le n} |S_k|),$$

while $M_+^{(j)}(n)$ is the j-th largest in the sequence

$$(\mu_1^+, \mu_2^+, \dots, \mu_{\xi(n)}^+, \max_{\rho_{\xi(n)} \le k \le n} S_k).$$

We define $M_+^{(j)}(n) = M^{(j)}(n) = L^{(j)}(n) = 0$ if $j > \xi(n) + 1$.
Let $\{W(t),\, t \ge 0\}$ be a standard one-dimensional Brownian motion starting from 0. For $t > 0$ denote by

$$V^{(1)}(t) \ge V^{(2)}(t) \ge \dots \ge V^{(n)}(t) \ge \dots > 0,$$

the ranked lengths of the countable excursions of W over $[0, t]$. We mention that this sequence includes the length $t - g(t)$ of the incomplete excursion $(W(s), g(t) \le s \le t)$, where $g(t) := \sup\{s \le t : W(s) = 0\}$.
Let furthermore

$$H_+^{(1)}(t) \ge H_+^{(2)}(t) \ge \dots H_+^{(n)}(t) \ge \dots > 0$$

and

$$H^{(1)}(t) \ge H^{(2)}(t) \ge ...H^{(n)}(t) \ge ... > 0$$

denote the ranked heights of countable positive and all excursions, resp. of W over $[0,t]$. These sequences include the heights $\sup_{g(t)\le s\le t} W(s)$ and $\sup_{g(t)\le s\le t} |W(s)|$ of the incomplete excursion $(W(s), g(t)\le s\le t)$.
The properties of these quantities for Brownian motion were investigated by Wendel [15], Knight [11], Pitman and Yor [13, 14] and their strong limit properties were studied in [8, 9], [3, 4, 5]. In [2] the properties of $L^{(1)}$ were investigated. For random walk excursions exact and limiting distributions were studied in [6].
In this paper we prove strong invariance principles for ranked lengths and heights and discuss certain consequences for limit theorems.

2 Invariance principle

We shall approximate the heights and lengths of random walk excursion by those of Brownian motion, using Skorokhod embedding.
Define $\sigma(0)=0$ and

$$\sigma(n)=\inf\{t>\sigma(n-1): |W(t)-W(\sigma(n-1))|=1\}, \qquad n\ge 1.$$

Take $S_i := W(\sigma(i))$. Then $\{S_i\}_{i=0}^{\infty}$ is a simple random walk obtained by Skorokhod embedding and we make use of the notations $(\xi(n), M_+^{(j)}(n), M^{(j)}(n), L^{(j)}(n))$ introduced in Section 1.

Theorem 2.1 *Almost surely, we have*

$$\limsup_{n\to\infty} \frac{1}{(\log n)^{1/2}(\log\log n)^{1/4}} \max_{1\le j\le \xi(n)} |M_+^{(j)}(n)-H_+^{(j)}(n)| \le 3, \tag{2.1}$$

$$\limsup_{n\to\infty} \frac{1}{(\log n)^{1/2}(\log\log n)^{1/4}} \max_{1\le j\le \xi(n)} |M^{(j)}(n)-H^{(j)}(n)| \le 3, \tag{2.2}$$

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \max_{1\le j\le \xi(n)} |L^{(j)}(n)-V^{(j)}(n)| \le 6. \tag{2.3}$$

We state below some known results as facts:

Fact 2.2 Csörgő and Révész ([7], Theorem 1.2.1) *Let a_t be a non-decreasing function of t such that $0<a_t\le t$ and t/a_t is non-decreasing. Then*

$$\limsup_{t\to\infty} \frac{1}{\sqrt{2a_t(\log(t/a_t)+\log\log t)}} \sup_{0\le u\le t-a_t} \sup_{0\le s\le a_t} |W(u+s)-W(u)| = 1, \qquad a.s.$$

Fact 2.3 *We have*

$$P(\sigma(1)\ge x)=\frac{4}{\pi}\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}\exp\left(-\frac{\pi^2}{8}(2k+1)^2x\right), \quad x\ge 0, \tag{2.4}$$

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \max_{1\le j\le n} |\sigma(j)-j| = \sqrt{2\ Var(\sigma(1))} = \frac{2}{\sqrt{3}}, \quad a.s., \tag{2.5}$$

$$\lim_{n\to\infty} \frac{1}{\log n} \max_{1\le j\le n} (\sigma(j)-\sigma(j-1)) = \frac{8}{\pi^2}, \quad a.s. \tag{2.6}$$

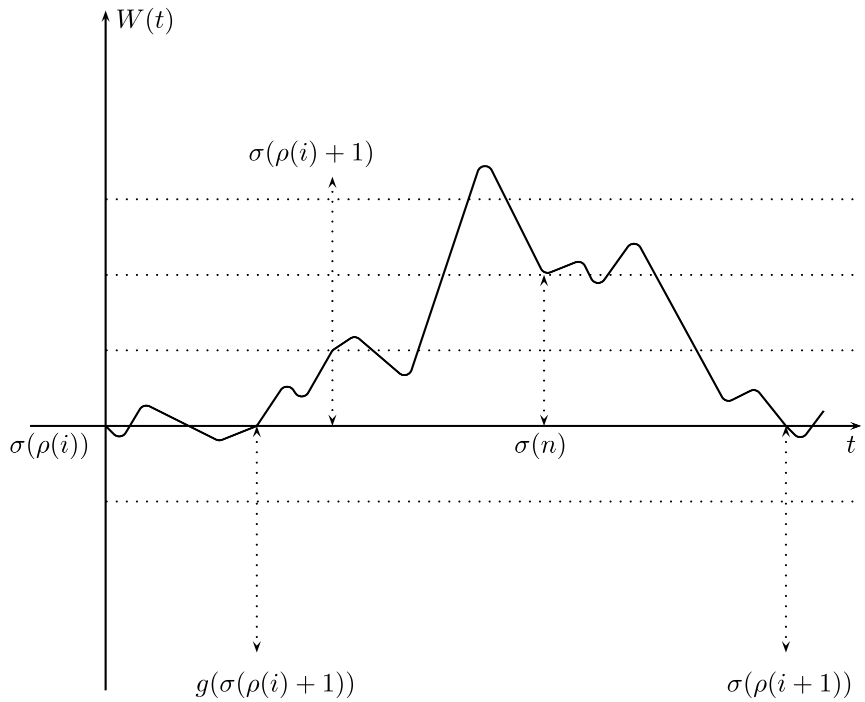

Figure 1: The Skorokhod embedding $S_n = W(\sigma(n))$

For (2.4) see e.g. Knight [10], Theorem 4.1.1, while the two estimates (2.5) and (2.6) follow from the usual law of the iterated logarithm and the standard extreme value theory, resp.

Proof of Theorem 2.1: Let us firstly prove the invariance on the heights (2.2).
The two intervals $[\rho(i), \rho(i+1) \wedge n]$ and $[g(\sigma(\rho(i)+1)), \sigma(\rho(i+1) \wedge n)]$ are respectively excursion interval of S and of W. Observe that

$$\Big| \max_{\rho(i) \wedge n \le k < \rho(i+1) \wedge n} |S_k| - \sup_{\sigma(\rho(i) \wedge n) \le s < \sigma(\rho(i+1) \wedge n)} |W(s)| \Big| \le 1,$$

where we adopt the convention $\max_\emptyset = \sup_\emptyset = 0$. Let $j \ge 1$ and $x > 1$. The event $\{1 \le M^{(j)}(n) < x\}$ yields that there are at most $j-1$ index $i \ge 0$ such that

$$\max_{\rho(i) \wedge n \le k < \rho(i+1) \wedge n} |S_k| \ge x,$$

which implies that there are at most $j-1$ index i such that

$$\sup_{\sigma(\rho(i) \wedge n) \le s < \sigma(\rho(i+1) \wedge n)} |W(s)| \ge x+1.$$

In other words,

$$\{1 \le M^{(j)}(n) < x\} \subset \{H_j(\sigma(n)) < x+1\}.$$

Similarly,

$$\{M^{(j)}(n) > x\} \subset \{H_j(\sigma(n)) > x-1\}.$$

Hence

$$\max_{1\le j\le \xi(n)} \left| M^{(j)}(n) - H_j(\sigma(n)) \right| \le 1.$$

Now, we observe that for any $0 \le s < t$

$$|H_j(t) - H_j(s)| \le \sup_{s\le u\le v\le t} |W(v) - W(u)|$$

which in view of Fact 2.2 and (2.5) imply (2.2).
The proof of (2.1) is similar.
To compare the lengths, we adopt the similar ω-by-ω argument: Roughly saying, the longest lengths of the excursions of W till $\sigma(n)$ are $\{\sigma(n\wedge\rho(i+1)) - \sigma(n\wedge\rho(i)), i \ge 1\} = \{n\wedge\rho(i+1) - n\wedge\rho(i)\}$ with error term bounded by $\max_{1\le j\le n}(\sigma(j)-\sigma(j-1)) = O(\sqrt{n\log\log n})$.
In fact, let $x > 4\sqrt{n\log\log n}$. It follows from (2.5) that

$$\limsup_{n\to\infty} \frac{1}{\sqrt{n\log\log n}} \max_{0\le i\le j\le n} \left|\sigma(j) - \sigma(i) - (j-i)\right| \le \frac{4}{\sqrt{3}}. \tag{2.7}$$

Consider a typical ω and $n \ge n_0(\omega)$ sufficiently large such that (2.6) and (2.7) hold. For any $1 \le j \le \xi(n)$, $L^{(j)}(n) > x$ yields that there are at least j index $i \ge 0$ such that $n\wedge\rho(i+1)-\rho(i) > x$, hence $\sigma(n\wedge\rho(i+1)) - \sigma(\rho(i)) > x - 3\sqrt{n\log\log n}$ and therefore $\sigma(n\wedge\rho(i+1)) - \sigma(\rho(i)+1) > x - 4\sqrt{n\log\log n}$. We have obtained

$$\{L^{(j)}(n) > x\} \subset \{V_j(\sigma(n)) > x - 4\sqrt{n\log\log n}\},$$

and in a similar way,

$$\{2 \le L^{(j)}(n) \le x\} \subset \{V_j(\sigma(n)) \le x + 4\sqrt{n\log\log n}\}.$$

Hence almost surely for all large $n \ge n_0(\omega)$, we have

$$\max_{1\le j\le \xi(n)} \left| L^{(j)}(n) - V_j(\sigma(n)) \right| \le 4\sqrt{n\log\log n}.$$

Note that $V_j(t) - V_j(s) \le t - s$ for any $s \le t$. This together with (2.5) yield (2.3), completing the whole proof of Theorem 2.1. $\square$

3 Limit theorems

It follows from our Theorem 2.1 that the limit results proved for heights and (or) lengths of excursions for the case of Brownian motion remain valid for similar quantities of simple symmetric random walk and vice versa. So the limiting distributions derived in [6] in random walk case are equivalent with the corresponding distributions in Brownian motion case.

Fact 3.1 [6]

$$\lim_{n\to\infty} P(M_+^{(j)}(n) \ge y\sqrt{n}) = 2(1 - \Phi((2j-1)y)) \tag{3.1}$$

$$\lim_{n\to\infty} P(M^{(j)}(n) \ge y\sqrt{n}) = 2^{j+1} \sum_{k=0}^{\infty} (-1)^k \binom{k+j-1}{k} (1 - \Phi((2k+2j-1)y)) \tag{3.2}$$

Hence it follows from Theorem 2.1 and scaling

Corollary 3.2

$$P(H_+^{(j)}(t) \geq y\sqrt{t}) = 2(1 - \Phi((2j-1)y)) \tag{3.3}$$

$$P(H^{(j)}(t) \geq y\sqrt{t}) = 2^{j+1} \sum_{k=0}^{\infty} (-1)^k \binom{k+j-1}{k} (1 - \Phi((2k+2j-1)y)). \tag{3.4}$$

Another form of the above distributions and further distributional results can be found in Pitman and Yor [12, 13, 14], and Wendel [15].
Furthermore, we mention some almost sure results proved for Brownian motion case, remaining valid also for random walk case.

Fact 3.3 [**3**] *Let $f > 0$ be a nondecreasing function. For $k \geq 2$, we have*

$$P(H^{(k)}(t) > \sqrt{t} f(t), \text{ i.o.}) = 0 \quad \text{or} \quad 1$$

according as

$$\int_1^\infty \frac{f(t)}{t} \exp\left(-\frac{(2k-1)^2 f^2(t)}{2}\right) dt < \infty \quad \text{or} \quad = \infty.$$

Here i.o. means that there is a sequence $\{t_i\}_{i=1}^\infty$ such that $\lim_{i\to\infty} t_i = \infty$ and $H^{(k)}(t_i) > \sqrt{t_i} f(t_i)$.
Theorem 2.1 and Fact 3.3 clearly imply

Corollary 3.4 *Let $f > 0$ be a nondecreasing function. For $k \geq 2$, we have*

$$P(M^{(k)}(n) > \sqrt{n} f(n), \text{ i.o.}) = 0 \quad \text{or} \quad 1$$

according as

$$\sum_{n=1}^\infty \frac{f(n)}{n} \exp\left(-\frac{(2k-1)^2 f^2(n)}{2}\right) < \infty \quad \text{or} \quad = \infty.$$

We note that Fact 3.3 and Corollary 3.4 remain true if $H^{(k)}(t)$ and $M^{(k)}(n)$ are replaced by $H_+^{(k)}(t)$ and $M_+^{(k)}(n)$, resp.

Fact 3.5 [**8**] *For any fixed integer $k \geq 2$ and nondecreasing function $\phi > 0$,*

$$P\left(V^{(k)}(t) > \frac{t}{k}\left(1 - \frac{1}{\phi(t)}\right), \text{ i.o.}\right) = 0 \quad \text{or} \quad 1$$

according as

$$\int_1^\infty \frac{dt}{t(\phi(t))^{3k/2-2}} \, dt < \infty \quad \text{or} \quad = \infty.$$

Theorem 2.1 and Fact 3.5 imply

Corollary 3.6 *For any fixed integer $k \geq 2$ and nondecreasing function $\phi > 0$,*

$$P\left(L^{(k)}(n) > \frac{n}{k}\left(1 - \frac{1}{\phi(n)}\right), \text{ i.o.}\right) = 0 \quad \text{or} \quad 1$$

according as

$$\sum_{n=1}^\infty \frac{1}{n(\phi(n))^{3k/2-2}} < \infty \quad \text{or} \quad = \infty.$$

This Corollary solves Problem 1 in [2] for random walk. For further strong limit theorems concerning excursion heights and lengths, we refer to [1], [2], [3], [4], [5], [8], [9].

Acknowledgements. Cooperation between the authors was supported by the joint French-Hungarian Intergovernmental Grant "Balaton" no. F-39/00.

References

[1] K.L. Chung and P. Erdős, On the application of the Borel-Cantelli lemma, *Trans. Amer. Math. Soc.* **72** (1952), 179–186.

[2] E. Csáki, P. Erdős and P. Révész, On the length of the longest excursion, *Z. Wahrsch. verw. Gebiete* **68** (1985), 365–382.

[3] E. Csáki and Y. Hu, Asymptotic properties of ranked heights in Brownian excursions, *J. Theoret. Probab.* **14**, (2001), 77–96.

[4] E. Csáki and Y. Hu, On the joint asymptotic behaviours of ranked heights of Brownian excursions, in *Limit Theorems in Probability and Statistics*, Balatonlelle, Hungary, 1999. I. Berkes et al. (eds.), J. Bolyai Mathematical Society, Budapest, 2002, Vol. I, pp. 347–363.

[5] E. Csáki and Y. Hu, On the ranked excursion heights of a Kiefer process, *J. Theoret. Probab.* **17** (2004), 145–163.

[6] E. Csáki and Y. Hu, Lengths and heights of random walk excursions, in *Discrete Random Walks, DRW'03*, Cyril Banderier and Christian Krattenthaler, (eds.), *Discrete Mathematics and Theoretical Computer Science Proceedings* AC (2003), pp. 45–52.

[7] M. Csörgő and P. Révész, *Strong Approximations in Probability and Statistics.* Academic Press, New York, 1981.

[8] Y. Hu and Z. Shi, Extreme lengths in Brownian and Bessel excursions, *Bernoulli* **3** (1997), 387–402.

[9] Y. Hu and Z. Shi, Shortest excursion lengths, *Ann. Inst. H. Poincaré Probab. Statist.* **35** (1999), 103–120.

[10] F.B. Knight, *Essentials of Brownian motion and diffusion.* American Mathematical Society, Providence, Rhode Island, 1981.

[11] F.B. Knight, On the duration of the longest excursion, *Seminar on stochastic processes, 1985 (Gainesville, Fla., 1985), Progr. Probab. Statist.*, **12**, Birkhäuser Boston, Boston, MA, 1986, pp. 117–147.

[12] J. Pitman and M. Yor, The two-parameter Poisson-Dirichlet distribution derived from a stable subordinator, *Ann. Probab.* **25** (1997), 855-900.

[13] J. Pitman and M. Yor, Ranked functionals of Brownian excursions, *C. R. Acad. Sci. Paris Ser. I* **326** (1998), 93–97.

[14] J. Pitman and M. Yor, On the distribution of ranked heights of excursions of a Brownian bridge, *Ann. Probab.* **29** (2001), 361–384.

[15] J.G. Wendel, Zero-free intervals of semi-stable Markov processes, *Math. Scand.* **14** (1964), 21–34.