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FINITE DIMENSIONAL DETERMINANTS AS CHARACTERISTIC FUNCTIONS OF QUADRATIC WIENER FUNCTIONALS

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Abstract
We show a method and the structure to calculate the characteristic functions of quadratic Wiener functionals by using the classical Weierstrass-Hadamard theory of entire functions. We also examine the idea by an example for Gaussian processes with multiple Markovian property.

1 Introduction

The aim of this article is to show a method and the structure of exact calculations for the characteristic functions of quadratic Wiener functionals, that is, the oscillatory integral type expectations with quadratic phase functions $F[w]$ on the classical Wiener space (W, P):

$$C(\lambda) = E[e^{\sqrt{-1}\lambda F}] = \int_W \exp\left(\sqrt{-1}\lambda F[w]\right) P(dw). \tag{1}$$

This problem has a long history since the fundamental works by P. Lévy, M. Kac, R. H. Cameron and W. T. Martin. It was recently revisited by studies on the concrete examples in the context of Malliavin calculus and its application to infinite dimensional stationary phase methods.
There is a correspondence between a quadratic Wiener functional and the symmetric Hilbert-Schmidt operator on the Cameron-Martin subspace H in W. Therefore the closed form of $C(\lambda)$ is computed by solving the associated eigenvalue problem for the operator and by calculating the infinite dimensional determinant. However, both steps are often hard, though $C(\lambda)$ is written by *finite* dimensional determinants in many cases. For such recent examples with new motivations, there is a series of works by N. Ikeda-S. Kusuoka-S. Manabe [4, 5, 6, 7], and also H. Matsumoto-S. Taniguchi [9]. The finite dimensional property of the determinants is studied

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as an analogue of Van Vleck formula in [7] and its functional analytic aspect is studied in [5]. Applications to the other fields, for example, filtering, solitons, etc., are found in N. Ikeda and the author [3], and also in Ikeda-Taniguchi [8].
In this article, we shall show a direct relation between these two infinite and finite determinants and also examine a shortcut of the calculation by the relation with a new example. The idea is twofold: the framework of boundary value problems of linear differential equations and Weierstrass-Hadamard's classical theory of entire complex functions.
The author should remark that S. Coleman [2] essentially used the relation to calculate the fundamental solutions of Schrödinger equations in the context of quantum physics, and also that similar observations are found in Ikeda-Kusuoka-Manabe [5] and T. Chan [1] for their own special cases.

2 Framework and results

Let $T > 0$ be fixed and consider the classical n dimensional Wiener space $(W, P) = (W_T, P_T)$, that is, the pair consisting of the space $W = W_T$ of continuous functions $w : [0, T] \to \mathbf{R}^n$ starting at the origin and the Wiener measure $P = P_T$ on W. Let $H = H_T$ be the Cameron-Martin subspace in W. The each element $h = {}^T(h^1, \ldots, h^n) \in H$ is absolutely continuous and has the square integrable derivative $\dot{h} = {}^T(\dot{h}^1, \ldots, \dot{h}^n)$ (where the symbol ${}^T M$ means the transpose of a matrix M). The inner product in H is given by

$$\langle h_1, h_2 \rangle_H = \int_0^T \sum_{j=1}^n \dot{h_1}^j(t)\, \dot{h_2}^j(t) dt, \quad (h_1, h_2 \in H).$$

We consider a functional $F : W \to \mathbf{R}$ that is quadratic in the following sense. We have the Wiener-Itô decomposition of the square integrable functionals on (W, P):

$$L^2(W, P) = \bigoplus_{j=0}^{\infty} \mathcal{C}_j,$$

where $\mathcal{C}_j$ is the jth ordered chaos. A functional F is called quadratic if F belongs to $\mathcal{C}_0 \oplus \mathcal{C}_1 \oplus \mathcal{C}_2$. In other words, it is an L^2 functional F with $\nabla^3 F = 0$ (∇ means Malliavin's derivative). However, we use the word 'quadratic' only for $F \in \mathcal{C}_2$ throughout this article because the $\mathcal{C}_0 \oplus \mathcal{C}_1$ part is easy to handle and out of our focus (cf. Matsumoto and Taniguchi [9]).
For each quadratic functional $F \in \mathcal{C}_2$, there exists a unique symmetric Hilbert-Schmidt operator B such that $F[h] = \langle Bh, h \rangle_H$ for $h \in H$. Let $\{\mu_n\}_{n=1}^{\infty}$ be the eigenvalues of B and $\sigma(B)$ the spectral radius of B. Then, as pointed out in Ikeda-Kusuoka-Manabe [4, 5], we have the fundamental relation:

$$\int_W \exp(zF[w]) P(dw) = \frac{1}{\sqrt{\prod_{j=1}^{\infty} (1 - 2z\mu_n) e^{2z\mu_n}}} \tag{2}$$

for every $z \in \mathbf{C}$ such that $2|\Re z|\sigma(B) < 1$.
Therefore, our problem is reduced to the eigenvalue problem of the symmetric Hilbert-Schmidt operator B and the calculation of the following infinite product, called a generalized determinant, or Carleman-Fredholm's determinant:

$$D(z) := \det{}_2(I - 2zB) = \prod_{j=1}^{\infty} (1 - 2z\mu_j) e^{2z\mu_j}, \tag{3}$$

where I is the identity operator. Since the operator B is of Hilbert-Schmidt type, the infinite product above converges absolutely and the complex function $D(z)$ is analytic in $|z| < \infty$. In other words, $D(z)$ is an entire function with the genus 1 in the sense of Weierstrass.
The eigenvalue problem has the general form with the corresponding symmetric integral kernel $F(t,s) = \{F^{ij}(t,s)\}_{i,j=1,\ldots,n}$ as follows.

$$\frac{d}{dt}(B[h])^i(t) = \sum_{j=1}^{n} \int_0^T F^{ij}(t,s)\dot{h}^j(s)ds = \mu \dot{h}^i(t), \quad (i = 1, \ldots, n) \tag{4}$$

for $\mu \in \mathbf{C}$ and $h(t) = {}^T(h^1(t), \ldots, h^n(t)) \in H$. We make the following assumption (A1).

(A1): the integral equation is rewritten in the following linear differential equation with the boundary condition:

$$L\dot{h} - \nu\dot{h} = \mathbf{0}, \tag{5}$$

$$\boldsymbol{b}[\dot{h}] = \mathbf{0}, \tag{6}$$

where $\nu = 1/\mu$, $\mathbf{0} = {}^T(0, \ldots, 0)$ is the origin of $\mathbf{R}^n$, and L is the formal linear differential operator:

$$L = \left(\frac{d}{dt}\right)^N + \sum_{k=1}^{N} p_k(t) \left(\frac{d}{dt}\right)^{N-k} \tag{7}$$

and $\boldsymbol{b}$ is the linear boundary operator:

$$(\boldsymbol{b}[x])_j = \sum_{k=1}^{N} q_{jk} \left(\frac{d}{dt}\right)^{k-1} x(0) + \sum_{k=1}^{N} r_{jk} \left(\frac{d}{dt}\right)^{k-1} x(T), \quad (j = 1, \ldots, N). \tag{8}$$

Here $p_1, \ldots, p_n$ are continuous $(N \times N)$-matrix valued functions on $[0,T]$. $\{q_{jk}\}$ and $\{r_{jk}\}$ are some constant $(N \times N)$-matrices.
In other words, we assume the kernel $F(t,s)$ is the Green function of the differential equation above. By the symmetry of our original problem (4), our differential equation (5) with (6) is self adjoint. So the operator L and its Lagrange's adjoint L^* for $\langle\, , \rangle_H$ coincide and the dimension N is even. Therefore the eigenvalues are real, not zero, and have their independent eigenfunctions. We remark that another equivalent expression is the variational problem, which is closely related to an analogue of Van Vleck formula (cf. Ikeda-Manabe [7]).
By the general theory of ordinary differential equations, we can choose a system of general solutions $\{\phi_k(t;\nu)\}_{k=1}^N$ of the above equation (5) such that they are analytic as complex functions of ν. (However, they are not necessarily entire.) Then the general solution of (5) has the following form:

$$\varphi(t;\nu) = c_1\phi_1(t;\nu) + \cdots + c_N\phi_N(t;\nu)$$

with constants $c_1, \ldots, c_N$. Because it satisfies the boundary condition (6), we have the following equation:

$$(\boldsymbol{b}[\varphi])_j = \sum_{k=1}^{N} q_{jk} \left(\frac{d}{dt}\right)^{k-1} \varphi(0;\nu) + \sum_{k=1}^{N} r_{jk} \left(\frac{d}{dt}\right)^{k-1} \varphi(T;\nu) = \mathbf{0}, \quad (j = 1, \ldots, N).$$

We consider the above equation as a linear equation for the variables c_j:

$$(\boldsymbol{b}[\varphi])_j = \sum_{i=1}^{N} (\boldsymbol{b}[\phi_i])_j c_i = \mathbf{0}, \quad j = 1, \ldots, N. \tag{9}$$

If a nontrivial solution $(c_1, \ldots, c_N)$ exists, all $(N \times N)$-minors of $(nN \times N)$-matrix of the above linear equation (9) vanish. Therefore, setting

$$d(\nu) := \sum{}' \left(\det((\widetilde{\boldsymbol{b}[\varphi]}))_{1 \le i,j \le N} \right)^2 ,$$

we have

$$d(\nu) = 0,$$

where the symbol $\sum'$ means the sum for all $(N \times N)$-minors $\det \widetilde{b}$ and that it is multiplied by the power of ν such that $d(0) \neq 0$. (See N. Ikeda-K. Hara [3] for the Grassmannian structure related to the minors above). Therefore the eigenvalues of our original problem (4) correspond to the zeros of $d(\nu)$. We claim that this $d(\nu)$ and our target $D(z)$ is the same one as a complex function with some correction terms under some assumptions. So we need not to calculate the eigenvalues and their infinite product. The key to prove the claim is Weierstrass-Hadamard's theory of entire functions.
Recall that the entire function $D(z)$ has the zeros at $z = 1/(2\mu_n)\,(n = 1, 2, \ldots)$. On the other hand, $d(2z)$ has the same zeros at $z = 1/(2\mu_n)\,(n = 1, 2, \ldots)$ with the same multiplicities. Therefore, if we can modify these two functions into entire ones with the same zeros and the same asymptotic behaviour, we can conclude that the two modified functions coincide by Weierstrass-Hadamard's theory. Now we need the assumption (A2) below for the modification. The point is the existence of a suitable uniformizing parameter of $d(z)$.
We suppose that

(A2): there is a polynomial $q(z)$ of $z \in \mathbf{C}$ such that $\phi_i(q(z))$ is entire, and

$$\sup_{|z| \le r} |\phi_i(q(z))| \le \exp(r^{1+\epsilon}) \tag{10}$$

for every $\epsilon > 0$, $i = 1, \ldots, N$, and large enough $r > 0$.

This assumption is trivially satisfied if the coefficients $p_j(t)\,(j = 1, \ldots, N)$ are constant scalar matrices, because we can simply choose the characteristic polynomial of the differential equation $L\phi = 0$ as $q(z)$.

Now we can state the main theorem under our assumptions.

Theorem 2.1 *Under the assumptions (A1) and (A2), we have*

$$D(q(z)) = e^{\widetilde{q(z)}}\, d(q(2z)),$$

where $\widetilde{q}(z)$ is a polynomial whose degree is less than or equal to the degree of $q(z)$. Furthermore, if the operator B is nuclear, we have

$$\widetilde{q}(z) = \alpha z + \beta + \gamma q(z), \quad (\alpha, \beta, \gamma \in \mathbf{C}).$$

Proof. By the assumption (A2), we have the estimate:

$$\sup_{|z|=r} |d(q(z))| \le \exp(r^{1+\epsilon}),$$

for every $\epsilon > 0$ and a large enough r. Therefore the order of $d(q(z))$ is 1 in the sense of Hadamard, and so the genus is 1 at most. Recall that $d(2z)$ has the zeros at $(1/(2\mu_n))_{n=1}^{\infty}$. Then, $d(q(2z))$ has the zeros:

$$z_m \in Z := \{z : q(z) = 1/2\mu_n,\ n = 1, 2, \dots\}, m = 1, 2, \dots.$$

Therefore, $e^{q(z)}d(2q(z))$ has the same zeros and the order is equal to the degree of $q(z)$, say $\deg q$. On the other hand, $D(q(z))$ is an entire function with the same zeros and also its order is the same $\deg q$, because we have a simple estimate:

$$\sup_{|z|=r} |D(q(z))| \leq \exp(q(r)^{1+\epsilon}).$$

Thus both entire functions have the same Weierstrass's canonical form as follows and the infinite product absolutely converges:

$$e^{g(z)} \prod_{z_m \in Z} \left(1 - \frac{z}{z_m}\right) \exp\left\{\frac{z}{z_n} + \dots + \frac{1}{\deg q}\left(\frac{z}{z_n}\right)^{\deg q}\right\},$$

where $g(z)$ is a polynomial such that $\deg g \leq \deg q$. Since the only difference is the term $e^{g(z)}$, we have

$$D(q(z)) = e^{\tilde{q}(z)} d(q(2z)),$$

where $\tilde{q}$ is a polynomial such that $\deg \tilde{q} \leq \deg q$.
Furthermore, if the operator B is nuclear, we can explicitly calculate as follows.

$$\begin{aligned} d(q(2z)) &= e^{az+b} \prod_n \prod_{q(z_m)=1/(2\mu_n)} \left(1 - \frac{z}{z_m}\right) e^{z/z_m} \\ &= e^{az+b} \prod_n (1 - 2\mu_n q(z)) e^{z \sum_{q(z_m)=1/(2\mu_n)} 1/z_m} \\ &= e^{az+b} \prod_n (1 - 2\mu_n q(z)) e^{2a_1 \mu_n z} \\ &= e^{az+b} \prod_n (1 - 2\mu_n q(z)) e^{2\mu_n q(z)} e^{-2\mu_n\{q(z) - a_1 z\}}, \end{aligned}$$

where a_1 is the coefficient of the term z of the polynomial $q(z)$. Therefore there are constants α, β, and γ such that

$$D(q(z)) = e^{\alpha z + \beta + \gamma q(z)} d(q(2z)).$$

q.e.d.

Remark 1. Since the both sides of the equation in the theorem are entire and the infinite products absolutely converge, we can differentiate them to determine the polynomial.

Remark 2.
Since $d(z)$ is a function of the minors, it is expressed by Plücker's coordinates. Then, it lies in a Grassmannian and it is naturally related to the theory of solitons. See Ikeda-Hara [3], also refer to Ikeda-Taniguchi [8] for another approach by Ricatti's equations and Girsanov's formula.

On the other hand, $d(z)$ is also expressed by the boundary data of the differential equation, which has an equivalent variational problem. See Ikeda-Manabe [7] for another aspect as an analogue of Van Vleck formula.

Remark 3.
We can trace the same line in the conditioned cases by using projection to a subspace H_0 in Cameron-Martin space H. In fact, we know the following formula for such cases including the pinned processes version:

$$\int_W e^{zF[w]}\delta_0(\langle \eta, w\rangle)P(dw) = \left(\frac{1}{2\pi}\right)^{m/2}\frac{1}{\sqrt{D^*(z)\det V}},$$

where δ_0 is the pullback of the Dirac delta function at the origin, $\eta = (\eta_1, \ldots, \eta_m)$ is a base of H_0, D^* is the Carleman-Fredholm determinant modified by the projection, and a correction term $\det V$ is a finite dimensional determinant that we can calculate by η. For the precise information about the formula above, see Ikeda-Kusuoka-Manabe [5].

3 Examples — Gaussian processes with multiple Markovian property

First we examine our framework by familiar examples before our main topic in this section. Consider the squared norm of the one dimensional Wiener process $w(t)$ starting at the origin:

$$F[w] = \int_0^1 |w(t)|^2 dt.$$

The corresponding differential equation is the following.

$$\varphi'' + \frac{1}{\mu}\varphi = 0, \quad \varphi(0) = \varphi'(1) = 0.$$

The general solution of the equation is expressed by

$$\varphi(t) = c_1 e^{i\sqrt{\nu}t} + c_2 e^{-i\sqrt{\nu}t}, \quad (\nu = 1/\mu).$$

Then the boundary conditions are equivalent to

$$\begin{pmatrix} 1 & 1 \\ e^{i\sqrt{\nu}} & -e^{-i\sqrt{\nu}} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \mathbf{0}.$$

The determinant of the matrix of the left hand side is zero, and it is nothing but the quantity that we need. Setting

$$d(\nu) = 1\cdot e^{-i\sqrt{\nu}} - (-1)\cdot e^{i\sqrt{\nu}} = 2\cos\sqrt{\nu}$$

and $q(z) = z^2$, all assumptions are satisfied and we have the known result:

$$E[e^{i\lambda F[w]}] = 1/\sqrt{e^{\tilde{q}}\, d(2i\lambda)} = 1/\sqrt{\cos\sqrt{2i\lambda}}, \quad \lambda \in \mathbf{R}.$$

Another interesting example is Lévy area for the two dimensional Wiener processes (w_1, w_2) starting at the origin:

$$S[w] = \int_0^1 w_1 dw_2 - w_2 dw_1.$$

The eigenvalue problem is well-known as follows:

$$\begin{pmatrix} \varphi_1' \\ \varphi_2' \end{pmatrix} + \frac{1}{\mu} \begin{pmatrix} \varphi_2 - \varphi_2(1)/2 \\ -\varphi_1 + \varphi_1(1)/2 \end{pmatrix} = \mathbf{0}, \quad \begin{pmatrix} \varphi_1(0) \\ \varphi_2(0) \end{pmatrix} = \mathbf{0},$$

We can trace the computation above to get the determinant associated with the boundary conditions.

$$d(\nu) = (e^{i\nu} + e^{-i\nu})^2 = 4\cos^2\nu, \quad (\nu = 1/\mu).$$

Note that this is already an entire function (so, we choose $q(z) = z$ simply) . Applying our theorem, we recover the familiar result:

$$E[e^{i\lambda S[w]}] = 1/\cosh\lambda, \quad \lambda \in \mathbf{R}.$$

Now we apply our framework to a new example. This is also a generalization of the first case above.
Again, let $w(t)$ be the one dimensional Wiener process starting at the origin. Let us consider the Gaussian processes $X_N(t)$ defined by

$$X_0(t) = w(t), \qquad X_N(t) = \int_0^t X_{N-1}(s)ds, \quad (N = 1, 2, \ldots).$$

N. Ikeda, S. Kusuoka, and S. Manabe [4] gave Lévy area formula for the two dimensional version of the processes $X_N(t)$.
We will calculate the characteristic function

$$C(z) = E\left[e^{zF_N(w)}\right],$$

where $F_N(w)$ is the following quadratic functional:

$$F_N(w) = (-1)^N \int_0^1 |X_N(t,w)|^2 dt.$$

The corresponding eigenvalue problem is

$$\int_0^1 M(t,s)\phi(t)dt = \mu\phi(s)$$

with

$$M(t,s) = \frac{1}{(N!)^2} \int_0^{\min(t,s)} (t-u)^N (s-u)^N du.$$

The kernel $M(t,s)$ is symmetric and the integral operator is Hilbert-Schmidt type. The equivalent differential equation is

$$\phi^{(2N+2)}(t) = \nu\phi(t) \qquad (\nu = 1/\mu),$$

with the boundary conditions:

$$\begin{array}{rcl}\phi(0)=0, \quad \phi^{(n)}(0) &=& 0 \quad (n=1,2,\dots,N),\\ \phi^{(n)}(1) &=& 0 \quad (n=N+1,N+2,\dots,2N+1).\end{array}$$

The solution to the equation above can be expressed by a linear combination of bases as follows:

$$\phi(t)=\sum_{k=0}^{2N+1} c_k e^{\xi_k t},$$

where ξ_k is

$$\xi_k=\omega_k\nu^{1/(2N+2)}, \quad \omega_k=\exp\left(\frac{2\pi k}{2N+2}i\right), \quad k=0,1,\dots,2N+1,$$

and $\lambda^{1/(2N+2)}$ is chosen in a suitable branch. Then we can express the boundary conditions by the linear equation as follows:

$$\begin{pmatrix} 1 & 1 & \dots & 1\\ \xi_0 & \xi_1 & \dots & \xi_{2N+1}\\ \vdots & \vdots & & \vdots\\ \xi_0^N & \xi_1^N & \dots & \xi_{2N+1}^N\\ \xi_0^{N+1}e^{\xi_0} & \xi_1^{N+1}e^{\xi_1} & \dots & \xi_{2N+1}^{N+1}e^{\xi_{2N+1}}\\ \vdots & \vdots & & \vdots\\ \xi_0^{2N+1}e^{\xi_0} & \xi_1^{2N+1}e^{\xi_1} & \dots & \xi_{2N+1}^{2N+1}e^{\xi_{2N+1}} \end{pmatrix}\begin{pmatrix} c_0\\ c_1\\ \vdots\\ c_{N-1}\\ c_N\\ \vdots\\ c_{2N+1}\end{pmatrix}=\mathbf{0}.$$

The eigenvalues are $\{\nu_j\}$ that give the non-trivial solutions of the equation above, that is, the zeros of the determinant of the matrix above. Then, setting $q(z)=z^{2N+2}$, we have $q(\nu^{1/(2N+2)})=\nu$ and the equivalent equation as follows:

$$d(\nu)=\det\Xi(q(\nu))=\det\begin{pmatrix} 1 & 1 & \dots & 1\\ \omega_0 & \omega_1 & \dots & \omega_{2N+1}\\ \vdots & \vdots & & \vdots\\ \omega_0^N & \omega_1^N & \dots & \omega_{2N+1}^N\\ \omega_0^{N+1}e^{\nu\omega_0} & \omega_1^{N+1}e^{\nu\omega_1} & \dots & \omega_{2N+1}^{N+1}e^{\nu\omega_{2N+1}}\\ \vdots & \vdots & & \vdots\\ \omega_0^{2N+1}e^{\nu\omega_0} & \omega_1^{2N+1}e^{\nu\omega_1} & \dots & \omega_{2N+1}^{2N+1}e^{\nu\omega_{2N+1}} \end{pmatrix}=0.$$

We can express $d(z)$ explicitly as follows.

Lemma 3.1 *Let the index set $I=\{0,1,\dots,2N+1\}$ be partitioned into two sets Γ and Γ^c such that $\Gamma\cup\Gamma^c=I$ and $\sharp\Gamma=\sharp\Gamma^c=N+1$. Let $|\Gamma|$ be the signature of the permutation such that maps*

$$(\gamma_0,\gamma_1,\dots,\gamma_N,\gamma_{N+1},\dots,\gamma_{2N+1}) \quad \textit{to} \quad (0,1,\dots,2N,2N+1)$$

where $\gamma_j\in\Gamma$ for $0\le j\le N$ and $\gamma_j\in\Gamma^c$ for $N+1\le j\le 2N+1$. Then,

$$d(z)=\sum_{\Gamma}\left\{(-1)^{|\Gamma|}\exp\left(z\sum_{k\in\Gamma}\omega_k\right)\prod_{l\in\Gamma}\omega_l^{N+1}\mathbf{\Pi}(\Gamma)\mathbf{\Pi}(\Gamma^c)\right\},$$

where

$$\mathbf{\Pi}(\Gamma) = \prod_{j,k\in\Gamma,k>j} (\omega_k - \omega_j),$$

and the first summation runs over all choices Γ.
In the special case,

$$d(0) = \prod_{0\le j<k\le 2N+2} (\omega_k - \omega_j).$$

Proof. By the definition of the determinant, we can express $d(z)$ in the following linear combination:

$$d(z) = \det \Xi(z) = \sum_{\Gamma} A_\Gamma \exp\left(z \sum_{k\in\Gamma} \omega_k \right).$$

We can reduce the constant A_Γ to the product of two Vandermond's type determinants by the suitable permutation of the columns. The special case $d(0)$ is just a Vandermond's determinant. q.e.d.

By the lemma above, we can check our assumptions on the asymptotics on $d(z)$. Then we have the formula for the Carleman-Fredholm determinant $D(z)$ of $C(z)$ in this case.

Corollary 3.1

$$D(z^{2N+2}) = e^{G(z)} d((2z)^{2N+2}),$$

where $G(z)$ *is a polynomial whose degree is* $2N+2$ *at most.*

We can calculate the explicit form of the polynomial $G(z) = \sum_{k=0}^{2N+2} a_k z^k$ by the special values of $D(z), d(z)$ and their derivatives. In fact, the constant term a_0 is

$$a_0 = -\log \prod_{1\le j<k\le 2N+2} (\omega_k - \omega_j),$$

because $D(0) = e^a d(0)$. By Leibnitz's rule, we have also

$$a_1 = a_2 = \cdots = a_{2N+1} = 0$$

and

$$a_{2N+2} = N!(D'(0) - d'(0)/d(0)).$$

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