DOI: 10.1214/ECP.v8-1096

Elect. Comm. in Probab. **8** (2003) 155–166

ELECTRONIC COMMUNICATIONS in PROBABILITY

PATH TRANSFORMATIONS OF FIRST PASSAGE BRIDGES

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Submitted 4 July 2003 , *accepted in final form* 5 November 2003

AMS 2000 Subject classification: 60J65 (60G09 60G17)
Keywords: first passage bridge, path transformation, time of the minimum, uniformly distributed time, exchangeable increments

Abstract
We define the first passage bridge from 0 to λ as the Brownian motion on the time interval [0,1] conditioned to first hit λ at time 1. We show that this process may be related to the Brownian bridge, the Bessel bridge or the Brownian excursion via some path transformations, the main one being an extension of Vervaat's transformation. We also propose an extension of these results to certain bridges with cyclically exchangeable increments.

1 Introduction

Let $B = (B(t), t \geq 0)$ denote a standard real Brownian motion started from $B(0) = 0$ and $T_x = \inf\{t : B(t) > x\}$ the first passage time above level $x \geq 0$. Then for $\ell > 0$ and $x \neq 0$, the process

$$(B(t), 0 \leq t \leq \ell) \quad \text{conditioned on } T_x = \ell$$

may be called the *Brownian first passage bridge of length ℓ from* 0 *to* x.
It is well known in the Brownian folklore (cf. e.g. [5]) that such first passage bridges may be represented in terms of the bridges of a 3-dimensional Bessel process. More precisely, for each

$\lambda > 0$ there is the identity in distribution

$$\begin{aligned} & (B(t), 0 \le t \le \ell \mid T_\lambda = \ell) \\ \stackrel{d}{=} \quad & (\lambda - BES_3(t), 0 \le t \le \ell \mid BES_3(0) = \lambda, BES_3(\ell) = 0) \end{aligned}$$

where BES_3 stands for a 3-dimensional Bessel process. But this representation obscures a number of fundamental properties of Brownian first passage bridges which follow directly from their interpretation in terms of a one-dimensional Brownian motion.
The point of this note is to record some of these properties and to suggest that Brownian first passage bridges should be regarded as basic processes from which other more complex Brownian processes can be derived by simple path transformations.
We begin with a section devoted to the discrete time analysis of the random walk case, which is based on elementary combinatorial principles. This leads in Section 3 to an extension of Vervaat's transformation for Brownian first passage bridges and other related identities in distribution. In a fourth part, we shall indicate extensions of these properties to a large class of bridges with exchangeable increments.

2 The main result in discrete time

Fix two integers λ and n such that $1 \le \lambda \le n$. Let $S = (S_i)_{0 \le i \le n}$ be a random chain such that $S_0 = 0$, $S_n = \lambda$. Suppose moreover that the increments, $\Delta S_i = S_i - S_{i-1}$, $i = 1, \dots, n$ take their values in the set $\{-1, +1\}$ and are cyclically exchangeable, that is, for any $k = 1, \dots, n$,

$$(\Delta S_1, \dots, \Delta S_n) \stackrel{d}{=} (\Delta S_{k+1}, \dots, \Delta S_n, \Delta S_1, \dots, \Delta S_k)\,.$$

Note that the latter property is equivalent to the fact that for any $k = 0, 1, \dots, n-1$, the shifted chain:

$$\theta_k(S)_i = \left\{ \begin{array}{ll} S_{i+k} - S_k, & \text{if } i \le n-k \\ S_{k+i-n} + S_n - S_k, & \text{if } n-k \le i \le n \end{array} \right. , \quad i = 0, 1, \dots, n\,.$$

has the same law as S.
A fundamental example of such chain is provided by the simple random walk conditioned to be equal to λ at time n.
For $k = 0, 1, \dots, \lambda - 1$, define the first time at which S reaches its maximum minus k as follows:

$$m_k(S) = \inf\{i : S_i = \max_{0 \le j \le n} S_j - k\}\,.$$

When no confusion is possible, we denote $m_k(S) = m_k$.

Theorem 1 *Let ν be a random variable which is independent of S and uniformly distributed on $\{0, 1, \dots, \lambda - 1\}$. The chain $\theta_{m_\nu}(S)$ has the law of S conditioned by the event $\{m_0 = n\}$. Moreover, the index m_ν is uniformly distributed on $\{0, 1, \dots, n-1\}$ and independent of $\theta_{m_\nu}(S)$.*

The proof of Theorem 1 relies on a simple combinatorial argument. In this direction, denote by Λ the support of the law of S. In particular, Λ is a subset of

$$\{(s_0, \dots, s_n) \in \mathbb{R}^{n+1} : s_0 = 0,\ s_n = \lambda,\ \Delta s_i \in \{-1 + 1\},\ i = 1, \dots, n\}\,.$$

Lemma 2 *For every $s \in \Lambda$, define the set*

$$\Lambda(s) = \{s, \theta_1(s), \dots, \theta_{n-1}(s)\}\,.$$

Then for any $s \in \Lambda$, the set $\overline{\Lambda}(s)$ of the paths in $\Lambda(s)$ which first hit their maximum at time n, i.e. $\overline{\Lambda}(s) = \{x \in \Lambda(s) : m_0(x) = n\}$, contains exactly λ elements and may be represented as

$$\overline{\Lambda}(s) = \{\theta_{m_0}(s), \theta_{m_1}(s), \dots, \theta_{m_{\lambda-1}}(s)\}\,. \tag{1}$$

Proof: We can easily see from a picture that for any $k = 0, 1 \dots, \lambda - 1$, the path $\theta_{m_k}(s)$ is contained in $\overline{\Lambda}(s)$. To obtain the other inclusion, it is enough to observe that if $i \neq m_k$, for $k = 0, 1, \dots, \lambda - 1$, then the maximum of $\theta_i(s)$ is reached before time n. ■

Remark. Lemma 2 is closely related to a combinatorial lemma in Feller [10] XII.6, p.412, also known as the ballot Theorem, see [9]. Here we complete Feller's result by associating a path transformation to the combinatorial result. Note also that it may be extended to any chain with exchangeable increments, see [2].

Proof of Theorem 1. For every bounded function f defined on $\{0, 1, \dots, n-1\}$ and every bounded function F defined on $\mathbf{Z}^{n+1}$, we have

$$E\left(F(\theta_{m_\nu}(S))f(m_\nu)\right) = \sum_{s \in \Lambda} P(S = s)\frac{1}{\lambda}\sum_{j=0}^{\lambda-1} F(\theta_{m_j}(s))f(m_j)\,.$$

But Lemma 2 allows us to write for any $s \in \Lambda$,

$$\sum_{j=0}^{\lambda-1} F(\theta_{m_j}(s))f(m_j) = \sum_{k=0}^{n-1} F(\theta_k(s))f(k)\mathbb{1}_{\{m_0(\theta_k(s))=n\}}\,,$$

so that

$$\begin{aligned} E\left(F(\theta_{m_\nu}(S))f(m_\nu)\right) &= \sum_{s \in \Lambda} P(S = s)\frac{1}{\lambda}\sum_{k=0}^{n-1} F(\theta_k(s))f(k)\mathbb{1}_{\{m_0(\theta_k(s))=n\}} \\ &= \frac{n}{\lambda}E\left(F(\theta_U(S))f(U)\mathbb{1}_{\{m_0(\theta_U(S))=n\}}\right)\,, \end{aligned}$$

where U is uniform on $\{0, 1, \dots, n-1\}$ and independent of S. Finally, it follows from the cyclic exchangeability that the chain $\theta_U(S)$ has the same law as S and is independent of U, hence we have:

$$E\left(F(\theta_{m_\nu}(S))f(m_\nu)\right) = E(F(S) \,|\, m_0(S) = n)E(f(U))\,,$$

which proves our result. ■

The following transformation could be viewed as the converse of that in Theorem 1; however, it is actually a sightly weaker result.

Corollary 3 *Let U be uniformly distributed on $\{0, 1, \dots, n-1\}$ and independent of S. Conditionally on the event $\{m_0(S) = n\}$, the chain $\theta_U(S)$ has the same law as S.*

3 Main results in the Brownian setting

By obvious Brownian scaling, it is enough to discuss bridges of unit length. So let

$$\left(F_x^{br}(t), 0 \le t \le 1\right) \stackrel{d}{=} \left(B(t), 0 \le t \le 1 \mid T_x = 1\right).$$

We refer to P. Lévy [16], Theorem 42.5 for a proper definition of this conditioning. The following fact is fundamental, and obvious by either random walk approximation or Brownian excursion theory. Write for an arbitrary real-valued process $(X(t), t \ge 0)$ and $y \ge 0$

$$T_y(X) = \inf\{t : X(t) > y\} \quad \text{and} \quad \overline{X}_t = \sup_{s \le t} X_s .$$

Lemma 4 *For each fixed $\lambda > 0$, the first passage process*

$$\left(T_a(F_\lambda^{br}), 0 \le a \le \lambda\right)$$

is a process with exchangeable increments, distributed as

$$\left(T_a(B), 0 \le a \le \lambda \mid T_\lambda(B) = 1\right),$$

where $(T_a(B), a \ge 0)$ is a stable$(1/2)$ subordinator; more precisely

$$\mathbf{E}\left(\exp(-\alpha T_a(B))\right) = \exp(-a\sqrt{2\alpha}), \qquad \alpha \ge 0 .$$

Moreover, conditionally given the process $(T_a(F_\lambda^{br}), 0 \le a \le \lambda)$, or, what is the same, given the past supremum process $\left(\overline{F}_\lambda^{br}(t), 0 \le t \le 1\right)$ of F_λ^{br}, the excursions of $\overline{F}_\lambda^{br} - F_\lambda^{br}$ away from 0 are independent Brownian excursions whose lengths are the lengths of the flat stretches of $\overline{F}_\lambda^{br}$ and correspond to the jumps of the first passage process $\left(T_a(F_\lambda^{br}), 0 \le a \le \lambda\right)$.

The next statement follows immediately from Lemma 4.

Proposition 5 *Let*

$$R_\lambda^{br}(t) := \overline{F}_\lambda^{br}(t) - F_\lambda^{br}(t), \qquad 0 \le t \le 1 .$$

Then

$$\left(R_\lambda^{br}(t), 0 \le t \le 1\right) \stackrel{d}{=} \left(|B^{br}(t)|, 0 \le t \le 1 \mid L_1^0(B^{br}) = \lambda\right)$$

where B^{br} is a standard Brownian bridge and $(L_t^0(B^{br}), 0 \le t \le 1)$ is the usual process of local times of B^{br} at level 0. Indeed, the above holds jointly with

$$\left(\overline{F}_\lambda^{br}(t), 0 \le t \le 1\right) \stackrel{d}{=} \left(L_t^0(B^{br}), 0 \le t \le 1 \mid L_1^0(B^{br}) = \lambda\right).$$

We point out that Proposition 5 can also be deduced from different well known path transformations. For instance, the transformation between $|B^{br}|$ and the Brownian meander B^{me} due to Biane and Yor [5]. Specifically, we know that if we define $B^{me} = (|B^{br}(s)| + L_s^0(B^{br}),\ 0 \le s \le 1)$, then B^{me} is a Brownian meander with $\min_{t \le s \le 1} B^{me}(s) = L_t^0(B^{br})$. We may then use the remark of Imhof [13] that B^{me} given $B^{me}(1) = \lambda$ is a 3-dimensional Bessel bridge from 0 to λ, so $(\lambda - B^{me}(1-s),\ 0 \le s \le 1)$ given $B^{me}(1) = \lambda$ is a copy of F_λ^{br}. The fundamental result of Lévy: $(\overline{B} - B, \overline{B}) \stackrel{d}{=} (|B|, L^B)$, where L^B is the local time at 0 of B yields also Proposition 5.

The reflecting Brownian motion conditioned on its local time at level 0 up to time 1, and various transformations of it, have turned out to be of interest in a number of studies. See Lemma 46 and the discussion below Definition 109 in [19], and the references cited in [19]. Proposition 5 allows a number of results obtained by Chassaing and Janson in [6] to be reformulated in a simple way with the Brownian first passage bridge regarded as the fundamental process from which other similar processes can be constructed.
For convenience in formulation of these results, the following notation is useful. First, just as in the discrete case, for $\omega \in \mathcal{C}([0,1])$ such that $\omega(0)=0$ and $u\in[0,1]$, let $\theta_u\circ\omega$ be the path starting at 0 whose increments are those of ω with a *cyclic shift* by u. That is

$$\theta_u\circ\omega(t) \;=\; \begin{cases} \omega(t+u)-\omega(u) & \text{if } t+u\le 1\,,\\ \omega(t+u-1)+\omega(1)-\omega(u) & \text{if } t+u\ge 1\,. \end{cases}$$

Observe the identity

$$\theta_u\circ\theta_v \;=\; \theta_w\,, \qquad \text{where } w=u+v \,\mathrm{mod}[1].$$

Second, for $\omega\in\mathcal{C}([0,1])$ and $\lambda>0$, let

$$\delta_\lambda\circ\omega(t) \;=\; \lambda t-\omega(t)\,, \qquad t\in[0,1]\,,$$

so that $\delta_\lambda\circ\omega$ is the path $-\omega$ *dragged up with drift* λ. Note that the operator δ_λ is an involution, i.e.

$$\delta_\lambda\circ\delta_\lambda\circ\omega \;=\; \omega\,,$$

and that θ_u and δ_λ commute,

$$\theta_u\circ\delta_\lambda \;=\; \delta_\lambda\circ\theta_u\,.$$

The first proposition, just to illustrate the notation, is a weaker form of Lemma 4, with the assertion of exchangeable increments weakened to cyclic exchangeability of increments.

Proposition 6 *Let $T_a:=T_a(F_\lambda^{br})$, $0\le a\le\lambda$. Then for each fixed a,*

$$\theta_{T_a}\circ F_\lambda^{br} \;\stackrel{d}{=}\; F_\lambda^{br}\,.$$

Next, we present the analogs for continuous time of Theorem 1 and Corollary 3. We denote by B_λ^{br} the Brownian bridge from 0 to λ and recall its classical construction:

$$B_\lambda^{br}\stackrel{d}{=}\delta_\lambda\circ B^{br}\,.$$

Theorem 7 *Let ν be uniformly distributed over $[0,\lambda]$ and independent of B_λ^{br}. Define the r.v. $U=\inf\{t: B_\lambda^{br}(t)=\sup_{0\le s\le 1}B_\lambda^{br}(s)-\nu\}$. Then the process $\theta_U(B_\lambda^{br})$ has the law of the first passage bridge F_λ^{br}. Moreover, U is uniformly distributed over $[0,1]$ and independent of $\theta_U(B_\lambda^{br})$.*

The following consequence of Theorem 7 may be interpreted as a converse of the transformation although the latter is obviously not invertible.

Corollary 8 *If U is uniformly distributed over $[0,1]$ and independent of F_λ^{br}, then the process $\theta_U(F_\lambda^{br})$ is a Brownian bridge from 0 to λ.*

The proof of Theorem 7 follows naturally from the discrete time result treated at the previous section and some conditioned versions of Donsker's invariance principle due to Liggett [17] and Iglehart [12] which we first recall. Let $(S_n)_{n\ge 0}$ be the simple random walk, that is $S_0=0$ and $S_n=\sum_{k=1}^n \xi_k$, where ξ_k, $k\ge 1$ are i.i.d. symmetric, $\{+1,-1\}$ valued random variables. It follows, as a particular case of Theorem 4 in [17], that the càdlàg process $(n^{-1/2}S_{[nt]},\, 0\le t\le 1)$ conditioned on the event $\{S_n=[n^{1/2}\lambda]\}$ converges weakly towards the Brownian bridge B_λ^{br}, in the space $\mathcal{D}([0,1])$ of càdlàg functions. For convenience in the sequel, we will deal with the continuous process $(n^{-1/2}(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}),\, 0\le t\le 1)$ rather than with $(n^{-1/2}S_{[nt]},\, 0\le t\le 1)$. The following lemma is straightforward from Liggett's convergence result.

Lemma 9 *The process*

$$\begin{aligned}&(B_\lambda^{(n)}(t),\, 0\le t\le 1)\\ \stackrel{(def)}{=}\;&(n^{-1/2}(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}),\, 0\le t\le 1 \,|\, S_n=[n^{1/2}\lambda])\,,\end{aligned}$$

converges weakly, as n goes to ∞, towards the Brownian bridge B_λ^{br}, in the space $\mathcal{C}([0,1])$ of continuous functions.

The next step is to define a "discrete" first passage bridge just as we defined the discrete bridge $B_\lambda^{(n)}$, and to check that it converges weakly towards F_λ^{br}. We did not find such result in the literature. However, with $m_0=\inf\{i: S_i=\sup_{0\le j\le n} S_j\}$, an invariance principle result due to Iglehart [12] asserts that (up to time reversal), the conditioned process

$$(n^{-1/2}(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}),\, 0\le t\le 1 \,|\, m_0=n)\,,$$

converges weakly towards $(B_1^{me}-B_{1-t}^{me}\,, 0\le t\le 1)$, where we recall that $(B_t^{me},\, 0\le t\le 1)$ is the Brownian meander. From the discussion after Proposition 5 and the introduction of this paper, it follows that conditionally on $\{B_1^{me}=\lambda\}$, the process $(B_1^{me}-B_{1-t}^{me}\,, 0\le t\le 1)$ has the law of the first passage bridge F_λ^{br}.
The next lemma is obtained by following the same arguments as in the proofs of Theorems (2.23) and (3.4) in Iglehart [12].

Lemma 10 *The process*

$$\begin{aligned}&(F_\lambda^{(n)}(t),\, 0\le t\le 1)\\ \stackrel{(def)}{=}\;&(n^{-1/2}(S_{[nt]}+(nt-[nt])\xi_{[nt]+1}),\, 0\le t\le 1 \,|\, S_n=[n^{1/2}\lambda], m_0=n)\,,\end{aligned}$$

converges weakly, as n goes to ∞, towards the first passage bridge F_λ^{br}, in the space $\mathcal{C}([0,1])$ of continuous functions.

We are now able to end the proof of Theorem 7.
Proof of Theorem 7: Fix an integer $n\ge 1$. Let (ν_i) be a sequence of r.v.'s which is independent of (S_i) and such that each ν_i is uniformly distributed on $\{0,1,\dots,[i^{1/2}\lambda]-1\}$. From Theorem 1, if $U_n=\inf\{t: B_\lambda^{(n)}(t)=\sup_{0\le s\le 1} B_\lambda^{(n)}(s)-n^{-1/2}\nu_n\}$, then the process $\theta_{U_n}(B_\lambda^{(n)})$ has the same law as $F_\lambda^{(n)}$. Moreover, U_n is uniformly distributed on $\left\{0,\frac{1}{n},\dots,1-\frac{1}{n}\right\}$ and independent of $\theta_{U_n}(B_\lambda^{(n)})$. Then Lemmas 9 and 10 enable us to conclude. ∎

Note that the weak limit of the law of F_λ^{br} as $\lambda \to 0+$ is that of $-B^{ex}$, where B^{ex} stands for a standard Brownian excursion (i.e. a bridge with unit length from 0 to 0 of a 3-dimensional Bessel process). Thus we also have as a corollary:

Corollary 11 (Vervaat [23], Imhof [14], Biane [4]) *Let U be uniform $[0,1]$ and independent of B^{ex}. Then*

$$\theta_U \circ B^{ex} \stackrel{d}{=} B^{br}\,.$$

Moreover, B^{ex} can be recovered from $B^{br} := \theta_U \circ B^{ex}$ as $B^{ex} = \theta_m \circ B^{br}$ where m is the a.s. unique time at which B^{br} attains its minimum, and then $m = 1-U$ is independent of B^{ex}.

Since for any $u \in [0,1]$, the shift θ_u preserves the amplitude of the trajectories, we deduce from Theorem 7 that the maximum of the three dimensional Bessel bridge (or equivalently, the amplitude of the first passage bridge) has the same law as the amplitude of the Brownian bridge. The latter observation, as well as the following proposition, may be deduced from general calculations for the law of the maximum of diffusion bridges, which may be found for instance in Kiefer [15] or Pitman and Yor [20], [21].
In [8], Durret and Iglehart obtained the joint law of the maximum and the terminal value of the Brownian meander:

$$P\left(\sup_{0\le t\le 1} B_t^{me} \le x,\, B_1^{me} \le \lambda\right) = \sum_{k=-\infty}^{\infty}\left(e^{-(2kx)^2/2} - e^{-(2kx+\lambda)^2/2}\right),\;\; 0<\lambda\le x\,.$$

Taking the derivative in the variable λ, we obtain the distribution of the maximum of the Brownian meander conditionally on its terminal value which is a 3-dimensional Bessel bridge, according to our discussion after Proposition 5. From above and Theorem 7, we now deduce the following.

Proposition 12 *For $0<\lambda\le x$:*

$$\begin{aligned} P\left(\inf_{0\le t\le 1} F_\lambda^{br}(t) \ge \lambda - x\right) &= P\left(\sup_{0\le t\le 1} B_\lambda^{br}(t) - \inf_{0\le t\le 1} B_\lambda^{br}(t) \le x\right) \\ &= \frac{e^{\lambda^2/2}}{\lambda}\sum_{k=-\infty}^{\infty}(2kx+\lambda)e^{-(2kx+\lambda)^2/2}\,. \end{aligned}$$

Letting λ tend to 0 in the above expression, we find the law of the maximum of the normalized Brownian excursion

$$P\left(\sup_{0\le t\le 1} B^{ex}(t) \le x\right) = 1 + 2\sum_{k=1}^{\infty}[1-(2kx)^2]e^{-(2kx)^2/2}$$

which is mentioned to be the same as the law of the amplitude of the Brownian bridge in Vervaat [23] and in earlier works.
Another easy consequence of Theorem 7 is:

Corollary 13 *Let μ be the a.s. unique instant when the process $\delta_\lambda \circ F_\lambda^{br}$ attains its minimum. Then*

$$\theta_\mu \circ \delta_\lambda \circ F_\lambda^{br} \stackrel{d}{=} B^{ex}\,,$$

and consequently

$$\theta_\mu \circ F_\lambda^{br} \stackrel{d}{=} \delta_\lambda \circ B^{ex}\,.$$

Proof: Defined $B^{br} = \theta_U \circ \delta_\lambda \circ F_\lambda^{br}$, where U is independent of F_λ^{br}. As the operators θ and δ commute and δ is an involution, we have $\theta_U \circ F_\lambda^{br} = \delta_\lambda \circ B^{br}$, and we know from Corollary 8 that B^{br} is a standard Brownian bridge.
On the other hand, if m (respectively, μ) is the instant when B^{br} (respectively, $\delta_\lambda \circ F_\lambda^{br}$) attains its minimum, then it is clear pathwise that

$$m + U \;=\; \mu \ \mathrm{mod}[1]\,,$$

and therefore

$$\theta_\mu \circ \delta_\lambda \circ F_\lambda^{br} \;=\; \theta_m \circ \theta_U \circ \delta_\lambda \circ F_\lambda^{br} \;=\; \theta_m \circ B^{br}.$$

We conclude applying Vervaat's identity (cf. Corollary 11) which ensures that the right-hand side above is distributed as B^{ex}. ∎

Next, recall the construction of the reflecting Brownian motion conditioned on its local time which is given in Proposition 5. A variation of arguments used above enable us to recover the collection of results obtained by Chassaing and Janson [6] via consideration of discrete approximation by parking schemes. As a typical example, we focus on Theorem 2.6 (i) in [6] which we now re-state.

Corollary 14 (Chassaing and Janson [6]) *Set* $B_\lambda^{ex} = \delta_\lambda \circ B^{ex}$, $\overline{B}_\lambda^{ex}(t) = \max_{0\le s\le t} B_\lambda^{ex}(s)$, *and*

$$R_\lambda^{ex}(t) \;:=\; \overline{B}_\lambda^{ex}(t) - B_\lambda^{ex}(t)\,, \qquad t \in [0,1]\,.$$

Next, in the notation of Proposition 5, let $(\ell(t), 0 \le t \le 1)$ *be the process of the local times at level* 0 *of* R_λ^{br}*, normalized so that* $\ell(1) = \lambda$*. Finally, let* μ *be the a.s. unique time at which* $\delta_\lambda \circ \ell$ *attains its minimum. Then,*

$$\theta_\mu \circ R_\lambda^{br} \;\stackrel{d}{=}\; R_\lambda^{ex}\,.$$

Proof: By Proposition 5, $R_\lambda^{br} = \overline{F}_\lambda^{br} - F_\lambda^{br}$ and $\ell = \overline{F}_\lambda^{br}$. A variation of the classical argument of Skorohod (see e.g. Lemma 2 in [3]) shows the identity

$$\min_{0\le s\le t}(\lambda s - \overline{F}_\lambda^{br}(s)) \;=\; \min_{0\le s\le t}(\lambda s - F_\lambda^{br}(s))\,, \qquad \text{for every } t \in [0,1]\,.$$

This entails that the instant of the minimum of $\delta_\lambda \circ \overline{F}_\lambda^{br}$ coincides with the instant of the minimum of $\delta_\lambda \circ F_\lambda^{br}$, and thus the notation μ for this instant is coherent with Corollary 13. Now we have

$$\theta_\mu \circ R_\lambda^{br} \;=\; \theta_\mu \circ \overline{F}_\lambda^{br} - \theta_\mu \circ F_\lambda^{br}\,. \tag{2}$$

Because μ is an instant at which $\overline{F}_\lambda^{br}$ increases, we have $R_\lambda^{br}(\mu) = 0$, and thus the left-hand side in (2) is always nonnegative. On the other hand, the process $\overline{F}_\lambda^{br}$ is increasing, and increases only at instants t such that $\overline{F}_\lambda^{br}(t) = F_\lambda^{br}(t)$. By cyclic permutation, $\theta_\mu \circ \overline{F}_\lambda^{br}$ is an increasing process that increases only when $\theta_\mu \circ \overline{F}_\lambda^{br} = \theta_\mu \circ F_\lambda^{br}$. By Skorohod Lemma, we see that

$$\theta_\mu \circ \overline{F}_\lambda^{br}(t) \;=\; \max_{0\le s\le t} \theta_\mu \circ F_\lambda^{br}(s)\,.$$

An application of Corollary 13 completes the proof of our claim. ∎

Remark. Assuming Proposition 5, Corollary 13 could also be derived from Corollary 14. As another example of application, we point out the following identity in distribution which appeared in [3].

Corollary 15 (Bertoin [3]) *The laws of ranked lengths of excursions of* R_λ^{br} *and* R_λ^{ex} *away from* 0 *are identical.*

Proof: This is immediate from Corollary 14. ■

This law of ranked lengths of excursions, and the associated laws of partitions of n for $n = 1, 2, \ldots$ derived by random sampling from the lengths, are described explicitly in Aldous and Pitman [1] and Pitman [19]; see especially Section 4.5 in [19]. Other proofs of Corollary 15 have been given by Schweinsberg [22] and Miermont [18]. Their analysis gives the deeper result that the length of the first excursion of R_λ^{ex} is a size-biased pick from the ranked lengths of excursions of R_λ^{ex}.

4 Extension to certain bridges with cyclically exchangeable increments

In this section, we indicate a generalization of the previous results in the Brownian setting to a large class of processes in continuous time with cyclically exchangeable increments. We start by introducing some canonical notation.
Let $\lambda > 0$ be fixed and Ω denote the space of càdlàg paths $\omega : [0,1] \to \mathbf{R}$ such that ω has no positive jumps, $\omega(0) = 0$ and $\omega(1) = \lambda$. In other words, Ω is the space of upwards skip-free bridges with unit length from 0 to 1. We shall use the standard notation $\overline{\omega}(t) = \sup_{[0,t]} \omega$ for the (continuous) supremum path.
For every $v, t \in [0,1]$, we set

$$\theta_v \circ \omega(t) \;=\; \begin{cases} \omega(t+v) - \omega(v) & \text{if } t+v \leq 1\,, \\ \omega(t+v-1) + \lambda - \omega(v) & \text{if } t+v \geq 1\,, \end{cases}$$

and denote by m_v the largest instant of the overall maximum of $\theta_v \circ \omega$, viz.

$$\begin{aligned} &\theta_v \circ \omega(m_v -) \geq \theta_v \circ \omega(t) \text{ for all } t \in [0,1] \\ \text{and} \quad &\theta_v \circ \omega(m_v -) > \theta_v \circ \omega(t-) \text{ for all } t \in]m_v, 1[. \end{aligned}$$

In particular, m_0 is the largest instant when ω reaches its maximum.
We consider a probability measure $\mathbf{P}$ on Ω for which the canonical process ω has cyclically exchangeable increments. Informally, we should like to define the law corresponding to first passage bridges, that is to conditioning ω to be maximal at time 1. In this direction, we recall that the process of first (upwards) passage times of ω is denoted by

$$T_x \;=\; T_x(\omega) \;:=\; \inf\{t \in [0,1] : \omega(t) > x\}\;, \qquad x > 0$$

(with the convention that $\inf \emptyset = \infty$). We introduce an independent uniform random variable U and set $Y = \overline{\omega}(1) - \lambda U$, so that conditionally on ω, Y is uniformly distributed on $[\overline{\omega}(1) -$

$\lambda, \overline{\omega}(1)]$. Finally, anticipating notational coherence, we define $\mathbf{P}(\cdot \mid m_0 = 1)$ as the distribution of $\theta_{T_Y} \circ \omega$ under $\mathbf{P}$, i.e.

$$\mathbf{P}(\cdot \mid m_0 = 1) := \int_0^1 \mathbf{P}(\theta_{T_{\overline{\omega}(1)-\lambda u}} \circ \omega \in \cdot) du \,.$$

Under a simple additional hypothesis, we are able to define rigorously the law of ω conditioned on reaching its overall maximum at time 1.

Proposition 16 *Introduce $\mathcal{L} := \{t : \omega(t) = \overline{\omega}(t)\}$, the ascending ladder time set of ω, and for every $\varepsilon > 0$, denote by*

$$\mathcal{L}_\varepsilon := ([0, \varepsilon] + \mathcal{L}) \cap [0, 1]$$

the ε-right neighborhood of $\mathcal{L}$ (in $[0,1]$). Suppose that there are real numbers $b_\varepsilon > 0$ such that for every $t \in [0,1]$,

$$\lim_{\varepsilon \to 0+} b_\varepsilon \int_0^t \mathbf{1}_{\{s \in \mathcal{L}_\varepsilon\}} ds = \overline{\omega}(t) \,, \qquad \textit{in } L^1(\mathbf{P}).$$

Then, when $\varepsilon \to 0+$, $\mathbf{P}(\cdot \mid m_0 \geq 1-\varepsilon)$ converges in the sense of finite dimensional distributions towards $\mathbf{P}(\cdot \mid m_0 = 1)$.

Proof: It is easily seen that the set of times $v \in [0,1]$ such that the path $\theta_v \circ \omega$ remains bounded from above by λ can be expressed as

$$\{v \in [0,1] : m_v = 1\} = \{v \in \mathcal{L} : \overline{\omega}(1) - \overline{\omega}(v) \leq \lambda\} \,.$$

Similarly, one readily checks that

$$\{v \in [0,1] : m_v \geq 1 - \varepsilon\} = \{v \in \mathcal{L}_\varepsilon : \overline{\omega}(1) - \overline{\omega}(v) \leq \lambda\} \,.$$

This enables us to provide the following simple representation for the law of ω conditioned on $m_0 \geq 1 - \varepsilon$. Specifically, introduce the probability measure $\mathbf{Q}_\varepsilon$ on $\Omega \times [0,1]$ given by

$$\mathbf{Q}_\varepsilon(d\omega, dv) = c_\varepsilon \mathbf{1}_{\{v \in \mathcal{L}_\varepsilon\}} \mathbf{1}_{\{\overline{\omega}(1) - \overline{\omega}(v) \leq \lambda\}} dv \mathbf{P}(d\omega) \,,$$

where $c_\varepsilon > 0$ stands for the normalizing constant. The cyclic exchangeability of the increments of ω obviously implies that the distribution of $\theta_v \circ \omega$ under $\mathbf{Q}_\varepsilon$ can be identified as the conditional law $\mathbf{P}(\cdot \mid m_0 \geq 1 - \varepsilon)$.
Now the hypothesis of Lemma 4 ensures that for every bounded measurable functional $F : \Omega \times [0,1] \to \mathbf{R}$ such that the random map $v \to F(\omega, v)$ is continuous at v for $d\overline{\omega}(\cdot)$ almost every v, $\mathbf{P}(d\omega)$-a.s., we have

$$\lim_{\varepsilon \to 0+} \mathbf{Q}_\varepsilon \left(F(\omega, v) \right) = \mathbf{Q} \left(F(\omega, v) \right) \,,$$

where

$$\mathbf{Q}(d\omega, dv) := \frac{1}{\lambda} \mathbf{1}_{\{\overline{\omega}(1) - \overline{\omega}(v) \leq \lambda\}} d\overline{\omega}(v) \, \mathbf{P}(d\omega) \,.$$

The statement follows. ■

Proposition 16 readily entails the generalization of results of Section 3. For instance, consider an independent uniform $[0,1]$ variable V. By definition, the distribution of $\theta_V \circ \omega$ under

$\mathbf{P}(\cdot \mid m_0 = 1)$ is the same as that of $\theta_V \circ \theta_{T_Y} \circ \omega$ under $\mathbf{P}$. Since $V + T_Y$ mod [1] is again uniformly distributed on $[0,1]$ and independent of ω, the latter is the same as the law of $\theta_V \circ \omega$ under $\mathbf{P}$, which is $\mathbf{P}$ by the cyclic exchangeability of the increment. In other words, we have the analog of Corollary 8. The analog of Proposition 6 can be derived by even simpler arguments.

Remark. We point out that the hypothesis of Proposition 16 is fulfilled in particular when $\mathbf{P}$ is the distribution of a bridge from 0 to λ of a Lévy process with no positive jumps. Indeed, recall that the supremum process $\overline{X}$ of a Lévy process with no positive jumps X can be viewed as the local time at 0 for the reflected process $\overline{X} - X$, which is Markovian. Using techniques of excursion theory as in Fristedt and Taylor [11], one readily checks that X fulfills the condition of Proposition 16, and thus the same still holds when conditioning on $X(1) = \lambda$ (at least for almost-every $\lambda > 0$ with respect to the law of $X(1)$; and the word 'almost' may even be dropped under some regularity condition on the latter).

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