DOI: 10.1214/ECP.v7-1048

Elect. Comm. in Probab. **7** (2002) 67–78

ELECTRONIC COMMUNICATIONS in PROBABILITY

A MONOTONICITY RESULT FOR HARD-CORE AND WIDOM–ROWLINSON MODELS ON CERTAIN D-DIMENSIONAL LATTICES

OLLE HÄGGSTRÖM[1]
Mathematical Statistics, Chalmers University of Technology and Göteborg University, 412 96 Göteborg, Sweden
email: `olleh@math.chalmers.se`

submitted June 20, 2001 *Final version accepted* February 2, 2002

AMS 2000 Subject classification: 60K35 82B20 82B26
Hard-core model, Widom-Rowlinson model, Gibbs measures, monotonic phase transition, site-random-cluster model

Abstract

For each $d \geq 2$, we give examples of d-dimensional periodic lattices on which the hard-core and Widom–Rowlinson models exhibit a phase transition which is monotonic, in the sense that there exists a critical value λ_c for the activity parameter λ, such that there is a unique Gibbs measure (resp. multiple Gibbs measures) whenever $\lambda < \lambda_c$ (resp. $\lambda > \lambda_c$). This contrasts with earlier examples of such lattices, where the phase transition failed to be monotonic. The case of the cubic lattice $\mathbf{Z}^d$ remains an open problem.

1 Introduction

This paper is concerned with Gibbs measures for hard-core and Widom–Rowlinson lattice gas models; we refer to Georgii, Häggström and Maes [2] for a gentle introduction to these models, and (unless otherwise indicated) for the known results quoted in this section.

In the hard-core lattice gas model, 0's and 1's are assigned randomly to the vertices of a graph G, in such a way that pairs of adjacent 1's do not occur. This is supposed to model a gas where particles have non-negligible radii and cannot overlap. When G is finite, the hard-core model arises by first letting each vertex independently take value 0 or 1 with probabilities $\frac{1}{\lambda+1}$ and $\frac{\lambda}{\lambda+1}$, where $\lambda > 0$ is the so-called activity parameter, and then conditioning on the event that no two vertices sharing an edge both take value 0. When the graph is infinite, the corresponding event to condition on has probability 0, so we instead apply the standard DLR (Dobrushin–Lanford–Ruelle) definition of infinite-volume Gibbs measures:

Definition 1.1 *Let $G = (V, E)$ be a finite or countably infinite locally finite graph, and fix $\lambda > 0$. A probability measure ν on $\{0, 1\}^V$ is said to be a* **Gibbs measure for the hard-core model on G at activity** λ*, if it admits conditional probabilities such that for all $v \in V$ and* *all $\xi \in \{0,1\}^{V \setminus \{v\}}$, a $\{0,1\}^V$-valued random object X with distribution ν satisfies*

$$\nu(X(v) = 1 \,|\, X(V \setminus \{v\}) = \xi) = \begin{cases} \frac{\lambda}{\lambda+1} & \textit{if } \xi(w) = 0 \textit{ for all } w \in V \textit{ with } \langle v, w\rangle \in E \\ 0 & \textit{otherwise.} \end{cases}$$

It is easy to see that the definition agrees with the description above in the case of finite graphs. For infinite graphs, it follows from standard compactness arguments that at least one Gibbs measure exists for a given G and a given λ. The obvious next question is the following: *For given G and λ, can there be more than one Gibbs measure?*

Consider the important special case $G = \mathbf{Z}^d$, $d \geq 2$, which we write as short for the graph with vertex set $\mathbf{Z}^d$ and edges connecting Euclidean nearest neighbors. For this graph, it is known that there exist constants $0 < \lambda_1 < \lambda_2 < \infty$ (depending on d), such that the hard-core model on $\mathbf{Z}^d$ at activity λ has

$$\begin{cases} \text{a unique Gibbs measure} & \text{if } \lambda < \lambda_1 \\ \text{multiple Gibbs measures} & \text{if } \lambda > \lambda_2. \end{cases} \tag{1}$$

Furthermore, there exist, in the Gibbs measure multiplicity region of the parameter space, two particular Gibbs measures ν^λ_{even} and ν^λ_{odd} which arise as perturbations of the even and odd "checkerboard patterns" (the even checkerboard pattern is obtained by placing 1's precisely at those vertices whose Cartesian coordinates sum to 0 mod 2, and similarly for the odd checkerboard pattern). From (1), it is tempting to conjecture the stronger statements that there exists a critical value λ_c (again depending on d) such that the hard-core model on $\mathbf{Z}^d$ has

$$\begin{cases} \text{a unique Gibbs measure} & \text{if } \lambda < \lambda_c \\ \text{multiple Gibbs measures} & \text{if } \lambda > \lambda_c. \end{cases} \tag{2}$$

However, no proof of the monotonicity statement contained in (2) − that multiple Gibbs measures at activity λ implies the same thing at all higher values of the activity − is known. In our first main result of this paper, we obtain the threshold behavior in (2), not for $\mathbf{Z}^d$, but for certain other lattices in d-dimensional Euclidean space.

Theorem 1.2 *For each $d \geq 2$, there exists a d-dimensional periodic lattice G, and a $\lambda_c > 0$, such that the hard-core model on G has*

$$\begin{cases} \textit{a unique Gibbs measure} & \textit{for all } \lambda < \lambda_c \\ \textit{multiple Gibbs measures} & \textit{for all } \lambda > \lambda_c. \end{cases} \tag{3}$$

For the precise definition of a "d-dimensional periodic lattice", see Definition 2.1. Intuitively, a d-dimensional periodic lattice is a transitive graph that is periodically embedded in $\mathbf{R}^d$; examples include the usual $\mathbf{Z}^d$ lattice, as well as the triangular and hexagonal lattices in $d = 2$. The examples we will work with are somewhat more involved.

A statement analogous to that in Theorem 1.2 has previously been obtained only for the hard-core model on regular trees, whose recursive structure allow exact calculation of λ_c; see Kelly [5]. Ours is the first example where the desired behavior is obtained for lattices that can be embedded in a nice way in Euclidean space, and is also the first example where (3) is obtained by more abstract arguments that do not involve calculating the critical value.

Let us now move on to the Widom–Rowlinson model. This is a lattice gas model where vertices take values in $\{-1, 0, 1\}$. For a finite graph, the model at activity λ arises by letting each vertex independently take value -1, 0 or $+1$ with respective probabilities $\frac{\lambda}{2\lambda+1}$, $\frac{1}{2\lambda+1}$, and $\frac{\lambda}{2\lambda+1}$, and then conditioning on the event that no -1 shares an edge with a $+1$ anywhere in the graph. We may think of $+1$'s and -1's as two types of particles that cannot coexist at close distance. The corresponding DLR definition is as follows.

Definition 1.3 *Fix $\lambda > 0$ and a finite or countably infinite locally finite graph $G = (V, E)$. A probability measure μ on $\{-1, 0, 1\}^V$ is said to be a* **Gibbs measure for the Widom–Rowlinson model on G at activity λ**, *if it admits conditional probabilities such that the following holds for all $v \in V$ and all $\xi \in \{-1, 0, 1\}^{V \setminus \{v\}}$. For a $\{-1, 0, 1\}^V$-valued random object X with distribution μ, the conditional distribution of $X(v)$ on $\{-1, 0, 1\}$, given that $X(V \setminus \{v\}) = \xi$, is*

$$\begin{cases} (0,1,0) & \text{if } \xi \in A^+, \xi \in A^- \\ (0, \frac{1}{\lambda+1}, \frac{\lambda}{\lambda+1}) & \text{if } \xi \in A^+, \xi \notin A^- \\ (\frac{\lambda}{\lambda+1}, \frac{1}{\lambda+1}, 0) & \text{if } \xi \notin A^+, \xi \in A^- \\ (\frac{\lambda}{2\lambda+1}, \frac{1}{2\lambda+1}, \frac{\lambda}{2\lambda+1}) & \text{if } \xi \notin A^+, \xi \notin A^- . \end{cases}$$

Here A^+ (resp. A^-) is the set of configurations in $\{-1, 0, 1\}^{V \setminus \{v\}}$ in which at least one neighbor of v in G take value $+1$ (resp. -1).

As for the hard-core model, the existence of some Gibbs measure for the Widom–Rowlinson model for given G and λ is standard, and the main question is whether or not it is unique. For $G = \mathbf{Z}^d$, $d \geq 2$, it is known that we have a unique Gibbs measure for λ sufficiently small but not for λ sufficiently large. In particular, for large λ, there exists a Gibbs measure μ_+^λ which is concentrated on the event that the limiting large-scale fraction of $+1$'s is strictly greater than that of -1's (thus breaking the ± 1 symmetry of the model), and an analogous Gibbs measure μ_-^λ in which the -1's form a majority over the $+1$'s. Again, it is natural to expect that the threshold phenomenon in (2) should hold, but just like for the hard-core model this has not been demonstrated for any other graphs than regular trees, for which the critical value λ_c has been calculated (see Wheeler and Widom [6]). We shall prove the following Widom–Rowlinson analogue of Theorem 1.2.

Theorem 1.4 *For each $d \geq 2$, there exists a d-dimensional periodic lattice G, and a $\lambda_c > 0$, such that the Widom–Rowlinson model on G has*

$$\begin{cases} \textit{a unique Gibbs measure} & \textit{for all } \lambda < \lambda_c \\ \textit{multiple Gibbs measures} & \textit{for all } \lambda > \lambda_c . \end{cases}$$

It is interesting to compare this result to the (somewhat surprising) result of Brightwell, Häggström and Winkler [1, p 428] that there are other d-dimensional periodic lattices for which having multiple Gibbs measures at some λ does *not* imply the same property for higher values of λ. Similar examples (contrasting Theorem 1.2) for the hard-core model are also easily obtained by the ideas reviewed in Section 2. Hence, different periodic lattices in d dimensions give rise to qualitatively quite different behavior, both in the hard-core model and in the Widom–Rowlinson model. This is perhaps a bit surprising, and in any case it demonstrates that these models do not exhibit the sort of "universality" – that qualitative features of the model should only depend on the dimension d and not on the details of the lattice – that is generally expected to hold in, for instance, the Ising model and Bernoulli percolation (see, e.g., Grimmett [3]).

The remaining sections of this paper are devoted to the task of proving Theorems 1.2 and 1.4. A brief outline is as follows.

In Section 2, we show how the task of proving Theorem 1.2 can be reduced to that of proving Theorem 1.4. This is done using an observation from [1], that the Widom–Rowlinson model is equivalent to the hard-core model on a different lattice; it turns out that this can be exploited to translate the example with the desired monotonicity property for the Widom–Rowlinson model (witnessing Theorem 1.4) into an analogous example for the hard-core model.

Then, in Section 3, we recall some facts about our main tool for analyzing the Widom-Rowlinson model: the so-called site-random-cluster model. This model arises by identifying $+1$'s and -1's in the Widom–Rowlinson model as a single spin, denoted 1. The resulting probability measure on $\{0,1\}^V$ is i.i.d. measure, perturbed by a weighting factor $2^{k(\eta)}$, where $k(\eta)$ is the number of connected components of 1's in $\eta \in \{0,1\}^V$. This extra factor is due to the fact that each such connected component has two possible values in the Widom–Rowlinson model: $+1$ or -1.

In Section 4 we then prove Theorem 1.4. A natural approach – which suggests itself by the corresponding proof of monotonicity of phase transition in the Ising model by means of its Fortuin–Kasteleyn random-cluster representation – is to use Holley's inequality (Lemma 4.1) to show that the site-random-cluster measures are stochastically increasing in the activity parameter λ. However, the desired stochastic monotonicity fails in general, which is essentially a consequence of the fact that $k(\eta)$ fails to be decreasing in η, unlike in the Fortuin–Kasteleyn random-cluster model. Comparison with the latter model (which also weights i.i.d. measure by $2^{k(\eta)}$, but lives on edges rather than on vertices) suggests that we should consider a covering lattice, as defined in (8). It turns out that this does not quite work, because isolated vertices are weighted differently in the resulting model compared to the Fortuin–Kasteleyn model. We therefore modify the covering graph using a certain "decoration" (i.e., addition of certain vertices) which causes the weighting of isolated vertices (in the original lattice) to increase, in such a way that Holley's inequality can be invoked to finally deduce the desired stochastic monotonicity.

2 Reduction of the hard-core result

We first need to make Theorems 1.2 and 1.4 precise by defining the class of lattices referred to in the theorems.

Definition 2.1 *An infinite locally finite graph $G = (V, E)$ is said to be a d-***dimensional periodic lattice** *if the following conditions hold:*

(A) $V = \{v + z : v \in \{v_1, \ldots, v_n\}, z \in \mathbf{Z}^d\}$ *for some finite set* $\{v_1, \ldots, v_n\} \subset \mathbf{R}^d$,

(B) $E = \{\langle x + z, y + z\rangle : \langle x, y\rangle \in \{\langle x_1, y_1\rangle, \langle x_2, y_2\rangle, \ldots, \langle x_m, y_m\rangle\}, z \in \mathbf{Z}^d\}$ *for some finite set* $\{(x_1, y_1), \ldots, (x_m, y_m)\} \subset V^2$,

(C) *G is connected.*

Conditions (A) and (B) capture the intuitive meaning of a periodic lattice. Condition (C) is included to avoid examples such as the graph obtained by taking the 3-dimensional cubic lattice $\mathbf{Z}^3$ and deleting all vertical edges: the resulting graph decomposes into infinitely many connected components, each of which is essentially 2-dimensional.

Brightwell et al [1, Section 5] noted the following connection between the hard-core and Widom–Rowlinson models. Let $G = (V, E)$ be any finite or countably infinite locally finite graph, and construct another graph $G^* = (V^*, E^*)$ as follows. Let $V^* = V \times \{-1, 1\}$, and let two vertices (x, i) and (y, j) be linked by an edge in E^* if either

(a) $x = y$ and $i = -j$, or

(b) $\langle x,y\rangle \in E$ and $i=-j$.

Suppose now that the $\{0,1\}^{V^*}$-valued random object X is distributed according to a Gibbs measure for the hard-core model on G^* at activity λ, and define $Y \in \{-1,0,+1\}^V$ by setting

$$Y(x) = \begin{cases} -1 & \text{if } X(x,-1)=1 \\ +1 & \text{if } X(x,+1)=1 \\ 0 & \text{otherwise,} \end{cases}$$

for each $x \in V$. A direct calculation using Definitions 1.1 and 1.3 shows that the distribution of Y then becomes a Gibbs measure for the Widom–Rowlinson model on G at the same activity λ. Conversely, if $Y \in \{-1,0,+1\}^V$ is distributed according to any Gibbs measure for the Widom–Rowlinson model on G, and $X \in \{0,1\}^{V^*}$ is obtained by setting

$$X(x,i) = \begin{cases} 1 & \text{if } Y(x)=i \\ 0 & \text{otherwise} \end{cases}$$

for each $(x,i) \in V^*$, then X is distributed according to the hard-core model on G^* at the same activity. It is easy to see that these mappings between Gibbs measures for the Widom–Rowlinson model on G and Gibbs measures for the hard-core model on G^* form a bijection. This is what we need in order to reduce the proof of Theorem 1.2 to that of Theorem 1.4:

Proof of Theorem 1.2 from Theorem 1.4: Fix the dimension d, and let G and λ_c be as in Theorem 1.4. Construct G^* from G as above. Clearly, by relabelling the vertices of G^*, we can represent it as a d-dimensional periodic lattice. Furthermore, by the above bijection between Gibbs measures, the hard-core model on G^* at activity λ has a unique Gibbs measure whenever $\lambda < \lambda_c$, and multiple Gibbs measures whenever $\lambda > \lambda_c$. □

3 Background on the Widom–Rowlinson model

In this section we review some background on the Widom–Rowlinson model; all of it can be found in more detail in [2].

Let $G=(V,E)$ be countably infinite and locally finite. In the introduction we mentioned the Gibbs measures μ_-^λ and μ_+^λ for the Widom–Rowlinson model on G at activity λ; these can be constructed as follows. Let $V_1 \subset V_2 \subset \dots$ be an increasing sequence of finite subsets of V, converging to V in the sense that each $v \in V$ is in all but finitely many of the V_n's. Define the (inner) boundary of V_n as

$$\partial V_n = \{x \in V_n : \exists y \in V \setminus V_n \text{ such that } \langle x,y\rangle \in E\}\,.$$

Also define the graphs $G_n = (V_n, E_n)$ where

$$E_n = \{\langle x,y\rangle \in E : x,y \in V_n\}\,.$$

Let the probability measure $\mu_{+,n}^\lambda$ on $\{-1,0,+1\}^{V_n}$ be given by the Widom–Rowlinson model on G_n with so-called "plus boundary condition", meaning that we condition on the event that all vertices on the boundary ∂V_n take value $+1$. More precisely, $\mu_{+,n}^\lambda$ is the probability measure which to each $\xi \in \{-1,0,+1\}^{V_n}$ assigns probability

$$\mu_{+,n}^\lambda(\xi) = \frac{1}{Z_n^\lambda} \prod_{v\in V_n} \lambda^{|\xi(v)|} I_{\{\xi(x)\xi(y)\ge 0 \text{ for all } \langle x,y\rangle\in E_n\}} I_{\{\xi(x)=+1 \text{ for all } x\in\partial V_n\}} \tag{4}$$

where I_A denotes the indicator function of the event A, and Z_n^λ is a normalizing constant. We will also identify $\mu_{+,n}^\lambda$ with the probability measure on $\{-1,0,+1\}^V$ that corresponds to setting $X(x) = +1$ for all $x \in V \setminus V_n$ and picking $X(V_n)$ according to (4). With this interpretation in mind, it is well-known (and can be shown by standard stochastic monotonicity arguments) that the measures $\mu_{+,n}^\lambda$ converge to a limiting probability measure μ_+^λ on $\{-1,0,+1\}^V$ as $n \to \infty$, in the sense that

$$\lim_{n\to\infty} \mu_{+,n}^\lambda(A) = \mu_+^\lambda(A)$$

for any cylinder event A. Moreover, the limiting measure does not depend on the choice of the vertex sets $\{V_n\}_{n=1}^\infty$, and it is a Gibbs measure for the Widom–Rowlinson model on G at activity λ.

Analogously, the measure μ_-^λ on $\{-1,0,+1\}^V$ is obtained as a limit of measures $\mu_{-,n}^\lambda$, which are defined as $\mu_{+,n}^\lambda$ except that we put -1's instead of $+1$'s on the boundary. The following statements are known to be equivalent.

(i) The Widom–Rowlinson model on G has a unique Gibbs measure.

(ii) $\mu_-^\lambda = \mu_+^\lambda$

(iii) $\mu_+^\lambda(X(x) = +1) = \mu_+^\lambda(X(x) = -1)$ for all $x \in V$.

In order to analyze when (i)–(iii) hold, it is useful to consider the projection from $\{-1,0,+1\}^{V_n}$ to $\{0,1\}^{V_n}$ obtained by taking absolute values at each vertex: Suppose that we pick $X \in \{-1,0,+1\}^{V_n}$ according to $\mu_{n,+}^\lambda$ and obtain $Y \in \{0,1\}^{V_n}$ by setting

$$Y(x) = |X(x)| \quad \text{for each } x \in V_n\,.$$

The distribution of Y on $\{0,1\}^{V_n}$ is denoted ϕ_n^λ, and is called the **wired site-random-cluster measure** for G_n at activity λ. (Readers familiar with random-cluster analysis of Ising and Potts models may note below that site-random-cluster measures play a similar role for the Widom–Rowlinson model as the usual (Fortuin–Kasteleyn) random-cluster measures do for Ising and Potts models.) A direct calculation shows that $\phi_n^\lambda(\eta)$ for $\eta \in \{0,1\}^{V_n}$ is given by

$$\frac{2^{k(\eta)}}{Z_n^\lambda} \prod_{v\in V_n} \lambda^{\eta(v)} I_{\{\eta(x)=1 \text{ for all } x\in\partial V_n\}} \tag{5}$$

where $k(\eta)$ is the number of connected components not intersecting ∂V_n of the set of 1's in η, and Z_n^λ is as in (4). Furthermore, the conditional distribution of X given Y can be described as follows. X has 0's at precisely the same vertices as Y, and $+1$'s at all vertices that take value 1 in Y and sit in a connected component of 1's intersecting ∂V_n; all other connected components of 1's in Y are independently assigned "all $+1$'s" or "all -1's" with probability $\frac{1}{2}$ each. Hence, for $x \in V_n$,

$$\mu_{+,n}^\lambda(X(x) = +1) - \mu_{+,n}^\lambda(X(x) = -1) = \phi_n^\lambda(x \leftrightarrow \partial V_n)$$

where $\{x \leftrightarrow \partial V_n\}$ is the event that there is a connected component of 1's containing x and intersecting ∂V_n. It follows that conditions (i)–(iii) above are equivalent to

(iv) $\lim_{n\to\infty} \phi_n^\lambda(x \leftrightarrow \partial V_n) = 0$ for all $x \in V$

and this is the condition that we will analyze directly in the next section.

4 Proof of the Widom–Rowlinson result

The purpose of this section is to prove Theorem 1.4. What we need to show is that if condition (iv) above fails for $\lambda = \lambda_1$ for some given λ_1, then it fails for all $\lambda > \lambda_1$. The natural way to try to do this is to show that the measures $\{\phi_n^\lambda\}_{\lambda>0}$ are stochastically increasing in λ, so we need to recall the concept of stochastic domination. For $\eta, \eta' \in \{0,1\}^S$, where S is an arbitrary finite set, we write $\eta \preceq \eta'$ if $\eta(s) \leq \eta'(s)$ for all $s \in S$. A function $f : \{0,1\}^S \to \mathbf{R}$ is said to be increasing if $f(\eta) \leq f(\eta')$ whenever $\eta \preceq \eta'$. For two probability measures π and π' on $\{0,1\}^S$, we say that π is stochastically dominated by π', writing $\pi \overset{\mathcal{D}}{\preceq} \pi'$, if

$$\int_{\{0,1\}^S} f \, d\pi \leq \int_{\{0,1\}^S} f \, d\pi' \tag{6}$$

for all increasing $f : \{0,1\}^S \to \mathbf{R}$.

A standard tool for establishing stochastic domination is the following result; see, e.g., [2] for a proof.

Lemma 4.1 (Holley's inequality) *Let S be a finite set and let π and π' be probability measures on $\{0,1\}^S$ that both put positive probability on all elements of $\{0,1\}^S$. Let X and X' be $\{0,1\}^S$-valued random elements with distributions π and π'. If, for all $s \in S$ and all $\eta, \eta' \in \{0,1\}^{S \setminus \{s\}}$ such that $\eta \preceq \eta'$, we have*

$$\pi(X(s) = 1 \,|\, X(S \setminus \{s\}) = \eta) \leq \pi'(X'(s) = 1 \,|\, X'(S \setminus \{s\}) = \eta')$$

then $\pi \overset{\mathcal{D}}{\preceq} \pi'$.

Consider now the wired site-random-cluster measure ϕ_n^λ in condition (iv). Suppose that we could establish, for any $x \in V_n$, that

$$\phi_n^\lambda(Y(x) = 1 \,|\, Y(V_n \setminus \{x\}) = \eta) \tag{7}$$

is increasing both in λ and in η. Then Lemma 4.1 would show that $\phi_n^{\lambda_1} \overset{\mathcal{D}}{\preceq} \phi_n^{\lambda_2}$ whenever $\lambda_1 \leq \lambda_2$. Applying (6) with $f = I_{\{x \leftrightarrow \partial V_n\}}$ (which is obviously an increasing function) would then give that

$$\phi_n^{\lambda_1}(x \leftrightarrow \partial V_n) \leq \phi_n^{\lambda_2}(x \leftrightarrow \partial V_n)$$

so that by letting $n \to \infty$ and using the equivalence between (iv) and (i)–(iii), we would arrive at the desired conclusion: if there are multiple Gibbs measures at $\lambda = \lambda_1$, then this is the case at $\lambda = \lambda_2$ as well, whenever $\lambda_2 \geq \lambda_1$.

Unfortunately this approach does not quite work, due to the fact that although the expression (7) is always increasing in λ, it is sometimes *not* increasing in η. This feature of the site-random-cluster model (which is discussed further in [2]) distinguishes it from the ordinary (Fortuin–Kasteleyn) random-cluster model, for which the above-sketched monotonicity argument does work. Since the latter model lives on the edges of a graph, rather than on the vertices, this immediately suggests the following approach in searching for a lattice that will exemplify Theorem 1.4: Given a d-dimensional periodic lattice $G = (V, E)$, consider its covering lattice (also known as the line graph) $G' = (E', V')$ defined by $V' = E$ and

$$E' = \{\langle x, y \rangle : \, x, y \in V', \text{ the edges } x \text{ and } y \text{ share a vertex in } G\}. \tag{8}$$

Unfortunately again, the site-random-cluster model on G' does not work quite the same as the Fortuin–Kasteleyn random-cluster model on G, because it turns out that whereas the latter gives the same weighting factor 2 to all connected components (relative to i.i.d. measure; cf the factor $2^{k(\eta)}$ in (5)), the former gives a different weighting factor for isolated vertices. To deal with this problem, we introduce a variation of a covering lattice which is tailored for our purposes. For a graph $G = (V, E)$, define another graph $G^* = (V^*, E^*)$ by setting

$$V^* = V^*_{(1)} \cup V^*_{(2)}$$

where $V^*_{(1)} = E$ and $V^*_{(2)} = V \times \{1, 2\}$, and

$$E^* = E^*_{(1)} \cup E^*_{(2)}$$

where

$$E^*_{(1)} = \{\langle x, y\rangle : x, y \in V^*_{(1)}, \text{ the edges } x \text{ and } y \text{ share a vertex in } G\}$$

and

$$E^*_{(2)} = \{\langle x, (y, i)\rangle : x \in V^*_{(1)}, (y, i) \in V^*_{(2)}, \text{ the edge } x \text{ is incident to the vertex } y \text{ in } G\}.$$

In other words, G^* is obtained by first taking the covering graph $G' = (V', E')$, and then adding two extra vertices corresponding to each vertex x in G, where each such extra vertex gets an edge in G^* to each vertex $y \in V'$ that correspond to an edge in G that is incident to x.

The point of this construction is that the vertices in $V^*_{(2)}$ compensate for the above-mentioned problem with isolated vertices in an edge configuration on G, by getting a greater amount of freedom to "choose" their value in case of such isolation; see (10) below. This additional freedom shows up as the extra term $\lambda I_{\{v \text{ is isolated w.r.t. } \xi\}}$ in (11).

The following result is a more specific variant of Theorem 1.4. For an infinite graph G, we let $p_c(G, bond)$ denote the critical value for i.i.d. bond percolation on G, i.e.,

$$\begin{aligned} p_c(G, bond) \quad = \quad & \inf\{p \in [0,1] : \text{ i.i.d. bond percolation on } G \text{ with retention para-} \\ & \text{meter } p \text{ produces an infinite cluster with positive probability.}\} \end{aligned}$$

The critical value $p_c(G, site)$ for site percolation is defined analogously.

Proposition 4.2 *Let $d \geq 2$, let $G = (V, E)$ be a d-dimensional periodic lattice such that $\sqrt{2} - 1 \leq p_c(G, bond) < 1$, and define $G^* = (V^*, E^*)$ from G as above. Then there exists a $\lambda_c > 0$ such that the Widom–Rowlinson model on G has*

$$\begin{cases} \textit{a unique Gibbs measure} & \textit{for all } \lambda < \lambda_c \\ \textit{multiple Gibbs measures} & \textit{for all } \lambda > \lambda_c. \end{cases} \tag{9}$$

Remark. The requirement that $p_c(G, bond) < 1$ is in fact superfluous; it is possible to show, by a standard renormalization argument, that $p_c(G, bond) < 1$ holds for any d-dimensional periodic lattice with $d \geq 2$. This observation is, however, not needed in our proof of Theorem 1.4.

Before proving Proposition 4.2, we first show how it implies Theorem 1.4.

Proof of Theorem 1.4 from Proposition 4.2: It is easy to see that if G is a d-dimensional periodic lattice, then we can relabel the vertices of G^* to make it a d-dimensional periodic lattice as well. It therefore only remains to show that for each d there exists a d-dimensional periodic lattice with $p_c(G, bond) \in [\sqrt{2}-1, 1)$. For $d = 2$, we may take $G = \mathbf{Z}^2$, because $p_c(\mathbf{Z}^2, bond) = \frac{1}{2}$; see, e.g., [3]. For $d \geq 3$, we have $p_c(\mathbf{Z}^d, bond) \in (0, 1)$; see [3] again. If we now let $G_{d,n}$ denote the lattice obtained by replacing each edge in $\mathbf{Z}^d$ by n edges in series, then, clearly,

$$p_c(G_{d,n}, bond) = (p_c(\mathbf{Z}^d, bond))^{1/n} .$$

This critical value tends to 1 as $n \to \infty$, whence we may take $G = G_{d,n}$ with n large enough so that $(p_c(\mathbf{Z}^d, bond))^{1/n} \geq \sqrt{2} - 1$. □

For the proof of Proposition 4.2, it is useful to isolate the following lemma.

Lemma 4.3 *Let $d \geq 2$, let $G = (V, E)$ be a d-dimensional periodic lattice satisfying $p_c(G, bond) \in [\sqrt{2}-1, 1)$, and define G^* from G as above. Then the Widom–Rowlinson model on G^* has*

(a) *a unique Gibbs measure for all $\lambda < \frac{1}{\sqrt{2}}$, and*

(b) *multiple Gibbs measures for all sufficiently large λ.*

Proof: We begin with part (b). From the construction of G^*, it is clear that i.i.d. site percolation on G^* with parameter p produces an infinite cluster with positive probability if and only if the same holds for i.i.d. bond percolation on G at the same parameter value. Hence $p_c(G^*, site) = p_c(G, bond) \in [\sqrt{2}-1, 1)$. In particular, $p_c(G^*, site) < 1$, which, in combination with the observation that G^* has bounded degree (by the definition of a d-dimensional periodic lattice), allows us to invoke [4, Theorem 1.1] to deduce that (b) holds.
Moving on to (a), take $\lambda < \frac{1}{\sqrt{2}}$, and consider the conditional distribution of the +1-particles given the positions of the −1-particles. Under any Gibbs measure for the Widom–Rowlinson model on G at activity λ – specifically, the plus measure μ_+^λ – this conditional distribution is simply that each vertex that is not occupied by or adjacent to a −1 independently takes value 0 or +1 with respective probabilities $\frac{1}{\lambda+1}$ and $\frac{\lambda}{\lambda+1}$. Hence the (unconditional) distribution of the +1-particles is stochastically dominated by i.i.d. site percolation on G with parameter $\frac{\lambda}{\lambda+1}$. Since

$$\frac{\lambda}{\lambda+1} < \frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}+1} = \sqrt{2} - 1 \leq p_c(G^*, site) ,$$

we get that the i.d.d. site percolation forms no infinite cluster almost surely, and therefore there are μ_+^λ-a.s. no infinite clusters of +1's. Reversing the roles of +1's and −1's in this argument shows that there are also no infinite clusters of −1's μ_+^λ-a.s., so that in fact all connected components of +1's or −1's are finite. This implies condition (iii), which in turn implies that there is only one Gibbs measure. □

Proof of Proposition 4.2: Fix a G^* as in the proposition. In view of Lemma 4.3, we only need to show the monotonicity part of (9) – that having multiple Gibbs measure at some activity λ implies the same thing at all higher activities.
Define a sequence $\{G_n^* = (V_n^*, E_n^*)\}_{n=1}^\infty$ of finite subgraphs of G^* in the same manner as the sequence $\{G_n\}_{n=1}^\infty$ was defined in Section 3, but with the additional requirement that $\partial V_n^* \subset V_{(1)}^*$ for each n. Let $\mu_{+,n}^\lambda$ denote the probability measure on $\{-1, 0, +1\}^{V_n^*}$ corresponding to the Widom–Rowlinson model on G_n^* with activity λ and plus boundary condition; hence $\mu_{+,n}^\lambda$ is given by (4) with E_n^* and ∂V_n^* in place of E_n and ∂V_n. Also let $\overline{\mu}_{+,n}^\lambda$ denote the projection of $\mu_{+,n}^\lambda$ on $\{-1, 0, +1\}^{V_n^* \cap V_{(1)}^*}$.

Given a configuration $\xi \in \{-1,0,+1\}^{V_n^* \cap V_{(1)}^*}$, call a vertex $x \in V_n^* \cap V_{(2)}^*$ **isolated** if $\xi(y) = 0$ for all $y \in V_n^* \cap V_{(1)}^*$ such that $\langle x, y\rangle \in E_n^*$. Note that all neighbors y of x are neighbors of each other (see the definition of G^*), so that if any of the neighbors take value $+1$, then none of them take value -1, and vice versa. Hence, the set of possible values of $X(x)$ given $X(V_n^* \cap V_{(1)}^*) = \xi$ is

$$\begin{cases} \{-1,0,+1\} & \text{if } x \text{ is isolated} \\ \{0,+1\} & \text{if } x \text{ is not isolated, and has a neighbor with value } +1 \\ \{-1,0\} & \text{if } x \text{ is not isolated, and has a neighbor with value } -1, \end{cases} \tag{10}$$

irrespective of the values of all other vertices in $V_n^* \cap V_{(2)}^*$. We can therefore integrate out $\{X(v)\}_{x \in V_n^* \cap V_{(2)}^*}$ in $\mu_{+,n}^\lambda$, and using (4) we get that $\overline{\mu}_{+,n}^\lambda$ is given by

$$\begin{aligned} \overline{\mu}_{+,n}^\lambda(\xi) &= \frac{1}{Z_n^\lambda} \prod_{v \in V_n^* \cap V_{(1)}^*} \lambda^{|\xi(v)|} \prod_{v \in V_n^* \cap V_{(2)}^*} \Big(\lambda + 1 + \lambda I_{\{v \text{ is isolated w.r.t. } \xi\}}\Big) \\ &\quad \times I_{\{\xi(x)\xi(y) \geq 0 \text{ for all } \langle x,y\rangle \in E_n\}} I_{\{\xi(x)=+1 \text{ for all } x \in \partial V_n\}} \end{aligned} \tag{11}$$

for each $\xi \in \{-1,0,+1\}^{V_n^* \cap V_{(1)}^*}$.

Next let ϕ_n^λ be the wired site-random-cluster measure for G_n^* at activity λ, and let $\overline{\phi}_n^\lambda$ be the projection of ϕ_n^λ on $\{0,1\}^{V_n^* \cap V_{(1)}^*}$. By (11), we have that the probability assigned by $\overline{\phi}_n^\lambda$ to each $\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^*}$ is given by

$$\begin{aligned} \overline{\phi}_n^\lambda(\eta) &= \frac{2^{k(\eta)}}{Z_n^\lambda} \prod_{v \in V_n^* \cap V_{(1)}^*} \lambda^{\eta(v)} \prod_{v \in V_n^* \cap V_{(2)}^*} \Big(\lambda + 1 + \lambda I_{\{v \text{ is isolated w.r.t. } \xi\}}\Big) \\ &\quad \times I_{\{\eta(x)=1 \text{ for all } x \in \partial V_n\}} \end{aligned} \tag{12}$$

where $k(\eta)$ is the number of connected components not intersecting ∂V_n^* of the set of 1's in η. It is to this probability measure $\overline{\phi}_n^\lambda$ that we will now be able to apply Holley's inequality (Lemma 4.1) to obtain a useful stochastic comparison between the behaviors at different values of λ.

Fix an $x \in V_n^* \cap V_{(1)}^* \setminus \partial V_n^*$. Write y_1 and y_2 for the two vertices in G that x connect when viewed as an edge in G, and write B_1 (resp. B_2) for the set of vertices in $V_n^* \cap V_{(1)}^* \setminus \{x\}$ whose corresponding edge in G has y_1 (resp. y_2) as an endpoint. For $\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}}$, consider the connected components of 1's in η, and note that at most one such component intersects B_1 (because all pairs of vertices in B_1 share an edge in G_n^*). We define $C_1(\eta)$ to be this connected component if it exists; otherwise we set $C_1(\eta) = \emptyset$. $C_2(\eta)$ is defined analogously. Finally in this long sequence of definitions, we partition $\{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}}$ into four subsets A, A', A'' and A''' as follows. Let

$$\begin{aligned} A &= \{\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}} : C_1(\eta) = C_2(\eta) = \emptyset\}, \\ A' &= \{\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}} : \text{ exactly one of the components } C_1(\eta) \text{ and } C_2(\eta) \\ &\qquad \text{is empty}\}, \\ A'' &= \{\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}} : \text{ neither } C_1(\eta) \text{ nor } C_2(\eta) \text{ is empty, } C_1(\eta) \neq C_2(\eta), \\ &\qquad \text{and at most one of them intersects } \partial V_n^*\}, \\ A''' &= \{\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}} : \text{ neither } C_1(\eta) \text{ nor } C_2(\eta) \text{ is empty, and we have} \\ &\qquad \text{either that } C_1(\eta) = C_2(\eta) \text{ or that both components intersect } \partial V_n^*\}. \end{aligned}$$

Let Y be a $\{0,1\}^{V_n^* \cap V_{(1)}^*}$-valued random object with distribution $\overline{\phi}_n^\lambda$. By direct application of (12), we get, for any $x \in V_n^* \cap V_{(1)}^* \setminus \partial V_n^*$ and any $\eta \in \{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \{x\}}$ such that $\eta(\partial V_n^*) \equiv 1$, that

$$\frac{\overline{\phi}_n^\lambda(Y(x) = 1 \,|\, Y(V_n^* \cap V_{(1)}^* \setminus \{x\}) = \eta)}{\overline{\phi}_n^\lambda(Y(x) = 0 \,|\, Y(V_n^* \cap V_{(1)}^* \setminus \{x\}) = \eta)} = \begin{cases} \dfrac{2\lambda(\lambda+1)^4}{(2\lambda+1)^4} & \text{if } \eta \in A \\[2ex] \dfrac{\lambda(\lambda+1)^2}{(2\lambda+1)^2} & \text{if } \eta \in A' \\[2ex] \dfrac{\lambda}{2} & \text{if } \eta \in A'' \\[2ex] \lambda & \text{if } \eta \in A'''\,. \end{cases} \tag{13}$$

A crucial observation now is that if we increase η (meaning that we change some of the 0's in η to 1's), then we can only move down the list of events in (13) – from A towards A''' – but not the other way around. For $\lambda \geq \frac{1}{\sqrt{2}}$ we have $\left(\frac{2\lambda+1}{\lambda+1}\right)^2 \geq 2$, so that

$$\frac{2\lambda(\lambda+1)^4}{(2\lambda+1)^4} \leq \frac{\lambda(\lambda+1)^2}{(2\lambda+1)^2} \leq \frac{\lambda}{2} \leq \lambda$$

and we can deduce that the left-hand-side of (13) is increasing in η whenever $\lambda \geq \frac{1}{\sqrt{2}}$. A straightforward calculation also shows that all four expressions in the right-hand-side of (13) are increasing in λ for all positive λ. Hence, the left-hand-side of (13) is increasing both in η and in λ as long as $\lambda \geq \frac{1}{\sqrt{2}}$. It follows that

$$\begin{array}{l} \overline{\phi}_n^\lambda(Y(x) = 1 \,|\, Y(V_n^* \cap V_{(1)}^* \setminus \{x\}) = \eta) \text{ is increasing} \\ \text{in } \eta \text{ and in } \lambda \text{, whenever } \lambda \geq \frac{1}{\sqrt{2}}. \end{array} \tag{14}$$

We now claim that for all $\lambda_1, \lambda_2 \in [\frac{1}{\sqrt{2}}, \infty)$, we have

$$\overline{\phi}_n^{\lambda_1} \overset{\mathcal{D}}{\preceq} \overline{\phi}_n^{\lambda_2}\,. \tag{15}$$

To see this, first note that both measures put probability one on the "all 1's" configuration on the boundary ∂V_n^*, and then note that Lemma 4.1 in combination with (14) shows that the projection of $\overline{\phi}_n^{\lambda_1}$ on $\{0,1\}^{V_n^* \cap V_{(1)}^* \setminus \partial V_n^*}$ is stochastically dominated by the same projection of $\overline{\phi}_n^{\lambda_2}$. Hence, (15) is established.
To finish the proof, we need to show (recall condition (iv) in Section 3) that if $\lambda_1 \leq \lambda_2$ and

$$\limsup_{n\to\infty} \phi_n^{\lambda_1}(x \leftrightarrow \partial V_n^*) > 0 \;\text{ for some } x \in V^*\,, \tag{16}$$

then also

$$\limsup_{n\to\infty} \phi_n^{\lambda_2}(x \leftrightarrow \partial V_n^*) > 0 \;\text{ for some } x \in V^*\,, \tag{17}$$

To this end, suppose that $\lambda_1 \leq \lambda_2$ and that (16) holds. By Lemma 4.3, we have $\lambda_1 \geq \frac{1}{\sqrt{2}}$. We may assume that $x \in V_{(1)}^*$, because if not (i.e., if $x \in V_{(2)}^*$), then it is easy to see that (16) holds for some nearest neighbor of x, which is necessarily in $V_{(1)}^*$. Having made this assumption, note that if $\xi \in \{0,1\}^{V_n^*}$ contains a path of 1's from x to ∂V_n^*, then it also contains such a path with the additional property that all vertices on the path are in $V_{(1)}^*$ (such a path can be obtained by just skipping the vertices in $V_{(2)}^*$ from the first path). Hence $\overline{\phi}_n^{\lambda_1}(x \leftrightarrow \partial V_n^*) = \phi_n^{\lambda_1}(x \leftrightarrow \partial V_n^*)$, so that

$$\limsup_{n\to\infty} \overline{\phi}_n^{\lambda_1}(x \leftrightarrow \partial V_n^*) > 0 \,.$$

Furthermore, (15) implies that $\overline{\phi}_n^{\lambda_1}(x \leftrightarrow \partial V_n^*) \leq \overline{\phi}_n^{\lambda_2}(x \leftrightarrow \partial V_n^*)$, so that

$$\limsup_{n\to\infty} \overline{\phi}_n^{\lambda_2}(x \leftrightarrow \partial V_n^*) > 0 \,.$$

This implies (17), so the proof is complete. □

Acknowledgement. Some valuable advice from the referee allowed me to make the paper more reader-friendly.

References

[1] Brightwell, G., Häggström, O. and Winkler, P. (1999) Nonmonotonic behavior in hard-core and Widom–Rowlinson models, *J. Statist. Phys.* **94**, 415–435.

[2] Georgii, H.-O., Häggström, O. and Maes, C. (2001) The random geometry of equilibrium phases, *Phase Transitions and Critical Phenomena, Volume 18* (C. Domb and J.L. Lebowitz, eds), pp 1–142, Academic Press, London.

[3] Grimmett, G.R. (1999) *Percolation* (2nd edition), Springer, New York.

[4] Häggström, O. (2000) Markov random fields and percolation on general graphs, *Adv. Appl. Probab.* **32**, 39–66.

[5] Kelly, F.P. (1985) Stochastic models of computer communication systems, *J. Roy. Statist. Soc. B* **47**, 379–395.

[6] Wheeler, J.C. and Widom, B. (1970) Phase equilibrium and critical behavior in a two-component Bethe-lattice gas or three-component Bethe-lattice solution, *J. Chem. Phys.* **52**, 5334–5343.