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EDGE OCCUPATION MEASURE FOR A REVERSIBLE MARKOV CHAIN

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Abstract:
In this note, we study the Gaussian fluctuations of the edge occupation measure for a reversible Markov chain and give a nice description of the covariance matrix. Then we give some large deviations results concerning this occupation measure.

Introduction

Consider a connected, finite dimensional graph (V, E) with V a set of n vertices. Each edge $(i, j) \in E$ has a nonnegative weight denoted by $w^{(ij)}$. Let $(X_n)_{n\in\mathbb{N}}$ be the reversible Markov chain with state space V and transition probabilities

$$p(i,j) = \mathbb{P}(X_{n+1} = j | X_n = i) = \frac{w^{(ij)}}{\sum_{l\in N(i)} w^{(il)}},$$

where $N(i)$ is the set of neighbours of the vertex i. Define the (unoriented) edge occupation measure associated to a path, and defined for an edge $(i, j) \in E$ by

$$\mu_n\left((i,j)\right) = \mu_n^{(ij)} = \left|\left\{1 \le k \le n-1 \ : \ X_{k+1} = j,\ X_k = i \text{ or } X_{k+1} = i,\ X_k = j\right\}\right|.$$

The aim of this note is to study the random sequence of edge occupation measures $\mu_n = (\mu_n^{(ij)})_{(ij)\in E}$ associated to the sample path $(X_k)_{k\le n}$, $n \in \mathbb{N}$ for the Markov chain introduced above.
Because we have restricted our attention to finite Markov chains, it is clear that $\frac{1}{n}\mu_n$ converges almost surely to $\frac{1}{\|w\|}w$ and $\frac{1}{\sqrt{n}}\left(\mu_n - \frac{w}{\|w\|}\right)$ converges in law to $\mathcal{G}(0, C)$ where $\|w\| = \sum_{(ij)\in E} w^{(ij)}$ and $\mathcal{G}(m, C)$ is an $|E|$-dimensional Gaussian distribution with vector mean m

and covariance matrix C. Our interest is to obtain explicit results concerning the covariance C and also some large deviations results.
Such results are well known for the site occupation measure, where it is also easy to generalise to the continuous case, but seem new for the edge process. The author's attention was directed to the problem as it turns out to be an essential initial step in understanding various path dependent stochastic evolution proccesses.

1. The main calculation

Vertex occupation measures for Markov processes are well understood even in the non-reversible case. So consider the expanded Markov chain $(Y_n)_{n\in\mathbb{N}}$ with state space the directed edges $\tilde{E} = \{(ij) \in V^2;\ p(i,j) > 0\}$ and transition probabilities

$$P((ij),(kl)) = \mathbb{P}(Y_{n+1} = (kl)|Y_n = (ij)) = p(j,l)\delta_{j,k}.$$

This process is an irreducible Markov Chain, and has stationary law:

$$\nu(ij) = \frac{w^{(ij)}}{\sum_{(ij)\in\tilde{E}} w^{(ij)}}.$$

Then, for every function f defined on $\tilde{E}$,

$$\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} f(Y_k) = \nu(f) = \sum_{(ij)\in\tilde{E}} f(ij)\nu(ij) \quad \text{a.s.}$$

Applying this result to the function $1_{(ij)} + 1_{(ji)}$ and noting that $\mu_n(ij) = \sum_{k=1}^{n}(1_{(ij)}(Y_k) + 1_{(ji)}(Y_k))$ gives us the pointwise convergence of the occupation measure.

Let A be the $|\tilde{E}| \times |\tilde{E}|-$matrix with coefficients

$$A((ij),(i'j')) = \nu(i'j').$$

Let Z be the Green function (or fundamental matrix) of the Markov chain $(Y_n)_{n\in\mathbb{N}}$ defined by

$$Z = (I - P + A)^{-1}$$

and P^* be the adjoint of P for the inner product $(.,.)$, in fact $P^*((ij),(i'j')) = \frac{\nu(i'j')}{\nu(ij)}P((i'j'),(ij))$.

Proposition 1. *With these notations*

$$\frac{1}{\sqrt{n}}(\mu_n(ij) - 2n\nu(ij))_{(ij)\in\tilde{E}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{G}(0,C)$$

where $C((ij),(i'j')) = \big(Z(1_{(ij)} + 1_{(ji)}), (I - P^*P)Z(1_{(i'j')} + 1_{(j'i')})\big)$

Let us recall the well-known central limit theorem for the finite Markov chains (see Kemeny and Snell [1] and [2]). Let $(X_n)_{n\in\mathbb{N}}$ be an irreducible Markov chain with finite state space E with a transition matrix denoted by p and its stationary law by μ. Let z be the Green function of this Markov chain. Let $f_1, \ldots, f_m$ be functions defined on E. Then,

$$\frac{1}{\sqrt{n}}(\sum_{k=1}^{n} f_l(X_k) - n\mu(f_l))_{l=1,\ldots,m} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{G}(0,C)$$

where C is a $m \times m-$matrix with coefficients

$$C(l,l') = \sum_{i,j\in E} f_l(i)c_{ij}f_{l'}(j)$$

with

$$c_{ij} = \mu(i)z(i,j) + \mu(j)z(j,i) - \mu(i)\delta_{ij} - \mu(i)\mu(j)$$

Now we apply this result to the Markov chain $(Y_n)_{n\in\mathbb{N}}$ with the functions defined on $\tilde{E}$

$$f_{(ij)} = 1_{(ij)} + 1_{(ji)}$$

The proposition 1 is proved by making the substitution $PZ = A - I + Z$ in the covariance terms.

Proposition 2. *The self-adjoint operator $I - P^*P$ is an orthogonal projection, it has only eigenvalues 0 and 1 with multiplicities n and $m_n = 2|E| - |V|$ respectively. (The graph being connected, m_n is strictly positive if $n \geq 3$)*

First, using the definitions of the Markov chains $(X_n)_n$ and $(Y_n)_n$, for every function f defined on $\tilde{E}$,

$$\begin{aligned} P^*Pf(kj) &= \sum_{m,l} \frac{\nu(mk)}{\nu(kj)} f(kl)p(k,l)p(k,j) \\ &= \sum_{l\in V} p(k,l)f(kl) \end{aligned}$$

So, the function f is an eigenfunction of $I - P^*P$ for the eigenvalue 0 if and only if for every $k, j \in V$,

$$\sum_{l\in V} p(k,l)f(kl) = f(kj).$$

It is easy to see that for $i = 1, \ldots, n$ the functions defined by $f_i = \sum_{(kl)\in\tilde{E}} \delta_{i,k} 1_{(kl)}$ verify this equation and form an orthogonal n-dimensional basis of the eigenspace associated to the eigenvalue 0. Now, let us construct the m_n eigenfunctions of $I - P^*P$ for the eigenvalue 1. The function f is an eigenfunction of $I - P^*P$ for the eigenvalue 1 if and only if for all i,

$$\sum_{l\in N(i)} w^{(il)} f(il) = 0.$$

For any given $i \in V$, the dimension of the space of the functions satisfying the previous equation and such that $f(kl) = 0, k \neq i$ is $|N(i)| - 1$, these functions are denoted by $f_i^{(j)}, 1 \leq j \leq |N(i)| - 1$. We can find a basis of this space orthogonal for the inner product $(.,.)$. And, if $i \neq i'$, the functions $(f_i^{(j)})_j$ and $(f_{i'}^{(j)})_j$ are orthogonal to each other. Then, we have an orthogonal m_n-dimensional basis of the eigenspace associated to the eigenvalue 1. All these eigenfunctions can be renormalized to be an orthonormal basis $(f_i)_i$ of $L^2(\tilde{E}, \nu)$. So, we can rewrite the covariance as

$$C((ij),(i'j')) = \sum_{l=1}^{m_n} \left(Z(1_{(ij)} + 1_{(ji)}), f_l\right) \left(Z(1_{(i'j')} + 1_{(j'i')}), f_l\right).$$

2. Large Deviations

The fact that $\frac{1}{n}\mu_n$ converges in probability to $\frac{1}{\|w\|}w$ as $n\to\infty$ indicates that $\frac{1}{n}\mu_n$ is a good candidate for a large deviations principle in the space of probability measures $\mathcal{M}_1(E)$.

Proposition 3. *The sequence of measures* $\frac{1}{n}\mu_n$ *satisfies a large deviations principle with the good rate function*

$$I'(q) = \sum_{(ij)\in E} \frac{q(ij)}{2} \log\left(\frac{q^2(ij)}{\sum_l q(il)\sum_l q(jl) p(i,j)p(j,i)}\right)$$

where $q(ij) = \mu(ji) + \mu(ij)$. *This means that for every* $\Gamma \in \mathcal{M}_1(E)$,

$$\begin{aligned} -\inf_{q\in\Gamma^o} I'(q) &\leq \liminf_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{\mu_n}{n} \in \Gamma) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{P}(\frac{\mu_n}{n} \in \Gamma) \leq -\inf_{q\in\bar{\Gamma}} I'(q) \end{aligned}$$

The directed edges occupation measure for the expanded Markov chain $(Y_n)_{n\in\mathbb{N}}$ is very well understood (see Dembo and Zeitouni [1] for example) and an explicit formula for the good rate function can be given. For any $\mu \in \mathcal{M}_1(\tilde{E})$, let us define

$$\mu_1(i) = \sum_{j\in V} \mu(ij) \text{ and } \mu_2(i) = \sum_{j\in V} \mu(ji).$$

Then, $\mu \in \mathcal{M}_1(\tilde{E})$ is *shift invariant (s.i.)* if $\mu_1 = \mu_2$. The good rate function is then

$$I(\mu) = \begin{cases} \displaystyle\sum_{(ij)\in\tilde{E}} \mu(ij) \log(\frac{\mu(ij)}{\mu_1(i)p(i,j)}) \text{ if } \mu \text{ is shift invariant,} \\ \\ \infty \text{ otherwise.} \end{cases}$$

Applying the contraction principle with the continuous function f which at each measure μ in $\mathcal{M}_1(\tilde{E})$ associates the new measure $(\mu(ij)+\mu(ji))_{(ij)\in E}$ in $\mathcal{M}_1(E)$.

$$I'(q) = \inf_{\mu \ s.i.; f(\mu)=q} I(\mu).$$

We can rewrite $I(\mu)$ as

$$I(\mu) = \sum_{i\in V}\sum_{l\in N(i)} \mu(il)\log(\mu(il)) - \sum_{i\in V}\sum_{l\in N(i)} \mu(il)\log(\sum_{k\in N(i)} \mu(ik)p(i,l)).$$

Using the fact that μ is shift invariant and $\mu(ij) = q(ij) - \mu(ji)$ for every (ij), the marginals of μ are determined by q and then the second double sum does not depend on μ and as we differentiate with respect to a (directed) edge (ij), only two terms in the first double sum are involved: $\mu(ij)$ and $\mu(ji) = q(ij) - \mu(ij)$. After some calculations, it appears that the infimum is attained at μ defined as $\mu(ij) = \mu(ji) = \frac{1}{2}q(ij)$ which is a strict minimum. So

$$I'(q) = \sum_{(ij)\in E} \frac{q(ij)}{2} \log\left(\frac{q^2(ij)}{\sum_l q(il)\sum_l q(jl) p(i,j)p(j,i)}\right).$$

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