DOI: 10.1214/ECP.v4-1003

Elect. Comm. in Probab. **4** (1999) 31–37

ELECTRONIC COMMUNICATIONS in PROBABILITY

CONSTRUCTIONS OF A BROWNIAN PATH WITH A GIVEN MINIMUM

JEAN BERTOIN
Laboratoire de Probabilités, tour 56, Université Pierre et Marie Curie, 4 Place Jussieu, F-75252 Paris Cedex 05, France
email: jbe@ccr.jussieu.fr

JIM PITMAN
Department of Statistics, University of California, 367 Evans Hall # 3860, Berkeley, CA 94720-3860, USA
email: pitman@stat.Berkeley.edu

JUAN RUIZ DE CHAVEZ
Depto. de Matemáticas UAM-I, Apdo. Postal 55-534, C.P. 09340 México D.F., México
email: jrch@xanum.uam.mx

submitted February 25, 1999; *accepted* July 9, 1999

AMS subject classification: 60J65
Conditioned Brownian motion, path transformations

Abstract

We construct a Brownian path conditioned on its minimum value over a fixed time interval by simple transformations of a Brownian bridge.

Path transformations have proved useful in the study of Brownian motion and related processes, by providing simple constructions of various conditioned processes such as Brownian bridge, meander and excursion, starting from an unconditioned Brownian motion. As well as providing insight into the structure of these conditioned processes, path constructions assist in the computation of various conditional laws of Brownian functionals, and in the simulation of conditioned processes.

Starting from a standard one-dimensional Brownian motion $B = (B_t)_{0 \le t \le 1}$ with $B_0 = 0$, one well known construction of a Brownian bridge of length 1 from 0 to x, denoted $B^{\mathrm{br},x}$, is the following:

$$B_u^{\mathrm{br},x} := B_u - uB_1 + ux \qquad (0 \le u \le 1). \tag{1}$$

Then a Brownian meander of length 1 starting at 0 and conditioned to end at $r \ge 0$, denoted $B^{\mathrm{me},r}$, can be constructed from three independent copies $(B_{i,u}^{\mathrm{br},0})_{0 \le u \le 1}, i = 1, 2, 3$ of the *standard Brownian bridge* $B^{\mathrm{br},0}$ as

$$B_u^{\mathrm{me},r} := \sqrt{(ru + B_{1,u}^{\mathrm{br},0})^2 + (B_{2,u}^{\mathrm{br},0})^2 + (B_{3,u}^{\mathrm{br},0})^2} \qquad (0 \le u \le 1). \tag{2}$$

So $B^{\mathrm{me},r}$ is identified with the *three-dimensional Bessel bridge from* 0 *to* r, the case $r = 0$ yielding the *standard Brownian excursion.* The *standard Brownian meander* is recovered as $B^{\mathrm{me}} := B^{\mathrm{me},\rho}$, where $\rho = B_1^{\mathrm{me}}$ is independent of the three bridges with the *Rayleigh density*

$$P(\rho \in dx)/dx = xe^{-\frac{1}{2}x^2} \qquad (x > 0).$$

The above descriptions of $B^{\mathrm{me},r}$ and B^{me} are read from [20, 12]. See also [8, 3, 6, 17, 18] for further background. Many other path transformations relating these processes are known. For instance, the transformation of Vervaat [19] (see also Biane [4] and Imhof [13]) shows that the standard Brownian excursion can be obtained by transposing the pre-minimum and the post-minimum parts of a standard Brownian bridge. Analogously, reversing the pre-minimum part and then tacking on the post-minimum part of a standard Brownian bridge from 0 to 0 yields a standard Brownian meander, as shown by Bertoin [2]. We refer to Biane and Yor [5], Bertoin and Pitman [3], Chaumont [7] and Yor [22] for many further results in this vein.

The work of Williams [21] and Denisov [10] shows how the path of B over $[0, 1]$ decomposes at the a.s. unique time μ of its minimum on $[0, 1]$ into two path fragments, which given μ are are two independent Brownian meanders of lengths μ and $1-\mu$ respectively, put back-to-back. Combined with any of the constructions of Brownian meander mentioned above, this gives an explicit construction of the path of B given μ, the time of its minimum on $[0, 1]$. The main purpose of this note is to present the following construction of B conditioned instead on B_μ, the level of the minimum:

Theorem 1 *For each* $x \le 0$ *there is the equality of distributions on the path space* $C[0, 1]$

$$(B \,|\, B_\mu = x) \stackrel{d}{=} B^{(max-reflect,x)} \tag{3}$$

where the left side denotes the unique determination of the conditional law of $(B_t)_{0\le t\le 1}$ *given* $B_\mu = x$ *that is weakly continuous in* x*, and the process on the right side is constructed as follows from a Brownian bridge* $B^{\mathrm{br},x}$ *from* 0 *to* x*:*

$$B_t^{(max-reflect,x)} = \begin{cases} B_t^{\mathrm{br},x} & \text{if } 0 \le t \le T_x^{\mathrm{br},x} \\ 2\left(\max_{T_x^{\mathrm{br},x} \le u \le t} B_u^{\mathrm{br},x}\right) - B_t^{\mathrm{br},x} & \text{if } T_x^{\mathrm{br},x} < t \le 1 \end{cases} \tag{4}$$

where $T_x^{\mathrm{br},x}$ *is the first hitting time of* x *by* $B^{\mathrm{br},x}$*. The path-transformation* $B^{\mathrm{br},x} \to B^{(max-reflect,x)}$ *is depicted in Figure 1 below.*

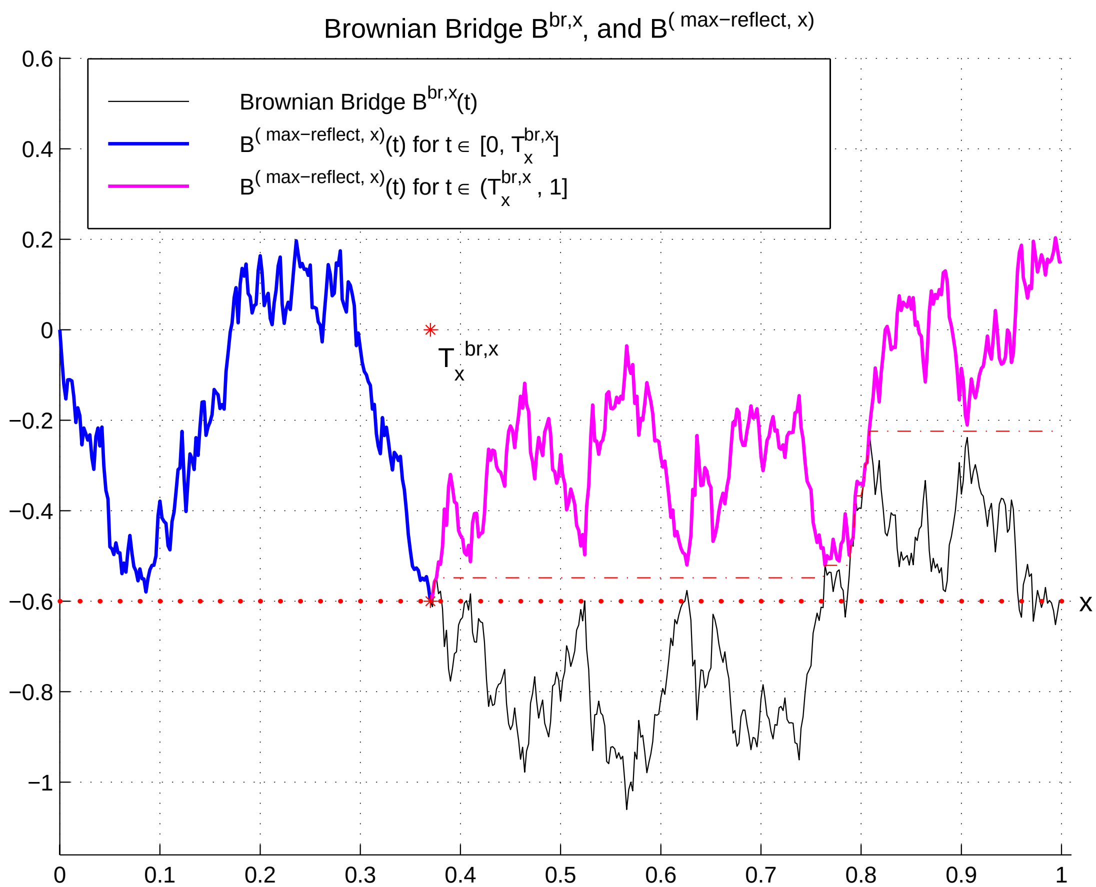

Fig. 1

Proof: The path decomposition at time μ due to Williams [21] and Denisov [10] as formulated in Proposition 2 of [1] states that the process $((B_{s-t} - x)_{0 \le t \le s} \,|\, \mu = s \text{ and } B_\mu = x)$ has the same distribution as a three-dimensional Bessel bridge of length s from 0 to $|x|$. It then follows from a classical time-reversal identity observed by Williams [21] that there is the equality in distribution of processes

$$((B_t)_{0 \le t \le s} \,|\, \mu = s \text{ and } B_\mu = x) \stackrel{d}{=} ((B_t)_{0 \le t \le s} \,|\, T_x = s) \tag{5}$$

where T_x denotes the first hitting time of x by the Brownian motion B.

On the other hand, it is elementary, and implicit in well known results [9, 17], that for $0 < t < 1$ and $x < 0$

$$P(\mu \in ds, B_\mu \in dx) = 2P(T_x \in ds, B_1 \in dx). \tag{6}$$

This can be understood in terms of random walks, using the basic random walk duality lemma of Feller [11]. Or it can be justified by excursion theory, using the fact underlying the arcsine density of μ, that $1/\sqrt{t}$ is up to constant factors both the rate of Brownian excursions of length

$> t$, and the density of returns to 0 at time t. It follows now readily from (5) and (6) that

$$((B_t)_{0\le t\le\mu} \mid B_\mu = x) \stackrel{d}{=} (B_t^{\mathrm{br},x})_{0\le t\le T_x^{\mathrm{br},x}} \tag{7}$$

Again by decomposition at the minimum, the remaining path of B on the interval $[\mu, 1]$ given $(B_t)_{0\le t\le\mu}$ is a Brownian meander of length $1-\mu$ shifted to start at (μ, x). On the other hand, the rest of the bridge $B^{\mathrm{br},x}$ given $(B_t^{\mathrm{br},x})_{0\le t\le T_x^{\mathrm{br},x}}$ is a Brownian bridge of length $1 - T_x^{\mathrm{br},x}$ from $(T_x^{\mathrm{br},x}, x)$ to $(1, x)$. The conclusion of the theorem now follows by a simple scaling of the construction of the standard meander from a standard bridge which is the known special case $x = 0$ of the theorem, recalled in the following lemma rephrasing Theorem 4.3 in [3]. ■

Lemma 2 *In the notation of Theorem 1,*

$$(B \mid B_\mu = 0) \stackrel{d}{=} B^{\mathrm{me}} \stackrel{d}{=} B^{(max-reflect,0)} \tag{8}$$

where

$$B_t^{(max-reflect,0)} := 2\left(\max_{0\le u\le t} B_u^{\mathrm{br},0}\right) - B_t^{\mathrm{br},0} \qquad (0 \le t \le 1).$$

We now point out another construction of the conditioned process considered in Theorem 1 which involves a time-reversal. See also [16] for another application of a very similar construction. Given a path $\omega = (\omega_t)_{0\le t\le 1}$ and a real number $x \in \{\omega_t, t \in [0,1]\}$, let

$$\ell(x) \;:=\; \sup\{t \in [0,1] : \omega_t = x\}\,,$$

denote the last passage time of ω at level x. Consider the path transformation `Reverse` defined by time-reversing the portion of ω before its last passage time at level x, then tacking on the part after $\ell(x)$. That is

$$\texttt{Reverse}(\omega, x)_t \;:=\; \begin{cases} \omega_{\ell(x)-t} - x & \text{if } t \le \ell(x)\,, \\ \omega_t - 2x & \text{if } \ell(x) \le t \le 1\,. \end{cases}$$

Theorem 3 *With notation as Theorem 1, for each $x < 0$, there is the following equality of distributions on $C[0,1]$:*

$$(B \mid B_\mu = x) \stackrel{d}{=} (\texttt{Reverse}(B^{\mathrm{me}}, |x|) \mid B_1^{\mathrm{me}} > |x|). \tag{9}$$

The path-transformation `Reverse`$(B^{\mathrm{me}}, |x|)$ *is depicted in Figure 2 below.*

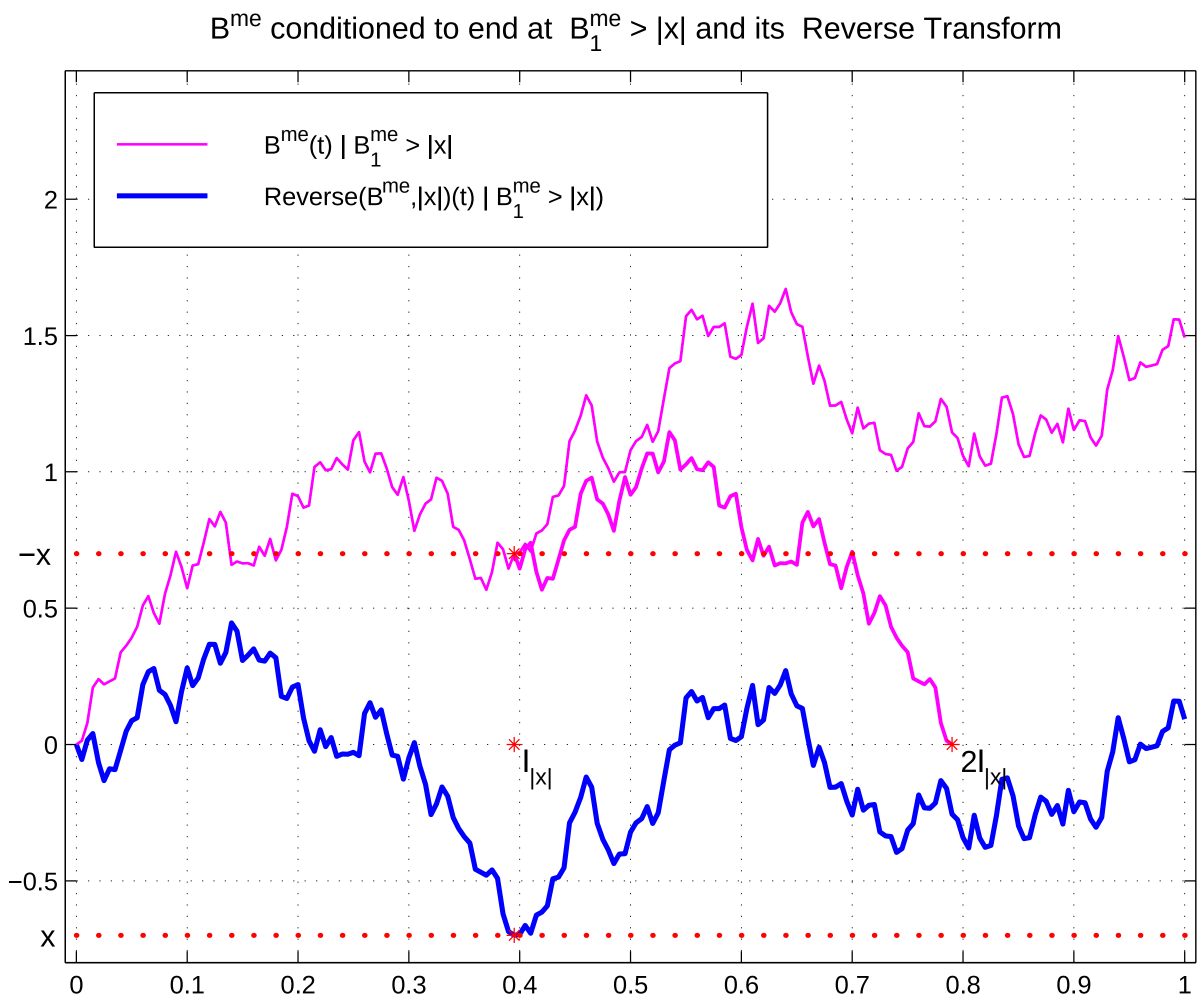

Figure 2

Proof: This is the weak limit, by standard approximation arguments, of a corresponding bijection between the set of n-step lattice paths as in [11], starting at $(0,0)$ whose minimum value is x, and the set of $n+1$ step lattice paths starting at $(0,0)$ which remain strictly positive and terminate at level $|x|+1$ or higher. ■

We point out that the Brownian meander conditioned on having a terminal value greater than $|x|$ which appears in Theorem 3 can be constructed from a Brownian bridge with length 1 from 0 to x by a path transformation similar to that in Lemma 2; see the remark after Theorem 4.3 in [3].

In the same vein, we record also the following result, which is related to Corollary 4 in

[2].

Theorem 4 *Let* $(R_t)_{0\le t\le 1}$ *be a* 3*-dimensional Bessel process, started at* $R_0 = 0$*, and* U *an independent random variable with uniform distribution on* $[0,1]$*. Set*

$$B := \texttt{Reverse}(R, UR_1)\,.$$

Then B *is a standard* 1*-dimensional Brownian motion and*

$$\min_{0\le t\le 1} B_t = -UR_1\,.$$

Proof: This is a variation of the result of [15] that if (R_t) is constructed from a Brownian motion B as $R_t := 2M_t - B_t$, where $M_t := \max_{0\le s\le t} B_s$, then $M_1 = UR_1$ where U is uniform on $[0,1]$ independent of R. There is an exact analog for lattice walks, which can be given a bijective proof and then passed to the limit as in [15] and [14]. ■

Theorem 4 can also be deduced from excursion theory, or by the techniques developed by Biane and Yor [5].

Acknowledgement. We thank Jean-François Le Gall for posing the problem of finding a simple construction of Brownian motion conditioned on its minimum.

References

[1] S. Asmussen, P. Glynn, and J. Pitman. Discretization error in simulation of one-dimensional reflecting Brownian motion. *Ann. Applied Prob.*, 5:875–896, 1995.

[2] J. Bertoin. Décomposition du mouvement brownien avec dérive en un minimum local par juxtaposition de ses excursions positives et negatives. In *Séminaire de Probabilités XXV*, pages 330–344. Springer-Verlag, 1991. Lecture Notes in Math. 1485.

[3] J. Bertoin and J. Pitman. Path transformations connecting Brownian bridge, excursion and meander. *Bull. Sci. Math. (2)*, 118:147–166, 1994.

[4] Ph. Biane. Relations entre pont et excursion du mouvement Brownien réel. *Ann. Inst. Henri Poincaré*, 22:1–7, 1986.

[5] Ph. Biane and M. Yor. Valeurs principales associées aux temps locaux Browniens. *Bull. Sci. Math. (2)*, 111:23–101, 1987.

[6] Ph. Biane and M. Yor. Quelques précisions sur le méandre brownien. *Bull. Sci. Math.*, 112:101–109, 1988.

[7] L. Chaumont. An extension of Vervaat's transformation and its consequences. Prépublication 402, Laboratoire de Probabilités, Université Paris VI, 1997.

[8] K. L. Chung. Excursions in Brownian motion. *Arkiv fur Matematik*, 14:155–177, 1976.

[9] E. Csáki, A. Földes, and P. Salminen. On the joint distribution of the maximum and its location for a linear diffusion. *Annales de l'Institut Henri Poincaré, Section B*, 23:179–194, 1987.

[10] I. V. Denisov. A random walk and a Wiener process near a maximum. *Theor. Prob. Appl.*, 28:821–824, 1984.

[11] W. Feller. *An Introduction to Probability Theory and its Applications,* Vol 1,3rd ed. Wiley, New York, 1968.

[12] J. P. Imhof. Density factorizations for Brownian motion, meander and the three-dimensional Bessel process, and applications. *J. Appl. Probab.*, 21:500–510, 1984.

[13] J. P. Imhof. On Brownian bridge and excursion. *Studia Sci. Math. Hungar.*, 20:1–10, 1985.

[14] J. F. Le Gall. Une approche élémentaire des théorèmes de décomposition de Williams. In *Séminaire de Probabilités XX*, pages 447–464. Springer, 1986. Lecture Notes in Math. 1204.

[15] J. Pitman. One-dimensional Brownian motion and the three-dimensional Bessel process. *Advances in Applied Probability*, 7:511–526, 1975.

[16] J. Pitman. The distribution of local times of Brownian bridge. Technical Report 539, Dept. Statistics, U.C. Berkeley, 1998. To appear in *Séminaire de Probabilités XXXIII.* Available via http://www.stat.berkeley.edu/users/pitman.

[17] J. Pitman and M. Yor. Decomposition at the maximum for excursions and bridges of one-dimensional diffusions. In N. Ikeda, S. Watanabe, M. Fukushima, and H. Kunita, editors, *Itô's Stochastic Calculus and Probability Theory*, pages 293–310. Springer-Verlag, 1996.

[18] D. Revuz and M. Yor. *Continuous martingales and Brownian motion.* Springer, Berlin-Heidelberg, 1999. 3rd edition.

[19] W. Vervaat. A relation between Brownian bridge and Brownian excursion. *Ann. Probab.*, 7:143–149, 1979.

[20] D. Williams. Decomposing the Brownian path. *Bull. Amer. Math. Soc.*, 76:871–873, 1970.

[21] D. Williams. Path decomposition and continuity of local time for one dimensional diffusions I. *Proc. London Math. Soc. (3)*, 28:738–768, 1974.

[22] M. Yor. *Local Times and Excursions for Brownian Motion: a concise introduction*, volume 1 of *Lecciones en Matemáticas.* Postgrado de Matemáticas, Facultad de Ciencias, Universidad Central de Venezuela, Caracas, 1995.