

UNIFORM UPPER BOUND FOR A STABLE MEASURE OF A SMALL BALL

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Abstract

The authors of [1] stated the following conjecture: Let μ be a symmetric α -stable measure on a separable Banach space and B a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b)$, depending only on b , such that $\mu(tB) \leq R(b)t\mu(B)$ for all $0 < t < 1$. We prove that the above inequality holds but the constant R must depend also on α .

Recently, the authors of [1] proved the following (Theorem 6.4 in [1]):

Let μ be a symmetric α -stable measure, $0 < \alpha \leq 2$, on a separable Banach space, fix $b < 1$, and let B denote a centered ball such that $\mu(B) \leq b$. Then there exists a constant $R(b) = \frac{3}{b\sqrt{1-b}}$, depending only on b , such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq R(b)t^{\alpha/2}\mu(B). \quad (1)$$

Of course, for small values of t , the quantity $t^{\alpha/2}$ is much larger than t . The authors of [1] stated in their Conjecture 7.4 that (1) is true for all symmetric α -stable measures with t instead of $t^{\alpha/2}$ and some $R(b)$ depending only on b .

In our earlier paper [3], we also gave an estimate of a stable measure of a small ball. Namely, we proved the following.

Let μ be a symmetric α -stable measure, $0 < \alpha \leq 2$, on a separable Banach space, put $B = \{x : \|x\| \leq 1\}$, let $0 < r < \alpha$ and suppose that μ is so normalized that $\int \|x\|^r \mu(dx) = 1$. Then there exists a constant $K = K(\alpha, r)$ such that for all $0 \leq t \leq 1$

$$\mu(tB) \leq K(\alpha, r)t. \quad (2)$$

Some estimates of $K(\alpha, r)$ were also given in [3], we recall one of them in the final Remark. Some normalization of μ is needed, as we will show in the sequel (see Example), in the paper [3] we chose the normalizing condition $\int \|x\|^r \mu(dx) = 1$. But proving the inequality (2), we also obtained the inequality

$$\mu(tB) \leq K(\alpha, r)[1 - \mu(B)]^{-1/r} t. \quad (3)$$

In this note we will show that using (3) we can prove an estimate that is very close to the above-mentioned conjecture, however, the constant $R(b)$ must depend also on α .

The following is a generalization of (1).

Theorem 1 *Let μ be a symmetric α -stable measure, $0 < \alpha \leq 2$, on a separable Banach space F . Then for every closed, symmetric, convex set $B \subset F$ and for each $b < 1$ there exists $R(\alpha, b)$ such that for all $0 \leq t \leq 1$*

$$\mu(tB) \leq R(\alpha, b)t\mu(B), \quad \text{if } \mu(B) \leq b. \quad (4)$$

First we show that the constant R must depend on α .

Example. Suppose that there exists positive function $R(b)$ that fulfills (4), does not depend on α and is bounded on every closed subinterval of $(0, 1)$. Let X_α be an α -stable random variable with the characteristic function $e^{-|t|^\alpha}$. It is known (see e.g. [4]) that

$$|X_\alpha|^\alpha \xrightarrow{d} \frac{1}{W}, \quad \text{as } \alpha \rightarrow 0+, \quad (5)$$

where W is a random variable having the exponential distribution with mean 1. Consider one-dimensional ball $B = [-1, 1]$. From (5) we infer that

$$b_\alpha = P(X_\alpha \in B) = P(-1 \leq X_\alpha \leq 1) = P(|X_\alpha|^\alpha \leq 1) \xrightarrow{\alpha \rightarrow 0} P\left(\frac{1}{W} \leq 1\right) = \frac{1}{e}.$$

Denote by μ the distribution of X_α . It is easy to compute the value of the density of μ at zero:

$$p_\alpha(0) = \frac{1}{\pi} \int_0^\infty e^{-t^\alpha} dt = \frac{1}{\pi} \Gamma\left(\frac{1}{\alpha}\right).$$

Now

$$\lim_{\alpha \rightarrow 0} \lim_{t \rightarrow 0+} \frac{1}{t} \mu(tB) = \lim_{\alpha \rightarrow 0} \lim_{t \rightarrow 0+} \frac{1}{t} \int_0^t p(x) dx = \lim_{\alpha \rightarrow 0} p_\alpha(0) = \lim_{\alpha \rightarrow 0} \frac{1}{\pi} \Gamma\left(\frac{1}{\alpha}\right) = \infty,$$

and

$$\lim_{\alpha \rightarrow 0} R(b_\alpha) b_\alpha = R\left(\frac{1}{e}\right) \frac{1}{e},$$

contradicting the inequality (4).

This implies that $R(b)$ must also depend on α .

The proof of the theorem is almost the same as the proof of (1) in the paper [1], the difference is that instead of Kanter inequality we use our estimate (3). For the sake of completeness we repeat this proof.

We start with two lemmas.

Lemma 1. Let μ be a symmetric α -stable measure, $0 < \alpha \leq 2$, on a separable Banach space F . Fix $0 < r < \alpha$. Then there exists a constant $K(\alpha, r) \geq 2$ such that for every convex, symmetric, closed set $B \subset F$, every $y \in F$ and all $t \in [0, 1]$ there holds

$$\mu(tB + y) \leq K(\alpha, r) R t \mu(2B + y),$$

where $R = (\mu(B))^{-1}(1 - \mu(B))^{-1/r}$.

Proof. It is well-known that symmetric stable measures are conditionally Gaussian [2], hence they satisfy the Anderson property.

Case 1. If $y \in B$ then $B \subset 2B + y$ so that $\mu(B) \leq \mu(2B + y)$, hence by the Anderson property and (3)

$$\mu(tB + y) \leq \frac{K(\alpha, r)}{(1 - \mu(B))^{1/r}} t \leq \frac{K(\alpha, r)\mu(B)}{\mu(B)(1 - \mu(B))^{1/r}} t \leq \frac{K(\alpha, r)}{\mu(B)(1 - \mu(B))^{1/r}} t \mu(2B + y).$$

Case 2. If $y \notin B$ then take $r = [t^{-1} - 2^{-1}]$. Then for $k = 0, 1, \dots, r$ the balls $\{y_k + tB\}$ are disjoint and contained in $y + 2B$, where $y_k = (1 - 2t\|y\|^{-1}k)y$. By the Anderson property $\mu(y_k + tB) \geq \mu(y + tB)$ for $k = 0, 1, \dots, r$. Therefore

$$\begin{aligned} \mu(tB + y) &\leq (r + 1)^{-1} \mu(2B + y) \leq \frac{2t}{2 - t} \mu(2B + y) \\ &\leq \frac{K(\alpha, r)}{(1 - \mu(B))^{1/r}} \mu(2B + y) t, \end{aligned}$$

because we assumed that $K(\alpha, r) > 2$ and $2 - t \geq 1 > (1 - \mu(B))^{1/r}$.

Lemma 2. With the same assumptions as in Lemma 1, we have for all $0 \leq \kappa, t \leq 1$

$$\mu(\kappa t B) \leq R' t \mu(\kappa B),$$

where $R' = \frac{2K(\alpha, r)}{\mu(B/2)(1 - \mu(B/2))^{1/r}}$.

Proof. For $0 \leq t \leq 1$ define a measure μ_t by the formula $\mu_t(C) = \mu(tC) = P(X/t \in C)$, where X is a symmetric α -stable random variable with the distribution μ . Then μ_t is also α -stable and we have the following equality:

$$\mu * \mu_s(C) = P(X + X'/s \in C) = P((1 + s^{-\alpha})^{1/\alpha} X \in C) = \mu_t(C),$$

where $t = (1 + s^{-\alpha})^{-1/\alpha}$ and X' is an independent copy of X . Now by Lemma 1

$$\mu(\kappa(tB)) = \mu(t(\kappa B)) = P(X/t \in \kappa B) = \mu * \mu_s(\kappa B) = \int_F \mu\left(\frac{2\kappa B}{2} + y\right) \mu_s(dy)$$

$$\leq \frac{K(\alpha, r)2\kappa}{\mu(B/2)(1 - \mu(B/2))^{1/r}} \mu_t(B) = \frac{2K(\alpha, r)}{\mu(B/2)(1 - \mu(B/2))^{1/r}} \kappa \mu(tB).$$

Proof of the Theorem. Fix B with $\mu(B) \leq b$ and take $s \geq 1$ such that $\mu(sB) = b$. Now, in Lemma 2, put $\kappa = t$ and $t = \frac{1}{2s}$. Then

$$\begin{aligned} \mu(tB) &= \mu\left(t \cdot \frac{1}{2s} \cdot (2sB)\right) \leq t \frac{K(\alpha, r)2}{\mu(sB)(1 - \mu(sB))^{1/r}} \mu\left(\frac{1}{2s} \cdot 2sB\right) \leq \\ &\frac{K(\alpha, r)2}{\mu(sB)(1 - \mu(sB))^{1/r}} t \mu(B) = R(b)K(\alpha, r)t \mu(B), \end{aligned}$$

where $R(b) = 2b^{-1}(1 - b)^{-1/r}$. Taking different values of $r \in (0, \alpha)$ we get different values of $K(\alpha, r)$. If, for simplicity, we take $r = \alpha/2$ we get $R(\alpha, b) = K(\alpha, \alpha/2) \frac{2}{b(1-b)^{1-\alpha/2}}$. This ends the proof of the theorem.

Remark. Let us recall some estimates of $K(\alpha, r)$ which were given in the paper [3]. If we take $r = \alpha/2$ then

$$K\left(\alpha, \frac{\alpha}{2}\right) = \frac{1}{2^{1/\alpha}\sqrt{\pi}} \Gamma^{\frac{2}{\alpha}}\left(\frac{\alpha}{4} + \frac{1}{2}\right) \Gamma\left(1 + \frac{2}{\alpha}\right) \inf_{x>0} \frac{1}{x^{2/\alpha}(1 - \Phi(x))},$$

where Φ is the distribution function of a standard normal variable. For different values of r other estimates are possible, it could be interesting to find the least value of $K(\alpha, r)$. Of course, if we consider $\alpha \geq \varepsilon > 0$ then we can find

$$R(b) = \sup_{\varepsilon \leq \alpha \leq 2} R(\alpha, b) < \infty$$

and then for all $0 \leq t \leq 1$ and $\alpha \geq \varepsilon$

$$\mu(tB) \leq R(b)t \mu(B), \quad \text{if } \mu(B) \leq b.$$

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