Electron. Commun. Probab. **20** (2015), no. 68, 1–13.
DOI: 10.1214/ECP.v20-4252
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

The mean spectral measures of random Jacobi matrices related to Gaussian beta ensembles*

Trinh Khanh Duy† Tomoyuki Shirai†

Abstract

An explicit formula for the mean spectral measure of a random Jacobi matrix is derived. The matrix can be regarded as the limit of Gaussian beta ensemble (GβE) matrices as the matrix size N tends to infinity with the constraint that $N\beta$ is a constant.

Keywords: random Jacobi matrix ; Gaussian beta ensemble ; spectral measure ; self-convolutive recurrence.
AMS MSC 2010: Primary 47B80, Secondary 15A52 ; 44A60 ; 47B36.
Submitted to ECP on April 22, 2015, final version accepted on September 21, 2015.
Supersedes arXiv:1504.06904v1.

1 Introduction

The paper studies spectral measures of random (symmetric) Jacobi matrices of the form

$$J_\alpha = \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & & \\ \tilde\chi_{2\alpha} & \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \quad (\alpha > 0),$$

where the diagonal is an i.i.d. (independent identically distributed) sequence of standard Gaussian $\mathcal{N}(0,1)$ random variables, the off diagonal is another i.i.d. sequence of $\tilde\chi_{2\alpha}$-distributed random variables. Here $\tilde\chi_{2\alpha} = \chi_{2\alpha}/\sqrt{2}$ with $\chi_{2\alpha}$ denoting the chi distribution with 2α degrees of freedom. As explained later, J_α is regarded as the limit of Gaussian beta ensembles (GβE for short) as the matrix size N tends to infinity and the parameter β also varies with the constraint that $N\beta = 2\alpha$.

Let us explain some of the terminology and introduce the main results of the paper. A (semi-infinite) Jacobi matrix is a symmetric tridiagonal matrix of the form

$$J = \begin{pmatrix} a_1 & b_1 & & \\ b_1 & a_2 & b_2 & \\ & \ddots & \ddots & \ddots \end{pmatrix}, \text{ where } a_i \in \mathbb{R}, b_i > 0.$$

For a Jacobi matrix J, there is a probability measure μ on $\mathbb{R}$ such that

$$\int_{\mathbb{R}} x^k d\mu = \langle J^k e_1, e_1\rangle = J^k(1,1), \quad k = 0, 1, \dots,$$

*This work is partially supported by Kyushu University Interdisciplinary Programs in Education and Projects in Research Development(T.K.D.) and JSPS Grant-in-Aid for Scientific Research (B) 26287019(T.S.)

†Institute of Mathematics for Industry, Kyushu University. E-mail: trinh,shirai@imi.kyushu-u.ac.jp

where $e_1 = (1, 0, \dots)^T \in \ell^2$. Here $\langle u, v\rangle$ denotes the inner product of u and v in ℓ^2, while $\langle \mu, f\rangle := \int f d\mu$ will be used to denote the integral of a function f with respect to a measure μ. Then the measure μ is unique if and only if J, as a symmetric operator defined on $D_0 = \{x = (x_1, x_2, \dots) : x_k = 0 \text{ for } k \text{ sufficiently large}\}$, is essentially self-adjoint, that is, J has a unique self-adjoint extension in ℓ^2. When the measure μ is unique, it is called the spectral measure of J, or more precisely, the spectral measure of (J, e_1). It is known that the condition

$$\sum_{i=1}^{\infty} \frac{1}{b_i} = \infty$$

implies the essential self-adjointness of J, [8, Corollary 3.8.9].

For the random Jacobi matrix J_α, the above condition holds almost surely because its off diagonal elements are positive i.i.d. random variables. Thus its spectral measures μ_α are uniquely determined by the following relations

$$\langle \mu_\alpha, x^k\rangle = J_\alpha^k(1,1), \quad k = 0, 1, \dots.$$

The mean spectral measure $\bar\mu_\alpha$ is defined to be a probability measure satisfying

$$\langle \bar\mu_\alpha, f\rangle = \mathbb{E}[\langle \mu_\alpha, f\rangle],$$

for all bounded continuous functions f on $\mathbb{R}$. It then follows that

$$\langle \bar\mu_\alpha, x^k\rangle = \mathbb{E}[\langle \mu_\alpha, x^k\rangle], \quad k = 0, 1, \dots,$$

provided that the right hand side of the above equation is finite for all k.

The purpose of this paper is to identify the mean spectral measure $\bar\mu_\alpha$. Our main results are as follows.

Theorem 1.1 (Main result)**.**

(i) *The mean spectral measure $\bar\mu_\alpha$ coincides with the spectral measure of the non-random Jacobi matrix A_α, where*

$$A_\alpha = \begin{pmatrix} 0 & \sqrt{\alpha+1} & & \\ \sqrt{\alpha+1} & 0 & \sqrt{\alpha+2} & \\ & \ddots & \ddots & \ddots \end{pmatrix}.$$

(ii) *The measure $\bar\mu_\alpha$ has the following density function*

$$\bar\mu_\alpha(y) = \frac{e^{-y^2/2}}{\sqrt{2\pi}} \frac{1}{|\hat f_\alpha(y)|^2},$$

where

$$\hat f_\alpha(y) = \sqrt{\frac{2}{\pi}} \int_0^\infty f_\alpha(t) e^{iyt} dt, \quad f_\alpha(t) = \pi \sqrt{\frac{\alpha}{\Gamma(\alpha)}} t^{\alpha-1} \frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}}.$$

Let us sketch out main ideas for the proof of the above theorem. To show the first statement, the key idea is to regard the Jacobi matrix J_α as the limit of GβE as the matrix size N tends to infinity with $N\beta = 2\alpha$. More specifically, let $T_N(\beta)$ be a finite random Jacobi matrix whose components are (up to the symmetry constraints) independent and are distributed as

$$T_N(\beta) = \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{(N-1)\beta} & & \\ \tilde\chi_{(N-1)\beta} & \mathcal{N}(0,1) & \tilde\chi_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \tilde\chi_\beta & \mathcal{N}(0,1) \end{pmatrix}.$$

Then it is well known in random matrix theory that the eigenvalues of $T_N(\beta)$ are distributed as GβE, namely,

$$(\lambda_1, \dots, \lambda_N) \propto \prod_{l=1}^{N} e^{-\lambda_l^2/2} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta.$$

Moreover, by letting $N \to \infty$ with $\beta = 2\alpha/N$, the matrices $T_N(\beta)$ converge, in some sense, to J_α. That crucial observation together with a result on moments of GβE ([4, Theorem 2.8]) makes it possible to show that $\bar{\mu}_\alpha$ coincides with the spectral measure of A_α.

The next step is to establish the following self-convolutive recurrence for even moments of $\bar{\mu}_\alpha$,

$$u_n(\alpha) = (2n-1)u_{n-1}(\alpha) + \alpha \sum_{i=0}^{n-1} u_i(\alpha) u_{n-1-i}(\alpha),$$

where $u_n(\alpha)$ is the $2n$th moment of $\bar{\mu}_\alpha$. Note that its odd moments are all vanishing because the spectral measure of A_α is symmetric. Finally, the explicit formula for $\bar{\mu}_\alpha$ is derived by using the method in [6].

The paper is organized as follows. In the next section, we mention some known results on GβE needed in this paper. In Section 3, we introduce the matrix model and step by step, prove the main theorem.

2 A result on Gaussian β-ensembles

The Jacobi matrix model for GβE, a finite random Jacobi matrix, was discovered by Dumitriu and Edelman [3]. First of all, let us mention some preliminary facts about finite Jacobi matrices. Assume that J is a (symmetric) finite Jacobi matrix of order N (with the requirement that the off diagonal elements are all positive). Then the matrix J has exactly N distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_N$. Let $v_1, v_2, \dots, v_N$ be the corresponding eigenvectors which are chosen to be an orthonormal basis in $\mathbb{R}^N$. Then the spectral measure μ, which is well defined by $\langle \mu, x^k \rangle = J^k(1,1), k = 0, 1, \dots$, can be expressed as

$$\mu = \sum_{j=1}^{N} q_j^2 \delta_{\lambda_j}, \quad q_j = |v_j(1)|,$$

where δ_λ denotes the Dirac measure. It is known that a finite Jacobi matrix of order N is one-to-one correspondence with a probability measure supported on N points, or a set of Jacobi matrix parameters $\{a_i\}_{i=1}^N, \{b_j\}_{j=1}^{N-1}$ is one-to-one correspondence with the spectral data $\{\lambda_i\}_{i=1}^N, \{q_j\}_{j=1}^N$.

The Jacobi matrix model for GβE is defined as follows. Let $\{a_i\}_{i=1}^N$ be an i.i.d. sequence of standard Gaussian $\mathcal{N}(0,1)$ random variables and $\{b_j\}_{j=1}^{N-1}$ be a sequence of independent random variables having $\tilde{\chi}$ distributions with parameters $(N-1)\beta, (N-2)\beta, \dots, \beta$, respectively, which is independent of $\{a_i\}_{i=1}^N$. Here $\tilde{\chi}_k$, for $k > 0$, denotes the distribution with the following probability density function

$$\frac{2}{\Gamma(k/2)} u^{k-1} e^{-u^2}, u > 0,$$

which is nothing but $\chi_k/\sqrt{2}$, or the square root of the gamma distribution with parameter $(k/2, 1)$. We form a random Jacobi matrix $T_N(\beta)$ from $\{a_i\}_{i=1}^N$ and $\{b_j\}_{j=1}^{N-1}$ as follows,

$$T_N(\beta) = \begin{pmatrix} \mathcal{N}(0,1) & \tilde{\chi}_{(N-1)\beta} & & \\ \tilde{\chi}_{(N-1)\beta} & \mathcal{N}(0,1) & \tilde{\chi}_{(N-2)\beta} & \\ & \ddots & \ddots & \ddots \\ & & \tilde{\chi}_\beta & \mathcal{N}(0,1) \end{pmatrix}.$$

Then the eigenvalues $\{\lambda_i\}_{i=1}^N$ and the weights $\{q_j\}_{j=1}^N$ are independent, with the distribution of the former given by

$$(\lambda_1, \lambda_2, \dots, \lambda_N) \propto \prod_{l=1}^{N} e^{-\lambda_l^2/2} \prod_{1 \le j < k \le N} |\lambda_k - \lambda_j|^\beta,$$

and the distribution of the latter given by

$$(q_1, q_2, \dots, q_N) \propto \frac{1}{q_N} \prod_{i=1}^{N} q_i^{\beta-1}, \quad (q_i > 0, \sum_{i=1}^{N} q_i^2 = 1).$$

It is also known that $q = (q_1, \dots, q_N)$ is distributed as a vector $(\tilde{\chi}_\beta, \dots, \tilde{\chi}_\beta)$ with i.i.d. components, normalized to unit length.

The trace of $T_N(\beta)^n$ and $T_N(\beta)^n(1,1)$ can be expressed in term of the spectral data as

$$\mathrm{Tr}(T_N(\beta)^n) = \sum_{j=1}^{N} \lambda_j^n, \quad T_N(\beta)^n(1,1) = \sum_{j=1}^{N} q_j^2 \lambda_j^n.$$

Consequently,

$$\begin{aligned} \mathbb{E}[T_N(\beta)^n(1,1)] &= \mathbb{E}[\sum_{j=1}^{N} q_j^2 \lambda_j^n] = \sum_{j=1}^{N} \mathbb{E}[q_j^2] \mathbb{E}[\lambda_j^n] = \frac{1}{N} \sum_{j=1}^{N} \mathbb{E}[\lambda_j^n] \\ &= \frac{1}{N} \mathbb{E}[\mathrm{Tr}(X_N(\beta)^n)]. \end{aligned}$$

In the rest of this section, for convenience, we use the parameter $\hat{\beta} = \beta/2$. Let $m_p(N, \hat{\beta}) = \mathbb{E}[T_N(2\hat{\beta})^{2p}(1,1)]$. It is clear that $m_p(N, \hat{\beta})$ is a polynomial of degree p in N, and thus $m_p(N, \hat{\beta})$ is defined for all $N \in \mathbb{R}$. Then a result for the trace of $T_N(\beta)^n$ can be rewritten for $m_p(N, \hat{\beta})$ as follows.

Theorem 2.1 (cf. [4, Theorem 2.8] and [9, Theorem 2])**.** *It holds that*

$$m_p(N, \hat{\beta}) = (-1)^p \hat{\beta}^p m_p(-\hat{\beta} N, \hat{\beta}^{-1}).$$

Observe that $\tau^{-p} m_p(N, \tau)$ is the expectation of the $2p$th moment of the spectral measure of the following Jacobi matrix

$$\frac{1}{\sqrt{\tau}} T_N(2\tau) = \frac{1}{\sqrt{\tau}} \begin{pmatrix} \mathcal{N}(0,1) & \tilde{\chi}_{(N-1)2\tau} & & \\ \tilde{\chi}_{(N-1)2\tau} & \mathcal{N}(0,1) & \tilde{\chi}_{(N-2)2\tau} & \\ & \ddots & \ddots & \ddots \\ & & \tilde{\chi}_{2\tau} & \mathcal{N}(0,1) \end{pmatrix}.$$

As $\tau \to \infty$, it holds that

$$\frac{\mathcal{N}(0,1)}{\sqrt{\tau}} \to 0, \quad \frac{\tilde{\chi}_{k2\tau}}{\sqrt{\tau}} = \left(\frac{\Gamma(k\tau, 1)}{\tau} \right)^{1/2} \to \sqrt{k} \text{ (in } L^q \text{ for any } q \ge 1).$$

The convergences also hold almost surely. Therefore as $\tau \to \infty$,

$$\frac{1}{\sqrt{\tau}} T_N(2\tau) \to \begin{pmatrix} 0 & \sqrt{N-1} & & \\ \sqrt{N-1} & 0 & \sqrt{N-2} & \\ & \ddots & \ddots & \ddots \\ & & 1 & 0 \end{pmatrix} =: H_N.$$

Here the convergence of matrices means the convergence (in L^q) of their elements. Let $h_p(N) = H_N^{2p}(1,1)$ for $N > p$. Then $h_p(N)$ is a polynomial of degree p in N so that $h_p(N)$ is defined for all $N \in \mathbb{R}$. The above convergence of matrices implies that for fixed p and fixed N,

$$h_p(N) = \lim_{\tau\to\infty} \tau^{-p} m_p(N,\tau). \tag{2.1}$$

Let

$$A_\alpha = \begin{pmatrix} 0 & \sqrt{\alpha+1} & & \\ \sqrt{\alpha+1} & 0 & \sqrt{\alpha+2} & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

and let $u_p(\alpha) = A_\alpha^{2p}(1,1)$. Then $u_p(\alpha)$ is also a polynomial of degree p in α. In addition, it is easy to see that

$$u_p(\alpha) = (-1)^p h_p(-\alpha). \tag{2.2}$$

As a direct consequence of Theorem 2.1 and relations (2.1) and (2.2), we get the following result.

Proposition 2.2. *As $N \to \infty$ with $\hat\beta = \hat\beta(N) = \alpha/N$,*

$$m_p(N, \hat\beta) \to u_p(\alpha) = A_\alpha^{2p}(1,1).$$

3 Random Jacobi matrices related to Gaussian β ensembles

3.1 A matrix model and proof of Theorem 1.1(i)

Recall that the random Jacobi matrix J_α,

$$J_\alpha = \begin{pmatrix} \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & & \\ \tilde\chi_{2\alpha} & \mathcal{N}(0,1) & \tilde\chi_{2\alpha} & \\ & \ddots & \ddots & \ddots \end{pmatrix},$$

consists of two i.i.d. sequence of random variables, one for the diagonal and the other for the off diagonal. Thus the spectral measure μ_α of J_α exists and is unique almost surely because

$$\sum_{j=1}^\infty \frac{1}{b_j} = \infty \text{(almost surely)}.$$

Here $\{b_j\}$ denotes the off diagonal elements.

The mean spectral measure $\bar\mu_\alpha$ is defined to be a probability measure satisfying

$$\langle \bar\mu_\alpha, f\rangle = \mathbb{E}[\langle \mu_\alpha, f\rangle],$$

for all bounded continuous functions f on $\mathbb{R}$. Then Theorem 1.1(i) states that the measure $\bar\mu_\alpha$ coincides with the spectral measure of (A_α, e_1).

Proof of Theorem 1.1(i). Note that the spectral measure of A_α, a probability measure μ satisfying

$$\langle \mu, x^k\rangle = A_\alpha^k(1,1), \quad k = 0,1,\dots,$$

is unique because

$$\sum_{j=1}^\infty \frac{1}{\sqrt{\alpha+j}} = \infty.$$

Also, it is clear that

$$\langle \bar\mu_\alpha, x^k\rangle = \mathbb{E}[\langle \mu_\alpha, x^k\rangle], \quad k = 0,1,\dots,$$

because $\mathbb{E}[\langle \mu_\alpha, |x|^k \rangle] < \infty$ for all $k = 0, 1, \dots$. Therefore, our task is now to show that for all $k = 0, 1, \dots,$

$$\langle \bar{\mu}_\alpha, x^k \rangle = A_\alpha^k(1,1). \tag{3.1}$$

We consider the case of even k first. For any fixed j, all moments of the $\tilde{\chi}_{(N-j)2\hat{\beta}}$ distribution converge to those of the $\tilde{\chi}_{2\alpha}$ distribution as $N \to \infty$ with $\hat{\beta} = \alpha/N$. Thus for fixed p, as $N \to \infty$ with $\hat{\beta} = \alpha/N$,

$$m_p(N, \hat{\beta}) = \mathbb{E}[T_N(2\hat{\beta})^{2p}(1,1)] \to \mathbb{E}[J_\alpha^{2p}(1,1)] = \mathbb{E}[\langle \mu_\alpha, x^{2p} \rangle].$$

Consequently, for even k, namely, $k = 2p$,

$$\langle \bar{\mu}_\alpha, x^k \rangle = A_\alpha^k(1,1),$$

by taking into account Proposition 2.2.

For odd k, both sides of the equation (3.1) are zeros. Indeed, $A_\alpha^k(1,1) = 0$ when k is odd because the diagonal of A_α is zero. Also all odd moments of $\bar{\mu}_\alpha$ are vanishing,

$$\langle \bar{\mu}_\alpha, x^{2p+1} \rangle = \mathbb{E}[\langle \mu_\alpha, x^{2p+1} \rangle] = 0,$$

because the expectation of odd moments of any diagonal element of J_α are zero. The proof is completed. □

3.2 Moments of the spectral measure of A_α

Recall that

$$u_n(\alpha) = A_\alpha^{2n}(1,1), n = 0, 1, \dots.$$

Proposition 3.1.

(i) $u_n(\alpha)$ *is a polynomial of degree* n *in* α *and satisfies the following relations*

$$\begin{cases} u_n(\alpha) = (\alpha+1)\sum_{i=0}^{n-1} u_i(\alpha+1)u_{n-1-i}(\alpha), \quad n \ge 1, \\ u_0(\alpha) = 1. \end{cases} \tag{3.2}$$

(ii) $\{u_n(\alpha)\}_{n=0}^\infty$ *also satisfies the following relations*

$$\begin{cases} u_n(\alpha) = (2n-1)u_{n-1}(\alpha) + \alpha\sum_{i=0}^{n-1} u_i(\alpha)u_{n-1-i}(\alpha), \quad n \ge 1, \\ u_0(\alpha) = 1. \end{cases} \tag{3.3}$$

Remark 3.2. The sequences $\{u_n(\alpha)\}_{n\ge 0}$, for $\alpha = 1$ and $\alpha = 2$, are the sequences A000698 and A167872 in the On-line Encyclopedia of Integer Sequences [7], respectively. Relations (3.2) and (3.3) as well as many interesting properties for those sequences can be found in the above reference. In the proof below, we give another explanation of $u_n(\alpha)$ as the total sum of weighted Dyck paths of length $2n$.

Proof. In this proof, for convenience, let the index of the matrix A_α start from 0. Since the diagonal of A_α is zero, it follows that

$$A_\alpha^{2n}(0,0) = \sum_{\{i_0, i_1, \dots, i_{2n}\} \in \mathfrak{D}_{2n}} \prod_{j=0}^{2n-1} A_\alpha(i_j, i_{j+1}),$$

where $\mathfrak{D}_{2n}$ denotes the set of indices $\{i_0, i_1, \dots, i_{2n}\}$ satisfying that

$$i_0 = 0, i_{2n} = 0, i_j \ge 0,$$
$$|i_{j+1} - i_j| = 1, j = 0, 1, \dots, 2n-1.$$

Each element in $\mathfrak{D}_{2n}$ corresponds to a path of length $2n$ consisting of rise steps or rises and fall steps or falls which starts at $(0,0)$ and ends at $(2n,0)$, and stays above the x-axis, called a Dyck path. We also use $\mathfrak{D}_{2n}$ to denote the set of all Dyck paths of length $2n$.

A Dyck path p is assigned a weight $w(p)$ as follows. We assign a weight $(\alpha+k+1)$ for each rise step from level k to $k+1$, and the weight $w(p)$ is the product of all those weights. Then

$$u_n(\alpha) = A_\alpha^{2n}(0,0) = \sum_{p\in\mathfrak{D}_{2n}} w(p).$$

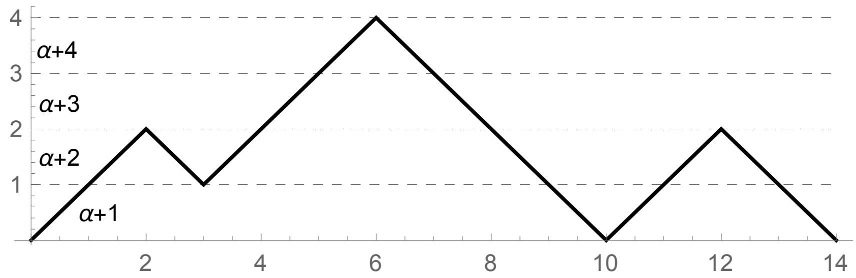

Figure 1: A Dyck path p with weight $w(p) = (\alpha+1)^2(\alpha+2)^3(\alpha+3)(\alpha+4)$.

Let $\mathfrak{D}_{2n}^*$ be the set of all Dyck paths of length $2n$ which do not meet the x-axis except the starting and the ending points. Let

$$v_n(\alpha) = \sum_{p\in\mathfrak{D}_{2n}^*} w(p).$$

Since each Dyck path $p=(i_0,i_1,\dots,i_{2n-1},i_{2n})\in\mathfrak{D}_{2n}^*$ is one-to-one correspondence with a Dyck path $q=(i_1-1,i_2-1,\dots,i_{2n-1}-1)$ of length $2(n-1)$, it follows that

$$v_n(\alpha) = (\alpha+1)u_{n-1}(\alpha+1).$$

Moreover, let $2i$ be the first time that the Dyck path p meets the x-axis. Then either $i=n$ or the Dyck path p is the concatenation of a Dyck path in $\mathfrak{D}_{2i}^*,(1\le i<n)$, and another Dyck path of length $2(n-i)$. Thus,

$$\begin{aligned}u_n(\alpha) &= v_n(\alpha) + \sum_{i=1}^{n-1} v_i(\alpha)u_{n-i}(\alpha)\\ &= (\alpha+1)u_{n-1}(\alpha+1) + \sum_{i=1}^{n-1}(\alpha+1)u_{i-1}(\alpha+1)u_{n-i}(\alpha)\\ &= (\alpha+1)\sum_{i=0}^{n-1} u_i(\alpha+1)u_{n-1-i}(\alpha).\end{aligned}$$

The proof of (i) is complete. We will prove the second statement after the next lemma. □

Lemma 3.3. *Let $\alpha\ge 0$ be fixed. Let $\{a_n\}$ be a sequence defined recursively by*

$$\begin{cases} a_n = (2n-1)a_{n-1} + \alpha\sum_{i=0}^{n-1} a_i a_{n-1-i}, & n\ge 1,\\ a_0 = 1.\end{cases} \tag{3.4}$$

Let $\{b_n\}$ be a sequence defined by the following relations $b_0=1$,

$$a_n = (\alpha+1)\sum_{i=0}^{n-1} b_i a_{n-1-i}, \quad n\ge 1. \tag{3.5}$$

Then $\{b_n\}$ satisfies an analogous recursive relation as $\{a_n\}$,

$$\begin{cases} b_n = (2n-1)b_{n-1} + (\alpha+1)\sum_{i=0}^{n-1} b_i b_{n-1-i}, \quad n \ge 1, \\ b_0 = 1. \end{cases} \tag{3.6}$$

Proof. Consider the field of formal Laurent series over $\mathbb{R}$, denoted by $\mathbb{R}((X))$,

$$\mathbb{R}((X)) = \left\{ f(X) = \sum_{n\in\mathbb{Z}} c_n X^n : c_n \in \mathbb{R}, c_n = 0 \text{ for } n < n_0 \right\}.$$

The addition is defined as usual and the multiplication is well defined as

$$f(X)g(X) = \sum_{n\in\mathbb{Z}} \left(\sum_{i\in\mathbb{Z}} c_i d_{n-i} \right) X^n,$$

for $f(X) = \sum c_n X^n, g(X) = \sum d_n X^n \in \mathbb{R}((X))$. The quotient $f(X)/g(X)$ is understood as $f(X)g(X)^{-1}$ for $g(X) \neq 0$. The formal derivative is also defined as

$$f'(X) = \sum_{n\in\mathbb{Z}} c_n n X^{n-1} \in \mathbb{R}((X)).$$

Now let

$$f(X) = \sum_{n=0}^{\infty} a_n X^n, \quad g(X) = \sum_{n=0}^{\infty} b_n X^n.$$

It is straightforward to show that the recursive relation (3.4) is equivalent to the following equation

$$f(X) - 1 = 2X^2 f'(X) + Xf(X) + \alpha X f^2(X).$$

In addition, the relation (3.5) leads to

$$g(X) = \frac{f(X)-1}{(\alpha+1)Xf(X)}.$$

Finally, we can easily check that $g(X)$ satisfies

$$g(X) - 1 = 2X^2 g'(X) + Xg(X) + (\alpha+1)Xg^2(X),$$

which is equivalent to the recursive relation (3.6). The proof is complete. □

Proof of Proposition 3.1(ii). When $\alpha = 0$, it is well known that $u_n(0)$ is the $2n$th moment of the standard Gaussian distribution, and is given by

$$u_n(0) = (2n-1)!!.$$

Consequently, the conditions in Lemma 3.3 are satisfied for $a_n = u_n(0), b_n = u_n(1)$ and $\alpha = 0$. It follows that the recursive relation (3.3) then holds for $\alpha = 1$. Continue this way, it follows that the recursive relation (3.3) holds for any $\alpha \in \mathbb{N}$. We conclude that it holds for all α because of the fact that $\{u_n(\alpha)\}$ is a polynomial of degree n in α. The proof is complete. □

3.3 Explicit formula for the spectral measure of A_α, proof of Theorem 1.1(ii)

In this section, by using the method of Martin and Kearney [6], we derive the explicit formula for the mean spectral measure $\bar\mu_\alpha$ from the relation (3.3),

$$\begin{cases} u_n(\alpha) = (2n-1)u_{n-1}(\alpha) + \alpha \sum_{i=0}^{n-1} u_i(\alpha)u_{n-1-i}(\alpha), \quad n \ge 1, \\ u_0(\alpha) = 1. \end{cases}$$

Recall that $u_n(\alpha) = \langle \bar\mu_\alpha, x^{2n}\rangle$ and $\bar\mu_\alpha$ is a symmetric probability measure.

Let us extract here the main result of [6]. The problem is to find a function ν for which

$$\int_0^\infty x^{n-1}\nu(x)dx = u_n, \quad n = 1, 2, \dots,$$

where the sequence $\{u_n\}$ is given by a general self-convolutive recurrence

$$\begin{cases} u_n = (\alpha_1 n + \alpha_2)u_{n-1} + \alpha_3 \sum_{i=1}^{n-1} u_i u_{n-i}, \quad n \ge 2, \\ u_1 = 1, \end{cases} \tag{3.7}$$

α_1, α_2 and α_3 being constants. Then the solution is given by (Eq. (13)–Eq. (16) in [6]),

$$\nu(x) = \frac{k(kx)^{-b}e^{-kx}}{\Gamma(a+1)\Gamma(a-b+1)} \frac{1}{U_R(kx)^2 + U_I(kx)^2},$$

where,

$$U_R(x) = e^{-x}\Bigg(\frac{\Gamma(1-b)}{\Gamma(a-b+1)} {}_1F_1(b-a; b; x) - (\cos\pi b)\frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1(1-a; 2-b; x)\Bigg),$$

$$U_I(x) = (\sin\pi b)e^{-x}\frac{\Gamma(b-1)}{\Gamma(a)} x^{1-b} {}_1F_1(1-a; 2-b; x),$$

and $k = 1/\alpha_1, a = \alpha_3/\alpha_1, b = -1 - \alpha_2/\alpha_1$, provided $\alpha_1 \ne 0$. Here ${}_1F_1(a; b; z)$ is the Kummer function.

The sequence $\{u_n(\alpha)\}_{n\ge 0}$ is a particular case of the self-convolutive recurrence (3.7) with parameters $\alpha_1 = 2, \alpha_2 = -3$ and $\alpha_3 = \alpha$. Note that our sequence $\{u_n(\alpha)\}$ starts from $n = 0$, and thus $\alpha_2 = -3$. By direct calculation, we get $k = 1/2, a = \alpha/2$, and $b = 1/2$. Therefore, the function $\nu_\alpha(x)$ for which $u_n(\alpha) = \int_0^\infty x^n d\nu_\alpha(x)dx, n = 0, 1, \dots,$ is given by

$$\nu_\alpha(x) = \frac{1}{\sqrt{2}\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac12)} \frac{1}{\sqrt{x}} e^{-\frac{x}{2}} \frac{1}{U_R(x/2)^2 + U_I(x/2)^2}, \quad x > 0,$$

where

$$U_R(x) = e^{-x}\frac{\Gamma(\frac12)}{\Gamma(\frac{\alpha}{2}+\frac12)} {}_1F_1(\frac12 - \frac{\alpha}{2}; \frac12; x), \tag{3.8}$$

$$U_I(x) = e^{-x}\frac{\Gamma(-\frac12)}{\Gamma(\frac{\alpha}{2})} x^{1/2} {}_1F_1(1 - \frac{\alpha}{2}; \frac32; x). \tag{3.9}$$

It is clear that $\nu_\alpha(x) > 0$ for any $x > 0$. Now it is easy to check that the function $\bar\mu_\alpha(y)$ defined by

$$\bar\mu_\alpha(y) = |y|\nu_\alpha(y^2), \quad y \in \mathbb{R},$$

satisfies the following relations

$$\int_{\mathbb{R}} y^{2n+1}\bar\mu_\alpha(y)dy = 0, \quad \int_{\mathbb{R}} y^{2n}\bar\mu_\alpha(y)dy = u_n(\alpha), \quad n = 0, 1, \dots,$$

In other words, $\bar{\mu}_\alpha(y)$ is the density of the mean spectral measure $\bar{\mu}_\alpha$ with respect to the Lebesgue measure.

We are now in a position to simplify the explicit formula of $\bar{\mu}_\alpha$. Let

$$\begin{aligned}V_R(y) &= \left(\frac{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^{1/2} U_R(y^2/2),\\ &= 2^{-\frac{\alpha}{2}}\Gamma(\alpha+1)^{\frac{1}{2}}\frac{\Gamma(\frac{1}{2})}{\Gamma(\frac{\alpha}{2}+\frac{1}{2})}e^{-\frac{y^2}{2}}{}_1F_1(\frac{1}{2}-\frac{\alpha}{2};\frac{1}{2};\frac{y^2}{2}),\\ V_I(y) &= -\left(\frac{\Gamma(\frac{\alpha}{2}+1)\Gamma(\frac{\alpha}{2}+\frac{1}{2})}{\Gamma(\frac{1}{2})}\right)^{1/2} U_I(y^2/2)\\ &= -2^{-\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\alpha+1)^{\frac{1}{2}}\frac{\Gamma(-\frac{1}{2})}{\Gamma(\frac{\alpha}{2})}ye^{-\frac{y^2}{2}}{}_1F_1(1-\frac{\alpha}{2};\frac{3}{2};\frac{y^2}{2}).\end{aligned}$$

Here, in the above expressions, we have used the following relation for the gamma function

$$\frac{\Gamma(\frac{\alpha}{2}+\frac{1}{2})\Gamma(\frac{\alpha}{2}+1)}{\Gamma(\frac{1}{2})} = 2^{-\alpha}\Gamma(\alpha+1). \tag{3.10}$$

Then $\bar{\mu}_\alpha(y)$ can be written as

$$\bar{\mu}_\alpha(y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}\frac{1}{V_R(y)^2+V_I(y)^2}.$$

Next, we will show that $V_R(y)$ and $V_I(y)$ are the Fourier cosine transform and Fourier sine transform of

$$f_\alpha(t) = \pi\sqrt{\frac{\alpha}{\Gamma(\alpha)}}t^{\alpha-1}\frac{e^{-\frac{t^2}{2}}}{\sqrt{2\pi}},$$

respectively. Let us now give definitions of Fourier transforms. The Fourier transform of a function $f\colon \mathbb{R}\to\mathbb{C}$ is defined to be

$$\mathcal{F}(f)(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)e^{iyt}dt, \quad (y\in\mathbb{R}),$$

and the Fourier cosine transform, the Fourier sine transform are defined to be

$$\begin{aligned}\mathcal{F}_c(f)(y) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\cos(yt)dt, \quad (y>0),\\ \mathcal{F}_s(f)(y) &= \sqrt{\frac{2}{\pi}}\int_0^\infty f(t)\sin(yt)dt, \quad (y>0),\end{aligned}$$

respectively. Then those transforms are related as follows

$$\begin{cases}\mathcal{F}(f)(y) = \mathcal{F}_c(f)(y), \quad (y\ge 0), & \text{if } f(t) \text{ is even},\\ \mathcal{F}(f)(y) = i\mathcal{F}_s(f)(y), \quad (y\ge 0), & \text{if } f(t) \text{ is odd}.\end{cases}$$

For $\alpha>0$, we have (cf. Formula 3.952(8) in [5])

$$\mathcal{F}_c(t^{\alpha-1}e^{-\frac{t^2}{2}}) = \frac{2^{\frac{\alpha}{2}-\frac{1}{2}}\Gamma(\frac{\alpha}{2})}{\sqrt{\pi}}e^{-\frac{y^2}{2}}{}_1F_1(\frac{1}{2}-\frac{\alpha}{2};\frac{1}{2};\frac{y^2}{2}).$$

Then by some simple calculations, we arrive at the following relation

$$V_R(y) = \mathcal{F}_c(f_\alpha(t))(y), \quad y\ge 0.$$

Similarly,

$$V_I(y) = \mathcal{F}_s(f_\alpha(t))(y), \quad y \geq 0,$$

by using Formula 3.952(7) in [5],

$$\mathcal{F}_s(t^{\alpha-1}e^{-\frac{t^2}{2}}) = \frac{2^{\frac{\alpha}{2}}\Gamma(\frac{\alpha}{2}+\frac{1}{2})}{\sqrt{\pi}} ye^{-\frac{y^2}{2}} {}_1F_1(1-\frac{\alpha}{2};\frac{3}{2};\frac{y^2}{2}).$$

By definitions, $V_R(y)$ is an even function and $V_I(y)$ is an odd function. Thus the following expression holds for all $y \in \mathbb{R}$,

$$\begin{aligned} V_R(y) + iV_I(y) &= \sqrt{\frac{2}{\pi}} \int_0^\infty f_\alpha(t)(\cos(yt) + i\sin(yt)dt \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty f_\alpha(t) e^{iyt} dt =: \hat{f}_\alpha(y). \end{aligned}$$

Consequently,

$$V_R(y)^2 + V_I(y)^2 = |\hat{f}_\alpha(y)|^2,$$

which completes the proof of Theorem 1.1(ii).

Remark 3.4. The measure $\bar{\mu}_\alpha$ was discussed as the probability measure of associated Hermite polynomials [2]. It was also investigated in [1] in connection with Gaussian beta ensembles by deriving a partial differential equation for its Stieljes transform. The authors would like to thank Professor Fumihiko Nakano for letting us know these references.

We plot the graph of the density $\bar{\mu}_\alpha(y)$ for several values α as in the following figure by using Mathematica. It follows from the Jacobi matrix form that the spectral measure of $\frac{1}{\sqrt{\alpha}} A_\alpha$ converges weakly to the semicircle law as α tends to infinity. Note that the semicircle law, the probability measure supported on $[-2, 2]$ with the density

$$\frac{1}{2\pi}\sqrt{4-x^2}, (-2 \leq x \leq 2),$$

is the spectral measure of the following Jacobi matrix

$$\begin{pmatrix} 0 & 1 & & & \\ 1 & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & \ddots & \ddots & \ddots \end{pmatrix}.$$

Remark 3.5. When α in a positive integer number, we can give even more explicit expressions for $V_R(y)$ and $V_I(y)$.

(i) $\alpha = 2n, n \in \mathbb{N}$. In this case, $f_\alpha(t)$ is an odd function. Therefore

$$V_I(y) = \mathcal{F}_s(f_\alpha(t)) = -i\mathcal{F}(f_\alpha(t)).$$

Note that

$$\mathcal{F}(e^{-\frac{t^2}{2}}) = e^{-\frac{y^2}{2}}.$$

Therefore, for integer $\alpha \geq 1$,

$$\mathcal{F}(t^{\alpha-1}e^{-\frac{t^2}{2}}) = (i)^{\alpha-1}\frac{d^{\alpha-1}}{dy^{\alpha-1}}(e^{-\frac{y^2}{2}}).$$

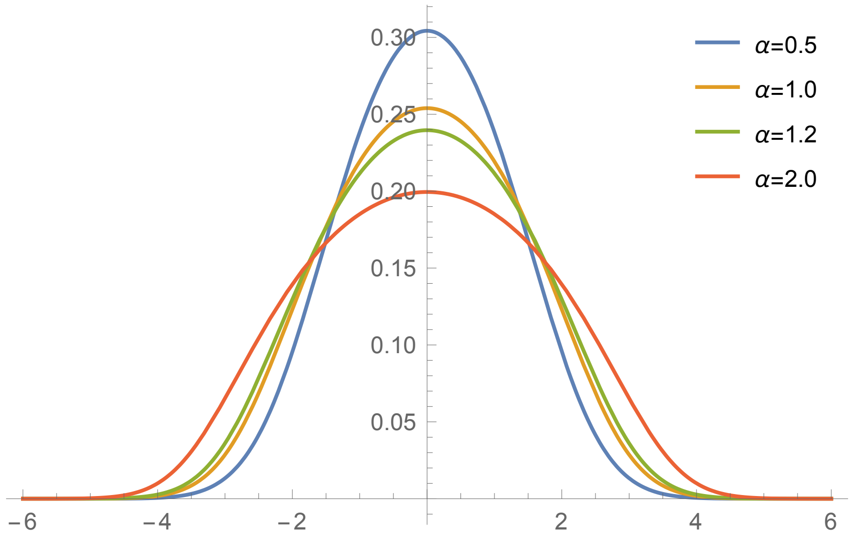

Figure 2: The density $\bar{\mu}_\alpha(y)$ for several values α.

Consequently,

$$\begin{aligned} V_I(y) &= -i^\alpha \pi \sqrt{\frac{\alpha}{\Gamma(\alpha)}} \frac{d^{\alpha-1}}{dy^{\alpha-1}} (e^{-\frac{y^2}{2}}) \frac{1}{\sqrt{2\pi}} \\ &= -i^\alpha \pi \sqrt{\frac{\alpha}{\Gamma(\alpha)}} e^{\frac{y^2}{2}} \frac{d^{\alpha-1}}{dy^{\alpha-1}} (e^{-\frac{y^2}{2}}) \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \\ &= -i^\alpha \pi \sqrt{\frac{\alpha}{\Gamma(\alpha)}} He_{\alpha-1} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}. \end{aligned}$$

Here He_m denotes probabilists' Hermite polynomials.

(ii) $\alpha = 2n+1$. This case is very similar. Since $f_\alpha(t)$ is an even function, it follows that

$$V_R(y) = \mathcal{F}_c(f_\alpha(t))(y) = \mathcal{F}(f_\alpha(t)) = i^{\alpha-1} \pi \sqrt{\frac{\alpha}{\Gamma(\alpha)}} He_{\alpha-1} \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}.$$

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Acknowledgments. The authors would like to thank the referee(s) for valuable comments.