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Up-to-constants bounds on the two-point Green's function for SLE curves*

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Abstract

The Green's function for the chordal Schramm-Loewner evolution SLE_κ for $0 < \kappa < 8$, gives the normalized probability of getting near points. We give up-to-constant bounds for the two-point Green's function.

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1 Introduction

The Schramm-Loewner evolution (SLE_κ) is a conformally invariant family of probability measures on curves originally given by Schramm as a candidate for the scaling limit of lattice models in statistical physics. The chordal Green's function gives the normalized probability that the path goes through a point and the two-point Green's functions gives the correlations for this quantity. While the one-point function is known (up to an arbitrary multiplicative constant in the definition), and the existence of the two-point function has been established, the exact form of the two-point function is not known. Estimates for the two-point function have proved to be important in analyzing fractal properties of the SLE curves, in particular the Hausdorff dimension and the Minkowski content. The goal of this paper is to give up-to-constant bounds valid for all pairs of points in a domain. It is still open to give a closed form for the function.

We start by reviewing the definition of SLE and giving the relevant known results. See [3] for more details. Suppose that $\gamma : (0, \infty) \to \mathbb{H} = \{x + iy : y > 0\}$ is a curve with $\gamma(0+) \in \mathbb{R}$ and $\gamma(t) \to \infty$ as $t \to \infty$. Let H_t be the unbounded component of $\mathbb{H} \setminus \gamma(0, t]$. Using the Riemann mapping theorem, one can see that there is a unique conformal transformation $g_t : H_t \longrightarrow \mathbb{H}$ satisfying $g_t(z) - z \to 0$ as $z \to \infty$. For any $a > 0$, it can be parametrized so that as $z \to \infty$,

$$g_t(z) = z + \frac{at}{z} + O(|z|^{-2}).$$

The conformal maps g_t satisfy the chordal Loewner equation

$$\dot{g}_t(z) = \frac{a}{g_t(z) - U_t}, \quad g_0(z) = z, \tag{1.1}$$

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where $U_t = g_t(\gamma(t))$ is a continuous real-valued function. The *Schramm-Loewner evolution* (SLE_κ) is obtained by choosing $a = 2/\kappa$ and U_t to be a standard (one-dimensional) Brownian motion. In this paper we will consider only $0 < \kappa < 8$ and let $a = 2/\kappa > 1/4$. We write

$$Z_t(z) = g_t(z) - U_t.$$

For $z \in \mathbb{H} \setminus \{0\}$, the function $t \mapsto g_t(z)$ is well defined up to time $T_z := \sup\{t : \mathrm{Im}[g_t(z)] > 0\}$. Rohde and Schramm [8] showed that for $\kappa < 8$ the Loewner equation above generates a random curve γ, which is also called SLE_κ, and they showed in a weak sense that the dimension of the path is

$$d = 1 + \frac{\kappa}{8}. \tag{1.2}$$

If $\kappa \geq 8$, the curve exists but is plane filling and is not relevant for this paper. If $0 < \kappa \leq 4$, the paths are simple with $\gamma(0,\infty) \subset \mathbb{H}$ while there are double points and $\gamma(0,\infty) \cap \mathbb{R} \neq \emptyset$ for $4 < \kappa < 8$. Moreover, if H_t denotes the unbounded component of $\mathbb{H} \setminus \gamma(0,t]$, then

$$H_t = \{z \in \mathbb{H} : T_z > t\}.$$

Their starting point to compute (1.2) was to assume that there exists a function G and a constant $\hat{c}$ such that

$$\lim_{\epsilon \to 0} \epsilon^{d-2}\, \mathbb{P}\{\mathrm{dist}(z,\gamma) < \epsilon\} = \hat{c}\, G(z), \tag{1.3}$$

where $\gamma = \gamma(0,\infty)$. Although did not establish the limit, they did note that if such a function exists, then the conformal Markov property of SLE_κ implies that

$$M_t(z) = |g_t'(z)|^{2-d}\, G(Z_t(z)), \tag{1.4}$$

must be a local martingale. From this one can determine the only possible value of d is that given in (1.2), and the function G must be a multiple of

$$G(z) = \mathrm{Im}(z)^{d-2}\, [\sin \arg(z)]^{4a-1}. \tag{1.5}$$

We call G (with this choice of constant) the SLE_κ Green's function.

In [2] it was proved that the Hausdorff dimension of the path is indeed d, and in [5] it was established that the d-dimensional Minkowski content of $\gamma[0,t]$ is finite and nonzero. In [4], the limit was shown to exist if we replace distance with the conformal radius of z in the domain $\mathbb{H} \setminus \gamma$. More recently, [5] established the existence of the limit as given although the value of the constant $\hat{c}$ is unknown.

The two-point Green's function is defined by

$$\lim_{\epsilon \to 0, \delta \to 0} \epsilon^{(d-2)}\; \delta^{(d-2)}\, \mathbb{P}\{\mathrm{dist}(z,\gamma) < \epsilon, \mathrm{dist}(w,\gamma) < \epsilon\} = \hat{c}^2\, G(z,w). \tag{1.6}$$

The existence of the limit with conformal radius replacing distance was established in [6] and the limit with distance was proved in [5]. As shown in [5], if $\Theta(Y)$ denotes the d-dimensional Minkowski content of $V \cap \gamma$, then

$$\mathbb{E}[\Theta(V)] = \hat{c} \int_V G(z)\, dA(z), \quad \mathbb{E}[\Theta(V)^2] = \hat{c}^2 \int_V \int_V G(z,w)\, dA(z)\, dA(w),$$

respectively. Unlike the one-point case, no exact expression has been given for $G(z,w)$. The goal of this paper is to give up-to-constants functions by proving the following theorem.

Theorem 1.1. *There exist $0 < c_1 < c_2 < \infty$ such that if $z, w \in \mathbb{H}$ with $|z| \le |w|$, then*

$$c_1\, q^{d-2}\, [S(w) \vee q]^{-\beta} \le \frac{G(z,w)}{G(z)\, G(w)} \le c_2\, q^{d-2}\, [S(w) \vee q]^{-\beta},$$

where

$$S(w) = \sin[\arg(w)], \quad q = \frac{|w-z|}{|w|} \le 2, \quad \beta = \frac{\kappa}{8} + \frac{8}{\kappa} - 2 > 0.$$

Two important estimates exist in the literature now. In [6], and implicitly in [2] although it was not phrased in this way, it was shown that if V is a bounded domain in $\mathbb{H}$ bounded away from the real line and $z, w \in V$, then

$$G(z,w) \asymp_V |z-w|^{d-2},$$

where $\asymp_V$ indicates that the implicit constant depends on V. In [7], it was shown that there exists c such that for all z, w,

$$G(z,w) \ge c\, G(z)\, G(w).$$

While we have defined the Green's function in terms of SLE in $\mathbb{H}$, it can easily be extended to simply connected domains D. To be more precise, suppose that D is a simply connected domain and w_1, w_2 are distinct points in ∂D. Let $F : \mathbb{H} \to D$ be a conformal transformation of $\mathbb{H}$ onto D with $F(0) = w_1, F(\infty) = w_2$. Then the distribution of

$$\tilde{\gamma}(t) = F \circ \gamma(t),$$

is that of SLE_κ in D from w_1 to w_2. Although the map F is not unique, the scaling invariance of SLE_κ in $\mathbb{H}$ shows that the distribution is independent of the choice. The Green's functions $G_D(F(z); w_1, w_2), G_D(F(z), F(w); w_1, w_2)$ can be defined by conformal covariance,

$$G(z) = |F'(z)|^{2-d}\, G_D(F(z); w_1, w_2),$$
$$G(z,w) = |F'(z)|^{2-d}\, |F'(w)|^{2-d}\, G_D(F(z), F(w), w_1, w_2),$$

and the corresponding limits (1.3) and (1.6) hold. We can write

$$G_D(F(z); w_1, w_2) = \Upsilon_D(F(z))^{d-2}\, S_D(F(z); w_1, w_2)^{4a-1},$$

Here $\Upsilon_D(F(z)) = \mathrm{Im}(z)|F'(z)|/2$ denotes $(1/2)$ times the conformal radius of D with respect to $F(z)$ and $S_D(F(z); w_1, w_2) = \sin\arg[z]$. If ∂_1, ∂_2 denote the two components of $\partial D \setminus \{w_1, w_2\}$, then

$$S_D(F(z); w_1, w_2) \asymp \min\left\{\mathrm{hm}_D(F(z), \partial_1), \mathrm{hm}_D(F(z), \partial_2)\right\}. \tag{1.7}$$

Here, and throughout this paper, hm will denote harmonic measure; that is, $\mathrm{hm}_D(z, K)$ is the probability that a Brownian motion starting at z exits D at K.

Using the Schwarz lemma and the Koebe $(1/4)$-theorem, we see that

$$\frac{\Upsilon_D(z)}{2} \le \mathrm{dist}(z, \partial D) \le 2\, \Upsilon_D(z). \tag{1.8}$$

If $\gamma(t)$ is an SLE_κ curve with transformations g_t and driving function U_t, we write $\gamma_t = \gamma(0,t], \gamma = \gamma_\infty$. If $z \in \mathbb{H}$ and $t < T_z$, we let

$$Z_t(z) = g_t(z) - U_t, \quad S_t(z) = \sin\left[\arg Z_t(z)\right], \quad \Upsilon_t(z) = \frac{\mathrm{Im}[g_t(z)]}{|g_t'(z)|}. \tag{1.9}$$

It is easy to check that if $t < T_z$, then $\Upsilon_t(z)$ as given in (1.9) is the same as $\Upsilon_{H_t}(z)$. Also, if $z \not\in \gamma$, then $\Upsilon(z) := \Upsilon_{T_z-}(z) = \Upsilon_D(z)$ where D denotes the connected component of $\mathbb{H} \setminus \gamma$ containing z. Similarly, if w_1, w_2 are distinct boundary points on a simply connected domain D and $z \in D$, we define

$$S_D(z; w_1, w_2) = \sin[\arg f(z)],$$

where $f : D \to \mathbb{H}$ is a conformal transformation with $f(w_1) = 0, f(w_2) = \infty$. If $t < T_z$, we set $S_t(z) = S_{H_t}(z; \gamma(t), \infty)$. If $f : D \to f(D)$ is a conformal transformation, then it is easy to show that

$$S_D(z; w_1, w_2) = S_{f(D)}(f(z); f(w_1), f(w_2)).$$

We extend the definition (1.5) as follow. If D is a simply connected domain with distinct $w_1, w_2 \in \partial D$, we define

$$G_D(z; w_1, w_2) = \Upsilon_D(z)^{d-2}\, S_D(z; w_1, w_2)^{4a-1}.$$

Under this definition $G(z) = G_{\mathbb{H}}(z; 0, \infty)$. The Green's function satisfies the conformal covariance rule

$$G_D(z; w_1, w_2) = |f'(z)|^{2-d}\, G_{f(D)}(f(z); f(w_1), f(w_2)).$$

Note that if $t < T_z$, then

$$M_t(z) = G_{H_t}(z; \gamma(t), \infty).$$

The local martingale $M_t(z)$ is not a martingale because it "blows up" at time $t = T_z$. If we stop it before that time, it is actually a martingale. To be precise, suppose that

$$\tau = \tau_{\epsilon,z} = \inf\{t : \Upsilon_t(z) \le \epsilon\}. \tag{1.10}$$

Then for every $\epsilon > 0$, $M_{t\wedge\tau}(z)$ is a martingale. The following is proved in [4] (the proof there is in the upper half plane, but it immediately extends by conformal invariance).

Proposition 1.2. *Suppose $\kappa < 8$, $z \in D, w_1, w_2 \in \partial D$ and γ is a chordal SLE_κ path from w_1 to w_2 in D. Let D_∞ denote the component of $D \setminus \gamma$ containing z. Then, as $\epsilon \downarrow 0$,*

$$\mathbb{P}\{\Upsilon_{D_\infty}(z) \le \epsilon\} \sim c_*\, \epsilon^{2-d}\, G_D(z), \quad c_* = 2\, \left[\int_0^\pi \sin^{4a} x\, dx\right]^{-1}.$$

Let us sketch the proof of the Theorem 1.1. By scaling, it suffices to prove the theorem for $w = x_w + iy_w$ with $|w| = 1$, in which case the conclusion can be written as

$$\frac{G(z,w)}{G(z)\, G(w)} \asymp |z - w|^{d-2}\, [y_w \vee |z - w|]^{-\beta}.$$

Here and for the reminder of this paper we write $\asymp$ to indicate that quantities are bounded by constants where the constants depend only on κ. Let us give a heuristic description of this estimate to show where this comes from. The goal of this paper is to justify this heuristic. Let ϵ be very small and let E_z, E_w denote the events that $\mathrm{dist}(\gamma, z) < \epsilon$ and $\mathrm{dist}(\gamma, w) < \epsilon$, respectively.

- The hardest part of the proof is to show that if $|z - w| \asymp 1$, then E_z and E_w are independent events up to constants, that is, $\mathbb{P}(E_z \cap E_w) \asymp \mathbb{P}(E_z)\, \mathbb{P}(E_w)$.
- Suppose $|z - w|$ is small and $y_w > 2|z - w|$. Then $G(z) \asymp G(w) = y_w^{4a-1}\, y_w^{d-2} = y_w^\beta$. Let E' be the event that the path gets within distance $2|z - w|$ of w. It is known that

$$\mathbb{P}(E') \asymp G(w)\, |z - w|^{2-d} \asymp y_w^\beta\, |z - w|^{2-d}.$$

Given E', E_z and E_w are conditionally independent up to a multiplicative constant, with

$$\mathbb{P}(E_z \mid E') \asymp \mathbb{P}(E_w \mid E') \asymp \left[\frac{\epsilon}{|z-w|}\right]^{2-d}.$$

Therefore, as $\epsilon \downarrow 0$,

$$\epsilon^{2(2-d)}\, G(z,w) \asymp \mathbb{P}(E_z \cap E_w) \asymp \mathbb{P}(E')\, \mathbb{P}(E_z \mid E')\, \mathbb{P}(E_w \mid E')$$
$$\asymp \epsilon^{2(2-d)}\, y^{\beta}\, |z-w|^{d-2} \asymp \epsilon^{2(2-d)}\, y^{-\beta}\, G(z)\, G(w)\, |z-w|^{d-2}.$$

- Suppose $|z-w|$ is small and $y_w \leq 2|z-w|$. Again, let E' be the event that the path gets within distance $2|z-w|$ of w. In this case

$$\mathbb{P}(E') \asymp |z-w|^{4a-1}.$$

Given E', E_z and E_w are conditionally independent up to a multiplicative constant. If $\zeta = x_\zeta + iy_\zeta \in \{z,w\}$, then

$$\mathbb{P}(E_\zeta \mid E') \asymp \left[\frac{y_\zeta}{|z-w|}\right]^{4a-1} \left[\frac{\epsilon}{y_\zeta}\right]^{2-d} \asymp G(\zeta)\, \epsilon^{2-d}\, |z-w|^{(d-2)+(1-4a)}.$$

Therefore, as $\epsilon \downarrow 0$,

$$\epsilon^{2(2-d)}\, G(z,w) \asymp \mathbb{P}(E_z \cap E_w) \asymp \mathbb{P}(E')\, \mathbb{P}(E_z \mid E')\, \mathbb{P}(E_w \mid E')$$
$$\asymp \epsilon^{2(2-d)}\, G(z)\, G(w) |z-w|^{1-4a}\, |z-w|^{2(d-2)} \asymp \epsilon^{2(2-d)}\, G(z)\, G(w) |z-w|^{-\beta}\, |z-w|^{d-2}.$$

2 Proof of the theorem

We fix $0 < \kappa < 8, a = 2/\kappa$, $\beta = \frac{\kappa}{8} + \frac{8}{\kappa} - 2 = (4a-1) - (2-d) > 0$. Let γ denote an SLE_κ curve and

$$\gamma_t = \gamma(0,t], \quad \Delta_t(z) = \mathrm{dist}(z, \gamma_t), \quad \Delta(z) = \Delta_\infty(z).$$

In [6] it is shown that for each z, w, there exist ϵ_z, δ_w such that if $\epsilon < \epsilon_z, \delta < \delta_w$,

$$\mathbb{P}\{\Delta(z) \leq \epsilon\} \asymp G(z)\, \epsilon^{2-d}, \quad \mathbb{P}\{\Delta(w) \leq \delta\} \asymp G(w)\, \delta^{2-d}, \tag{2.1}$$

$$\mathbb{P}\{\Delta(z) \leq \epsilon, \Delta(w) \leq \delta\} \asymp G(z,w)\, \epsilon^{2-d}\, \delta^{2-d}. \tag{2.2}$$

When estimating $\mathbb{P}\{\Delta(z) \leq \epsilon\}$ there are two regimes. The interior or bulk regime, where $\epsilon \leq \mathrm{Im}(z)$ can be estimated using Proposition 1.2 since in this case $\Delta(z) \asymp \Upsilon(z)$. However for the boundary regime $\epsilon > \mathrm{Im}(z)$, one needs the following estimate.

Lemma 2.1. *There exists $0 < c_1 < c_2 < \infty$ such that if $0 < y \leq 1/4$ and $\sigma = \inf\{t : |\gamma(t) - 1| \leq 2y\}$, then*

$$c_1 y^{4a-1} \leq \mathbb{P}\{\sigma < \infty, S_\sigma(1+iy) \geq 1/10\} \leq \mathbb{P}\{\sigma < \infty\} \leq c_2 y^{4a-1}.$$

Proof. The bound $\mathbb{P}\{\sigma < \infty\} \asymp y^{4a-1}$ can be found in a number of places. See [1]. Another proof which includes a proof of the first inequality can be found in [7]. The first inequality is Lemma 2.10 of that paper. □

In particular, the lemma implies that if $\eta : (0,1) \to \mathbb{H}$ is a curve with $\eta(0+), \eta(1-) \in (0,\infty)$ and $\eta = \eta(0,1)$, then

$$\mathbb{P}\{\gamma \cap \eta \neq \emptyset\} \leq c \left[\frac{\mathrm{diam}(\eta)}{\mathrm{dist}(0,\eta)}\right]^{4a-1}.$$

One way to estimate the right-hand side is in terms of (Brownian) excursion measure (see [6, 4.1] for definitions and similar estimates). We recall that if D is a simply connected domain and V_1, V_2 are two arcs in ∂D, then the excursion measure (of the set of excursions from V_1 to V_2 in D) is given by

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \int_{V_2} H_{\partial D}(z, w)\, |dz|\, |dw|,$$

where $H_{\partial D}$ denotes the boundary Poisson kernel (normal derivative of the Green's function). We can also write this as

$$\mathcal{E}_D(V_1, V_2) = \int_{V_1} \partial_n \phi_2(z)\, |dz|\, |dw| = \int_{V_2} \partial_n \phi_1(z)\, |dz|\, |dw|,$$

where ϕ_j is the harmonic function on D with boundary value 1_{V_j} and ∂_n denotes normal derivative. These formulas assume that V_1, V_2 are smooth; however, this quantity is a conformal invariant so one can define this for nonsmooth boundaries. A standard calculation shows that if $\mathrm{diam}(\eta) \le \mathrm{dist}(0, \eta)$, and H denotes the unbounded component of $\mathbb{H} \setminus \eta$, then

$$\mathcal{E}_H(\eta, (-\infty, 0]) \asymp \frac{\mathrm{diam}(\eta)}{\mathrm{dist}(0, \eta)}.$$

Suppose $\eta' : (0,1) \to \mathbb{H}$ is a curve in $\mathbb{H}$ with $\eta'(0+) = 0, \eta'(1-) > 0$ that separates η from ∞ in H. Let H' be the bounded component of $H \setminus \eta'$. Then monotonicity of the excursion measure implies that

$$\mathcal{E}_{H'}(\eta, \eta') \ge \mathcal{E}_H(\eta, (-\infty, 0]).$$

The upshot of this is that if we can find such an η', then

$$\mathbb{P}\{\gamma \cap \eta \neq \emptyset\} \le c\, \mathcal{E}_{H'}(\eta, \eta')^{4a-1}. \tag{2.3}$$

We will prove Theorem 1.1 in a sequence of propositions. We assume $|z| \le |w|$ and let

$$\beta = (4a - 1) - (2 - d) = 4a + \frac{1}{4a} - 2 > 0.$$

It will be useful to define a quantity that allows us to consider the boundary and interior cases simultaneously. Let

$$\Phi_t(z) = \Delta_t(z)^{4a-1} \quad \text{if} \quad \Delta_t(z) \ge \mathrm{Im}(z),$$

$$\Phi_t(z) = \mathrm{Im}(z)^{4a-1} \left[\frac{\Delta_t(z)}{\mathrm{Im}(z)}\right]^{2-d} \quad \text{if} \quad \Delta_t(z) \le \mathrm{Im}(z),$$

and let $\Phi(z) = \Phi_\infty(z)$. Note that $\Phi_0(z) = |z|^{4a-1}$, and scaling implies that the distribution of $\Phi(rz)$ is the same as that of $r^{4a-1}\Phi(z)$. Since $4a - 1 > 2 - d$, we see that

$$\Delta_t(z)^{4a-1} \le \Phi_t(z). \tag{2.4}$$

The next lemma combines the interior and boundary estimates into one estimate.

Lemma 2.2. *There exist $0 < c_1 < c_2 < \infty$ such that for all $z \in \mathbb{H}$ and $0 < \epsilon \le 1$,*

$$c_1 \epsilon \le \mathbb{P}\{\Phi(z) \le \epsilon\, \Phi_0(z)\} \le c_2 \epsilon. \tag{2.5}$$

Proof. Let $z = x + iy$. By scaling we may assume that $|z| = 1$ and hence $\Phi_0(z) = 1, S(z) = y$. Let $\Delta = \Delta_\infty(z), \Phi = \Phi_\infty(z)$. Proposition 1.2 and Lemma 2.1 imply that

$$\mathbb{P}\{\Delta \le \epsilon\} \asymp \epsilon^{4a-1}, \quad \epsilon \ge y,$$

$$\mathbb{P}\{\Delta \le \epsilon\} \asymp y^{4a-1}\,[\epsilon/y]^{2-d}, \quad \epsilon \le y.$$

If $\epsilon \ge y$, then

$$\mathbb{P}\{\Phi \le \epsilon^{4a-1}\} = \mathbb{P}\{\Delta \le \epsilon\} \asymp \epsilon^{4a-1}.$$

If $\epsilon \le y$, then if $u = (4a-1)/(2-d)$,

$$\mathbb{P}\{\Phi \le \epsilon^{4a-1}\} = \mathbb{P}\{y\,(\Delta/y)^{\frac{2-d}{4a-1}} \le \epsilon\} = \mathbb{P}\{\Delta \le y\,(\epsilon/y)^u\} \asymp y^{4a-1}\,[(\epsilon/y)^u]^{2-d} = \epsilon^{4a-1}.$$

□

The hardest step in estimating the two-point Green's function is to show that if two points are not very close to each other, then the events that the paths get close to the two points are independent at least up to a multiplicative constant. The next proposition gives a precise version of this statement in terms of the quantity $\Phi(z)$.

Proposition 2.3. *There exists $c < \infty$ such that if $|z| \le 4|w|$, and $0 < \epsilon_z, \epsilon_w \le 1$, then*

$$\mathbb{P}\{\Phi(z) \le \epsilon_z\,\Phi_0(z), \Phi(w) \le \epsilon_w\,\Phi_0(w)\} \le c\,\epsilon_z\,\epsilon_w.$$

The proof is similar to proofs in [6]. The details are somewhat technical so let us sketch the basic strategy. The idea is to show that if one is going to get very close to both z and w, then one is likely to get very close to one of them first without getting too close to the other and then one goes to the other point. In other words, one does not keep going back and forth between smaller and smaller neighborhoods of z and w. The way that one establishes this is to fix a curve I between z and w and consider excursions of the SLE paths from I. What one shows is that if γ is already very close to z, then it is unlikely that γ will get even closer to z and return to I. There are two different possibilities. Suppose that I_t is a crosscut of H_t contained in I and $\gamma(t) \in I_t$. If z is in the bounded component of $H_t \setminus I_t$, then $S_{H_t}(z;\gamma(t),\infty)$ is small, and the SLE path does not want to get closer to z. If z is in the unbounded component of $H_t \setminus I_t$, then the SLE path can get closer to z, but then it is unlikely to return to I_t. The proof makes this idea precise.

To prove Proposition 2.3 we start with a lemma that gives an upper bound for the probability that an SLE path gets close to a point and subsequently returns to a given crosscut. It is a generalization of Lemmas 4.10 and 4.11 of [6], and we use ideas from those proofs. Before stating the lemma, we set up some notation. Suppose $\eta : (0,1) \to \mathbb{H}$ is a simple curve with $\eta(0+) = 0, \eta(1-) > 0$ and write $\eta = \eta(0,1)$. Let V_1, V_2 denote respectively the bounded and unbounded components of $\mathbb{H} \setminus \eta$ and assume that $z = x_z + iy_z \in V_1, w = x_w + iy_w \in V_2$. Recall that H_t is the unbounded component of $\mathbb{H} \setminus \gamma_t$. We will let I_t be a decreasing collection of subarcs of η that are crosscuts of H_t separating z and w. To be more specific, one can show (see [6, Appendix A]) that there is a collection of open subarcs $\{I_t : t < T_z \wedge T_w\}$ of η with the following properties.

- $I_0 = \eta$.
- $I_t \subset H_t$. Moreover, $H_t \setminus I_t$ has two connected components, one containing z and the other containing w.
- If $s < t$, then $I_t \subset I_s$. Moreover, if $\gamma(s,t] \cap I_s = \emptyset$, then $I_t = I_s$.

If $\zeta \in \{z, w\}$, define stopping times σ_k, σ, τ depending on ζ by

$$\sigma_k = \inf\{t : \Phi_t(\zeta) = 2^{-k}\,\Phi_0(\zeta)\}, \quad \sigma = \sigma_1,$$

$$\tau = \inf\{t \ge \sigma : \gamma(t) \in \overline{I_\sigma}\} = \inf\{t \ge \sigma : \gamma(t) \in I_\sigma\}.$$

Here $\tau = \infty$ if $\sigma = \infty$ and the second equality holds with probability one. If $\tau < \infty$, let

$$J = \frac{\Phi_\tau(\zeta)}{\Phi_0(\zeta)}.$$

Lemma 2.4. *There exists $c < \infty$ such that under the setup above, if $0 < \epsilon \leq 1/2$ and $\alpha = 2a - \frac{1}{2} > 0$,*

$$\mathbb{P}\{\tau < \infty, J \leq \epsilon\} \leq c\,\epsilon, \quad \textit{if } \zeta = z,$$

$$\mathbb{P}\{\tau < \infty, J \leq \epsilon\} \leq c\,\epsilon \left[\frac{\operatorname{diam}(\eta)}{|w|}\right]^{\alpha}, \quad \textit{if } \zeta = w.$$

Proof. The first inequality follows immediately from (2.5), as does the second if $|w| \leq 4\operatorname{diam}(\eta)$. Therefore, using scaling, we may assume that $\operatorname{diam}(\eta) = 1, |w| \geq 4, \zeta = w$. Let C denote the half-circle of radius $\sqrt{|w|}$ in $\mathbb{H}$ centered at the origin. Let k_0 be the largest integer such that $2^{-k_0} \geq S(w) = \operatorname{Im}(w)/|w|$. Let ρ be the first time t that w is not in the unbounded component of $H_t \setminus C$. Note that if $\rho < T_w$, then $\gamma(\rho) \in C$. Let

$$\hat{J} = \frac{\Phi_\rho(w)}{\Phi_0(w)}.$$

Then, if k is a positive integer and $\hat{\sigma} = \sigma_k$,

$$\mathbb{P}\{\tau < \infty, J \leq 2^{-k}\} \leq \mathbb{P}\{\hat{\sigma} < \rho \wedge \tau, \tau < \infty\} + \sum_{j=1}^{k} \mathbb{P}\{\rho < \hat{\sigma} < \infty, 2^{-j} < \hat{J} \leq 2^{-j+1}\}.$$

We will now show that

$$\mathbb{P}\{\hat{\sigma} < \rho \wedge \tau, \tau < \infty\} \leq c\,2^{-k}\,|w|^{-\alpha}. \tag{2.6}$$

Let $H = H_{\hat{\sigma}}, I = I_{\hat{\sigma}}, g = g_{\hat{\sigma}}, U = U_{\hat{\sigma}}$. By (2.5),

$$\mathbb{P}\{\hat{\sigma} < \rho \wedge \tau\} \leq \mathbb{P}\{\hat{\sigma} < \infty\} \leq c\,2^{-k}.$$

Let H^* be the component of $H \setminus C$ containing w. On the event $\hat{\sigma} < \rho$, H^* is unbounded. Using simple connectedness of H, we can see that there is a subarc $l \in \partial H^* \cap C$ that is a crosscut of H and that separates w from I in H. Since l does not separate w from ∞, $g(l)$ is a crosscut of $\mathbb{H}$ that does not separate U from ∞; for ease let us assume that its endpoints are on $(-\infty, U]$. Since l separates w from I, l also separates I from ∞ in H. Therefore $g(l)$ separates $g(I)$ from U and ∞ in $\mathbb{H}$. We use excursion measure to estimate the probability that $\gamma[\hat{\sigma}, \infty)$ returns to I. The excursion measure between $g(I)$ and $[U, \infty)$ in $\mathbb{H} \setminus g(I)$ is bounded above by the excursion measure between $g(I)$ and $g(l)$ in $\mathbb{H} \setminus (g(I) \cup g(l))$ which by conformal invariance equals the excursion measure between I and l in $H \setminus (I \cup l)$. This in turn is bounded above by the excursion measure between C and $\partial\mathbb{D}$ in $\{\zeta \in \mathbb{H} : 1 < |\zeta| < \sqrt{|w|}\}$ which is $O(1/\sqrt{|w|})$. Given this, we can use (2.3) to see that the probability that an SLE_κ path from U to ∞ in $\mathbb{H}$ hits $g(I)$ is $O(|w|^{-(4a-1)/2})$. Using conformal invariance, we conclude that

$$\mathbb{P}\{\tau < \infty \mid \hat{\sigma} < \rho \wedge \tau\} \leq c\,|w|^{(1-4a)/2}$$

which gives (2.6).

We noted above that if $j \leq k_0$, then

$$\mathbb{P}\{\rho < \hat{\sigma} < \infty\} = 0.$$

We will now show that if $j > k_0$,

$$\mathbb{P}\{\rho < \hat{\sigma} < \infty, 2^{-j} < \hat{J} \leq 2^{-j+1}\} \leq c\,2^{-k}\,2^{-j/2}\,|w|^{-\alpha}. \tag{2.7}$$

The proposition then follows by summing over j. Consider the event

$$E_j = \{\rho < \infty, 2^{-j} < \hat{J} \leq 2^{-j+1}\}.$$

Using (2.5), we see that

$$\mathbb{P}(E_j) \le c\, 2^{-j}. \tag{2.8}$$

Let $H = H_\rho$. On the event E_j, there is a subarc l of $H \cap C$ that is a crosscut of H with one endpoint equal to $\gamma(\rho)$ such that l disconnects w from ∞ in H. Using this and the relationship between S and harmonic measure, we see that $S_\rho(w)$ is bounded above by the probability that a Brownian motion starting at w reaches C without leaving H. Using (2.4), we see that on the event E_j, $\mathrm{dist}(w, \partial H) \le 2^{-j/(4a-1)}|w|$. Using the Beurling estimate, we see that the probability a Brownian motion starting at w reaches distance $|w|/2$ from w without leaving H is $O(2^{-j/2(4a-1)})$. Given this, the probability that it reaches C without leaving $\mathbb{H}$ is bounded above by $O(1/|\sqrt{|w|})$. Therefore, on the event E_j,

$$S_\rho(w) \le c\, 2^{-j/2(1-4a)}\, |w|^{-1/2}.$$

Using the strong Markov property and (2.5), we see that

$$\mathbb{P}\{\hat{\sigma} < \infty \mid E_j\} \le c\, 2^{-j/2}\, |w|^{-\alpha}\, 2^{-(k-j)},$$

which combined with (2.8) gives (2.7).

□

Proof of Proposition 2.3. By scaling, we may assume that $|z| \le 1/2, |w| = 2$. We will consider crosscuts of H_t that are contained in the unit circle. To be more precise, we consider a decreasing collection of arcs $\{I_t : t < T_z \wedge T_w\}$ with the following properties.

- $I_0 = \{\zeta \in \mathbb{H} : |\zeta| = 1\}$.
- For each t, I_t is a crosscut of H_t that separates z from w in H_t.
- If $t > s$, then $I_t \subset I_s$. Moreover, if $\gamma(s,t] \cap \overline{I_s} = \emptyset$, then $I_t = I_s$.

We define a sequence of stopping times as follows.

$$\sigma_0 = 0,$$

$$\tau_0 = \inf\{t : |\gamma(t)| = 1\} = \inf\{t : \gamma(t) \in \overline{I_{\sigma_0}}\}.$$

Recursively, if $\tau_k < \infty$,

$$\sigma_{k+1} = \inf\left\{t > \tau_k : \Phi_t(w) = \frac{1}{2}\,\Phi_{\tau_k}(w) \text{ or } \Phi_t(z) = \frac{1}{2}\,\Phi_{\tau_k}(z)\right\},$$

and if $\sigma_{k+1} < \infty$,

$$\tau_{k+1} = \inf\{t \ge \sigma_{k+1} : \gamma(t) \in \overline{I_{\sigma_{k+1}}}\}.$$

If one of the stopping times takes on the value infinity, then all the subsequent ones are set equal to infinity. If $\sigma_{k+1} < \infty$, we set $R_k = z$ if $\Phi_{\sigma_{k+1}}(z) = \Phi_{\tau_k}(z)/2$. Note that in this case, $\Delta_{\sigma_{k+1}}(z) \le 2^{-\frac{1}{4a-1}}$, and $\Phi_t(w) > \Phi_{\tau_k}(w)/2$ for all $t \le \tau_{k+1}$. Likewise, we set $R_k = w$ if $\Phi_{\sigma_{k+1}}(w) = \Phi_{\tau_k}(w)/2$.

It follows immediately from (2.5) that for $r \le 1/2$,

$$\mathbb{P}\left\{\Phi_{\tau_0}(z) \le r\,\Phi_0(z)\right\} \le c\, r,$$

and for r sufficiently small

$$\mathbb{P}\left\{\Phi_{\tau_0}(w) \le r\,\Phi_0(w)\right\} = 0.$$

The key estimate, which we now establish, is the following.

- There exists c, α such that if $\tau_k < \infty, 0 < r \leq 1/2$ and $\zeta = x + iy \in \{z, w\}$, then

$$\mathbb{P}\left\{\tau_{k+1} < \infty, R_k = \zeta, \Phi_{\tau_{k+1}}(\zeta) \leq r\,\Phi_{\tau_k}(\zeta) \mid \gamma_{\tau_k}\right\} \leq c\,r\,\Phi_{\tau_k}(\zeta)^{\alpha}. \tag{2.9}$$

To prove, (2.9), let $H = H_{\tau_k}, I = I_{\tau_k}, \hat{g} = g_{\tau_k} - U_{\tau_k}, \hat{I} = \hat{g}(I), \hat{\zeta} = \hat{g}(\zeta), \Delta = \Delta_{\tau_k}(\zeta), \Phi = \Phi_{\tau_k}(\zeta), \lambda = |g'(\zeta)|$. Recall that $\Delta^{4a-1} \leq \Phi$. If $\Phi_t(\zeta) = r\Phi$ then $|\zeta - \gamma(t)| = \theta\,\Delta$ where

$$\theta = \left[\frac{y \wedge \Delta}{\Delta} \vee r\right]^{\frac{1}{4a-1}} \left[\frac{r\Delta}{y \wedge \Delta} \wedge 1\right]^{\frac{1}{2-d}}.$$

Note that if $r \leq 1/2$ then $\theta \leq 2^{-\frac{1}{4a-1}} < 1$.

Let V denote the closed disk of radius $2^{-\frac{1}{4a-1}}\Delta$ about ζ, $y_* = y \vee (\theta\Delta/2)$ and $\zeta_* = x + y_* i$. Note that $|\zeta - \zeta_*| \leq \theta\Delta/2 \leq 2^{-\frac{1}{4a-1}}\Delta/2$ and hence $\zeta_* \in V$. We consider g as a conformal transformation defined on the open disk of radius Δ about ζ; if $y < \Delta$, then we extend g by Schwarz reflection. By the distortion theorem, there exist $0 < c_1 < c_2 < \infty$ such that if $\zeta_1 \in V$,

$$c_1\,\lambda \leq |\hat{g}'(\zeta_1)| \leq c_2\,\lambda,$$

$$c_1\,\lambda\,|\zeta_1 - \zeta| \leq |\hat{g}(\zeta_1) - \hat{\zeta}| \leq c_2\,\lambda\,|\zeta_1 - \zeta|.$$

In particular,

$$c_1\,\lambda\,y \leq \mathrm{Im}\hat{\zeta} \leq c_2\,\lambda\,y.$$

Note that $\hat{I}$ is a crosscut of $\mathbb{H}$ with one endpoint equal to zero. We consider separately the cases where $\hat{\zeta}$ is in the bounded or unbounded component of $\mathbb{H} \setminus \hat{I}$.

Let E_1 denote the event that $\hat{\zeta}$ is in the bounded component. We claim that there exists $c < \infty$, such that for all $\hat{\zeta}' = \hat{g}(\zeta') \in \hat{g}(V)$,

$$S(\hat{\zeta}') = \frac{\mathrm{Im}(\hat{\zeta}')}{|\hat{\zeta}'|} \leq c\,\Delta^{1/2}. \tag{2.10}$$

To see this, assume for ease that $\mathrm{Re}[\hat{\zeta}'] \geq 0$ and let $\Theta = \arg\hat{\zeta}'$. Then $\mathrm{Im}(\hat{\zeta}')/|\hat{\zeta}'| = \sin\Theta \leq \Theta$ and Θ/π is the probability that a Brownian motion starting at $\hat{\zeta}'$ hits $(-\infty, 0]$ before leaving $\mathbb{H}$. This is bounded above by the probability that a Brownian motion starting at $\hat{\zeta}'$ hits $\hat{I}$ before leaving $\mathbb{H}$. By conformal invariance, this last probability is the same as the probability that a Brownian motion starting at ζ' hits I before leaving H. The Beurling estimate implies that this is bounded above by $c\Delta^{1/2}$. This gives (2.10). Therefore, there exists c such that if $|\zeta - \gamma(t)| = \theta\Delta$, then

$$\Phi(\hat{g}(\gamma(t))) \leq c\,\Delta^{(4a-1)/2}\,r\,|\gamma(t)| \leq c\,\sqrt{\Phi}\,r|\gamma(t)|.$$

Using (2.5), we see that

$$\mathbb{P}\{\Phi(\zeta) \leq r\,\Phi_{\tau_k}(\zeta), E_1 \mid \gamma_{\tau_k}\} \leq c\,\sqrt{\Phi}\,r.$$

We now suppose that $\hat{\zeta}$ is in the unbounded component. By the same argument, for every $\hat{\zeta}' := \hat{g}(\zeta') \in \hat{g}(V)$, the probability that a Brownian motion starting at $\hat{\zeta}' := \hat{g}(\zeta')$ hits $\hat{I}$ before leaving $\mathbb{H}$ is bounded above by $c\Delta^{1/2}$. We will split into two subcases. We first assume that

$$\mathrm{Im}(\hat{\zeta}') \leq \Delta^{1/4}\,|\hat{\zeta}'|, \quad \zeta' \in V.$$

In this case, we an argue as in the previous paragraph to see that the probability SLE_κ in $\mathbb{H}$ hits $\hat{g}(V)$ is bounded above by $c\,\Phi^{1/4}\,r$. For the other case we assume that

$\mathrm{Im}(\hat{\zeta}') \geq \Delta^{1/4}\,|\hat{\zeta}'|$ for some $\hat{\zeta}' \in \hat{g}(V)$. Using the Poisson kernel in $\mathbb{H}$, we can see that the probability that a Brownian motion starting at $\hat{\zeta}'$ hits $\hat{I}$ before leaving $\mathbb{H}$ is bounded below by a constant times

$$\frac{\mathrm{diam}(\hat{I})}{\Delta^{1/4}\,|\hat{\zeta}'|}.$$

From this we conclude that

$$\mathrm{diam}(\hat{I}) \leq c\,\Delta^{1/4}\,|\hat{\zeta}'|.$$

We appeal to Lemma 2.4 to say that the probability that SLE_κ in $\mathbb{H}$ hits $\hat{g}(V)$ and then returns to $\hat{I}$ is bounded above by a constant times

$$r\,[\mathrm{diam}(\hat{I})/|\hat{\zeta}'|]^{(4a-1)/2} \leq c\,r\,\Phi^{1/8}.$$

Given (2.9), the remainder of the proof proceeds in the same way as [6, Section 4.4] so we omit this.

□

Proposition 2.5. *There exist* $0 < c_1 < c_2 < \infty$ *such that if* $|z| \leq |w|/4$,

$$c_1\,G(z)\,G(w) \leq G(z,w) \leq c_2\,G(z)\,G(w).$$

Proof. The bound $G(z,w) \geq c\,G(z)\,G(w)$ was proved in [7] so we need only show the other inequality. Proposition 2.3 implies that for ϵ sufficiently small

$$\mathbb{P}\{\Delta(z) \leq \epsilon, \Delta(w) \leq \epsilon\} \leq c\,\mathbb{P}\{\Delta(z) \leq \epsilon\}\,\mathbb{P}\{\Delta(w) \leq \epsilon\}.$$

Hence (2.1) and (2.2) imply that $G(z,w) \leq c\,G(z)\,G(w)$. □

The next estimate will be important even though it is not a very sharp bound for large $|z|, |w|$.

Proposition 2.6. *For every* $\epsilon > 0$*, there exists* $c < \infty$ *such that if* $|z|, |w| \geq \epsilon$ *and* $|z-w| \geq \epsilon$*, then*

$$G(z,w) \leq c\,\mathrm{Im}(z)^{4a-1}\,\mathrm{Im}(w)^{4a-1}.$$

Proof. By scaling it suffices to prove the result when $\epsilon = 1$. This can be done as the proof of the proposition 2.3, so we omit the details. The key step is to choose an appropriate splitting curve I_0. We can choose I_0 either to be a half-circle with endpoints on $\mathbb{R}$ or a vertical line. We choose I_0 so that I_0 separates z and w and $\mathrm{dist}(z, I_0), \mathrm{dist}(w, I_0) \geq 1/4$. □

Proof of Theorem 1.1. By scaling, we may assume that $|w| = 1$ and hence $q = |w-z|$. If $q \geq 1/10$, the conclusion is

$$G(z,w) \asymp G(z)\,G(w).$$

The bound $G(z,w) \geq c\,G(z)\,G(w)$ was done in [7]. The other inequality can be deduced from Propositions 2.5 and 2.6, respectively, for $|z| \leq 1/4$ and $|z| \geq 1/4$. Here we use the fact that $G(z) \geq \mathrm{Im}(z)^{4a-1}$ for $|z| \leq 1$.

For the remainder of the proof we assume $q \leq 1/10$, and hence $9/10 \leq |z| \leq 1$. Let $z = x_z + iy_z, w = x_w + iy_w$, and $\zeta = x_w + i(y_w \vee q)$. Note that $G(w) \asymp y_w^{4a-1}, G(z) \asymp y_z^{4a-1}$. Let $\sigma = \inf\{t : |\gamma(t) - w| = 2q\}$, and on the event $\{\sigma < \infty\}$, let $h = \lambda[g_\sigma - U_\sigma]$ where the constant λ is chosen so that $\mathrm{Im}[h(\zeta)] = 1$. We write

$$h(\zeta) = \hat{\zeta} = \hat{x}_\zeta + i, \quad h(z) = \hat{z} = \hat{x}_z + i\hat{y}_z, \quad h(w) = \hat{w} = \hat{x}_w + i\hat{y}_w.$$

Recall that $Z_t = g_t - U_t$. Then

$$\begin{aligned} G(z,w) &= \mathbb{E}\left[|g_\sigma'(z)|^{2-d}\,|g_\sigma'(w)|^{2-d}\,G(Z_\sigma(z),Z_\sigma(w));\sigma<\infty\right] \\ &= \mathbb{E}\left[|g_\sigma'(z)|^{2-d}\,|g_\sigma'(w)|^{2-d}\,\lambda^{2(2-d)}\,G(\lambda Z_\sigma(z),\lambda Z_\sigma(w));\sigma<\infty\right] \\ &= \mathbb{E}\left[|h'(z)|^{2-d}\,|h'(w)|^{2-d}\,G(\hat z,\hat w);\sigma<\infty\right]. \end{aligned}$$

The Koebe $(1/4)$-theorem implies that $|h'(\zeta)| \asymp q^{-1}$. Distortion estimates (using Schwarz reflection if $y_w \leq 2q$) imply that

$$|h'(z)| \asymp |h'(w)| \asymp |h'(\zeta)| \asymp q^{-1},$$

$$|\hat z - \hat w| \asymp 1,$$

$$|\hat z|, |\hat w| \geq c,$$

$$\hat y_z \asymp (y_z \wedge q)\, q^{-1}, \quad \hat y_w \asymp (y_w \wedge q)\, q^{-1}.$$

These estimates hold regardless of the value of $S(\hat\zeta)$. If we also know that if $S(\hat\zeta) \geq 1/10$, then

$$|\hat\zeta| \asymp |\hat z| \asymp |\hat w| \asymp 1.$$

Hence, by Proposition 2.6, we see that

$$G(\hat z,\hat w) \leq c\,\left[\frac{(y_z\wedge q)\,(y_w\wedge q)}{q^2}\right]^{4a-1},$$

$$G(\hat z,\hat w) \geq c'\,\left[\frac{(y_z\wedge q)\,(y_w\wedge q)}{q^2}\right]^{4a-1}, \quad \text{if } S(\hat\zeta)\geq 1/10.$$

Lemma 2.1 implies that

$$\mathbb{P}\{\sigma<\infty\} \asymp \mathbb{P}\{\sigma<\infty, S(\hat\zeta)\geq 1/10\} \asymp \begin{cases} y_w^{4a-1}\,(q/y_w)^{2-d}, & y_w \geq q \\ q^{4a-1}, & y_w \leq q. \end{cases}$$

Therefore,

$$G(z,w) \asymp y_w^{4a-1}\,(q/y_w)^{2-d}\,q^{2(d-2)}\,\left[\frac{(y_z\wedge q)\,q}{q^2}\right]^{4a-1}, \quad y_w \geq q,$$

$$G(z,w) \asymp q^{4a-1}\,q^{2(d-2)}\,\left[\frac{(y_z\wedge q)\,y_w}{q^2}\right]^{4a-1}, \quad y_w \leq q.$$

If $q \leq y_w \leq 2q$ we can use either expression. If $y_w \leq 2q$, then $y_w \wedge q \asymp y_w, y_z\wedge q \asymp y_z, S(w)\vee q \asymp q$ and we can write

$$G(z,w) \asymp q^{2(d-2)}\,q^{1-4a}\,y_z^{4a-1}y_w^{4a-1} \asymp q^{d-2}\,[S(w)\vee q]^{-\beta}\,G(z)\,G(w).$$

If $y_w \geq 2q$, then $y_z \asymp y_w, y_z \wedge q \asymp q, S(w)\vee q \asymp y_w$, and we can write

$$G(z,w) \asymp y_w^{4a-1}\,q^{d-2}\,y_w^{d-2} = q^{d-2}\,y_w^{-\beta}\,y_w^{2(4a-1)} \asymp q^{d-2}\,[S(w)\vee q]^{-\beta}\,G(z)\,G(w).$$

□

3 Open problems

The obvious open problem is to determine the value of the Green's function $G(z,w)$. One can use the argument of Rohde and Schramm to determine a partial differential equation satisfied by G, see [6], but it is unknown whether or not there is an explicit solution.

One can also ask questions about the (directed) multi-point Green's function $\hat{G}(z_1,...z_n)$. The argument in [6] can be used to show that it exists and represents the normalized probability of hitting $n-$point $z_1,z_2,...,z_n$ in the *order* that we have them. More precisely,

$$\hat{c}^n\,\hat{G}(z_1,...z_n) = \lim_{\epsilon_1,\ldots,\epsilon_n\to 0} \mathbb{P}\{\tau^1 < \tau^2 < \cdots < \tau^n < \infty\},$$

where

$$\tau^j = \tau^j(\epsilon_j) = \inf\{t : \Delta_t(z_j) \le \epsilon_j\}.$$

As a starting point, we can ask the following questions.

- Does there exist $c < \infty$ such that for any n and $z_1,...,z_n \in \mathbb{H}$,

$$\hat{G}(z_1,...z_n) \le c^n \prod_{i=1}^{n} |z_i - z_{i+1}|^{d-2}\ ?$$

- Suppose V is a compact subset of $\mathbb{H}$ with $\mathrm{dist}(0,\mathbb{R}) > 0$. Is it true that

$$\hat{G}(z_1,...z_n) \asymp_{V,n} \prod_{i=1}^{n} |z_i - z_{i+1}|^{d-2}\ ?$$

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