Maximal displacement of $d$-dimensional branching Brownian motion

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Abstract

We consider a branching Brownian motion evolving in $\mathbb{R}^d$. We prove that the asymptotic behaviour of the maximal displacement is given by a first ballistic order, plus a logarithmic correction that increases with the dimension $d$. The proof is based on simple geometrical evidence. It leads to the interesting following side result: with high probability, for any $d \geq 2$, individuals on the frontier of the process are close parents if and only if they are geographically close.

Keywords: Branching Brownian motion; Brownian motion; branching process.

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1 Introduction

Let $d \geq 1$. A $d$-dimensional branching Brownian motion (or $d$-dim. BBM for short) is a particle process in which individuals displace according to independent Brownian motions and reproduce at rate 1. It starts with a unique individual positioned at point $0 \in \mathbb{R}^d$ at time 0. This individual displaces according to a $d$-dimensional Brownian motion until time $T$, distributed as an exponential random variable independent of the displacement. At time $T$, the individual dies giving birth to two children on its current position. These two particles then start independent $d$-dimensional branching Brownian motions.

For any $t \geq 0$, we write $\mathcal{N}_t$ for the set of individuals alive at time $t$ in the process. Let $u \in \mathcal{N}_t$ and $s \leq t$, we set $X_s(u)$ the position at time $s$ of the ancestor of $u$ that was alive at that time. The quantity of interest is $R_t = \max_{u \in \mathcal{N}_t} \|X_t(u)\|$. Bramson [5] proved the following asymptotic behaviour in dimension 1

$$R_t = \sqrt{2t} - \frac{3}{2\sqrt{2}} \log t + O_P(1),$$

where $O_P(1)$ is a process $(Y_t, t \geq 0)$ such that $\lim_{K \to +\infty} \sup_{t \geq 0} P(|Y_t| \geq K) = 0$. This process has been intensively studied in dimension 1, partly because of its links with the FKPP equation: $\partial_t u = \frac{1}{2} \partial_x^2 u + u(1 - u)$. The function

$$u : (t, x) \in (0, +\infty) \times \mathbb{R} \mapsto \mathbb{P}\left(\max_{u \in \mathcal{N}_t} X_t(u) \geq x\right),$$

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is a travelling wave solution of the F-KPP equation.

Gärtner [6] studied a $d$-dimensional version of the F-KPP equation, solved by e.g.,

$$w: (t, x) \in (0, +\infty) \times \mathbb{R}^d \mapsto \mathbf{P} (\exists u \in \mathcal{N}_t : \|X_t(u) - x\| \leq 1).$$

It is proved that for large $t$, the function $w(t, \cdot)$ admits a sharp cutoff located close to the ball of radius $\sqrt{2t} - \frac{d+2}{2\sqrt{2}} \log t$. Consequently the probability to find an individual $u \in \mathcal{N}_t$ within distance 1 of a given point $x$ is small if $\|x\| > \sqrt{2t} - \frac{d+2}{2\sqrt{2}} \log t$ and large if $\|x\| < \sqrt{2t} - \frac{d+2}{2\sqrt{2}} \log t$. Therefore if we replace every individual alive at time $t$ by a ball of radius 1 the radius $\rho_t$ of the percolation cluster containing the origin should verify $\rho_t = \sqrt{2t} - \frac{d+2}{2\sqrt{2}} \log t + O_P(1)$. Observe the logarithmic correction decreases as the dimension increases.

However the projection of the $d$-dim. BBM on any given line is a 1-dim. BBM. By (1.1), with high probability the maximal displacement in the process is larger than $\sqrt{2t} - \frac{d}{2\sqrt{2}} \log t$. Consequently, $R_t$ has a different behaviour than $\rho_t$. In particular, we prove in Theorem 1.1 below that while the first order does not depend on the dimension, the logarithmic correction of $R_t$ increases with $d$.

There have been few studies of quantities similar to $R_t$. In [10], the authors consider a complex, thus 2-dimensional, branching Brownian motion, but considered the process around its maximal value in 1 direction. Similarly, [4] also considered a model related to the $d$-dimensional branching Brownian motion, in which individuals diffuse as Brownian motions in one direction, and move at piecewise ballistic speed in the orthogonal direction. They described the extremal process, around the individual that travelled the farthest in the diffusive direction.

The main result of this article extends Bramson’s result on the asymptotic behaviour of the maximal displacement $R_t$ to any dimension $d \geq 1$.

**Theorem 1.1.** For any $d \geq 1$, we have

$$R_t = \sqrt{2t} + \frac{d-4}{2\sqrt{2}} \log t + O_P(1).$$

Moreover there exists $C > 0$ such that for any $t \geq 1$ and $y \in [1, t^{1/2}],$ 

$$ye^{-\sqrt{2y}}/C \leq \mathbf{P} \left( R_t \geq \sqrt{2t} + \frac{d-4}{2\sqrt{2}} \log t + y \right) \leq Cy e^{-\sqrt{2y}}. \quad (1.2)$$

In the rest of this article, $C$ stands for a generic positive constant, chosen large enough, that may change from line to line. Moreover, we write $x \wedge y$ for the minimum between $x$ and $y$ and $x \vee y$ as the maximum between $x$ and $0$.

**Remark 1.2.** Note that for any $v \in \mathbb{S}^{d-1}$ (the $d - 1$-dimensional sphere), the process $((X_t(u), v, u \in \mathcal{N}_t), t \geq 0)$ is a 1-dim. BBM, thus $\max_{u \in \mathcal{N}_t} X_t(u), v = \sqrt{2t} - \frac{3}{2\sqrt{2}} \log t + O_P(1)$ by (1.1). We expect that for large times $t$, the convex hull of the set of occupied positions at time $t$ looks like a ball of radius $\sqrt{2t} - \frac{3}{2\sqrt{2}} \log t$, with spikes of height $\frac{d-4}{2\sqrt{2}} \log t$, tossed uniformly at random on the surface of the ball.

The asymptotic behaviour of $R_t$ may be decomposed as follows:

$$R_t = \sqrt{2t} - \frac{3}{2\sqrt{2}} \log t + \frac{d-4}{2\sqrt{2}} \log t + O_P(1).$$

The term $-\frac{3}{2\sqrt{2}} \log t$ comes from the branching structure of the process. It is linked to the exponent of decay of the probability for a Brownian motion to make an excursion of length $t$ above 0\textsuperscript{1}. The term $\frac{d-4}{2\sqrt{2}} \log t$ comes from the fact that the frontier of the $d$-dim. BBM is the $d - 1$ dimensional sphere of radius $O(t)$.

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\textsuperscript{1}This is underlined in [3] for the closely related model of the branching random walk.

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directions that can be followed to reach the maximal displacement at time \( t \), yielding a term similar to the maximum of \( O(t^{(d-1)/2}) \) independent exponential random variables with parameter \( \sqrt{2} \).

The rest of the paper is organised as follows. In Section 2, we introduce the celebrated many-to-few lemma. We also recall some Brownian motion and geometric estimates, that precise the rough picture presented in Remark 1.2. Section 3 is then devoted to the proof of the upper bound of (1.2), that comes from a frontier argument; and Section 4 to its lower bound, using second moment methods. We end Section 4 completing the proof of Theorem 1.1.

2 Preliminary lemmas

2.1 The many-to-few lemmas

Let \( ((X_t(u), u \in \mathcal{N}_t), t \geq 0) \) be \( d \)-dim BBM. The many-to-one lemma links the mean of an additive function of the branching Brownian motion with a Brownian motion estimate. Its origins can be tracked back to the works of Peyrière [12] and Kahane and Peyrière [9]. This lemma has been enhanced, through the so-called spinal decomposition and stopping lines theory. However we only need in this article a simple version, corollary of [7, Lemma 1].

**Lemma 2.1** (Many-to-one lemma). For any \( t \geq 0 \) and measurable positive function \( f \), we have

\[
E \left[ \sum_{u \in \mathcal{N}_t} f(X_s(u), s \leq t) \right] = e^t E \left[ f(B_s, s \leq t) \right],
\]

where \( B \) is a \( d \)-dimensional Brownian motion.

This lemma is used to bound the mean number of individuals belonging to certain specific sets. To bound from below the probability for a set of individuals to be non-empty, we compute some second moments. This result, sometimes called in the literature the many-to-two lemma is also a consequence of [7, Lemma 1].

**Lemma 2.2** (Many-to-two lemma). Let \( B \) and \( B' \) be two independent \( d \)-dimensional Brownian motions. For \( s \geq 0 \) we set \( W(s) : t \in [0, +\infty) \mapsto B_t \wedge s + B'_{t-s} \), a Brownian path identical to \( B \) until time \( s \) and with independent increments afterwards. For any measurable positive functions \( f, g \) and \( t \geq 0 \), we have

\[
E \left[ \left( \sum_{u \in \mathcal{N}_t} f(X_s(u), s \leq t) \right) \left( \sum_{u \in \mathcal{N}_t} g(X_s(u), s \leq t) \right) \right] = E \left[ \sum_{u \in \mathcal{N}_t} f(X_s(u), s \leq t) g(X_s(u), s \leq t) \right] + \int_0^t e^{2t-s} E \left[ f(B_u, u \leq t) g(W_u^{(s)}, u \leq t) \right].
\]

2.2 Ballot theorem for the Brownian motion

We recall in this section some well-known Brownian motion estimates. Let \( \beta \) be a standard Brownian motion on \( \mathbb{R} \). The quantity \( I_t = \inf_{s \leq t} \beta_s \) has the same law as \( -|\beta_t| \). Consequently there exists \( C > 0 \) such that for any \( t \geq 1 \) and \( y \geq 1 \),

\[
\frac{y \wedge t^{1/2}}{C t^{1/2}} \leq P(\beta_s \geq -y, s \leq t) \leq \frac{C(y \wedge t^{1/2})}{t^{1/2}}.
\]  

We often call this equation the ballot theorem, for its similarities with the well-known random walk estimate (see [1]).
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Using the Girsanov theorem, it is an easy exercise to prove that for any $A \in \mathbb{R}$ and $\alpha < 1/2$, there exists $C > 0$ such that for all $t \geq 1$ and $y \geq 1$,

$$\frac{y \wedge t^{1/2}}{Ct^{1/2}} \leq P(\beta_s \geq -y + As^\alpha, s \leq t) \leq \frac{C(y \wedge t^{1/2})}{t^{1/2}}. \tag{2.2}$$

These estimates can be used to compute the probability for a Brownian motion to make an excursion above a given curve. Dividing the Brownian path on $[0, t]$ into three pieces, the first and last pieces being Brownian motion that stay above a given path, the middle part joining these two pieces, we obtain the following result$^2$. For any $A > 0$ and $\alpha < 1/2$, there exists $C > 0$ such that for any $t \geq 1$, for any function $f$ satisfying

$$\sup_{s \leq t} \frac{|f(s)|}{s^\alpha} + \frac{|f(t) - f(s)|}{(t - s)^\alpha} < A, \tag{2.3}$$

(which implies in particular $f(0) = f(t) = 0$) and for any $y, z \geq 1$, we have

$$\frac{(y \wedge t^{1/2})(z \wedge t^{1/2})}{Ct^{3/2}} \leq P\left(\beta_s \geq -y + f(s), s \leq t \quad \beta_t + y - f(t) \in [z, z + 1]\right) \leq \frac{C(y \wedge t^{1/2})(z \wedge t^{1/2})}{t^{3/2}}. \tag{2.4}$$

2.3 Geometry estimates

We conclude the section with an observation on the geometry of the sphere

$$S^{d-1} = \{(x_1, \ldots, x_d) \in \mathbb{R}^d : \|x\| := x_1^2 + \ldots + x_d^2 = 1\}.
$$

Using the fact that this is a manifold of dimension $d - 1$, we are able to exhibit $t^{(d-1)/2}$ “distinct” directions on the sphere of radius $t$.

**Lemma 2.3.** There exists $C > 0$ such that for any $R > 1$, there exists $U(R) \subset S^{d-1}$ verifying $#U(R) \leq CR^{(d-1)/2}$ and

$$\{x \in \mathbb{R}^d : \|x\| \geq R\} \subset \bigcup_{v \in U(R)} \{x \in \mathbb{R}^d : x.v \geq R - 1\}. \tag{2.5}$$

![Figure 1: It takes $R^{(d-1)/2}$ spherical caps of height 1 to cover a sphere of radius $R$.](image)

**Proof.** Let $R > 1$ and $x \in \mathbb{R}^d$ such that $\|x\| \geq R$. We set $y = R \frac{x}{\|x\|}$ its projection on the sphere of radius $R$. We note that for any $v \in S^{d-1}$, if $y.v \geq R - 1 > 0$ then $x.v \geq R - 1$. Therefore, it is enough to prove there exists $U(R)$ such that

$$\{x \in \mathbb{R}^d : \|x\| = R\} \subset \bigcup_{v \in U(R)} \{x \in \mathbb{R}^d : \|x\| = R, x.v \geq R - 1\}. \tag{2.6}$$

$^2$For a similar computation for random walks, see e.g. [3, 11].
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Let $v \in S^{d-1}$ and $x \in \mathbb{R}^d$, we have $\|x - Rv\|^2 = \|x\|^2 - 2Rx.v + R^2$, thus

$$\{x \in \mathbb{R}^d : \|x\| = R, x.v \geq R - 1\} = \{x \in \mathbb{R}^d : \|x\| = R, \|x - Rv\| \leq \sqrt{2R}\}.$$  

By homothetic transformation of ratio $R^{-1}$, a set $\mathcal{U}(R)$ satisfying (2.6) satisfies

$$S^{d-1} = \bigcup_{v \in \mathcal{U}(R)} \{u \in S^{d-1} : \|v - u\| \leq \sqrt{2/R}\}.$$  

For any $\epsilon > 0$ small enough, the sphere $S^{d-1}$ can be paved by $O(\epsilon^{1-d})$ balls of radius $\epsilon$, we write $\mathcal{U}(R)$ for the set of the center of such a tiling with balls of radius $\sqrt{2/R}$. $\square$

To explicitly construct a set $\mathcal{U}(R)$ solution of (2.5), one can take the union of the image of $R^{-1/2}Z^d$ by the stereographic projection of the northern and the southern hemisphere of $S^{d-1}$, which is a Lipschitz bijective mapping.

3 The upper bound of Theorem 1.1

Let $(X_t(u), u \in \mathcal{N}_t)_{t \geq 0}$ be a $d$-dim. BBM. We obtain in this section an upper bound for the tail of $R_t = \max_{u \in \mathcal{N}_t} \|X_t(u)\|$. We prove that for any $t \geq 1$ and $y \geq 1$, with high probability there exists no individual exiting at some time $s \leq t$ the ball of radius

$$f_s^{t,y} = \sqrt{2s + \frac{d-1}{2\sqrt{2}} \log(s+y)} - \frac{3}{2\sqrt{2}} \log \frac{t+1}{t-s} + y. \quad (3.1)$$

We set $\tilde{f}_s^{t,y} = f_s^{t,y} - \sqrt{2s}$. We observe that for any $\alpha < 1/2$ there exists $A > 0$ such that for any $t \geq 1$, $(\tilde{f}_s^{t,y} - f_s^{t,y}, s \leq t)$ satisfies (2.3). We prove in this section:

**Lemma 3.1.** There exists $C > 0$ such that for any $t \geq 1$ and $y \in [1, t^{1/2}]$,

$$\mathbb{P} \left[ \exists u \in \mathcal{N}_t, \exists s \leq t : \|X_s(u)\| \geq f_s^{t,y} \right] \leq Cy e^{-\sqrt{2y}}.$$

**Proof.** Let $t \geq 1$ and $y \in [1, t^{1/2}]$. To simplify notation, we assume that $t$ is an integer. For any $0 \leq k \leq t - 1$, we set

$$Z_k^{(t)}(y) = \sum_{u \in \mathcal{N}_{k+1}} 1_{\exists s \in [k,k+1] : \|X_s(u)\| \geq f_s^{t,y}} 1_{\{\|X_s(u)\| \leq f_s^{t,y}, s \leq k\}}.$$  

By the Markov inequality and the many-to-one lemma, we have

$$\mathbb{P} \left[ \exists s \leq t, u \in \mathcal{N}_t : \|X_s(u)\| \geq f_s^{t,y}(s) \right] \leq \sum_{k=0}^{t-1} \mathbb{E} \left( Z_k^{(t)}(y) \right) \leq \sum_{k=0}^{t-1} e^{k+1} \mathbb{P} \left[ \|B_s\| \leq f_s^{t,y}, s \leq k, \exists r \in [k,k+1] : \|B_r\| \geq f_r^{t,y} \right], \quad (3.2)$$

where $B$ is a $d$-dimensional Brownian motion.

As $s \mapsto f_s^{t,y}$ is increasing, applying the Markov property at time $k$ we have

$$\mathbb{P} \left[ \|B_s\| \leq f_s^{t,y}, s \leq k, \exists r \in [k,k+1] : \|B_r\| \geq f_r^{t,y} \right] \leq \mathbb{E} \left( \phi_k \left( \sup_{s \leq 1} \|B_s\| \right) \right),$$

where $\phi_k : x \in [0, +\infty) \mapsto \mathbb{P} \left[ \|B_k\| \geq f_k^{t,y} - x - 1, \|B_s\| \leq f_s^{t,y}, s \leq k \right]$. We now bound $\phi_k$ from above using Lemma 2.3. There exists $C > 0$ such that for any $k \leq t$ and $x \geq 0$, we have

$$\phi_k(x) \leq \sum_{v \in \mathcal{U}(f_s^{t,y} - x + 1)} \mathbb{P} \left( B_{k,v} \geq f_k^{t,y} - x - 1, B_{s,v} \leq f_s^{t,y}, s \leq k \right) \leq C (1 + (f_k^{t,y} - x)^{(d-1)/2} \mathbb{P} \left( \beta_k \geq f_k^{t,y} - x - 1, \beta_s \leq f_s^{t,y}, s \leq k \right) \leq C (1 + k + y)^{(d-1)/2} \mathbb{P} \left( \beta_k \geq f_k^{t,y} - x - 1, \beta_s \leq f_s^{t,y}, s \leq k \right),$$

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where $\beta$ is a standard Brownian motion, that has the same law as $B.v$ for any $v \in S^{d-1}$. Using the Girsanov transform then (2.4), we have

$$
P \left( \beta_k \geq f_{k}^{t,y} - x - 1, \beta_s \leq f_{s}^{t,y}, s \leq k \right) = E \left[ e^{-\sqrt{2}k - k \left( \beta_k + \sqrt{2}k \geq f_{k}^{t,y} - x - 1, \beta_s \leq \sqrt{2}k, s \leq k \right)} \right] \leq C e^{-k} e^{\sqrt{2}(x-f_{k}^{t,y})} P \left( \beta_k \geq f_{k}^{t,y} - x - 1, \beta_s \leq f_{s}^{t,y}, s \leq k \right) \leq C e^{-k} \frac{(t + 1)^{3/2} e^{\sqrt{2}(x-y)}}{(k + y)^{(d-1)/2} (t - k + 1)^{3/2}}. $$

We conclude that for any $x \geq 0$ and $k \leq t$,

$$
\phi_k(x) \leq C (1 + x) y e^{-k} e^{\sqrt{2}(x-y)} \frac{(t + 1)^{3/2}}{(k + y)^{(d-1)/2} (t - k + 1)^{3/2}}.
$$

As $\sup_{s \leq 1} \|B_s\|$ has Gaussian concentration, (3.2) yields

$$
P \left[ \exists s \leq t, u \in N_t : \|X(u)\| \geq f_{t,y}(s) \right] \leq C y e^{-\sqrt{2}y} \sum_{k=0}^{t-1} \frac{(t + 1)^{3/2}}{(k + y)^{(d-1)/2} (t - k + 1)^{3/2}} \leq C y e^{-\sqrt{2}y}.
$$

**Remark 3.2.** From proof of Lemma 3.1, observe the mean number of individuals hitting the frontier $f_{t,y}$ between times $k$ and $t-k$ is bounded from above by $C e^{-\sqrt{2}y} v_{-k} / k^{1/2}$.

By Lemma 3.1, we bound from above the tail of the maximal displacement at time $t$.

**Lemma 3.3.** There exists $C > 0$ such that for any $t \geq 1$ and $y \in [1, t^{1/2}]$,

$$
P \left( R_t \geq \sqrt{2}t + \frac{d-1}{2\sqrt{2}} \log t + y \right) \leq C y e^{-\sqrt{2}y}.
$$

**Proof.** Let $t \geq 1$ and $y \in [1, t^{1/2}]$. By continuity of the paths of the individuals we have

$$
P(R_t \geq f_{t,y}) \leq P \left( \exists u \in N_t : X_t(u) \geq f_{t,y} \right) \leq P \left( \exists u \in N_t, \exists s \leq t : X_s(u) \geq f_{s,y} \right),
$$

consequently Lemma 3.1 yields $P(R_t \geq f_{t,y}) \leq C y e^{-\sqrt{2}y}$.

As $f_{t,y} = \sqrt{2}t + \frac{d-1}{2\sqrt{2}} \log(t + 1) + y$, for any $t \geq 1$ large enough, for any $y \in [10, t^{1/2}]$, we have $f_{t,y}^5 \leq \sqrt{2}t + \frac{d-1}{2\sqrt{2}} \log t + y \leq f_{t,y}^5$; concluding the proof.

**4 A local lower bound on the maximal displacement**

For any $t \geq 1$, $y \in [1, t^{1/2}]$ and $v \in S^{d-1}$, we set

$$
A_{t,y}^v = \left\{ u \in N_t : \forall s \leq t, \|X_s(u)\| \leq f_{s,y}^v \right\} \quad \text{and} \quad A_{t,v}^y = \left\{ u \in A_{t,v} : X_t(u)v \geq f_{t,y}^v - 1 \right\}.
$$

By Lemma 3.1, we have $A_{t,y} = N_t$ with probability at least $1 - C y e^{-y}$. Thus with high probability, $A_{t,y}^v$ is the set of individuals that made a large displacement at time $t$ in direction $v$. To bound $N_t$ from below, we bound the probability that $A_{t,v}^y$ is non-empty, using the Cauchy-Scharz inequality. We start bounding the mean of $\# A_{t,v}^y$.

**Lemma 4.1.** There exists $C > 0$ such that for any $t \geq 1$, $y \in [1, t^{1/2}]$ and $v \in S^{d-1}$,

$$
y e^{-\sqrt{2}y} t^{(d-1)/2} / C \leq E \left[ \# A_{t,v}^y \right] \leq C y e^{-\sqrt{2}y} t^{(d-1)/2}.
$$
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**Proof.** Let $t \geq 1$, $y \in [1, t^{1/2}]$ and $v \in S^{d-1}$. The upper bound of $E(\# A_{t,y}^v)$ is a straightforward computation. Applying the many-to-one lemma, we have

$$
E \left[ \# A_{t,y}^v \right] = e^t P \left[ B_t, v \geq f^t_y - 1, \|B_s\| \leq f^t_y, s \leq t \right] \\
\leq e^t P \left[ \beta_t \geq f^t_y - 1, \beta_s \leq f^t_y, s \leq t \right],
$$

where $B$ is a $d$-dimensional Brownian motion, and $\beta = B_v$. By the Girsanov transform,

$$
E \left[ \# A_{t,y}^v \right] \leq E \left[ e^{-\sqrt{2} \beta_t} 1_{(\beta_t \geq f^t_y - 1, \beta_s \leq f^t_y, s \leq t)} \right] \leq Ce^{-\sqrt{2} y (1-d)/2},
$$

using (2.4), which ends the proof of the upper bound.

Let $B$ be a $d$-dimensional Brownian motion, we set $\beta = B_v$ and $B^\perp = B - \beta_v$. Note that $B^\perp$ is a $d-1$-dimensional Brownian motion independent of $\beta$. Let $\alpha < 1/2$, there exists $\lambda > 0$ such that for any $1 \leq s \leq t$ and $y \geq 1$,

$$
\left\{ x \in \mathbb{R}^d : \|x - (x,v)\| \leq 1/2 + \lambda \min(s^{\alpha+1/2}, t^{1/2}) \right\} \subseteq \{ x \in \mathbb{R}^d : \|x\| \leq f^t_y \}.
$$

Therefore, using Lemma 2.1, we have

$$
E \left[ \# A_{t,y}^v \right] = e^t P \left[ B_t, v \geq f^t_y - 1, \|B_s\| \leq f^t_y, s \leq t \right] \\
\geq e^t P \left[ \beta_t \geq f^t_y - 1, -1 \leq \beta_s \leq f^t_y - 1/2 - (s \wedge (t-s))^\alpha, s \leq t \right] \\
\times P \left[ \|B^\perp_s\| \leq 1/2 + \lambda \min(s^{1/2+\alpha}, t^{1/2}), s \leq t \right].
$$

By standard Brownian estimates, we have

$$
\inf_{t>0} P \left[ \|B^\perp_t\| \leq 1 + \lambda \min(s^{1/2+\alpha}, t^{1/2}), s \leq t \right] > 0.
$$

Moreover, using once again the Girsanov transform and (2.4), we have

$$
e^t P \left[ \beta_t \geq f^t_y - 1, -1 \leq \beta_s \leq f^t_y - 1/2 - (s \wedge (t-s))^\alpha, s \leq t \right] \\
\geq E \left[ e^{-\sqrt{2} \beta_t} 1_{(\beta_t \geq f^t_y - 1, -1 \leq \beta_s \leq f^t_y - 1/2 - (s \wedge (t-s))^\alpha, s \leq t)} \right] \geq e^{-\sqrt{2} y (1-d)/2}/C.
$$

We conclude that $E \left[ \# A_{t,y}^v \right] \geq e^{-\sqrt{2} y (1-d)/2}/C$. 

We now bound from above the second moment of $\# A_{t,y}^v$.

**Lemma 4.2.** There exists $C > 0$ such that for any $t \geq 1$, $y \in [1, t^{1/2}]$ and $v \in S^{d-1}$,

$$
E \left[ (\# A_{t,y}^v)^2 \right] \leq C y e^{-\sqrt{2} y (1-d)/2}.
$$

**Proof.** To compute this second moment, we use Lemma 2.2. Let $B$ and $B'$ be two independent Brownian motions, and $W^s : r \in [0, t] \mapsto B_{r+s} + B'_{(r-s)}$. We have

$$
E \left[ (\# A_{t,y}^v)^2 \right] \leq E \left[ \# A_{t,y}^v \right] + \int_0^t e^{2t-s} \left[ B_r, v \geq f^t_y - 1, \|B_r\| \leq f^t_y, r \leq t \right] \left[ W^s_r, v \geq f^t_y - 1, \|W^s_r\| \leq f^t_y, r \leq t \right] ds. \quad (4.1)
$$

By Lemma 4.1, we have $E \left[ \# A_{t,y}^v \right] \leq C y e^{-\sqrt{2} y (1-d)/2}$.

Let $0 \leq s \leq t$ and $\beta$ a standard Brownian motion. By the Markov property, we have

$$
e^{2t-s} \left[ B_r, v \geq f^t_y - 1, \|B_r\| \leq f^t_y, r \leq t \right] \left[ W^s_r, v \geq f^t_y - 1, \|W^s_r\| \leq f^t_y, r \leq t \right] \\
\leq e^{2t-s} \left[ B_r, v \geq f^t_y - 1, \|B_r\| \leq f^t_y, r \leq t \right] \left[ W^s_r, v \geq f^t_y - 1, \|W^s_r\| \leq f^t_y, r \leq t \right] \leq e^{2s} E \left[ \phi_s(\beta_s)^2 1_{(\beta_s \leq f^t_y, r \leq t)} \right],
$$

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where \( \phi_s : x \in \mathbb{R} \mapsto e^{t-s}\mathbb{P} \left[ \beta_{t-s} \geq f_{t}^{x,y}-1, \beta_{r} + x \leq f_{s+r}^{x,y}, r \leq t-s \right] \). By the Girsanov transform and (2.4) again, we have

\[
\phi_s(x) \leq C(1 + (f_{t}^{x,y} - x) + e^{\sqrt{2}(x-y-\sqrt{2}s)}(t+1)^{-1/2}(t-s)^{-3/2}.
\]

Consequently

\[
e^{2t-s}\mathbb{P}\left[ B_{t}v \geq f_{t}^{x,y}-1, \|B_{t}\| \leq f_{t}^{x,y}, r \leq t \right] \\
W_{t}^{x,v} \geq f_{t}^{x,y}-1, \|W_{t}\| \leq f_{t}^{x,y}, r \leq t \right] \\
\leq \frac{Ce^{-\sqrt{2}s}}{(t+1)^{1/2-(t+x+1)}}e^{s}E\left[ e^{\sqrt{2}(\beta_{s}-\sqrt{2}s)}(1 + (f_{t}^{x,y} - \beta_{s}))^{2} 1_{\{\beta_{r} \leq f_{s+r}^{x,y}, r \leq s\}} \right] \\
\leq \frac{Ce^{-\sqrt{2}s}}{(t+1)^{1/2-(t+x+1)}}E\left[ e^{\sqrt{2}\beta_{s}}(1 + (f_{t}^{x,y} - \beta_{s}))^{2} 1_{\{\beta_{r} \leq f_{s+r}^{x,y}, r \leq s\}} \right],
\]

using the Girsanov transform. By decomposition with respect to the value of \( \beta_{s} \) we have

\[
E\left[ e^{\sqrt{2}\beta_{s}}(1 + (f_{t}^{x,y} - \beta_{s}))^{2} 1_{\{\beta_{r} \leq f_{s+r}^{x,y}, r \leq s\}} \right] \\
\leq C\sum_{k=0}^{+\infty} e^{\sqrt{2}(k-1)^{2} + 1}P\left( \beta_{s} - f_{t}^{x,y} \leq [k-1, -k], \beta_{r} \leq f_{s+r}^{x,y}, r \leq s \right) \\
\leq Cy\sqrt{2}(s + y + 1)^{1/2}(t-s)^{1/2}(s+1)^{3/2}(t-s)^{-1/2}(s+1)^{-3/2}\sum_{k=0}^{+\infty}(k+1)^{3}e^{-\sqrt{2k}}.
\]

We conclude that for any \( 0 \leq s \leq t \),

\[
e^{2t-s}\mathbb{P}\left[ B_{t}v \geq f_{t}^{x,y}-1, \|B_{t}\| \leq f_{t}^{x,y}, r \leq t \right] \\
W_{t}^{x,v} \geq f_{t}^{x,y}-1, \|W_{t}\| \leq f_{t}^{x,y}, r \leq t \right] \\
\leq \frac{Cy}{(t+1)^{1/2-(t+s+1)}}e^{3/2} \left( s + y + 1 \right)^{(d-1)/2} \left( t-s \right)^{1/2} \left( s+1 \right)^{3/2} \left( t-s \right)^{-1/2}.
\]

Therefore, (4.1) yields \( E(\#A_{u}^{y})^{2} \leq Cy\sqrt{2}t^{-(d-1)/2} \).

**Remark 4.3.** By (4.2), a straightforward adaptation of Lemma 2.2, Lemma 2.3 and Lemma 3.1, the following side result holds: as soon as \( d \geq 2 \), for any \( 0 \leq R \leq t \)

\[
\mathbb{P}\left[ \exists u, u' \in \mathcal{N}_{t} : \|X_{t}(u)\| \geq f_{t}^{0,0}, \|X_{t}(u')\| \geq f_{t}^{0,0}, \|X_{t}(u) - X_{t}(u')\| \leq t^{1/2} \right] \leq C\log R \frac{R}{t^{1/2}},
\]

where MRCA\((u, u')\) is the time at which the most recent common ancestor of \( u \) and \( u' \) was alive. In other words, in a \( d \)-dim. BBM, two individuals on the frontier of the process that close to each other are close relatives. This result is well-known to fail in dimension 1, where individuals close to the edge of the process are either close relative, or their lineage had split within time \( O_{P}(1) \).

Using Lemmas 4.1 and 4.2 as well as the Cauchy-Schwarz inequality, for any \( t \geq 1, y \in [1, t^{1/2}] \) and \( v \in \mathbb{S}^{d-1} \), we have

\[
\mathbb{P}\left( \exists u \in \mathcal{N}_{t} : V(u), v \geq \sqrt{2}t + \frac{d-1}{2\sqrt{2}t^{1/2}} + y \right) \geq Cy\sqrt{2}t^{-(d-1)/2}/C.
\]

This is a lower bound of the probability there exists an individual making a large displacement in direction \( v \). To conclude the proof of the lower bound of Theorem 1.1, we bound from above the correlation between the existence of individuals making large displacements in two distinct directions \( v \) and \( v' \) at the same time.
Lemma 4.4. There exists $C > 0$ such that for any $t \geq 1$, $y \in [1, t^{1/2}]$ and $v, v' \in S^{d-1}$ such that $\|v - v'\| > Ct^{-1/2}$ and $\|v + v'\| \geq 3/2$, setting $\theta = \arccos(v.v')$ we have
\[
E \left[ \#A^{t,y}_v \#A^{t,y}_{v'} \right] \leq \frac{Cy \sqrt{2}y}{(t+1)(d-1)/2} \left[ \frac{1}{\theta^{d-2}} + e^{-\theta^2} \right].
\]

Figure 2: Path of individuals reaching the frontier of the branching Brownian motion.

Proof. We choose $C > 0$ large enough such that for any $t \geq 1$, $y \in [1, t^{1/2}]$ and $v, v' \in S^{d-1}$ verifying $\|v - v'\| > Ct^{-1/2}$, we have
\[
\{ x \in \mathbb{R}^d : \|x\| \leq f^{t,y}_x, x.v \geq f^{t,y}_x - 1, x.v' \geq f^{t,y}_x - 1 \} = \emptyset.
\]
Thus we assume in the rest of the proof that $A^{t,y}_v \cap A^{t,y}_{v'} = \emptyset$.

Let $v \neq v' \in S^{d-1}$ be such that $v.v' \geq 0$ and $\|v - v'\| > Ct^{-1/2}$. We set $w = \frac{v + v'}{\|v + v'\|}$, $w' = \frac{v - v'}{\|v - v'\|}$ and $\theta \in [0, \pi/4]$ such that $v = w \cos(\theta) + w' \sin(\theta)$. With the notation of Lemma 2.2, we have
\[
E \left[ \#A^{t,y}_v \#A^{t,y}_{v'} \right] \leq \int_0^t e^{2t-s} P \left[ B_t.v \geq f^{t,y}_x - 1, \|B_t.r\| \leq f^{t,y}_x, r \leq t, W_t.v' \geq f^{t,y}_x - 1, \|W_t.r\| \leq f^{t,y}_x, r \leq t \right],
\]
(4.5)
Applying the Markov property at time $s$, we have
\[
e^{2t-s} P \left[ B_t.v \geq f^{t,y}_x - 1, \|B_t.r\| \leq f^{t,y}_x, r \leq t, W_t.v' \geq f^{t,y}_x - 1, \|W_t.r\| \leq f^{t,y}_x, r \leq t \right] \leq e^{s} E \left[ \phi_{s,v}(B_s) \phi_{s,v}(B_s) 1_{\{\|B_s\| \leq f^{t,y}_x, r \leq s\}} \right],
\]
where $\phi_{s,v}(x) : x \in \mathbb{R}^d \mapsto e^{-s} P \left[ B_{t-s} + x, v \geq f^{t,y}_{t-s} - 1, \|B_{t-s}.r\| \leq f^{t,y}_{t-s}, r \leq t - s \right]$. For any $v \in S^{d-1}$ and $x \in \mathbb{R}^d$ such that $\|x\| \leq f^{t,y}_x$, setting $\beta = B.v$, we have
\[
\phi_{s,v}(x) \leq e^{-t-s} P \left[ \beta_{t-s} \geq f^{t,y}_{t-s} - 1 - x.v, \beta_r \leq f^{t,y}_{t-s} - x.v, r \leq t - s \right] \leq E \left[ e^{-\sqrt{2t}y \sqrt{3}(x.v - x.v)} (f^{t,y}_{t-s} - x.v) \right] \leq C (t-s+1)^{3/2},
\]
by the Girsanov transform and (2.4). Consequently, we have
\[
E^{2t-s} P \left[ B_t.v \geq f^{t,y}_x - 1, \|B_t.r\| \leq f^{t,y}_x, r \leq t, W_t.v' \geq f^{t,y}_x - 1, \|W_t.r\| \leq f^{t,y}_x, r \leq t \right] \leq C e^{-\sqrt{2t}y} e^{3t/2} E \left[ e^{-\sqrt{2t}y \sqrt{3}(f^{t,y}_{t-s} - B_s.v) + (f^{t,y}_{t-s} - B_s.v') 1_{\{\|B_s\| \leq f^{t,y}_x, r \leq s\}}} \right].
\]
We observe that for any $x \in \mathbb{R}^d$
\[
f^{t,y}_x - x.v = f^{t,y}_x - x.w \cos(\theta) - x.w' \sin(\theta) = f^{t,y}_x(1 - \cos(\theta)) + \cos(\theta)(f^{t,y}_x - x.w) - x.w' \sin(\theta).
\]
For any $a, b \in \mathbb{R}$ we have $(a + b) + (a - b) \leq 2a^2$ (both when $|b| < a$ and $|b| \geq a$), thus
\[
(f_s^{t,y} - x,v) + (f_s^{t,y} - x,v') \leq 2 \left( f_s^{t,y}(1 - \cos(\theta)) + \cos(\theta) \left( f_s^{t,y} - x,v \right)^2 \right) \\
\leq 4 \left( f_s^{t,y}(1 - \cos(\theta))^2 + \cos(\theta)^2 \left( f_s^{t,y} - x,v \right)^2 \right).
\]

As $\theta \in [0, \pi/4]$, there exists $C > 0$ such that $1 - \cos(\theta) \leq C\theta^2$. Therefore, we have \((f_s^{t,y} - x,v) + (f_s^{t,y} - x,v') \leq C \left( (s + y)^2 \theta^4 + (f_s^{t,y} - \beta_s)^2 \right)\), yielding
\[
e^{2t-s} P \left[ B_t,v \geq f_s^{t,y} - 1, \|B_r\| \leq f_s^{t,y}, r \leq t, W_s^t,x' \geq f_s^{t,y} - 1, \|W_s^t\| \leq f_s^{t,y}, r \leq t \right] \\
\leq C e^{-2t} \mathbb{E} \left[ e^{2\sqrt{2}(\beta_s \cos(\theta) - \sqrt{2}s)} \left[ (s + y)^2 \theta^4 + (f_s^{t,y} - \beta_s)^2 \right] \mathbf{1}_{\{\beta_s \leq f_s^{t,y}, r \leq s\}} \right].
\]

We use once again the Girsanov transform, we have
\[
e^s \mathbb{E} \left[ e^{2\sqrt{2}(\beta_s \cos(\theta) - \sqrt{2}s)} \left[ (s + y)^2 \theta^4 + (f_s^{t,y} - \beta_s)^2 \right] \mathbf{1}_{\{\beta_s \leq f_s^{t,y}, r \leq s\}} \right] \\
= e^{s(\cos(\theta) - 1)} \mathbb{E} \left[ e^{2\sqrt{2}(\cos(\theta) - 1)\beta_s} \left[ (s + y)^2 \theta^4 + (f_s^{t,y} - \beta_s)^2 \right] \mathbf{1}_{\{\beta_s \leq f_s^{t,y}, r \leq s\}} \right].
\]

For any $\theta < \pi/2$, we have $2 \cos(\theta) - 1 > 0.4$. Decomposing with respect to the value of $\beta_s$,
\[
\mathbb{E} \left[ e^{\sqrt{2}(2\cos(\theta) - 1)\beta_s} \left[ (s + y)^2 \theta^4 + (f_s^{t,y} - \beta_s)^2 \right] \mathbf{1}_{\{\beta_s \leq f_s^{t,y}, r \leq s\}} \right] \\
\leq \frac{Cy(s + y)^2 \theta^4 (s + y)^{(d-1)/2} (t - s + 1)^{1/2}}{(s + 1)^{3/2}} e^{\sqrt{2}y + 2\sqrt{2}(\cos(\theta) - 1)y},
\]

using (2.4). We conclude that for any $s \leq t$,
\[
e^{2t-s} P \left[ B_t,v \geq f_s^{t,y} - 1, \|B_r\| \leq f_s^{t,y}, r \leq t, W_s^t,x' \geq f_s^{t,y} - 1, \|W_s^t\| \leq f_s^{t,y}, r \leq t \right] \\
\leq C \frac{ye^{-\sqrt{2}y} (t + 1)^{1/2}}{(t + 1)^{(d-1)/2} (s + 1)^{3/2}} \theta^4 (s + y)^{(d+3)/2} (t - s + 1)^{1/2} e^{-1.1\theta^2(s+y)}. \tag{4.6}
\]

Note that for any $\lambda > 0$,
\[
\int_0^{t/2} \frac{(t + 1)^{1/2}}{(s + 1)^{3/2}} (s + y)^{(d+3)/2} (t - s + 1)^{1/2} e^{-\lambda(s+y)} ds \\
\leq \int_0^{\infty} s^{d/2} e^{-\lambda s} ds \\
\leq \frac{1}{\lambda^{d/2+1}(t + 1)^{(d-1)/2}}. 
\]

Moreover, for any $s > t/2$,
\[
e^{2t-s} P \left[ B_t,v \geq f_s^{t,y} - 1, W_s^t,x' \geq f_s^{t,y} - 1, \|B_r\| \leq f_s^{t,y}, \|W_s^t\| \leq f_s^{t,y}, r \leq t \right] \leq C e^{-1.1\theta^2t}.
\]

Therefore, (4.5) and (4.6) yield
\[
\mathbb{E} \left[ \#A_v^{t,y} \#A_v'^{t,y} \right] \leq \frac{Cye^{-\sqrt{2}y}}{(t + 1)^{(d-1)/2}} \left[ \frac{1}{\theta^{d-2}(t + 1)^{(d-1)/2}} + e^{-\theta^2 t} \right].
\]

The proof of Lemma 4.4 hints that with high probability, two individuals $u, u'$ alive at time $t$ close to the frontier of the process such that $\|X_t(u) - X_t(u')\| \geq Ct^{1/2}$ verify MRCA$(u, u') = \alpha_P(t)$. By Lemmas 4.1, 4.2 and 4.4, we bound from below $\mathbb{P}(R_t \geq f_s^{t,y})$. 

Maximal displacement of $d$-dimensional branching Brownian motion
Lemma 4.5. For any $\epsilon > 0$, there exists $C > 0$ such that for any $t \geq 1$, $v \in S^{d-1}$ and $y \in [1, t^{1/2}]$ we have
\[
P \left( \exists u \in \mathcal{N}_t : \|X_t(u)\| \geq \sqrt{8t} + \frac{d+1}{2 \sqrt{2}} \log t + y \frac{X_t(u)}{\|X_t(u)\|} \hat{w} > 1 - \epsilon \right) \geq ye^{-\sqrt{2}y} / C.
\]

Proof. Let $v \in S^{d-1}$, we set $v_2, \ldots, v_d$ such that $(v, v_2, \ldots, v_d)$ is an orthonormal basis of $R^d$. Let $\epsilon > 0$ and $t \geq 1$, we set
\[
\mathcal{L}_{t, \epsilon} = \left\{ w \in S^{d-1} : w.v > 1 - \frac{\epsilon}{2}, \frac{t^{1/2}}{\sqrt{2}} (w.v_j) \in Z, j \in \{2, \ldots, d\} \right\},
\]
where $C$ is a constant that we choose large enough such that Lemma 4.4 holds. Note there exists $K > 0$ such that $\frac{t^{(d-1)/2}}{K} \leq \# \mathcal{L}_{t, \epsilon} \leq K t^{(d-1)/2}$. We observe that for any $t \geq 1$ large enough and $y \in [1, t^{1/2}]$, we have
\[
P \left( \exists u \in \mathcal{N}_t : \|X_t(u)\| \geq f_t^{1/2} - 1, \frac{X_t(u)}{\|X_t(u)\|} \hat{w} > 1 - \epsilon \right) \geq P \left( \bigcup_{w \in \mathcal{L}_{t, \epsilon}} A_{t, w} \neq \emptyset \right),
\]
and we bound this probability using the Cauchy-Scharz inequality. We have
\[
P \left( \bigcup_{w \in \mathcal{L}_{t, \epsilon}} A_{t, w}^{1/2} \neq \emptyset \right) \geq P \left( \sum_{w \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \geq 1 \right) \geq \frac{E \left[ \sum_{w \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \right]^2}{E \left( \sum_{w \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \right)} \geq \left( \frac{t^{(d-1)/2}}{K} \right)^{1/2} \frac{1}{C}.
\]

By Lemma 4.1, we have $E \left[ \sum_{w \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \right] \geq \# \mathcal{L}_{t, \epsilon} \frac{t^{(d-1)/2}}{K} \geq \frac{t^{(d-1)/2}}{C}$. Similarly, using Lemma 4.2 we have $E \left[ \sum_{w \in \mathcal{L}_{t, \epsilon}} \left( \# A_{t, w}^{1/2} \right)^2 \right] \leq Cy e^{-\sqrt{2}y}$. As $w, w' \geq Ct^{-1/2}$ for any $w \neq w' \in \mathcal{L}_{t, \epsilon}$, we apply Lemma 4.4 to compute
\[
E \left[ \sum_{w \neq w' \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \# A_{t, w'}^{1/2} \right] \leq Cy e^{-\sqrt{2}y} \frac{t^{(d-1)/2}}{C} \sum_{w, w' \in \mathcal{L}_{t, \epsilon}} \frac{1}{\arccos(w, w')^2} \left( \frac{1}{(t+1)(t+1)} + e^{-\frac{\pi}{2} \arccos(w, w')^2} \right).
\]

We observe there exists $C > 0$ such that $\|w - w'\| \leq \arccos(w, w') \leq C\|w - w'\|$ for all $w, w' \in \mathcal{L}_{t, \epsilon}$. Consequently, setting $Z_{t,d-1} = Z_{d-1} \cap [-C t^{1/2}, C t^{1/2}]$, (4.8) becomes
\[
E \left[ \sum_{w \neq w' \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \# A_{t, w'}^{1/2} \right] \leq Cy e^{-\sqrt{2}y} \sum_{(k_2, \ldots, k_d) \in Z_{t,d-1}} \left( \frac{1}{\sum_{j=2}^{d-1} k_j^2} + e^{-\frac{1}{2} \sum_{j=2}^{d-1} k_j^2} \right) \left( \frac{1}{\sum_{j=2}^{d-1} k_j^2} + e^{-\frac{1}{2} \sum_{j=2}^{d-1} k_j^2} \right),
\]
where $(k_2, \ldots, k_d)$ are integers such that $(w - w').v_j = Ct^{-1/2} k_j$. Note that
\[
\sum_{(k_2, \ldots, k_d) \in Z_{t,d-1}} \frac{1}{\sum_{j=2}^{d-1} k_j^2} \leq Ct^{1/2} \quad \text{and} \quad \sum_{(k_2, \ldots, k_d) \in Z_{t,d-1}} e^{-\frac{1}{2} \sum_{j=2}^{d-1} k_j^2} \leq C,
\]
which yields $E \left[ \sum_{w \neq w' \in \mathcal{L}_{t, \epsilon}} \# A_{t, w}^{1/2} \# A_{t, w'}^{1/2} \right] \leq Cy e^{-\sqrt{2}y}$. Thus by (4.7), we have
\[
P \left( \exists u \in \mathcal{N}_t : \|X_t(u)\| \geq f_t^{1/2} - 1, \frac{X_t(u)}{\|X_t(u)\|} \hat{w} > 1 - \epsilon \right) \geq \frac{ye^{-\sqrt{2}y}}{C}.
\]
To conclude the proof, we observe that for any \( t \geq 1 \) large enough and \( y \in [1, t^{1/2}] \), we have \( f_t^{t,y-5} \leq \sqrt{2t} + \frac{4}{3\sqrt{2}} \log t + y \leq f_t^{t,y+5} \).

**Proof of Theorem 1.1.** We set \( r_t = \sqrt{2t} + \frac{4}{3\sqrt{2}} \log t \). The upper bound of Theorem 1.1 is a straightforward consequence of Lemma 3.3, as \( \lim_{y \to +\infty} \sup_{t \geq 0} P(R_t \geq r_t + y) = 0 \).

The lower bound is obtained using a standard cutting argument. We observe the process \((\#N_t, t \geq 0)\) is a standard Yule process. In particular, for any \( h > 0 \), \( \#N_h \) is a Geometric random variable with parameter \( e^{-h} \). By Lemma 3.3, we have \( P(R_h \geq \sqrt{2h} + h^{1/2}) \leq C h^{1/2} e^{-\sqrt{2h}^{1/2}} \). Applying the Markov property at time \( h \), on the event \( R_h \leq \sqrt{2h} + h^{1/2} \), the probability that \( R_{t+h} \leq r_t - 2h \) is bounded from above by the probability that none of the \( \#N_h \) individuals alive at time \( h \) have a descendent that made a displacement greater than \( r_t \). Thus

\[
P(R_{t+h} \leq r_t - 2h) \leq P(R_h \geq \sqrt{2h} + h^{1/2}) + e^{-h} \sum_{j=0}^{\infty} (1 - e^{-h})^j P(R_t \leq r_t + 1)^j \]

\[
\leq C h^{1/2} e^{-\sqrt{2h}^{1/2}} + \frac{e^{-h}}{1 - (1 - e^{-h})} P(R_t \leq r_t + 1).
\]

By Lemma 4.5, \( \sup_{t \geq 0} P(R_t \leq r_t + 1) < 1 \), yielding \( \lim_{h \to +\infty} \sup_{t \geq 0} P(R_{t+h} \leq r_t - 2h) = 0 \), which concludes the proof. \( \square \)

**References**


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