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Connective constant for a weighted self-avoiding walk on $\mathbb{Z}^2$

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Abstract

We consider a self-avoiding walk on the dual $\mathbb{Z}^2$ lattice. This walk can traverse the same square twice but cannot cross the same edge more than once. The weight of each square visited by the walk depends on the way the walk passes through it and the weight of the whole walk is calculated as a product of these weights. We consider a family of critical weights parametrized by angle $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. For $\theta = \frac{\pi}{3}$, this can be mapped to the self-avoiding walk on the honeycomb lattice. The connective constant in this case was proved to be equal to $\sqrt{2+\sqrt{2}}$ by Duminil-Copin and Smirnov in [8]. We generalize their result.

Keywords: weighted self-avoiding walks ; connective constant ; integrable weights ; Yang-Baxter equation ; parafermionic observable.
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1 Introduction

Self-avoiding walks (i.e. visiting each vertex at most once) were proposed by P. Flory and W. J. C. Orr [9, 18]. These walks turned out to be a very interesting object leading to rich mathematical theories, see [15, 3], and raising important challenges (it is difficult to understand because of its non-markovity). There are not so many rigorous statements in this field, one of the main conjectures being convergence to SLE$(8/3)$. Some progress in this direction was achieved by G. Lawler, O. Schramm and W. Werner, who proved in [14] that if the scaling-limit of self-avoiding walk exists and is conformally invariant, then it is SLE$(8/3)$. In 1984, B. Nienhuis nonrigorously derived in [16] that the connective constant for the hexagonal lattice equals to $\sqrt{2+\sqrt{2}}$. This has been proved recently by H. Duminil-Copin and S. Smirnov in [8]. Since the self-avoiding walk on the square lattice does not seem to be integrable, it is not reasonable to expect any explicit formula for the connective constant in this case. Nevertheless, one can study natural variations of the model, for instance by introducing additional weights.

We fix $\theta \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right]$ and consider the self-avoiding walk on Λ — the skewed $\mathbb{Z}^2$ lattice with edges having length 1 and all plaquets having angles θ and $\pi - \theta$.

To be precise this will be a curve starting and ending at the midpoints of edges, intersecting edges at right angles and having in each plaquet either one straight line connecting two opposite edges or two arcs surrounding opposite vertices or one arc or just no arcs (see fig. 1). Each rhombus has a weight according to the configuration of arcs inside it (see fig. 1):

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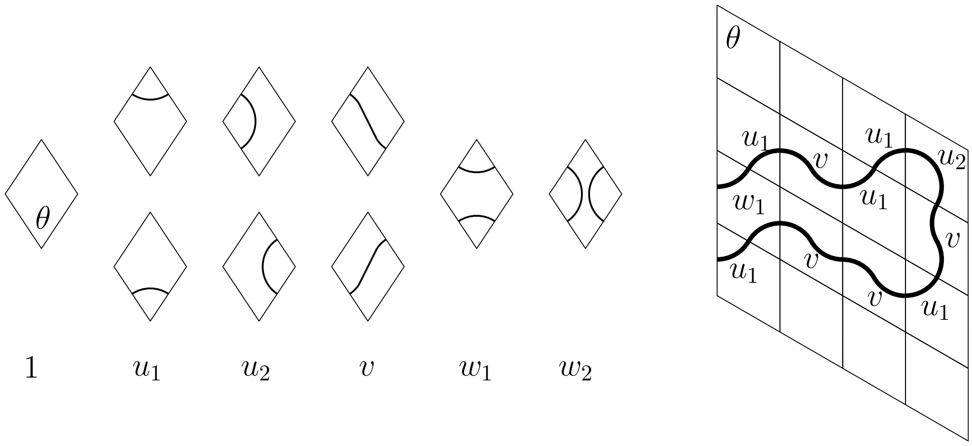

Figure 1: Different ways of passing a rhombus with their weights and an example of a walk of weight $u_1(\theta)^5 u_2(\theta) v(\theta)^4 w_1(\theta)$ and length 12.

- empty plaquet has weight 1,
- plaquet with an arc of angle θ has weight u_1,
- plaquet with an arc of angle $\pi - \theta$ has weight u_2,
- plaquet with a straight line has weight v,
- plaquet with two arcs of angle θ has weight w_1,
- plaquet with two arcs of angle $\pi - \theta$ has weight w_2.

The weight of the whole walk is calculated as the product of weights of the plaquets. Denote one of the mid-edges of the lattice by 0. The partition function is equal to the sum of the weights of all self-avoiding walks on Λ starting at 0:

$$\omega(\gamma) = \prod_{r-rhombus} \omega(r)\,,$$
$$Z(u_1, u_2, v, w_1, w_2) = \sum_{\gamma} \omega(\gamma)\,.$$

Let us consider

$$\tilde{c}_n = \frac{1}{u_1^n} \sum_{|\gamma|=n} \omega(\gamma),$$

where by $|\gamma|$ we mean the number of arcs in γ (straight passing of a rhombus counted as one arc). By definition, we find

$$Z(u_1, u_2, v, w_1, w_2) = \sum_{n=0}^{\infty} \tilde{c}_n u_1^n.$$

We are now in a position to state our main result:

Theorem 1.1. *There exists a family of weights* $(u_1, u_2, v, w_1, w_2)_\theta$ *parametrized by* $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$ *such that for these weights* $\lim_{n\to\infty} \sqrt[n]{\tilde{c}_n}$ *exists and is equal to* $\frac{1}{u_1}$*.*

Furthermore, these weights can be calculated explicitly:

$$u_1 = \frac{\sin(\frac{5\pi}{4})\sin(\frac{5\pi}{8}+\frac{3\theta}{8})}{\sin(\frac{5\pi}{4}+\frac{3\theta}{8})\sin(\frac{5\pi}{8}-\frac{3\theta}{8})}, \tag{1.1}$$

$$u_2 = \frac{\sin(\frac{5\pi}{4})\sin(\frac{3\theta}{8})}{\sin(\frac{5\pi}{4}+\frac{3\theta}{8})\sin(\frac{5\pi}{8}-\frac{3\theta}{8})}, \tag{1.2}$$

$$v = \frac{\sin(\frac{5\pi}{8}+\frac{3\theta}{8})\sin(-\frac{3\theta}{8})}{\sin(\frac{5\pi}{4}+\frac{3\theta}{8})\sin(\frac{5\pi}{8}-\frac{3\theta}{8})}, \tag{1.3}$$

$$w_1 = \frac{\sin(\frac{5\pi}{8}+\frac{3\theta}{8})\sin(\frac{5\pi}{4}-\frac{3\theta}{8})}{\sin(\frac{5\pi}{4}+\frac{3\theta}{8})\sin(\frac{5\pi}{8}-\frac{3\theta}{8})}, \tag{1.4}$$

$$w_2 = \frac{\sin(\frac{15\pi}{8}+\frac{3\theta}{8})\sin(-\frac{3\theta}{8})}{\sin(\frac{5\pi}{4}+\frac{3\theta}{8})\sin(\frac{5\pi}{8}-\frac{3\theta}{8})}. \tag{1.5}$$

Theorem 1.2. *Consider another way to define $|\gamma|$: a θ-arc has length 1, a $(\pi-\theta)-arc$ and a straight segment have any positive integer length (possibly different from each other). Then Theorem 1.1 remains true, i.e. the limit of $\sqrt[n]{\tilde{c}_n}$ is equal to $\frac{1}{u_1}$.*

Remark 1.3. The case $\theta = \frac{\pi}{3}$ corresponds to the honeycomb lattice and there is a way to define $|\gamma|$ in such a way that Theorem 1.2 computes the connective constant of the honeycomb lattice (see Section 2).

The weights (1.1)-(1.5) were discovered by B. Nienhuis [17] in 1990 as solutions of the Yang-Baxter equation. They were rediscovered by J. Cardy and Y. Ikhlef [12] in 2009 as the weights for which the parafermionic observable satisfies some particular equations. For the connection between these two approaches, see [1, 13]. See also [6] for the weights on the boundary. We should just mention here that in [17] and [12] a more general case is considered — the $O(n)$ model with $n \in [-2, 2]$ (the self-avoiding walk is a particular case of this model for $n = 0$). Unfortunately, the weights written there contain some minor misprints, so for completeness we include a correct version of the weights in Section 5.

In the case $\theta = \frac{\pi}{2}$ the weights are symmetric, i.e. $u_1 = u_2$ and $w_1 = w_2$. One can view a walk as a self-avoiding walk on $\mathbb{Z}^2$ which is allowed to touch itself but each time gets penalised by $w_1/u_1^2 \approx 0.675$ and that gets penalised by $v/u_1 \approx 0.785$ for each vertex it passes without a turn. Theorem 1.1 confirms the conjecture [2] that the asymptotics of $\sqrt[n]{\tilde{c}_n}$ is equal to

$$\frac{1}{u_1(\pi/2)} = \sqrt{3 + \frac{1}{2}\sqrt{26 + 7\sqrt{2}}} = 2.448\ldots$$

This is below the predicted [10] value ≈ 2.638 for a connective constant of $\mathbb{Z}^2$.

One can consider $\theta < \frac{\pi}{3}$ (or $\theta > \frac{2\pi}{3}$) but the weight w_2 (or w_1) becomes negative, so we do not address this question here.

Another interesting question is the value of the critical fugacities for walks in a half-plane interacting with the boundary. For the self-avoiding walk in the half-plane insertion of a fugacity means favouring each additional visit of the border. One can define the critical fugacity as the value of the fugacity above which the self-avoiding walk sticks to the border. In [4] it was proven that the critical fugacity for the self-avoiding on the hexagonal lattice is equal to $1 + \sqrt{2}$. It would be natural to generalize this computation. Though we conjecture that the same should hold, i.e. that the critical fugacity is equal to $1/(1 - 2u_1^2)$, we cannot prove this at the moment.

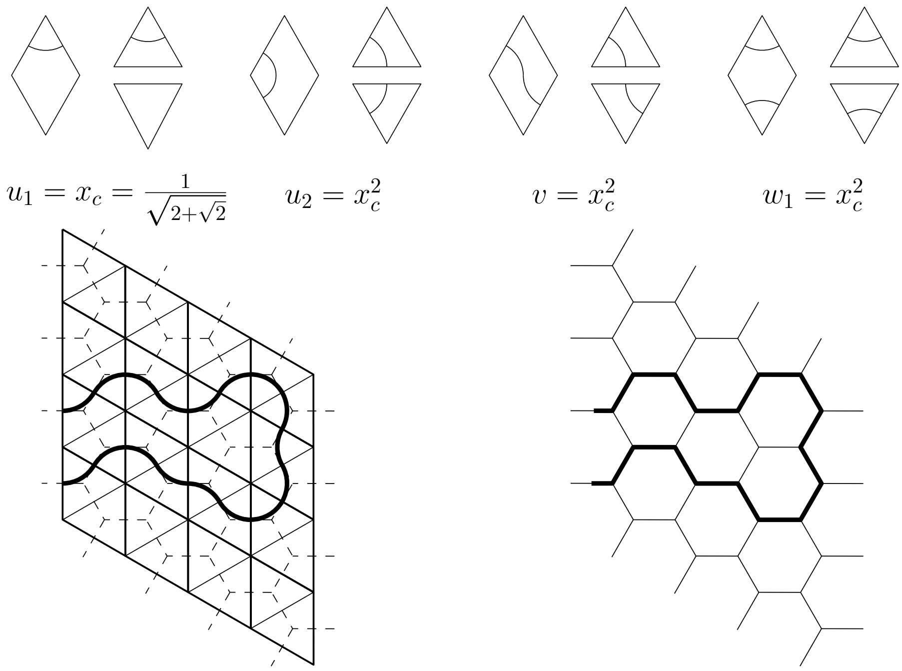

Figure 2: *Top*: a bijection between local configurations on rhombi with angle $\frac{\pi}{3}$ and on equilateral triangles. Weights are indicated just below the corresponding mapping. A rhombus with two $\frac{2\pi}{3}$-arcs is forbidden ($w_2 = 0$). *Bottom*: a self-avoiding walk drawn on the triangular and the hexagonal lattices. Its length is equal to 17 (compare to fig. 1).

2 Case $\theta = \frac{\pi}{3}$ and the sketch of the proof

For $\theta = \frac{\pi}{3}$ we have $u_1 = \frac{1}{\sqrt{2+\sqrt{2}}}$, $u_2 = v = w_1 = u_1^2$ and $w_2 = 0$. This case is in direct correspondence with the self-avoiding walk on the honeycomb lattice, and Theorem 1.2 specializes to [8]. We can divide a rhombus with angle $\frac{\pi}{3}$ into two equilateral triangles. Then, all possible states of a rhombus can be viewed as states of these two triangles (see fig. 2). Note that walking on the faces of the triangular lattice is the same as walking along the edges of its dual, i.e. of the hexagonal lattice. Each triangle with an arc of a walk inside it corresponds to a vertex of the hexagonal lattice visited by a walk. It is easy to see that the weight of a walk is equal to $\left(\frac{1}{\sqrt{2+\sqrt{2}}}\right)^{|\gamma|}$. Therefore, we just obtained the self-avoiding walk on the hexagonal lattice at criticality.

In order to get a natural length of a walk on the honeycomb lattice, one needs to fix the length of a $\frac{\pi}{3}$-arc at 1, and fix the length of a $\frac{2\pi}{3}$-arc and of a straight segment at 2.

Now we turn to the general case $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$. Let us give an outline of the proof stressing the differences from [8]:

— In Section 3 we define the parafermionic observable F_a in exactly the same way as in [8]. The only difference is that we consider a walk on a dual graph.

— In Lemma 3.1 we show that for the weights (1.1)-(1.5) the parafermionic observable satisfies a part of the discrete Cauchy-Riemann equations (3.2). This means that the contour integral of F_a around each rhombus is 0.

— In Lemma 4.1 (corresponds to Lemma 2 in [8]) we sum up this relation over rhombi contained in a big parallelogram Ω and obtain the relation (4.3) on the weights of walks going from the origin to different sides of Ω (see fig. 4).

— In Lemma 4.3 we show that for a long parallelogram Ω of a fixed width T the contribution of all walks going to the top and bottom sides is negligible.

— This leads to the relation (4.5) on the weights of arcs $A_T(x_c)$ and bridges $B_T(x_c)$ in a strip, where $x_c = \frac{1}{u_1}$ (corresponds to (5) in [8]).

— One can decompose an arc into two bridges and from (4.5) get a lower bound on $B_T(x_c)$. This is done in the same way as in [8] but there is a couple of subtleties. First of all, one is not allowed to do the symmetry around a line of a grid. In fact, one does not need an axial symmetry, the central symmetry is enough (and this we have). Another issue is that our walks are allowed to visit the same rhombus twice. In particular, a walk can visit a rhombus one time before the splitting point and one time after, and the weight of this walk will not be equal to the product of weights of two bridges. We need just an upper bound in terms of bridges, so the inequalities (4.1)-(4.2) save the situation. The last subtlety is that one needs to modify the endpoints of the bridges a little bit (see fig. 5).

— This leads to $Z(u_1, u_2, v, w_1, w_2) = \infty$, where the parameters are given by (1.1)-(1.5) (critical weights).

— In order to show that $Z < \infty$ in the subcritical case we do the classical bridges decomposition. One has to deal with a couple of subtleties that we have already mentioned. This finishes the proof.

3 Parafermionic observable and integrable weights

Throughout this section $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$.

To analyse the behaviour of the self-avoiding walk, we will use the *parafermionic* observable introduced in [19]. Let Ω be a parallelogram with angle θ divided into congruent rhombi (see fig. 4). Notations:

— $V(\Omega)$ is the set of all midpoints of the sides of the rhombi,

— $V(\partial\Omega)$ is the set of points in $V(\Omega)$ lying on $\partial\Omega$ (boundary of Ω).

Pick points $a \in V(\partial\Omega)$ and $z \in V(\Omega)$ and define:

$$F_a(z) = \sum_{\gamma: a \to z} \omega(\gamma) e^{-i\sigma W(\gamma)}, \tag{3.1}$$

where the sum runs over self-avoiding walks starting at a and ending at z. Above, $W(\gamma)$ denotes the winding of γ, i.e. the angle of rotation of γ going from a to z (a walk crosses all sides of the rhombi at the right angle). For instance, the arc from z_{SE} to z_{SW} on figure 3 has winding θ and the arc from z_{SE} to z_{NE} has winding $\theta - \pi$. The value σ will be fixed later. Observables for other models were introduced in [12, 5, 20], see also [7] for a survey.

We will find the weights for which our observable satisfies a half of discrete Cauchy-Riemann equation:

$$F_a(z_{SE}) - F_a(z_{NW}) = e^{i\theta}(F_a(z_{SW}) - F_a(z_{NE})),$$

where $SENW$ is any rhombus with angle θ (see fig. 3). We rewrite this equation:

$$F_a(z_{SE}) + e^{i\theta} F_a(z_{NE}) - F_a(z_{NW}) - e^{i\theta} F_a(z_{SW}) = 0\,. \tag{3.2}$$

The other half which is missing is a similar relation around each vertex.

Lemma 3.1. *If $\sigma = \frac{\ell}{8}$, where ℓ is some odd number, the unique weights such that*

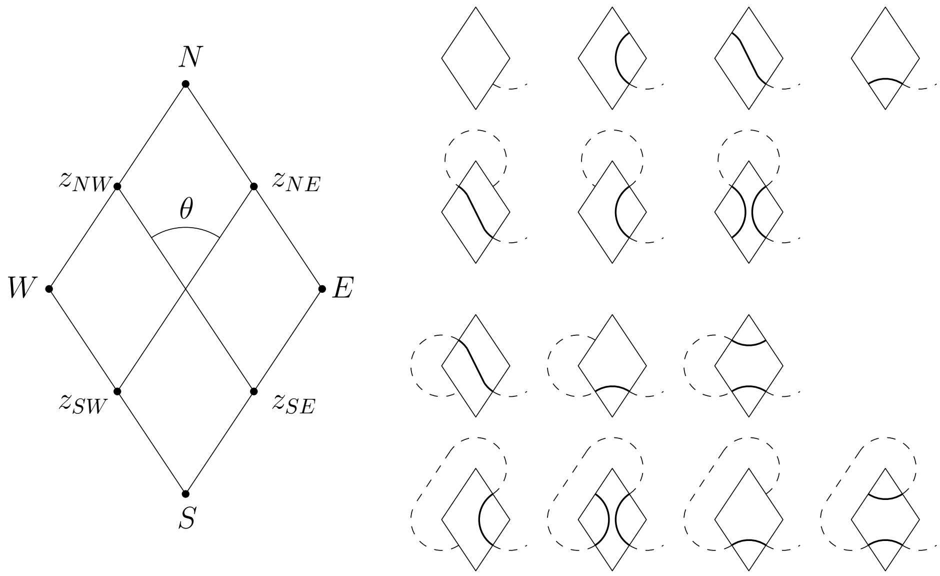

Figure 3: *Left:* A rhombus $SENW$ with angle θ and centres of the edges z_{SE}, z_{NE}, z_{NW}, z_{SW}. *Right:* Local changes of the path in different cases — each row of rhombi corresponds to one of the equations (3.11)-(3.14).

$F_a(u_1, u_2, v, w_1, w_2)$ *satisfies* (3.2) *are given by*

$$u_1 = \tfrac{1}{t}\sin[2\sigma\pi]\sin[(\sigma-1)(\pi-\theta)]\ , \tag{3.3}$$
$$u_2 = \tfrac{1}{t}\sin[2\sigma\pi]\sin[(\sigma-1)\theta]\ , \tag{3.4}$$
$$v = \tfrac{1}{t}\sin[(\sigma-1)\theta]\sin[(\sigma-1)(\pi-\theta)]\ , \tag{3.5}$$
$$w_1 = \tfrac{1}{t}\sin[(\sigma-1)(\pi-\theta)]\sin[(\sigma-1)(2\pi+\theta)]\ , \tag{3.6}$$
$$w_2 = \tfrac{1}{t}\sin[(\sigma-1)\theta]\sin[(\sigma-1)(3\pi-\theta)]\ , \tag{3.7}$$

where $t = \sin[(\sigma-1)(\pi+\theta)]\sin[(\sigma-1)(2\pi-\theta)]$.

If $\sigma = 1$ *then there is a one parameter family of weights such that* $F_a(u_1, u_2, v, w_1, w_2)$ *satisfies* (3.2)*:*

$$u_1 + u_2 = 1\,, \tag{3.8}$$
$$w_1 = u_1\,, \tag{3.9}$$
$$w_2 = u_2\,. \tag{3.10}$$

For all other values of σ *the weights such that* $F_a(u_1, u_2, v, w_1, w_2)$ *satisfies* (3.2) *exist only for some specific values of* θ.

The weights (3.3)-(3.7) give us the weights (1.1)-(1.5) if one takes $\sigma = \frac{5}{8}$.

Proof. Let us consider all paths contributing to Eq. (3.2) for a fixed rhombus $SENW$. Consider walks visiting $SENW$ first by z_{SE} (and possibly some other edges of $SENW$ afterwards). They can be divided into several groups such that walks in the same group differ only inside $SENW$ (see fig. 3):

- outside $SENW$ the walk γ is just a path from a to z_{SE};
- outside $SENW$ the walk γ is the union of a path from a to z_{SE} and a path between z_{NW} and z_{NE} (visited in any direction);

- outside $SENW$ the walk γ is the union of a path from a to z_{SE} and a path between z_{NW} and z_{SW} (visited in any direction);
- outside $SENW$ the walk γ is the union of a path from a to z_{SE} and a path between z_{SW} and z_{NE} (visited in any direction).

Note that if the total contribution of paths in each group is zero then F satisfies equation (3.2). At the same time, in each of these groups, paths differ one from another only inside the rhombus $SENW$. Hence, if the following equations hold, we obtain (3.2):

$$1 + \lambda\bar{\mu}e^{i\theta}u_2 - v - \lambda e^{i\theta}u_1 = 0\,, \tag{3.11}$$

$$\lambda\bar{\mu}^2 e^{i\theta}v - \mu u_2 - \lambda e^{i\theta}w_2 = 0\,, \tag{3.12}$$

$$-\lambda\mu e^{i\theta}v - \bar{\mu}u_1 + \lambda\bar{\mu}e^{i\theta}w_1 = 0\,, \tag{3.13}$$

$$-\lambda\mu e^{i\theta}u_2 - \mu^2 w_2 + \lambda\bar{\mu}^2 e^{i\theta}u_1 - \bar{\mu}^2 w_1 = 0\,, \tag{3.14}$$

where $\lambda = e^{-i\sigma\theta}$, $\mu = e^{-i\sigma\pi}$.

Moreover, it is not difficult to see that if (3.11)-(3.14) are not satisfied then (3.2) fails for some rhombi.

Now, if we consider walks visiting $SENW$ first by z_{SW}, we get the equations that are conjugates to (3.11)-(3.14) (i.e. the equations with all terms conjugated except the weights u_1, u_2, v, w_1, w_2). For walks visiting $SENW$ first by z_{NW} (or z_{NE}) one gets the same equations as for walks visiting $SENW$ first by z_{SE} (or z_{SW}).

Solving this linear system, we obtain that either $v = 0$ or $\sigma = \frac{\ell}{8}$ where ℓ is some odd number. For each $\sigma = \frac{\ell}{8}$ and θ parameters, weights satisfying equations (3.11)-(3.14) are given by (3.3)-(3.7). Details are given in the Appendix.

If $v = 0$, the solution exists for each θ if and only if $\sigma = 1$. In this case $w_1 + w_2 = 1$, $u_1 = w_1$ and $u_2 = w_2$. □

4 Proofs of Theorems 1.1-1.2

In order to have positive coefficients in (4.3) and to have the inequalities (4.1)-(4.2), we fix σ at the value $\frac{5}{8}$ till the end of the paper.

In this case, weights given by (3.3)-(3.7) can be rewritten as the weights given by (1.1)-(1.5). For $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$, all of them are non-negative and

$$u_1^2 \geq w_1\,, \tag{4.1}$$

$$u_2^2 \geq w_2\,. \tag{4.2}$$

For $\theta \in (0, \frac{\pi}{3}) \cup (\frac{2\pi}{3}, \pi)$, one of w_1 and w_2 is negative and one of the inequalities (4.1)-(4.2) fails.

Take any $\theta \in [\frac{\pi}{3}, \frac{2\pi}{3}]$.

We call a self-avoiding walk a *bridge* (fig. 4, walks from a to β) if it is contained in a strip such that both endpoints of the walk are contained in different borders of the strip and these borders go along the lines of the grid (i. e. contain sides of rhombi).

Consider $x > 0$. Let us take $x_c = u_1$, $u_1(x) = x$, $u_2(x) = x \cdot \frac{u_2}{x_c}$, $v(x) = x \cdot \frac{v}{x_c}$, $w_1(x) = x^2 \cdot \frac{w_1}{(x_c)^2}$ and $w_2(x) = x^2 \cdot \frac{w_2}{(x_c)^2}$. Denote by $\omega_x(\gamma)$ the weight of γ if the weights of the plaquets are x, $u_2(x)$, $v(x)$, $w_1(x)$ and $w_2(x)$. One can observe that for any x and any self-avoiding walk γ of length n holds $\omega_x(\gamma) = \left(\frac{x}{x_c}\right)^n \omega_c(\gamma)$, where $\omega_c(\gamma)$ is the weight of γ for $x = x_c$. In order to prove Theorem 1.1, we need to show that the radius of convergence of $Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x))$ is x_c.

Now, let us consider a parallelogram Ω with angles θ and $\pi - \theta$, and sides denoted by α, β, δ, ε (see fig. 4). Let $2L + 1$ be the number of rhombi touching α and T be the number of rhombi touching δ. The origin a will be in the middle of α. We will use the following notations:

$$A_{T,L}(x) = \sum_{\gamma:a\to z\in\alpha} \omega_x(\gamma), \qquad B_{T,L}(x) = \sum_{\gamma:a\to z\in\beta} \omega_x(\gamma),$$
$$D_{T,L}(x) = \sum_{\gamma:a\to z\in\delta} \omega_x(\gamma), \qquad E_{T,L}(x) = \sum_{\gamma:a\to z\in\varepsilon} \omega_x(\gamma).$$

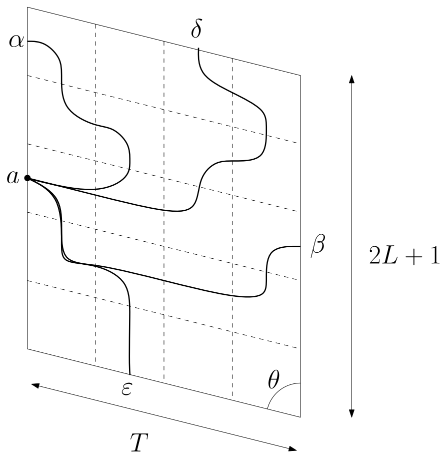

Figure 4: Parallelogram Ω and self-avoiding walks to its sides α, β, δ and ε.

Lemma 4.1. *For* $c_\alpha = \cos\frac{3\pi}{8}$, $c_\delta = \cos(\frac{3}{8}\theta)$ *and* $c_\varepsilon = \cos(\frac{3}{8}(\pi-\theta))$

$$c_\alpha A_{T,L}(x_c) + B_{T,L}(x_c) + c_\delta D_{T,L}(x_c) + c_\varepsilon E_{T,L}(x_c) = 1\,. \tag{4.3}$$

Proof. The relation (3.2) means that the contour integral of F_a over any rhombus is 0 (by the contour integral we mean the sum of the values of F_a along a counter-clockwise oriented contour multiplied by the direction of the corresponding edges). Thus, the contour integral of F_a over the whole Ω is 0. For the points on the boundary we know the winding. Taking the imaginary part, we get the desired relation on $A_{T,L}(x_c)$, $B_{T,L}(x_c)$, $D_{T,L}(x_c)$ and $E_{T,L}(x_c)$. The 1 on the right side is the contribution of the empty walk. □

Note that all three coefficients c_α, c_δ and c_ε are positive.

Remark 4.2. In fact, one can obtain a similar equation for any $k \in \mathbb{Z}$ and $\sigma = \frac{2k+1}{8}$, but the coefficients are not always positive. Also, it is interesting that we are using only the imaginary part of the relation. One can as well try to derive some information from its real part. A small difficulty is that now walks to the boundary α to the left and to the right of the origin will get different coefficients.

Lemma 4.3. *For* T *fixed,* $E_{T,L}(x_c) \to 0$ *and* $D_{T,L}(x_c) \to 0$ *as* $L \to \infty$.

Proof. Consider any walk γ contributing to $E_{T,L}(x_c)$ or $D_{T,L}(x_c)$. Let $\tilde{\gamma}$ be the walk ending at α obtained from γ by adding at the end one arc and several (at most $T-1$) straight segments going leftwards. It is easy to see that $\omega_c(\tilde{\gamma}) \geq c_T \omega_c(\gamma)$, where $c_T = v^{T-1}\min(u_1, u_2)$.

Note that $\tilde{\gamma}$ contributes to $A_{T,L+1}(x_c) - A_{T,L}(x_c)$ and $\tilde{\gamma}$ determines γ. Thus

$$A_{T,L+1}(x_c) - A_{T,L}(x_c) \geq c_T(E_{T,L}(x_c) + D_{T,L}(x_c))\,. \tag{4.4}$$

Obviously, the left-hand side is positive and by (4.3) $A_{T,L}(x_c)$ is bounded by 1. Thus, the left-hand side of (4.4) tends to 0 as L tends to ∞ (when T is fixed). □

For $x \le x_c$ consider $A_T(x)$ and $B_T(x)$:

$$A_T(x) = \lim_{L\to\infty} A_{T,L}(x)\,, \qquad\qquad B_T(x) = \lim_{L\to\infty} B_{T,L}(x)\,.$$

These limits exist because $A_{T,L}(x)$ and $B_{T,L}(x)$ are increasing in L and bounded by 1 (see (4.3)).

We thus obtain

$$c_\alpha A_T(x_c) + B_T(x_c) = 1\,. \tag{4.5}$$

The walks counted in B_T are self-avoiding bridges of width T.

Lemma 4.4. *The partition function is infinite at criticality: $Z(x_c) = \infty$.*

Proof. Note that $B_T(x_c) - B_{T+1}(x_c) = c_\alpha(A_{T+1}(x_c) - A_T(x_c))$. It is easy to see that $A_{T+1}(x_c) - A_T(x_c)$ is the sum of weights of all the self-avoiding paths in the strip of width $T+1$ beginning at a and ending on the left side of the strip, which also touch the right side. Each of these paths γ can be divided into a path γ_1 from the left side of the strip to the right one and path γ_2 from the right side of the strip to the left one (see fig. 5). More precisely, path γ_1 is defined as the part of γ from a up to (and including) the first visit to the rhombi on the right boundary of the strip, and $\gamma_2 = \gamma \setminus \gamma_1$.

At this point, one must be aware that the weight of γ is not the product of weights of γ_1 and γ_2 since rhombi containing two arcs of γ may contain one arc of γ_1 and one arc of γ_2. These rhombi contribute w_1 (or w_2) to $\omega_c(\gamma)$ and u_1^2 (or u_2^2 resp.) to $\omega_c(\gamma_1)\omega_c(\gamma_2)$. Nevertheless, the inequalities (4.1) and (4.2) imply that $\omega_c(\gamma) \le \omega_c(\gamma_1)\omega_c(\gamma_2)$.

Consider the last step of γ_1. It is the only time when γ_1 visits the right boundary of the strip. This implies that the last step in γ_1 is either a θ-arc or a $(\pi-\theta)$-arc (see fig. 5):

- If the last step in γ_1 is a $(\pi-\theta)$-arc, then we define γ_1' as the path obtained by adding a $(\pi-\theta)$-arc at the end of γ_1 and γ_2' as the path obtained by adding a θ-arc in the beginning of γ_2.
- If the last step in γ_1 is a θ-arc, then we define γ_1' as the path obtained by adding a θ-arc at the end of γ_1 and γ_2' as the path obtained by adding a $(\pi-\theta)$-arc in the beginning of γ_2.

Paths γ_1' and γ_2' are self-avoiding bridges of width $T+1$ starting at some particular points — γ_1' starts at a and γ_2' starts at the rhombus adjacent to the endpoint of γ_1'. This leads to the inequality

$$A_{T+1}(x_c) - A_T(x_c) \le (B_{T+1}(x_c))^2/(x_c u_2)\,.$$

Using (4.5), we obtain a lower bound on the growth of $B_T(x_c)$:

$$B_T(x_c) - B_{T+1}(x_c) \le \frac{c_\alpha}{x_c u_2}\cdot(B_{T+1}(x_c))^2$$

$$\frac{c_\alpha}{x_c u_2}\cdot(B_{T+1}(x_c))^2 + B_{T+1}(x_c) \ge B_T(x_c)\,.$$

The last inequality leads to the following bound on $B_{T+1}(x_c)$ in terms of B_T:

$$B_{T+1}(x_c) \ge \frac{1}{2}\left(-c + \sqrt{c^2 + 4cB_T(x_c)}\right) = \frac{B_T(x_c)}{\frac{1}{2} + \sqrt{\frac{1}{4} + \frac{B_T(x_c)}{c}}}\,, \tag{4.6}$$

where $c = \frac{x_c u_2}{c_\alpha}$. This gives the following bound on $B_{T+1}(x_c)$:

$$B_T(x_c) \ge \frac{1}{T}\min(B_1(x_c), c)\,. \tag{4.7}$$

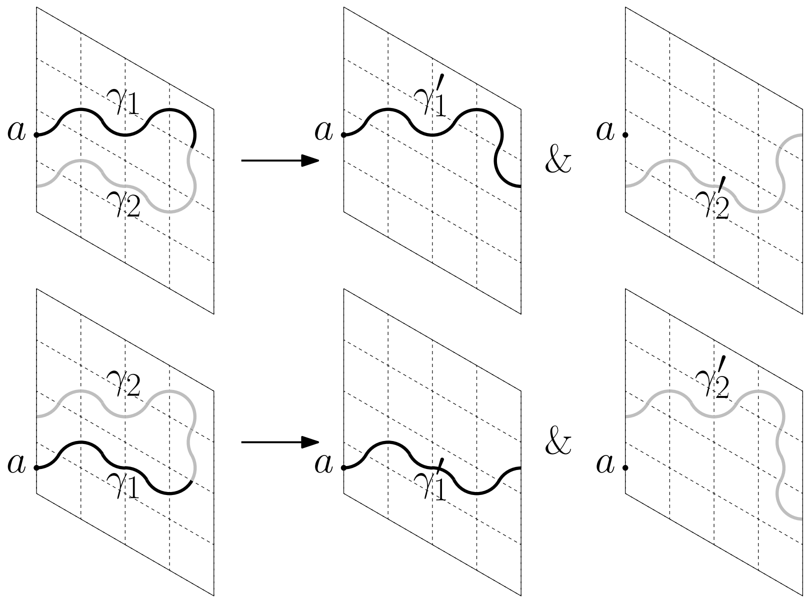

Figure 5: Two different cases of splitting a walk into γ_1 and γ_2, paths γ_1' and γ_2' in each of the cases.

The proof is done by induction, one just needs to use that the righthand side in (4.6) is increasing in B_T and to check that the denominator there is not greater than $\frac{T+1}{T}$ when $B_T(x_c)$ is replaced by the righthand side of (4.7).

Hence, $Z(x_c) \geq \sum_T B_T(x_c) = \infty$ since the harmonic series diverges. □

We give the proof of Theorem 1.1 below.

Proof. It is clear that $Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x)) = \sum_{n\geq 0} \tilde{c}_n x^n$. Because of $\tilde{c}_n \geq \frac{1}{u_1^n}(u_1 + v)^n$ (one can use only θ-arcs and straight segments), and the submultiplicativity $\tilde{c}_{n+m} \leq \tilde{c}_n \tilde{c}_m$ (any path of length $n + m$ can be divided into a path of length n and a path of length m: the weight of original path is not greater than the product of weights of these shorter paths by (4.1)-(4.2)), there exists $\tilde{\mu} \in (0, \infty)$ such that $\tilde{\mu} = \lim \sqrt[n]{\tilde{c}_n}$.

From Lemma 4.4, we obtain $Z(x_c) = \infty$. Thus $\tilde{\mu} \geq x_c^{-1}$. To get the upper bound on $\tilde{\mu}$ we need to show that $Z(x) < \infty$ for $x < x_c$.

Let us consider $x < x_c$. Any self-avoiding walk can be decomposed into self-avoiding bridges, no three of which have the same height (see [11]). In the original paper it was shown only for usual self-avoiding walks, but the proof goes through without any changes also in the case of weighted self-avoiding walks — decompose a walk into two half-space walks, then for each of them pick a bridge of a maximal width, remove them, note that we are left with two half-space walks of a smaller width, continue by induction. One just needs to modify bridges at their endpoints (in exactly the same way as in Lemma 4.4), so one gets an additional factor $(u_1 u_2)^{-1}$ that we denote by c.

At the same time, the weight of the walk is not greater than the product of weights of these bridges. Hence

$$Z(x) = Z(x, u_2(x), v(x), w_1(x), w_2(x)) \leq \prod_{T>0} (1 + cB_T(x))^2 .$$

It is clear that $B_T(x) \leq (\frac{x}{x_c})^T \cdot B_T(x_c) \leq (\frac{x}{x_c})^T$. Thus $Z(x) \leq \prod_{T>0}(1 + (\frac{x}{x_c})^T) < \infty$. Hence, $Z(x) < \infty$ for $x < x_c$ and the proof is finished. □

Note that we never use our particular choice of the definition $|\gamma|$, so Theorem 1.2 can be proven along the same lines.

5 Critical weights for the loop $O(n)$ model.

We consider a loop representation of the loop $O(n)$ model on any finite simply connected rhombic tiling. The configuration in each rhombus is one of those mentioned in fig. 1 and we consider only the configurations which can be decomposed into several loops. In this case the weight of the configuration is:

$$\omega(conf) = \prod_{\text{rhombus } r} \omega(r) \cdot n^{\#\text{loops}},$$

where $\omega(r)$ is the weight of rhombus r, i.e. either 1 or one of u_1, u_2, v, w_1, w_2. We can also add boundary conditions — allow paths going from one particular edge on the boundary to another.

Now let us take any s and $n = -2\cos\frac{4\pi}{3}s$. We consider the following family of weights parametrized by s and angle θ of the rhombus:

$$u_1 = \tfrac{1}{t} \cdot \sin{(\pi-\theta)s} \cdot \sin\tfrac{2\pi}{3}s\,, \tag{5.1}$$

$$u_2 = \tfrac{1}{t} \cdot \sin\theta s \cdot \sin\tfrac{2\pi}{3}s\,, \tag{5.2}$$

$$v = \tfrac{1}{t} \cdot \sin\theta s \cdot \sin{(\pi-\theta)s}\,, \tag{5.3}$$

$$w_1 = \tfrac{1}{t} \cdot \sin{(\tfrac{2\pi}{3}-\theta)s} \cdot \sin{(\pi-\theta)s}\,, \tag{5.4}$$

$$w_2 = \tfrac{1}{t} \cdot \sin{(\theta-\tfrac{\pi}{3})s} \cdot \sin\theta s\,, \tag{5.5}$$

where

$$t = \frac{\sin^3\frac{2\pi}{3}s}{\sin\frac{\pi}{3}s} + \sin{(\theta-\tfrac{\pi}{3})s} \cdot \sin{(\tfrac{2\pi}{3}-\theta)s}.$$

The weights given by (5.1)-(5.5) coincide with the weights given by (3.3)-(3.7) for any $\sigma = \frac{6k+5}{8}$, where $k \in \mathbb{Z}$, if one takes $s = \sigma - 1$. In particular, $s = -\frac{3}{8}$ gives (1.1)-(1.5).

One can define [12] the parafermionic observable for any n exactly in the same way as we did above for $n = 0$:

$$F_a(z) = \sum_{\gamma: a\to z} \omega(\gamma) e^{-i\sigma W(\gamma)},$$

where the sum runs over the configurations containing only loops and a path from a to z, $\omega(\gamma)$ stands for the weight of γ and $W(\gamma)$ denotes the winding of a path in γ going from a to z.

Another important tool is the Yang-Baxter equation (see [17]). Consider a symmetric equilateral hexagon H. Note that there are two different ways to tile it by 3 rhombi. Denote these two tilings by T_1 and T_2. We say that the model satisfies the Yang-Baxter equation if for any fixed configuration outside of H the sum of the weights of all its possible extensions to H is the same for tilings T_1 and T_2.

Proposition 5.1. *Let $s \in \mathbb{R}$ and take $n = -2\cos\frac{4\pi}{3}s$. Then the loop $O(n)$ model with the weights given by (5.1)-(5.5) satisfies Yang-Baxter equation and the parafermionic observable F with spin $\sigma = s+1$ satisfies the following equation on any rhombus $SENW$ (see fig. 3):*

$$F_a(z_{SE}) + e^{i\theta}F_a(z_{NE}) - F_a(z_{NW}) - e^{i\theta}F_a(z_{SW}) = 0.$$

The proof for the parafermionic observable can be done by local transformations in the same way as the proof of Lemma 3.1. One should just keep in mind that there are more different local configurations in this case because of loops.

For the connection between the parafermionic observable and the Yang-Baxter relation see [1].

Remark 5.2. The weights are symmetric in θ — if one takes $\pi - \theta$ instead of θ then v is the same, u_1 and u_2 are exchanged and w_1 and w_2 are exchanged.

One can see that for $\theta = \frac{\pi}{3}$ and any s the weights can be factorized, i. e. $w_1 = v = u_2 = u_1^2$, $w_2 = 0$ and $u_1 = \pm\frac{1}{\sqrt{2\pm\sqrt{2-n}}}$. So this is just the loop $O(n)$ model on the honeycomb lattice with the weight for each edge being equal to $\pm\frac{1}{\sqrt{2\pm\sqrt{2-n}}}$ (see fig. 2). Nienhuis nonrigorously derived [16] $\frac{1}{\sqrt{2+\sqrt{2-n}}}$ to be the critical value for the loop $O(n)$ model on the honeycomb lattice.

A Appendix

Computations in Lemma 3.1

Note that Eq. (3.11), (3.14) and the conjugates of Eq. (3.12)-(3.13) can be rewritten in the following way:

$$1 + \lambda\bar{\mu}e^{i\theta}u_2 - v - \lambda e^{i\theta}u_1 = 0\,, \tag{A.1}$$

$$\bar{\mu}^2(\lambda\bar{\mu}e^{i\theta}u_2 + w_2) = v\,, \tag{A.2}$$

$$\mu^2(-\lambda e^{i\theta}u_1 + w_1) = v\,, \tag{A.3}$$

$$\mu^2(\lambda\bar{\mu}e^{i\theta}u_2 + w_2) + \bar{\mu}^2(-\lambda e^{i\theta}u_1 + w_1) = 0\,. \tag{A.4}$$

It is easy to see that (A.2)-(A.4) are equivalent to (A.2)-(A.3) plus the following relation:

$$v(\mu^4 + \bar{\mu}^4) = 0. \tag{A.5}$$

First, consider the case $v \neq 0$. Then (A.5) gives us the desired condition on σ:

$$\cos(4\sigma\pi) = 0.$$

Equations (A.2)-(A.3) and their conjugates allow us to express everything in terms of v and some trigonometric functions:

$$u_1 = v \cdot \frac{\sin(-2\sigma\pi)}{\sin((1-\sigma)\theta)}\,,$$
$$u_2 = v \cdot \frac{\sin(-2\sigma\pi)}{\sin(\sigma\pi + (1-\sigma)\theta)}\,,$$
$$w_1 = v \cdot \frac{\sin((1-\sigma)\theta - 2\sigma\pi)}{\sin((1-\sigma)\theta)}\,,$$
$$w_2 = v \cdot \frac{\sin((1-\sigma)\theta + 3\sigma\pi)}{\sin(\sigma\pi + (1-\sigma)\theta)}\,.$$

Then (A.1) gives us a linear equation on v. It is straightforward to check that the solution is unique and given by (3.3)-(3.7).

If $v = 0$, equations are transformed into

$$u_1 = \lambda e^{i\theta}w_1\,,$$
$$u_2 = -\lambda\bar{\mu}e^{i\theta}w_2\,,$$
$$w_1 + w_2 = 1\,.$$

We know that the conjugated equations should also hold. Thus, in order to have a solution for each θ we have to set $\sigma = 1$. In this case $u_1 = w_1$ and $u_2 = w_2$.

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