

## The distribution of the supremum for spectrally asymmetric Lévy processes

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### Abstract

In this article we derive formulas for the probability  $\mathbb{P}(\sup_{t \leq T} X(t) > u)$ ,  $T > 0$  and  $\mathbb{P}(\sup_{t < \infty} X(t) > u)$  where  $X$  is a spectrally positive Lévy process with infinite variation. The formulas are generalizations of the well-known Takács formulas for stochastic processes with non-negative and interchangeable increments. Moreover, we find the joint distribution of  $\inf_{t \leq T} Y(t)$  and  $Y(T)$  where  $Y$  is a spectrally negative Lévy process.

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## 1 Introduction

Lévy processes arise in many areas of probability and play an important role among stochastic processes. The distribution of the supremum of a stochastic process especially of a Lévy process appears in many applications in finance, insurance, queueing systems and engineering. In this article we will consider spectrally asymmetric Lévy processes, that is, Lévy processes with Lévy measure concentrated on  $(0, \infty)$  or  $(-\infty, 0)$  (these have only positive jumps or only negative jumps, respectively). The problem to determine the distribution of supremum on finite and infinite time intervals has been investigated in many papers (see Bernyk et al. [2], Bertoin [3], Bertoin et al. [4], Furrer [8], Harrison [9], Hubalek and Kuznetsov [10], Kuznetsov [12], Simon [21], Takács [22], Zolotarev [23] and many others). In Zolotarev [23] the Laplace transform of the distribution of the supremum on the infinite time interval for spectrally positive Lévy processes is given. In the recent works of Bernyk et al. [2], Hubalek and Kuznetsov [10] and Kuznetsov [12] a series representation for the density function of the supremum on finite intervals for stable Lévy processes and a certain class of Lévy processes is given. In this article we determine formulas for the distribution of the supremum on finite and infinite intervals for spectrally positive Lévy processes which are generalizations of the pretty formulas of Takács [22] proven for Lévy processes with finite variation.

Let  $X = \{X(t) : t \geq 0\}$  be a spectrally positive Lévy process with characteristic function of the form

$$\mathbb{E} \exp\{iuX(t)\} = \exp \left\{ t \left[ iau - \frac{1}{2} \sigma^2 u^2 + \int_0^\infty (e^{iux} - 1 - iux \mathbb{I}(x < 1)) Q(dx) \right] \right\},$$

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where  $a \in \mathbb{R}$ ,  $\sigma^2 \geq 0$  and  $Q$  is a Lévy measure (thus,  $X$  can have Brownian component). We will investigate spectrally positive Lévy processes  $X$  with paths of infinite variation on finite intervals, which corresponds to the case that  $\int_0^1 x Q(dx) = \infty$  if Brownian component disappears. Some of the results we will phrase in terms of the spectrally negative Lévy process  $Y = -X$ . We assume throughout the article that  $u > 0$ .

The results of Takács for Lévy processes with non-negative increments are the following (originally derived in a slightly more general setting, that is, for processes with non-negative interchangeable increments; see Takács [22]).

**Theorem 1.1.** *If  $X$  is a Lévy process with non-negative increments and  $c > 0$  then*

$$\begin{aligned} & \mathbb{P}(\sup_{t \leq T} (X(t) - ct) > u) & (1.1) \\ &= \mathbb{P}(X(T) - cT > u) + \int_0^T \frac{\mathbb{E}(X(T-s) - c(T-s))^-}{c(T-s)} d_{cs} \mathbb{P}(X(s) - cs \leq u), \end{aligned}$$

where  $T > 0$ ,  $x^- = -\min\{x, 0\}$  and  $d_{cs} \mathbb{P}(X(s) - cs \leq u) = \mathbb{P}(u \leq X(s) - cs \leq u + cds)$ . If  $0 \leq \mathbb{E}X(1) < c$  then

$$\mathbb{P}(\sup_{t < \infty} (X(t) - ct) > u) = \frac{c - \mathbb{E}X(1)}{c} \int_0^\infty d_{cs} \mathbb{P}(X(s) - cs \leq u). \quad (1.2)$$

If  $\mathbb{E}X(1) \geq c$  then  $\mathbb{P}(\sup_{t < \infty} (X(t) - ct) > u) = 1$ .

We provide a rigorous definition of the measure  $d_{cs} \mathbb{P}(X(s) - cs \leq u) = \mathbb{P}(u \leq X(s) - cs \leq u + cds)$ .

**Definition 1.2.** *Let  $v \in \mathbb{R}$ ,  $c > 0$  and  $X = \{X(s), s \geq 0\}$  be a stochastic process. Let us assume that for  $a, b \in [0, \infty)$ ,  $b > a$ , the following limit exists*

$$m([a, b]) := \lim_{\delta(\mathcal{P}) \rightarrow 0} \sum_{k=1}^n (F(v + c \Delta s_{k-1}, s_{k-1}) - F(v, s_{k-1})),$$

where  $F(v, s) = \mathbb{P}(X(s) \leq v)$ ,  $\mathcal{P} = \{s_k\}_{k=0}^n$  is a partition of the interval  $[a, b]$ ,  $\Delta s_{k-1} = s_k - s_{k-1}$  and  $\delta(\mathcal{P})$  is the mesh of the partition  $\mathcal{P}$ . We assume that the limit exists regardless of the partition. Then  $d_{cs} \mathbb{P}(X(s) \leq v)$  is defined as a Borel measure  $m$  on  $[0, \infty)$  by

$$\int_a^b d_{cs} \mathbb{P}(X(s) \leq v) = m([a, b]), \quad a, b \in [0, \infty),$$

where  $m([a, b])$  is the measure of the interval  $[a, b]$ .

**Remark 1.3.** Note that if  $X$  has one-dimensional distributions that are absolutely continuous with respect to Lebesgue measure then the measure above exists and  $d_{cs} \mathbb{P}(X(s) \leq v) = cf(v, s) ds$ , where  $f(v, s)$  is a density function of the random variable  $X(s)$ . Moreover if the distribution of  $X(s)$  has a density function on the positive half-line (and e.g.  $X(s)$  has an atom at zero) then  $d_{cs} \mathbb{P}(X(s) \leq v) = cf(v, s) ds$  for  $v > 0$ , where  $f(v, s)$  is a density function of  $X(s)$  on the positive half-line. Thus if  $X(s)$  is a compound Poisson process with a density function on the positive half-line (a compound Poisson process has always an atom at zero) then eq. (1.1) gives the so-called Seal's formula (see Seal [20] and Prabhu [17] or e.g. Asmussen and Albrecher [1]). Moreover it is easy to notice that the measure  $d_{cs} \mathbb{P}(X(s) \leq v)$  exists if  $X(s)$  has a discrete distribution e.g. in the case of Poisson process or negative binomial Lévy process with a linear drift (see e.g. Sato [19] Example 4.6).

Further, if  $X(s)$  or  $Y(s)$  has a density function, their density functions will be denoted by  $f(v, s)$  and  $f(-v, s)$ , respectively. Theorem 1.1 completely solves the problem of

the supremum distribution for spectrally positive Lévy processes with finite variation. The proof of Theorem 1.1 uses a generalization of the classical ballot theorem and an approximation argument (see Takács [22]).

Similarly in Takács [22] a formula for the process  $ct - X(t)$  is derived (the process  $ct - X(t)$  is a spectrally negative Lévy process with finite variation).

**Theorem 1.4.** *If  $X$  is a Lévy process with non-negative increments and  $c > 0$  then*

$$\begin{aligned} \mathbb{P}(\sup_{t \leq T} (ct - X(t)) > u) &= \frac{u}{c} \int_{u/c}^T s^{-1} d_{cs} \mathbb{P}(X(s) - cs \leq -u) \\ &= \frac{u}{c} \int_0^T s^{-1} d_{cs} \mathbb{P}(cs - X(s) \leq u), \end{aligned} \quad (1.3)$$

where  $d_{cs} \mathbb{P}(X(s) - cs \leq -u) = \mathbb{P}(-u \leq X(s) - cs \leq -u + c ds)$  and  $d_{cs} \mathbb{P}(cs - X(s) \leq u) = \mathbb{P}(u \leq cs - X(s) \leq u + c ds)$ .

**Remark 1.5.** The formula (1.3) is the well-known identity of Kendall which is valid for any spectrally negative Lévy process not equal to a subordinator (see Kendall [11] or Borovkov and Burq [5] and references therein). Here we have Kendall's identity for a spectrally negative Lévy process  $ct - X(t)$  with finite variation.

Let us recall Kendall's identity which will be used in the proof of our main result Theorem 2.1. Let

$$S(z) = \inf\{t \geq 0 : Y(t) > z\}, \quad (1.4)$$

where  $z \geq 0$ .

**Theorem 1.6.** *For any spectrally negative Lévy process  $Y$  that is not a subordinator and  $t, z > 0$ , the following identity for measures on  $(0, \infty) \times (0, \infty)$  holds*

$$t \mathbb{P}(S(z) \in dt) dz = z \mathbb{P}(Y(t) \in dz) dt.$$

**Remark 1.7.** Under the condition that the one-dimensional distributions of  $Y$  are absolutely continuous with density functions  $f(-v, s)$ , the density of the random variable  $S(z)$  is given by

$$\frac{\mathbb{P}(S(z) \in dt)}{dt} = \frac{z}{t} f(-z, t).$$

## 2 The infinite variation case

We extend the results of Takács [22] to the case of Lévy processes with infinite variation.

**Theorem 2.1.** *If the one-dimensional distributions of  $X$  are absolutely continuous, then*

$$\begin{aligned} \mathbb{P}(\sup_{t \leq T} X(t) > u) \\ = \mathbb{P}(X(T) > u) + \int_0^T \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds, \end{aligned} \quad (2.1)$$

where  $x^- = -\min\{x, 0\}$  and  $f(u, s)$  is a density function of  $X(s)$  for  $s > 0$ .

**Remark 2.2.** Note that for spectrally positive Lévy processes  $\mathbb{E}(X(1))^- < \infty$  which gives that  $\mathbb{E}X(1) > -\infty$  (see e.g. Sato [19], Theorem 26.8).

*Proof.* The proof will be based on Kendall's identity, duality, strong Markov property and Hunt's switching identity. Let  $P_t$  and  $\hat{P}_t$  be Markov semigroups of the processes  $Y$  and  $-Y = X$ , respectively, killed upon entering the negative half-line  $(-\infty, 0)$  (see e.g. Bertoin [3], Sections 0.1 and II.1). By  $(\mathbb{P}_x, x \in \mathbb{R})$  we denote the family of measures

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conditioned on  $\{Y(0) = x\}$  with  $\mathbb{P}_0 = \mathbb{P}$ . Thus by Hunt's switching identity (see e.g. Bertoin [3], Theorem II.1.5) it holds for nonnegative measurable functions  $g, h$  and every  $t \geq 0$  that

$$\int_{\mathbb{R}} P_t h(x) g(x) dx = \int_{\mathbb{R}} h(x) \widehat{P}_t g(x) dx. \quad (2.2)$$

Moreover the right hand side of (2.2) is as follows

$$\begin{aligned} \int_{\mathbb{R}} h(x) \widehat{P}_t g(x) dx &= \int_{\mathbb{R}} h(x) \mathbb{E}_{-x}[g(-Y(t)) \mathbb{1}_{\{t < S(0)\}}] dx \\ &= \int_{\mathbb{R}} h(x) \mathbb{E}_{-x}[g(-Y(t))] dx \end{aligned} \quad (2.3)$$

$$- \int_{\mathbb{R}} h(x) dx \int_0^t \mathbb{E}_{-x}[g(-Y(t)) | S(0) = s] \mathbb{P}_{-x}(S(0) \in ds) \quad (2.4)$$

$$= \int_{\mathbb{R}} \mathbb{E}_x[h(Y(t))] g(x) dx - \int_{\mathbb{R}} h(x) dx \int_0^t \mathbb{E}[g(-Y(t-s))] \mathbb{P}(S(x) \in ds)$$

where in the last equality for the term (2.3) we use Bertoin [3], Proposition II.1.1, and for the term (2.4) we have  $\mathbb{P}_{-x}(S(0) \in ds) = \mathbb{P}(S(x) \in ds)$  and by the strong Markov property and by the fact  $Y(S(0)) = 0$  ( $Y$  does not jump upwards) we get

$$\mathbb{E}_{-x}[g(-Y(t)) | S(0) = s] = \mathbb{E}[g(-Y(t-s))]$$

for  $s < t$ . Taking  $h$  disappearing on the negative half-line and substituting Kendall's identity into the last subtrahend we obtain

$$\begin{aligned} &\int_{\mathbb{R}} h(x) dx \int_0^t \mathbb{E}[g(-Y(t-s))] \mathbb{P}(S(x) \in ds) \\ &= \int_0^\infty h(x) dx \int_0^t \mathbb{E}[g(-Y(t-s))] \frac{x}{s} f(-x, s) ds \\ &= \int_0^\infty h(x) dx \int_{\mathbb{R}} g(z) dz \int_0^t f(z, t-s) \frac{x}{s} f(-x, s) ds \\ &= \int_0^\infty h(x) dx \int_{\mathbb{R}} g(z) dz \int_0^t \frac{x}{t-s} f(z, s) f(-x, t-s) ds \\ &= \int_{\mathbb{R}} g(x) dx \int_0^\infty h(z) dz \int_0^t \frac{z}{t-s} f(x, s) f(-z, t-s) ds, \end{aligned}$$

where in the last equality we swapped  $x$  with  $z$  and changed the order of integrals. The left hand side of (2.2) is the following

$$\int_{\mathbb{R}} P_t h(x) g(x) dx = \int_{\mathbb{R}} g(x) dx \int_0^\infty h(z) \mathbb{P}_x(\inf_{s \leq t} Y(s) \geq 0, Y(t) \in dz).$$

Now returning to (2.2) we get the following identity for measures

$$\begin{aligned} &\mathbb{P}_x(\inf_{s \leq t} Y(s) \geq 0, Y(t) \in dz) dx \\ &= \mathbb{P}_x(Y(t) \in dz) dx - dz dx \int_0^t \frac{z}{t-s} f(x, s) f(-z, t-s) ds \end{aligned}$$

which gives

$$\mathbb{P}_x(\inf_{s \leq t} Y(s) < 0, Y(t) \in dz) = \int_0^t \frac{z}{t-s} \mathbb{P}(Y(t-s) \in dz) f(x, s) ds \quad (2.5)$$

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where  $t, x, z > 0$ . Thus we have

$$\mathbb{P}(\sup_{s \leq t} X(s) > x, x - X(t) \in dz) = \int_0^t \frac{z}{t-s} \mathbb{P}(-X(t-s) \in dz) f(x, s) ds$$

for  $z > 0$  and obviously

$$\mathbb{P}(\sup_{s \leq t} X(s) > x, x - X(t) \in dz) = \mathbb{P}(x - X(t) \in dz)$$

for  $z \leq 0$ . Integrating the last formula with respect to  $z$  we get the thesis of the theorem.  $\square$

**Remark 2.3.** In Theorem 2.1 we can assume that  $X(s)$  has an atom at zero that is the density function  $f(u, s)$  is not a proper density function.

The formula (2.1) was derived in Michna [14] for spectrally positive  $\alpha$ -stable Lévy processes (see also Furrer [7] Proposition 2.7).

In fact by (2.5) we proved a more general result which determines the joint distribution of  $\inf_{t \leq T} Y(t)$  and  $Y(T)$ .

**Theorem 2.4.** *If  $Y$  is a spectrally negative Lévy process that is not a subordinator and the one-dimensional distributions of  $Y$  are absolutely continuous, then*

$$\mathbb{P}(\inf_{t \leq T} Y(t) < -x, Y(T) + x \in dz) = dz \int_0^T \frac{z}{T-s} p(z, T-s) p(-x, s) ds,$$

where  $T, x, z > 0$  and here  $p(x, s)$  is a density function of  $Y(s)$  for  $s > 0$ .

Note that the formula (2.1) can also be obtained directly from Takács formula (1.1) using an approximation argument if  $X(s)$  does not have Brownian component. To outline the argument, we introduce for any  $\epsilon > 0$

$$N_\epsilon(t) = \sum_{s \leq t} \Delta X(s) \mathbb{I}(\Delta X(s) \geq \epsilon).$$

The process  $N_\epsilon$  is a compound Poisson process with positive jumps.

**Proposition 2.5.** *We have*

$$N_\epsilon(t) - \left( \int_\epsilon^1 x Q(dx) - a \right) t \rightarrow X(t), \tag{2.6}$$

as  $\epsilon \downarrow 0$  a.s. in the uniform topology.

*Proof.* The assertion follows from the proof of Lévy-Itô representation see e.g. Sato [19].  $\square$

Substituting  $X(t) = N_\epsilon(t)$  and  $c = \int_\epsilon^1 x Q(dx) - a$  to (1.1) and letting  $\epsilon$  tend to zero we arrive at the formula (2.1). To turn the sketched argument into a rigorous proof still requires a justification to take the limits under the integral.

We also note that taking  $T$  to infinity in (2.1) yields the following formula (the passage to the limit  $T \rightarrow \infty$  also needs a justification) for which we provide a proof under assumptions which are rather easy to check.

**Theorem 2.6.** *Let  $X$  be a spectrally positive Lévy process with a density function  $f(u, s)$  of  $X(s)$  for  $s > 0$  such that  $\mathbb{E}X(1) < 0$  and let a function  $g$  exist such that*

$$\frac{\mathbb{E}(X(t))^-}{t} \leq g(t), \quad t \in (0, 1) \tag{2.7}$$

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and

$$\int_0^1 g(t) dt < \infty. \tag{2.8}$$

Moreover, we assume

$$\sup_{t \geq 1} \frac{\mathbb{E}(X(t))^-}{t} < \infty \tag{2.9}$$

and

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}(X(t))^-}{t} = |\mathbb{E}X(1)| \tag{2.10}$$

and for every  $u > 0$

$$\lim_{s \rightarrow \infty} f(u, s) = 0. \tag{2.11}$$

Then

$$\mathbb{P}(\sup_{t < \infty} X(t) > u) = |\mathbb{E}X(1)| \int_0^\infty f(u, s) ds. \tag{2.12}$$

**Remark 2.7.** In Theorem 2.6 we can assume that  $X(s)$  has an atom at zero that is the density function  $f(u, s)$  is not a proper density function.

*Proof.* By the formula (2.1) and the assumption (2.10) and Fatou lemma we get

$$|\mathbb{E}X(1)| \int_0^\infty f(u, s) ds \leq \mathbb{P}(\sup_{t < \infty} X(t) > u). \tag{2.13}$$

Using (2.1) again we can write for  $1 \leq T_0 < T - 1$

$$\begin{aligned} & \mathbb{P}(\sup_{t \leq T} X(t) > u) \\ &= \mathbb{P}(X(T) > u) + \int_0^{T_0} \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds \\ & \quad + \int_{T_0}^{T-1} \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds + \int_{T-1}^T \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds. \end{aligned}$$

Thus if  $T \rightarrow \infty$  using (2.9), (2.10) and (2.13) we obtain

$$\begin{aligned} & \mathbb{P}(\sup_{t < \infty} X(t) > u) \\ & \leq |\mathbb{E}X(1)| \int_0^{T_0} f(u, s) ds \\ & \quad + \left( \sup_{t \geq 1} \frac{\mathbb{E}(X(t))^-}{t} \right) \int_{T_0}^\infty f(u, s) ds \\ & \quad + \lim_{T \rightarrow \infty} \int_{T-1}^T \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds. \end{aligned} \tag{2.14}$$

Applying the assumptions (2.7) and (2.8) and (2.11) we obtain

$$\lim_{T \rightarrow \infty} \int_{T-1}^T \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds = 0$$

because

$$\begin{aligned} \int_{T-1}^T \frac{\mathbb{E}(X(T-s))^-}{T-s} f(u, s) ds &= \int_0^1 \frac{\mathbb{E}(X(s))^-}{s} f(u, T-s) ds \\ &\leq \int_0^1 g(s) f(u, T-s) ds \\ &\leq \sup_{T-1 < s < T} f(u, s) \int_0^1 g(s) ds. \end{aligned}$$

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Hence taking  $T_0 \rightarrow \infty$  in (2.14) we obtain

$$\mathbb{P}(\sup_{t < \infty} X(t) > u) \leq |\mathbb{E}X(1)| \int_0^\infty f(u, s) \, ds$$

which together with (2.13) completes the proof.  $\square$

In the case  $\mathbb{E}X(1) \geq 0$  it is easy to show that  $\mathbb{P}(\sup_{t < \infty} X(t) > u) = 1$  (use the law of large numbers and Chung and Fuchs [6] in the case  $\mathbb{E}X(1) = 0$ ).

**Example 2.8.** Let us consider the spectrally positive  $\alpha$ -stable Lévy process  $Z_\alpha$  with  $1 < \alpha \leq 2$  (that is the skewness parameter  $\beta = 1$  and the shift parameter  $\mu = 0$ , see e.g. Samorodnitsky and Taqqu [18]). We will investigate the process  $X(t) = Z_\alpha(t) - ct$  with  $c > 0$  which is a spectrally positive Lévy process with  $\mathbb{E}X(t) = -ct$ . Note that

$$\begin{aligned} \frac{\mathbb{E}(X(t))^-}{t} &= \mathbb{E} \left( \frac{Z_\alpha(t)}{t} - c \right)^- \\ &= \mathbb{E} \left( t^{1/\alpha-1} Z_\alpha(1) - c \right)^- \\ &\leq t^{1/\alpha-1} \mathbb{E} (Z_\alpha(1) - c)^- \\ &= g(t) \end{aligned} \tag{2.15}$$

where in the second line we used the self-similarity of  $Z_\alpha$  and the inequality is valid for  $0 < t \leq 1$  providing the function  $g$  which satisfies the assumptions (2.7) and (2.8). Since

$$\left( t^{1/\alpha-1} Z_\alpha(1) - c \right)^- \leq |Z_\alpha(1)| + c \tag{2.16}$$

for  $t \geq 1$  and the right hand side of the last inequality is integrable we get

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}(X(t))^-}{t} = \lim_{t \rightarrow \infty} \mathbb{E} \left( t^{1/\alpha-1} Z_\alpha(1) - c \right)^- = |\mathbb{E}X(1)|$$

by Lebesgue dominated convergence theorem. By (2.15) and (2.16) the assumption (2.9) is satisfied. Moreover if  $f(x)$  is the density function of  $Z_\alpha(1)$  then the density function of  $Z_\alpha(s) - cs$  is

$$f(u, s) = s^{-1/\alpha} f(s^{-1/\alpha}(u + cs))$$

and it is clear that  $\lim_{s \rightarrow \infty} s^{-1/\alpha} f(s^{-1/\alpha}(u + cs)) = 0$ . Thus using (2.12) of Theorem 2.6 we get

$$\mathbb{P}(\sup_{t < \infty} (Z_\alpha(t) - ct) > u) = c \int_0^\infty s^{-1/\alpha} f(s^{-1/\alpha}(u + cs)) \, ds. \tag{2.17}$$

Applying a certain form of the density  $f(x)$  for  $1 < \alpha < 2$  for the parametrization as in Samorodnitsky and Taqqu [18] (see e.g. Nolan [16] and references therein) and the scale parameter  $\sigma = 1$  (that is  $Z_\alpha(1)$  has the scale parameter  $\sigma = 1$ ) we obtain

$$\begin{aligned} &\mathbb{P}(\sup_{t < \infty} (Z_\alpha(t) - ct) > u) \\ &= \frac{c}{\pi} \int_0^\infty ds s^{-1/\alpha} \int_0^\infty e^{-t^\alpha} \cos \left( ts^{-1/\alpha}(u + cs) - t^\alpha \tan \frac{\pi\alpha}{2} \right) dt \\ &= \sum_{n=0}^\infty \frac{(-a)^n}{\Gamma(1 + (\alpha - 1)n)} u^{(\alpha-1)n}, \end{aligned}$$

where  $a = c \cos(\pi(\alpha - 2)/2)$ ,  $c > 0$  and the last equality follows by comparing with the result of Furrer [8] (the last expression is the Mittag-Leffler function).

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For the standard Wiener process  $W(t)$  similarly as above we get the following identity

$$\mathbb{P}(\sup_{t < \infty} (W(t) - ct) > u) = \frac{c}{\sqrt{2\pi}} \int_0^\infty s^{-1/2} \exp\left(-\frac{(u + cs)^2}{2s}\right) ds = \exp(-2uc),$$

where  $c > 0$  and the last equality is the well-known result for the supremum distribution of Wiener process over the infinite time horizon (see e.g. Asmussen and Albrecher [1]).

Similarly one can consider the distribution of the supremum on finite intervals. By the formula (2.1) of Theorem 2.1 we derive (for simplicity we put  $c = 0$ , for  $c \neq 0$  the formula will be a little more complicated)

$$\begin{aligned} & \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u) \\ &= \mathbb{P}(Z_\alpha(T) > u) \\ & \quad + \frac{\mathbb{E}(Z(1))^-}{\pi} \int_0^T ds (T-s)^{1/\alpha-1} s^{-1/\alpha} \int_0^\infty e^{-t^\alpha} \cos\left(ts^{-1/\alpha}u - t^\alpha \tan \frac{\pi\alpha}{2}\right) dt, \end{aligned}$$

where

$$\mathbb{P}(Z_\alpha(T) > u) = \frac{T^{-1/\alpha}}{\pi} \int_u^\infty dx \int_0^\infty e^{-t^\alpha} \cos\left(tT^{-1/\alpha}x - t^\alpha \tan \frac{\pi\alpha}{2}\right) dt$$

and

$$\mathbb{E}(Z(1))^- = \frac{1}{\pi} \int_{-\infty}^0 dx x \int_0^\infty e^{-t^\alpha} \cos\left(tx - t^\alpha \tan \frac{\pi\alpha}{2}\right) dt;$$

compare the formula with Bernyk et al. [2] where they get a series representation for the density function of the supremum distribution, see also Bertoin et al. [4] and Hubalek and Kuznetsov [10].

**Example 2.9.** Assume that  $X(t)$  is a compound Poisson process with negative drift  $ct$  and nonnegative jumps perturbed by a spectrally positive  $\alpha$ -stable Lévy process  $Z_\alpha(t)$  with  $1 < \alpha \leq 2$ . Then

$$(X(t))^- \leq (Z_\alpha(t) - ct)^-$$

so using (2.15) and (2.16) one can easily check that the assumptions (2.7), (2.8), (2.9) and (2.10) are satisfied. Under mild conditions on the distribution of the compound Poisson process we can check the assumption (2.11).

In some cases the supremum distribution can be identified by using just the strong Markov property. Indeed, let us consider the spectrally negative  $\alpha$ -stable Lévy process  $Z_\alpha$  with  $1 < \alpha \leq 2$  without any drift (that is the skewness parameter  $\beta = -1$  and the shift parameter  $\mu = 0$ ). Thus, let  $\tau = \inf\{t > 0 : Z_\alpha(t) > u\}$  where  $u \geq 0$  then  $\{\sup_{t \leq T} Z_\alpha(t) > u\} = \{\tau < T\}$  a.s. Since the process  $Z_\alpha$  is spectrally negative (it has no positive jumps), we have  $Z_\alpha(\tau) = u$ . By the strong Markov property  $Z_\alpha^*(t) = Z_\alpha(t + \tau) - Z_\alpha(\tau)$  is a Lévy process with the same distribution as  $Z_\alpha$  and independent of  $\tau$ . We know that



## The distribution of the supremum

$\mathbb{P}(Z_\alpha(s) > 0) = 1/\alpha$ . Thus, for  $u \geq 0$  we have

$$\begin{aligned} & \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u) \\ &= \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u, Z_\alpha(T) > u) + \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u, Z_\alpha(T) \leq u) \\ &= \mathbb{P}(Z_\alpha(T) > u) + \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u, Z_\alpha^*(T - \tau) \leq 0) \\ &= \mathbb{P}(Z_\alpha(T) > u) + \mathbb{P}(\tau < T, Z_\alpha^*(T - \tau) \leq 0) \\ &= \mathbb{P}(Z_\alpha(T) > u) + \int_0^T \mathbb{P}(Z_\alpha^*(T - s) \leq 0) d_s \mathbb{P}(\tau < s) \\ &= \mathbb{P}(Z_\alpha(T) > u) + \left(1 - \frac{1}{\alpha}\right) \mathbb{P}(\tau < T) \\ &= \mathbb{P}(Z_\alpha(T) > u) + \left(1 - \frac{1}{\alpha}\right) \mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u), \end{aligned}$$

which gives

$$\mathbb{P}(\sup_{t \leq T} Z_\alpha(t) > u) = \alpha \mathbb{P}(Z_\alpha(T) > u);$$

compare the last formula with the result of Michna [15].

In this paper we studied the distribution of suprema for Lévy processes with jumps of single sign. The general case of Lévy processes with jumps of either sign (for example, symmetric Lévy processes) is much more complicated (see, for example, Kwaśnicki et al. [13]).

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