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Multivariate gamma distributions*

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Abstract

A representation is given for a large class of n-dimensional multivariate gamma random variables as defined by Verre-Jones. In particular, the probability density functions of all 2-dimensional gamma random variables are given explicitly and it is shown how to obtain the probability density functions of all 3-dimensional gamma random variables.

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1 Introduction

Vere-Jones [4] defines an n-dimensional multivariate gamma distribution to be the probability distribution of an R^n valued random variable $X = (X_1, \dots, X_n)$ that has Laplace transform

$$E\left(e^{-\sum_{i=1}^n s_i X_i}\right) = \frac{1}{|I+RS|^\alpha} \tag{1.1}$$

for some $n \times n$ matrix R and diagonal matrix S with entries s_i, $1 \leq i \leq n$, and $\alpha > 0$. This is not an unreasonable definition. A gamma random variable is one with probability density function

$$f(u, v; x) = \frac{v^u x^{u-1} e^{-vx}}{\Gamma(u)} \quad \text{for } x \geq 0 \text{ and } u, v > 0, \tag{1.2}$$

and equal to 0 for $x \leq 0$, where $\Gamma(u) = \int_0^\infty x^{u-1} e^{-x}\, dx$ is the gamma function. We use $\xi_{u,v}$ to denote a random variable with probability density function $f(u, v; x)$. The Laplace transform of $\xi_{u,v}$ is

$$\int_0^\infty \frac{v^u x^{u-1} e^{-(v+s)x}}{\Gamma(u)}\, dx = \frac{1}{\left(1 + \dfrac{s}{v}\right)^u} = \frac{v^u}{(v+s)^u}. \tag{1.3}$$

Therefore if R is a diagonal matrix with entries $1/v_i$, (1.1) is the Laplace transform of $(\xi_{\alpha,v_1}, \dots, \xi_{\alpha,v_n})$, in which all the components are independent.

For a less trivial example we note that when R is symmetric and positive definite and $\alpha = 1/2$, $X = (\eta_1^2/2, \dots, \eta_n^2/2)$, where $(\eta_1, \dots, \eta_n)$ is an n-dimensional normal random variable with mean zero and covariance matrix R. In this case the individual components

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of X are the gamma distributed random variables $\eta_i^2/2 = \xi_{1/2,R_{i,i}}$. Here we use the common notation $R = \{R_{i,j}\}_{i,j=1}^n$.

Two fundamental questions are apparent. One is, for which matrices R do there exist random variables X satisfying (1.1)? Vere-Jones answers this question but with criteria that are, in general, very difficult to verify. Nevertheless, using his criteria, Eisenbaum and Kaspi [1] show that it suffices that R is the inverse of an M-matrix; see Remark 2.1. Given this, the second, and perhaps more interesting question is: What is the distribution function of an n-dimensional multivariate gamma distribution?

In this paper we describe a large class of n-dimensional multivariate gamma distributions and give the probability density function of all multivariate gamma distributions in dimensions 2 and 3.

We assume that $|R| > 0$. Therefore, $A := R^{-1}$ exists and we write (1.4) as

$$E\left(e^{-\sum_{i=1}^n s_i X_i}\right) = \frac{|A|^\alpha}{|A+S|^\alpha}. \tag{1.4}$$

We explain in Remark 2.1 that without loss of generality, in dimension 2, we can take the matrix A to have the form

$$A_{(2)} = \begin{pmatrix} a & -\gamma \\ -\gamma & b \end{pmatrix} \tag{1.5}$$

where $a, b, \gamma > 0$ and $ab > \gamma^2$.

Theorem 1.1. *Let $X = (X_1, X_2)$ be a random variable determined by $A_{(2)}$ as in (1.4). Then its probability density function is*

$$\begin{aligned}\widetilde{g}(\alpha, A_{(2)}; (x_1, x_2)) \\ = \frac{(ab-\gamma^2)^\alpha}{\Gamma(\alpha)} \frac{\gamma^{1-\alpha}}{(x_1x_2)^{(1-\alpha)/2}} \mathcal{I}_{\alpha-1}\left(2\gamma\sqrt{x_1x_2}\,\right) e^{-(ax_1+bx_2)}\end{aligned} \tag{1.6}$$

on R_+^2, and zero elsewhere, where

$$\mathcal{I}_\nu(z) = \sum_{n=0}^\infty \frac{1}{\Gamma(n+\nu+1)\,n!}\left(\frac{z}{2}\right)^{2n+\nu} \tag{1.7}$$

is the modified Bessel function.

When $\gamma = 0$

$$\widetilde{g}(\alpha, A; (x_1, x_2)) = f(\alpha, a; x_1) f(\alpha, b; x_2). \tag{1.8}$$

If we take the limit as γ goes to zero in (1.6) we get (1.8) .

When $\alpha = 1$, we have a 2-dimensional exponential distribution and (1.6) is

$$\widetilde{g}(1, A; (x_1, x_2)) = (ab-\gamma^2)\mathcal{I}_0\left(2\gamma\sqrt{x_1x_2}\,\right) e^{-(ax_1+bx_2)}. \tag{1.9}$$

When $\alpha = 1/2$, we can write (1.6) as

$$\widetilde{g}(1/2, A; (x_1, x_2)) = \frac{(ab-\gamma^2)^{1/2}}{\pi\sqrt{x_1x_2}} \cosh\left(2\gamma\sqrt{x_1x_2}\,\right) e^{-(ax_1+bx_2)}. \tag{1.10}$$

This allows us to check these results since $\widetilde{g}(1/2, A; (x_1, x_2))$ is the probability density function of $(\eta_1^2/2, \eta_2^2/2)$ where (η_1, η_2) is a normal random variable with mean zero and covariance matrix A^{-1}.

We can describe (X_1, X_2) in Theorem 1.1 as a randomized sum of independent vectors with components that are independent 1-dimensional gamma random variables.

Theorem 1.2. *Let $X = (X_1, X_2)$ be a random variable with density function $\widetilde{g}(\alpha, A; (x_1, x_2))$. Then*

$$X \overset{law}{=} \sum_{n=0}^{\infty} I_n(Z)(\xi_{n+\alpha,a}, \xi'_{n+\alpha,b}), \tag{1.11}$$

where all the random variables are independent and

$$P\left(Z = n\right) = \frac{(ab - \gamma^2)^\alpha}{(ab)^\alpha} \frac{\Gamma(n+\alpha)\gamma^{2n}}{\Gamma(\alpha)\, n!(ab)^n}. \tag{1.12}$$

Note that when $\gamma = 0$, $P\left(Z = 0\right) = 1$. Also, we see that the probability distribution of Z depends only on α and γ^2/ab.

We show in [2] that in dimension 3 in order for (1.1) to define a random variable either R must be symmetric and positive definite or it must be the inverse of an M matrix, or both. Note that

$$\frac{|A|}{|A+S|} = \frac{|DAD^{-1}|}{|DAD^{-1}+S|}. \tag{1.13}$$

for all diagonal matrices D with strictly positive entries. Consequently (1.4) remains unchanged for many different matrices A. Using this observation we see that that the most general 3×3 non-singular M-matrix $A_{(3)}$ has the form

$$A_{(3)} = \begin{pmatrix} a_{1,1} & -b & -c \\ -b & a_{2,2} & -d'' \\ -c & -d' & a_{3,3} \end{pmatrix} \tag{1.14}$$

where where $a_{i,i}$, $i = 1,2,3$ and $b, c, d', d'' > 0$. Let $d'd'' = d^2$. In this case (1.4) is equal to

$$\frac{|A_{(3)}|^\alpha}{\left((a_{1,1}+s_1)(a_{2,2}+s_2)(a_{3,3}+s_3)\left(1 - Q(s_1,s_2,s_3)\right)\right)^\alpha} \tag{1.15}$$

where

$$\begin{aligned} Q(s_1,s_2,s_3) \quad = \quad & \frac{b^2}{(a_{1,1}+s_1)(a_{2,2}+s_2)} + \frac{c^2}{(a_{1,1}+s_1)(a_{3,3}+s_3)} \\ & + \frac{d^2}{(a_{2,2}+s_2)(a_{3,3}+s_3)} + \frac{bc(d'+d'')}{(a_{1,1}+s_1)(a_{2,2}+s_2)(a_{3,3}+s_3)}. \end{aligned} \tag{1.16}$$

Consequently we can write (1.4) as

$$\frac{|A_{(3)}|^\alpha}{\left((a_{1,1}+s_1)\cdots(a_{3,3}+s_3)\right)^\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!} Q^n(s_1,s_2,s_3). \tag{1.17}$$

One can use this to find the probability density function of the random variable determined by $A_{(3)}$, and, in analogy with Theorem 1.2, to obtain

Theorem 1.3. *Let $X = (X_1, X_2, X_3)$ be a random variable determined by $A_{(3)}$. Then*

$$X \overset{law}{=} \sum_{n_1,n_2,n_3}^{\infty} I_{n_1,n_2,n_3}(Z)(\xi_{n_1+\alpha,a_{1,1}}, \xi'_{n_{2_3}+\alpha,a_{2,2}}, \xi''_{n+\alpha,a_{3,3}}), \tag{1.18}$$

where all the random variables are independent and $Z = (Z_1, Z_2, Z_3)$ is an integer valued random variable. (Here the sum is taken over all sets of distinct integers in ${\mathbb{Z}_+}^3$.)

Example 1.4. As an explicit example we note that

$$(1-a^2-b^2)^\alpha (x_1x_2x_3)^{\alpha-1} e^{-(x_1+x_2+x_3)} \tag{1.19}$$
$$\sum_{k\le n, n=0}^{\infty} \frac{x_2^n}{\Gamma(\alpha) n!} \binom{n}{k} \frac{(a^2x_1)^k (b^2x_3)^{n-k}}{\Gamma(k+\alpha)\Gamma(n-k+\alpha)}$$

is the probability density function of the 3-dimensional multivariate gamma distribution determined by the M-matrix

$$A_{(3)} = \begin{pmatrix} 1 & -a & 0 \\ -a & 1 & -b \\ 0 & -b & 1 \end{pmatrix} \tag{1.20}$$

with $a, b > 0$ and $|A| > 0$ and inverse

$$R_{(3)} = \frac{1}{1-a^2-b^2} \begin{pmatrix} 1-b^2 & a & ab \\ a & 1 & b \\ ab & b & 1-a^2 \end{pmatrix}. \tag{1.21}$$

Other examples are given in Section 2.

With additional hypotheses we can describe a large class of multivariate gamma distributed random variables in higher dimensions. To begin let $\widetilde{B}$ be an $n \times n$ matrix. Let $D_{\widetilde{B}}$ denote the $n \times n$ diagonal matrix with entries consisting of the diagonal elements of $\widetilde{B}$. Let

$$\widetilde{B}_0 := \widetilde{B} - D_{\widetilde{B}}. \tag{1.22}$$

We define the property of a matrix having a strongly negative off diagonal determinant. Consider the $n \times n$ matrix A. Let M_i, $i = 1, \ldots, 2^n - (n+1)$, denote the principle sub-matrices of A of dimensions 2 to n, (i.e. A itself). We say that A has a strongly negative off diagonal determinant, if $|(M_i)_0| \le 0$ for all $i = 1, \ldots, 2^n - (n+1)$. (A principal sub-matrix of A is a matrix obtained from A by deleting the i_1 through i_p-th rows and columns of A, where $\{i_1, \ldots, i_p\} \subset \{1, \ldots, n\}$).

Let $A = \{a_{i,j}\}_{i,j=1}^n$. It is obvious that if $|A| > 0$ and A has a strongly negative off diagonal determinant, A has a diagonally dominant determinant, i.e.

$$\prod_{i=1}^n a_{i,i} > |A|. \tag{1.23}$$

Lemma 1.5. *Let A be an $n \times n$ matrix with $|A| > 0$, and with off diagonal elements that are less than or equal to zero, and that has a strongly negative off diagonal determinant. Then for $(s_1, \ldots, s_n) \subset [0, s_0]^n$, for some $s_0 > 0$,*

$$\frac{|A|^\alpha}{|A+S|^\alpha} = \frac{|A|^\alpha}{((a_{1,1}+s_1)\cdots(a_{n,n}+s_n))^\alpha} \sum_{q=0}^{\infty} \frac{\Gamma(q+\alpha)}{\Gamma(\alpha)\, q!} \tag{1.24}$$
$$\left(\sum_{m_1,\ldots,m_j} \frac{c_{m_1,\ldots,m_j}}{(a_{m_1,m_1}+s_{m_1})\cdots(a_{m_j,m_j}+s_{m_j})} \right)^q,$$

in which all coefficients the $c_{m_1,\ldots,m_j}$ are greater than or equal to zero. (Here the sum is taken over all sets of distinct integers in ${\mathbb{Z}_+}^n$.)

Under the hypotheses of Lemma 1.5

$$\frac{|A|^\alpha}{|A+S|^\alpha} = \sum_{k_1,\ldots,k_n} C_{k_1,\ldots,k_n} \prod_{i=1}^n \left(\frac{a_{i,i}}{a_{i,i}+s_i} \right)^{\alpha+k_i}, \tag{1.25}$$

where $C_{k_1,\dots,k_n} \geq 0$ and $\sum_{k_1,\dots,k_n} C_{k_1,\dots,k_n} = 1$.

It follows from (1.25) that an n-dimensional multivariate gamma distribution can be considered to be a randomized mixture of independent random vectors with components that are independent 1-dimensional gamma distributed random variables as we can see from the next theorem.

Theorem 1.6. *Set $Z = (Z_1, \dots, Z_n)$ with $P(Z = (k_1, \dots, k_n)) = C_{k_1,\dots,k_n}$ and $X = (X_1, \dots, X_n)$ with*

$$X = \sum_{k_1,\dots,k_n} I_{k_1,\dots,k_n}(Z) \left(\xi_{\alpha+k_1,a_{1,1}}, \dots, \xi_{\alpha+k_n,a_{n,n}}\right),$$

where Z and all the gamma distributed random variables, $\xi_{u,v}$ are independent and $\{a_i\}_{i=1}^n$ are the diagonal elements of A. Then

$$E\left(e^{-\sum_{i=1}^k s_i X_i}\right) = \frac{|A|^\alpha}{|A+S|^\alpha}. \tag{1.26}$$

A different representation for multivariate gamma distributions is given in [3]. It is better suited for studying permanental processes but not as useful for finding probability density functions in low dimensions.

2 Proofs and Examples

Proof of Lemma 1.5 We have

$$|A+S|^{-1} = \left|\prod_{i=1}^n (a_{i,i}+s_i) - \left(\prod_{i=1}^n (a_{i,i}+s_i) - |A+S|\right)\right|^{-1} \tag{2.1}$$

$$= \left(\prod_{i=1}^n (a_{i,i}+s_i)\right)^{-1} \left|1 - \left(1 - \frac{|A+S|}{\prod_{i=1}^n (a_{i,i}+s_i)}\right)\right|^{-1} \tag{2.2}$$

We want

$$\left|1 - \frac{|A+S|}{\prod_{i=1}^n (a_{i,i}+s_i)}\right| < 1, \tag{2.3}$$

or, equivalently,

$$0 < \frac{|A+S|}{\prod_{i=1}^n (a_{i,i}+s_i)} < 1. \tag{2.4}$$

By (1.23), $0 < |A| < \prod_{i=1}^n a_{i,i}$, therefore (2.4) holds when $S = 0$. Consequently, it holds for $(s_1, \dots, s_n) \subset [0, s_0]^n$ for some $s_0 > 0$.

We write

$$|A+S| = \prod_{i=1}^n (a_{i,i}+s_i) + \sum_{m_1,\dots,m_j} c_{m_1,\dots,m_j} \prod_{i=1}^{n-j} (a_{p_i,p_i} + s_{p_i}). \tag{2.5}$$

where $\{m_1, \dots, m_j\}$ and $\{p_1, \dots, p_{n-j}\}$ are partitions of $\{1, \dots, n\}$ and the sum runs over all disjoint subsets of $\{1, \dots, n\}$ of cardinality greater than or equal to 2. Consequently

$$1 - \frac{|A+S|}{\prod_{i=1}^n (a_{i,i}+s_i)} = -\sum_{m_1,\dots,m_j} \frac{c_{m_1,\dots,m_j}}{(a_{m_1,m_1}+s_{m_1}) \cdots (a_{m_j,m_j}+s_{m_j})} \tag{2.6}$$

and

$$\frac{|A|^\alpha}{|A+S|^\alpha} = \frac{|A|^\alpha}{((a_{1,1}+s_1)\cdots(a_{n,n}+s_n))^\alpha} \left(1-\sum_{m_1,\dots,m_j}\frac{c_{m_1,\dots,m_j}}{(a_{m_1,m_1}+s_{m_1})\cdots(a_{m_j,m_j}+s_{m_j})}\right)^{-\alpha} \tag{2.7}$$

Furthermore, by (2.3) and the hypotheses, the absolute value of the sum in (2.7) is less than one. Thus we get (1.24) in which the series converges.

We now show that all the coefficients $c_{m_1,\dots,m_j}$ in (1.24) are greater than or equal to zero. It should be clear from (2.5) that $c_{m_1,\dots,m_j}$ is the coefficient of the term in $|A+S|$ containing $(a_{p_1,p_1}+s_{p_1})\cdots(a_{p_{n-j},p_{n-j}}+s_{m_{n-j}})$. Therefore, $c_{m_1,\dots,m_j}$ is a member of the determinant of the $n-j\times n-j$ matrix obtained from A by eliminating the rows $m_1,\dots,m_j$ and columns $m_1,\dots,m_j$ from A. Let M denote this principle sub-matrix of A. It should be clear that $c_{m_1,\dots,m_j}$ does not contain any of the diagonal elements of M, since if it did there would be an additional factor of the form $(a_{m_i,m_i}+s_{m_i})$, for some $1\le i\le j$, when considering this term as part of the term $c_{m_1,\dots,m_j}(a_{p_1,p_1}+s_{p_1})\cdots(a_{p_{n-j},p_{n-j}}+s_{p_{n-j}})$ in $|A+S|$. In other words the terms $c_{m_1,\dots,m_j}$ that we want to be positive are terms in $-|M_0|$. The hypothesis $|M_0|\le 0$ is a statement that all the coefficients $c_{m_1,\dots,m_j}$ in (1.24) are greater than or equal to zero. □

Remark 2.1. Let $G=\{g_{i,j}\}_{1\le i,j\le n}$ be an $n\times n$ matrix. We call G a positive matrix and write $G\ge 0$ if $g_{i,j}\ge 0$ for all i,j.

A matrix C is said to be a nonsingular M-matrix if

(1) $c_{i,j}\le 0$ for all $i\ne j$.
(2) C is nonsingular and $C^{-1}\ge 0$.

Note that the diagonal elements of C are all positive. This is because the inner product of the i-th row of C and C^{-1} is equal to 1.

In dimension 2, for (1.1) to be a Laplace transform of a positive random variable the diagonal elements of R must be positive. Therefore, considering (1.4) it suffices to consider matrices of the form

$$A_{(2)}=\begin{pmatrix} a & -\gamma \\ -\gamma & b\end{pmatrix} \quad\text{or}\quad A'_{(2)}=\begin{pmatrix} a & -\gamma \\ \gamma & b\end{pmatrix} \tag{2.8}$$

where $a,b,\gamma>0$ and $ab>\gamma^2$. However, (1.4) with $A'_{(2)}$ is not a Laplace transform. Suppose it is the Laplace transform of (Y_1,Y_2). Then (1.4) with

$$\widetilde{A}=\begin{pmatrix} a & -\gamma & 0 \\ \gamma & b & 0 \\ 0 & 0 & 1\end{pmatrix} \tag{2.9}$$

would be the Laplace transform of (Y_1,Y_2,Y_3) for some random variable Y_3 that is independent of (Y_1,Y_2). However, as we point out on page 3, the most general matrix A that is not symmetric, that determines a Laplace transform by (1.4) has the form of (1.14).

Proof of Theorem 1.1 Rather than apply Lemma 1.5 to this case it is easier to repeat the proof of Lemma 1.5 in this simple case. Let A be as given in (1.5) and note that

$$\begin{aligned} |A+S| &= (a+s_1)(b+s_2)-\gamma^2 \\ &= (a+s_1)(b+s_2)\left(1-\frac{\gamma^2}{(a+s_1)(b+s_2)}\right). \end{aligned} \tag{2.10}$$

Since $|A| > 0$, the fraction in (2.10) is less than 1. Therefore, we have

$$\begin{aligned}\frac{|A|^\alpha}{|A+S|^\alpha} &= \frac{|A|^\alpha}{(a+s_1)^\alpha(b+s_2)^\alpha\left(1-\frac{\gamma^2}{(a+s_1)(b+s_2)}\right)^\alpha} \\ &= \frac{|A|^\alpha}{(a+s_1)^\alpha(b+s_2)^\alpha}\sum_{n=0}^\infty \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,n!}\frac{\gamma^{2n}}{(a+s_1)^n(b+s_2)^n}.\end{aligned} \tag{2.11}$$

Using (1.3) we see that (2.11) is the Laplace transform of

$$\begin{aligned}&\widetilde{g}(\alpha, A; (x_1, x_2)) \\ &= (ab-\gamma^2)^\alpha \sum_{n=0}^\infty \frac{\left(\gamma\sqrt{x_1, x_2}\right)^{2n}}{\Gamma(\alpha)\Gamma(n+\alpha)\,n!}(x_1x_2)^{\alpha-1}e^{-(ax_1+bx_2)} \\ &= \frac{(ab-\gamma^2)^\alpha}{(ab)^\alpha}\sum_{n=0}^\infty \frac{\Gamma(n+\alpha)\gamma^{2n}}{\Gamma(\alpha)\,n!(ab)^n} f(n+\alpha, a; x_1)f(n+\alpha, b; x_2).\end{aligned} \tag{2.12}$$

The expression in (1.6) follows easily from the first equation in (2.12). The expression in (1.9) is an obvious application of (1.6). The expression in (1.10) follows from (1.6) but, perhaps more easily, from the second equation in (2.12), since

$$\Gamma(n+1/2) = \frac{(2n-1)(2n-3)\cdots(1/2)\Gamma(1/2)}{2^n} \tag{2.13}$$

and

$$n!\Gamma(n+1/2) = \frac{(2n)!}{2^{2n}})\Gamma(1/2). \tag{2.14}$$

Since $\Gamma(1/2) = \sqrt{\pi}$, we get that (2.12) is equal to

$$\frac{(ab-\gamma^2)^{1/2}}{\pi}\sum_{n=0}^\infty \frac{\left(2\gamma\sqrt{x_1, x_2}\right)^{2n}}{(2n)!}(x_1x_2)^{-1/2}e^{-(ax_1+bx_2)} \tag{2.15}$$

which is (1.10). □

Proof of Theorem 1.3 Let $d'd'' = d^2$. We have

$$\begin{aligned}|A_{(3)}+S| &= (a_{1,1}+s_1)(a_{2,2}+s_2)(a_{3,3}+s_3) - b^2(a_{3,3}+s_3) \\ &\quad -c^2(a_{2,2}+s_2) - d^2(a_{1,1}+s_1) - bc(d'+d'').\end{aligned} \tag{2.16}$$

This gives (1.15) and consequently (1.17). □

Remark 2.2. We have

$$|A_{(3)}| = a_{1,1}a_{2,2}a_{3,3} - b^2 - c^2 - d^2 - bc(d'+d'') \tag{2.17}$$

and

$$(d'+d'') \geq 2d, \tag{2.18}$$

with equality if and only $d' = d''$. For an $n\times n$ matrix C we define $C_{sym} = \{(c_{i,j}c_{j,i})^{1/2}\}_{i,j=1}^n$. Since $|A_{(3)}|$ is maximized when $d' = d''$, we have $|(A_{(3)})_{sym}| \geq |A_{(3)}|$.

Since only the last term in (1.16) changes when we replace $A_{(3)}$ by $(A_{(3)})_{sym}$ and $|(A_{(3)})_{sym}| \geq |A_{(3)}|$, we see that in replacing $A_{(3)}$ by $(A_{(3)})_{sym}$ we put more weight on the terms in Q that are unchanged by the symmetrization.

Example 2.3. We give the probability density function of an explicit three dimensional multivariate gamma distribution determined by the symmetric M-matrix

$$A = \begin{pmatrix} 1 & -a & 0 \\ -a & 1 & -b \\ 0 & -b & 1. \end{pmatrix} \tag{2.19}$$

and kernel

$$R = \frac{1}{1-a^2-b^2} \begin{pmatrix} 1-b^2 & a & ab \\ a & 1 & b \\ ab & b & 1-a^2 \end{pmatrix}. \tag{2.20}$$

We have

$$Q(s_1,s_2,s_3) = \frac{a^2}{(1+s_1)(1+s_2)} + \frac{b^2}{(1+s_2)(1+s_3)}$$

and

$$Q^n(s_1,s_2,s_3) = \sum_{k=0}^{n} \binom{n}{k} \left(\frac{a^2}{(1+s_1)(1+s_2)}\right)^k \left(\frac{b^2}{(1+s_2)(1+s_3)}\right)^{n-k}. \tag{2.21}$$

Consequently

$$\begin{aligned} \frac{|A|^\alpha}{|A+S|^\alpha} &= \frac{(1-a^2-b^2)^\alpha}{((1+s_1)\cdots(1+s_3))^\alpha} \\ &\quad \sum_{k\le n,n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\,n!} \binom{n}{k} \frac{a^{2k}b^{2(n-k)}}{(1+s_1)^k(1+s_2)^n(1+s_3)^{n-k}}. \end{aligned} \tag{2.22}$$

The probability density functions of the random variables with these Laplace transforms are

$$\begin{aligned} &(1-a^2-b^2)^\alpha (x_1x_2x_3)^{\alpha-1} e^{-(x_1+x_2+x_3)} \\ &\quad \sum_{k\le n,n=0}^{\infty} \frac{x_2^n}{\Gamma(\alpha)n!} \binom{n}{k} \frac{(a^2x_1)^k (b^2x_3)^{n-k}}{\Gamma(k+\alpha)\Gamma(n-k+\alpha)}. \end{aligned} \tag{2.23}$$

Example 2.4. We consider another example in which the M-matrix is not symmetric. Let

$$A = \begin{pmatrix} 1 & -a & 0 \\ 0 & 1 & -b \\ -c & 0 & 1. \end{pmatrix} \tag{2.24}$$

with $abc < 1$, so that

$$R = \frac{1}{1-abc} \begin{pmatrix} 1 & a & ab \\ bc & 1 & b \\ c & ac & 1 \end{pmatrix}. \tag{2.25}$$

Since

$$|A+S| = (1+s_1)(1+s_2)(1+s_3) - abc \tag{2.26}$$

we can proceeding as in the proof of Theorem 1.1 to get

$$\begin{aligned} \widetilde{g}(\alpha, A; (x_1,x_2,x_3)) &= (1-abc)^\alpha \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)(abc)^n}{\Gamma(\alpha)\,n!} \\ &\quad f(n+\alpha,1;x_1) f(n+\alpha,1;x_2) f(n+\alpha,1;x_3). \end{aligned} \tag{2.27}$$

The dependence of the density function on the off diagonal elements of A is only through their product. Therefore, without loss of generality we can take $a = b = c = \delta$. If we take $\delta^3 = \gamma^2$ the 2-dimensional joint distributions of (2.27) are equal to those of (2.12).

This simple example easily extends to give the probability density function of an n-dimensional multivariate gamma distribution. Let $\widetilde{A}$ be an matrix $n \times n$ matrix with $a_i = 1$, $i = 1, \ldots, n$ and all other entries zero except for $a_{i,i+1}$, $i = 1, \ldots, n-1$ and $a_{n,1}$ which are strictly negative. It is easy to see that $|\widetilde{A}| = 1 - |a_{n,1}| \prod_{i=1}^{n-1} |a_{i,i+1}|$. Whatever the values $\{a_i\}$ the n-dimensional random variable determined by $\widetilde{A}$ is (1.4) is the same as that determined by

$$\widetilde{A}_n = \begin{pmatrix} 1 & -\delta & 0 & \cdots & 0 & 0 \\ 0 & 1 & -\delta & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & 1 & -\delta \\ -\delta & 0 & 0 & \cdots & 0 & 1 \end{pmatrix} \tag{2.28}$$

where $\delta = |a_{n,1}| \prod_{i=1}^{n-1} |a_{i,i+1}|$. Since we want $|\widetilde{A}_n| > 0$, we take $\delta < 1$.

We have

$$|\widetilde{A}_n + S| = (1+s_1)\cdots(1+s_n)\left(1 - \frac{\delta}{(1+s_1)\cdots(1+s_n)}\right). \tag{2.29}$$

Following the proof of Theorem 1.1 we see that a random variable $X_{(n)}$ with Laplace transform given by (1.4) with matrix $\widetilde{A}_n$ has density function

$$g(\alpha, \mathbf{1}; \mathbf{x}) = (1-\delta^n)^\alpha \sum_{m=0}^{\infty} \frac{\Gamma(m+\alpha)\delta^{nm}}{\Gamma(\alpha)\, m!} \prod_{i=1}^{n} f(m+\alpha, 1; x_i), \tag{2.30}$$

and can be written as

$$X_{(n)} \overset{law}{=} \sum_{m=0}^{\infty} I_m(Z)(\xi^{(1)}_{m+\alpha,1}, \ldots, \xi^{(n)}_{m+\alpha,1}), \tag{2.31}$$

where all the random variables are independent and

$$P\left(Z = m\right) = (1-\delta^n)^\alpha \frac{\Gamma(m+\alpha)\delta^{nm}}{\Gamma(\alpha)\, m!}. \tag{2.32}$$

The inverse of $\widetilde{A}_n$ is the n-dimensional Toeplitz matrix

$$\widetilde{R}_n = \frac{1}{1-\delta} \begin{pmatrix} 1 & \delta & \delta^2 & \cdots & \delta^{n-2} & \delta^{n-1} \\ \delta^{n-1} & 1 & \delta & \cdots & \delta^{n-3} & \delta^{n-2} \\ \delta^{n-2} & \delta^{n-1} & 1 & \cdots & \delta^{n-4} & \delta^{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \delta^2 & \delta^3 & \delta^4 & \ldots & 1 & \delta \\ \delta & \delta^2 & \delta^3 & \cdots & \delta^{n-1} & 1 \end{pmatrix}. \tag{2.33}$$

Example 2.5. The various density functions obtained for multivariate gamma distributions show that a 1-dimensional gamma distributed random variable can also be written as an infinite randomized choice of independent gamma distributed random variables. To see this more directly let X be the 1-dimensional gamma distributed random variable with Laplace transform is $|1 + s/\beta|^{-\alpha}$. We write this as

$$\begin{aligned} \frac{\beta^\alpha}{(\beta+s)^\alpha} &= \frac{\beta^\alpha}{(1+s-(1-\beta))^\alpha} \\ &= \frac{\beta^\alpha}{(1+s)^\alpha} \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)\, n!} \left(\frac{1-\beta}{1+s}\right)^n. \end{aligned} \tag{2.34}$$

When $\beta < 1$ we see that

$$X \stackrel{law}{=} \sum_{n=0}^{\infty} I_n(Z)\xi_{n+\alpha,1} \tag{2.35}$$

where

$$P(Z = n) = \frac{\Gamma(n+\alpha)\beta^{\alpha}(1-\beta)^n}{\Gamma(\alpha)\, n!}. \tag{2.36}$$

Note that for $1 < \beta < 2$ the sum in (2.34) also converges. However, it is difficult to see that the sum is a Laplace transform without recognizing that it is equal to $\beta^{\alpha}/(\beta+s)^{\alpha}$.

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