Electron. Commun. Probab. **19** (2014), no. 50, 1–3.
DOI: 10.1214/ECP.v19-3674
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A short proof of a symmetry identity for the q-Hahn distribution

Guillaume Barraquand*

Abstract

We give a short and elementary proof of a symmetry identity for the q-moments of the q-Hahn distribution arising in the study of the q-Hahn Boson process and the q-Hahn TASEP. This identity discovered by Corwin in "The q-Hahn Boson Process and q-Hahn TASEP", Int. Math. Res. Not., 2014, was a key technical step to prove an intertwining relation between the Markov transition matrices of these two classes of discrete-time Markov chains. This was used in turn to derive exact formulas for a large class of observables of both these processes.

Keywords: Markov duality; q-Hahn process.
AMS MSC 2010: 60J10;33D45.
Submitted to ECP on July 16, 2014, final version accepted on July 23, 2014.
Supersedes arXiv:1404.4265.

Introduction

Zero-range and exclusion processes are generic stochastic models for transport phenomena on a lattice. Integrability of these models is an important question. In a short letter [5], Evans, Majumdar and Zia considered spatially homogeneous discrete time zero-range processes on periodic domains. They adressed and solved the question of characterizing the jump distributions for which invariant measures are product measures. Povolotsky [7] further examined the most general form of jump distributions allowing solvability by Bethe ansatz, and found a family depending on three real parameters q, μ and ν, later called the q-Hahn distribution. In the same article [7], he also studied the corresponding q-Hahn Boson process and q-Hahn TASEP, and conjectured exact formulas for the models on the infinite lattice.

Using a Markov duality between the q-Hahn Boson process and the q-Hahn TASEP, Corwin [4] showed a variant of these formulas and provided a method to compute a large class of observables. This can be seen as a generalization of a similar work on q-TASEP and q-Boson process performed in [3, 2]. In his proof, the intertwining relation between the two Markov transition matrices essentially boils down to a symmetry identity verified by the q-moments of the q-Hahn distribution [4, Proposition 1.2]. The proof was adapted from [2, Lemma 3.7] which is the $\nu = 0$ case, and required the use of Heine's summation formula for the basic hypergeometric series ${}_2\phi_1$. In the following, we give a new proof of this identity.

*Université Paris-Diderot, France. E-mail: barraquand@math.univ-paris-diderot.fr

A symmetry property for the q-moments of the q-Hahn distribution

First, we define the three parameter deformation of the Binomial distribution introduced in [7].

Definition 0.1. *For $|q| < 1$, $0 \leqslant \nu \leqslant \mu < 1$ and integers $0 \leqslant j \leqslant m$, define the function*

$$\varphi_{q,\mu,\nu}(j|m) = \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{m-j}}{(\nu;q)_m} \begin{bmatrix} m \\ j \end{bmatrix}_q ,$$

where

$$\begin{bmatrix} m \\ j \end{bmatrix}_q = \frac{(q;q)_m}{(q;q)_j\,(q;q)_{m-j}}$$

are q-Binomial coefficients with, as usual,

$$(z;q)_n = \prod_{i=0}^{n-1} \left(1 - q^i z\right).$$

It happens that for each $m \in \mathbb{N} \cup \{\infty\}$, this defines a probability distribution on the set $\{0, \ldots, m\}$. The weights $\varphi_{q,\mu,\nu}(j|m)$ are very closely related to the weights associated with the q-Hahn orthogonal polynomials (see (7.2.22) in [6]), hence the use of the name q-Hahn.

Lemma 0.2 (Lemma 1.1, [4]). *For any $|q| < 1$ and $0 \leqslant \nu \leqslant \mu < 1$,*

$$\sum_{j=0}^{m} \varphi_{q,\mu,\nu}(j|m) = 1.$$

Proof. As shown in [4], this equation is equivalent to a specialization of some known summation formula for basic hypergeometric series ${}_2\phi_1$ (Heine's q-generalizations of Gauss' summation formula). □

We now state and prove the main identity.

Proposition 0.3 (Proposition 1.2, [4]). *Fix $|q| < 1$ and $0 \leqslant \nu \leqslant \mu < 1$. Let X (resp. Y) be a random variable following the q-Hahn distribution on $\{0, \ldots, x\}$ (resp. $\{0, \ldots, y\}$). We have*

$$\mathbb{E}\left[q^{xY}\right] = \mathbb{E}\left[q^{yX}\right].$$

Proof. Let $S_{x,y} := \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) q^{jy}$. We have to show that $S_{x,y} = S_{y,x}$ for all integers $x, y \geqslant 0$. Our proof is based on the fact that $S_{x,y}$ satisfies a recurrence relation which is invariant when exchanging the roles of x and y. First notice that by Lemma 0.2, $S_{x,0} = 1$ for all $x \geqslant 0$, and by definition $S_{0,y} = 1$ for all $y \geqslant 0$.

The Pascal identity for q-Binomial coefficients, (see 10.0.3 in [1]),

$$\begin{bmatrix} x+1 \\ j \end{bmatrix}_q = \begin{bmatrix} x \\ j \end{bmatrix}_q q^j + \begin{bmatrix} x \\ j-1 \end{bmatrix}_q ,$$

yields

$$\begin{aligned} S_{x+1,y} &= \sum_{j=0}^{x+1} \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{x+1-j}}{(\nu;q)_{x+1}} \begin{bmatrix} x \\ j \end{bmatrix}_q q^j q^{jy} + \sum_{j=0}^{x+1} \mu^j \frac{(\nu/\mu;q)_j(\mu;q)_{x+1-j}}{(\nu;q)_{x+1}} \begin{bmatrix} x \\ j-1 \end{bmatrix}_q q^{jy}, \\ &= \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \frac{1-\mu q^{x-j}}{1-\nu q^x} q^j q^{jy} + \sum_{j=0}^{x} \varphi_{q,\mu,\nu}(j|x) \mu \frac{1-\nu/\mu q^j}{1-\nu q^x} q^y q^{jy}. \end{aligned}$$

The last equation can be rewritten

$$\begin{aligned}(1-\nu q^x)S_{x+1,y} &= (S_{x,y+1}-\mu q^x S_{x,y}) + (\mu q^y(S_{x,y}-\nu/\mu S_{x,y+1}))\,, \\ &= (1-\nu q^y)S_{x,y+1}+\mu(q^y-q^x)S_{x,y}.\end{aligned}$$

Thus, the sequence $(S_{x,y})_{(x,y)\in\mathbb{N}^2}$ is completely determined by

$$\begin{cases}(1-\nu q^x)S_{x+1,y} = (1-\nu q^y)S_{x,y+1}+\mu(q^y-q^x)S_{x,y}, \\ S_{x,0}=S_{0,y}=1.\end{cases} \tag{0.1}$$

Setting $T_{x,y}=S_{y,x}$, one notices that the sequence $(T_{x,y})_{(x,y)\in\mathbb{N}^2}$ enjoys the same recurrence, which concludes the proof. □

Remark 0.4. *To completely avoid the use of basic hypergeometric series, one would also need a similar proof of the Lemma above. One can prove the result by recurrence on m (as in the proof of [2, Lemma 1.3]), but the calculations are less elegant when $\nu\neq 0$.*

More precisely, fix some m and suppose that for any $0\leqslant\nu\leqslant\mu<1$, $S_{m,0}(q,\mu,\nu):=\sum_{j=0}^m\varphi_{q,\mu,\nu}(j|m)=1$. Pascal's identity yields

$$\begin{aligned}S_{m+1,0}(q,\mu,\nu) &= \frac{1-\mu}{1-\nu}S_{m,0}(q,q\mu,q\nu)+\sum_{j=0}^m\varphi_{q,\mu,\nu}(j|m)\mu\frac{1-\nu/\mu q^j}{1-\nu q^m}, \\ &= \frac{1-\mu}{1-\nu}S_{m,0}(q,q\mu,q\nu)+\frac{\mu}{1-\nu q^m}\left(S_{m,0}(q,\mu,\nu)-\nu/\mu S_{m,1}(q,\mu,\nu)\right).\end{aligned}$$

Then, using the recurrence formula (0.1) for $S_{m,1}(q,\mu,\nu)$, and applying the recurrence hypothesis, one obtains $S_{m+1,0}(q,\mu,\nu)=1$.

References

[1] G. E. Andrews, R. Askey, and R. Roy, *Special Functions*, vol. 71, Cambridge University Press, 2001.

[2] A. Borodin and I. Corwin, *Discrete time q-TASEPs*, International Mathematics Research Notices (2013), rnt206.

[3] A. Borodin, I. Corwin, and T. Sasamoto, *From duality to determinants for q-TASEP and ASEP*, Ann. Probab., to appear, arXiv:1207.5035 (2012).

[4] I. Corwin, *The q-Hahn Boson Process and q-Hahn TASEP*, International Mathematics Research Notices (2014), rnu094.

[5] M. R. Evans, Satya N. Majumdar, and R. K. P. Zia, *Factorized steady states in mass transport models*, J. Phys. A **37** (2004), no. 25, L275–L280. MR-2073204

[6] George Gasper and Mizan Rahman, *Basic hypergeometric series*, Encyclopedia of Mathematics and its Applications, vol. 35, Cambridge University Press, Cambridge, 1990, With a foreword by Richard Askey. MR-1052153

[7] A. M. Povolotsky, *On the integrability of zero-range chipping models with factorized steady states*, J. Phys. A **46** (2013), no. 46, 465205, 25. MR-3126878

Acknowledgments. The author would like to thank his advisor Sandrine Péché for her support.