Weak convergence of the number of zero increments in the random walk with barrier

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Abstract

We continue the line of research of random walks with a barrier initiated by Iksanov and Möhle (2008). Assuming that the tail of the step of the underlying random walk has a power-like behavior at infinity with the exponent $-\alpha$, $\alpha \in (0,1)$, we prove that V_n the number of zero increments before absorption in the random walk with the barrier n, properly centered and normalized, converges weakly to the standard normal law. Our result complements the weak law of large numbers for V_n proved in Iksanov and Negadailov (2008).

Keywords: random walk with barrier; recursion with random indices; renewal process; undershoot.
AMS MSC 2010: 60C05; 60G09.
Submitted to ECP on July 4, 2014, final version accepted on October 28, 2014.

1 Introduction

Let $(\xi_k)_{k \in \mathbb{N}}$ be independent copies of a random variable ξ with distribution $p_k = \mathbb{P}\{\xi = k\}$, $k \in \mathbb{N}$. A random walk with the barrier $n \in \mathbb{N}$ is a sequence $(R_k^{(n)})_{k \in \mathbb{N}_0}$ (where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$) defined by

$$R_0^{(n)} := 0 \ \text{ and } \ R_k^{(n)} := R_{k-1}^{(n)} + \xi_k 1_{\{R_{k-1}^{(n)} + \xi_k < n\}}, \ \ k \in \mathbb{N}.$$

Plainly, $(R_k^{(n)})_{k \in \mathbb{N}_0}$ is a non-decreasing Markov chain which cannot reach the state n. In what follows we always assume that $p_1 > 0$ which implies that the random walk with the barrier n will eventually get absorbed in the state $n - 1$.

The equalities

$$M_n := \#\{k \in \mathbb{N} : R_{k-1}^{(n)} \neq R_k^{(n)}\} = \sum_{l=0}^{\infty} 1_{\{R_l^{(n)} + \xi_{l+1} < n\}};$$

$$T_n := \inf\{k \in \mathbb{N}_0 : R_k^{(n)} = n - 1\} = \sum_{l \geq 0} 1_{\{R_l^{(n)} < n-1\}};$$

$$V_n := T_n - M_n = \#\{i \leq T_n : R_{i-1}^{(n)} = R_i^{(n)}\} = \sum_{l=0}^{T_n - 1} 1_{\{R_l^{(n)} + \xi_{l+1} \geq n\}}$$

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define the number of jumps, the absorption time and the number of zero increments before the absorption in the random walk with the barrier n, respectively.

There is a large number of real life situations in which the random walk with barrier appears naturally. Imagine, for instance, a transport company that offers tours to the national park. The company uses buses with n seats. Various groups of people book seats. If the size of a group is less than remaining number of vacant seats, the request is satisfied, otherwise it is turned down. The quantities of interest are the total number of groups T_{n+1}, the number of accepted groups M_{n+1} and the number of rejections V_{n+1}.

In [9] (see also [8] for a particular case) it was shown that, if the law of ξ belongs to the domain of attraction of a stable law, M_n, properly normalized and centered, weakly converges. Furthermore, the set of limiting laws is comprised of stable laws and the laws of exponential subordinators. In [13] it was checked that the same group of results holds on replacing M_n by T_n. Finally, in [10] it was proved that: (a) if $\mathbb{E}\xi < \infty$ then V_n weakly converges (without normalization); (b) if the law of ξ belongs to the domain of attraction of an α-stable law with $\alpha \in (0,1]$ (for $\alpha = 1$ it is additionally assumed that $\mathbb{E}\xi = \infty$), equivalently if

$$\mathbb{P}\{\xi \geq n\} \sim n^{-\alpha}\ell(n), \quad n \to \infty, \tag{1.1}$$

for some ℓ slowly varying at infinity, then $V_n/\mathbb{E}V_n \xrightarrow{P} 1$ as $n \to \infty$.

To complete the picture, we treat weak convergence of V_n. Since $V_n = T_n - M_n$ and T_n and M_n are of the same order, the analysis of V_n calls for more delicate argument than that for M_n and/or T_n. As a consequence, the approach exploited in [9, 10] does not help in the present situation. Moreover, regular variation (1.1) alone does not seem to be sufficient for weak convergence of V_n, properly normalized and centered, and one has to impose a more restrictive "second-order" condition

$$\mathbb{P}\{\xi \geq n\} = cn^{-\alpha} + O(n^{-(\alpha+\varepsilon)}), \quad n \to \infty, \tag{1.2}$$

for some $c > 0$, $\alpha \in (0,1)$ and $\varepsilon > 0$.

In what follows we reserve notation η for a random variable with the beta $(1 - \alpha, \alpha)$ law, $\alpha \in (0,1)$, i.e.,

$$\mathbb{P}\{\eta \in \mathrm{d}x\} = \frac{\sin \pi\alpha}{\pi} x^{-\alpha}(1-x)^{\alpha-1} 1_{(0,1)}(x)\mathrm{d}x. \tag{1.3}$$

Further, we shall use

$$\mu_\alpha := \mathbb{E}|\log \eta| = \psi(1) - \psi(1 - \alpha)$$

and

$$\sigma_\alpha^2 := \mathrm{Var}\,(\log \eta) = \psi'(1 - \alpha) - \psi'(1),$$

where $\psi(x) = \Gamma'(x)/\Gamma(x)$ is the logarithmic derivative of the gamma function. Also, we put

$$S_0 = 0, \quad S_n = \xi_1 + \ldots + \xi_n, \quad n \in \mathbb{N},$$

and denote by $u_n := \sum_{k=0}^\infty \mathbb{P}\{S_k = n\}$ the corresponding renewal sequence.

The main result of this paper is given next.

Theorem 1.1. *Suppose* (1.2). *If* $\alpha \in (0, 1/2]$ *assume additionally that*

$$u_n \sim \frac{\sin(\pi\alpha)}{c\pi} n^{\alpha-1}, \quad n \to \infty. \tag{1.4}$$

Then

$$\frac{V_n - \mu_\alpha^{-1}\log n}{\sqrt{\sigma_\alpha^2 \mu_\alpha^{-3}\log n}} \xrightarrow{d} \mathcal{N}(0,1), \quad n \to \infty,$$

where $\mathcal{N}(0,1)$ *is a random variable with the standard normal law. Moreover, there is convergence of the first absolute moments.*

Remark 1.2. It is well-known (see Theorem 1 in [5]) that if (1.1) holds with $\alpha \in (1/2, 1)$ then

$$u_n \sim \frac{\sin(\pi\alpha)}{\ell(n)\pi} n^{\alpha-1}, \quad n \to \infty, \tag{1.5}$$

i.e., (1.2) implies (1.4). On the other hand, if $\alpha \in (0, 1/2]$, then (1.5) does not follow from (1.1) without further assumptions on p_n. It is known that (1.1) does entail (1.5) if one of the following conditions holds true

- $p_{n+1}p_{n-1} > p_n^2$ for $n = 2, 3, \ldots$ (see Theorem 1 in [2]);
- p_n is eventually non-increasing (see Corollary 3-A in [14]);
- $p_n \sim \alpha\ell(n)n^{-\alpha-1}$ as $n \to \infty$ (see Theorem 1.1. in [12]).

Our approach is based on the analysis of a random recursive equation for (V_n). It is shown that the sequence (V_n) can be approximated by a suitable renewal counting process, and the error of such an approximation is estimated in terms of an appropriate probability distance. A similar method has already been used in [7] to derive a weak convergence result for the number of collisions in the beta coalescents.

The rest of the paper is organized as follows. In Section 2 we define a renewal process that we use for approximation and point out random recursive equations for related quantities. The proofs are presented in Section 3. An auxiliary lemma is formulated and proved in Appendix.

2 Renewal process and recursion with random indices

Given the random walk $(S_k)_{k \geq 0}$, define the first passage process

$$N_n := \inf\{k \in \mathbb{N}_0 : S_k \geq n\}, \quad n \in \mathbb{N},$$

and the undershoot $Y_n := n - S_{N_n - 1}$. It was shown[1] in [10] that the sequence $(V_n)_{n \in \mathbb{N}}$ satisfies the following recursion with random index

$$V_1 = 0, \quad V_n \overset{d}{=} 1_{\{Y_n > 1\}} + V'_{Y_n}, \quad n \geq 2, \tag{2.1}$$

where $V'_k \overset{d}{=} V_k$ for all $k \in \mathbb{N}$ and $(V'_k)_{k \in \mathbb{N}}$ and Y_n are independent.

The recursion (2.1) can be slightly simplified by setting $X_n := V_n + 1_{\{n > 1\}}$, then

$$X_1 = 0, \quad X_n \overset{d}{=} 1 + X'_{Y_n}, \quad n \geq 2, \tag{2.2}$$

where likewise $X'_k \overset{d}{=} X_k$ for all $k \in \mathbb{N}$ and $(X'_k)_{k \in \mathbb{N}}$ and Y_n are independent. Clearly, the asymptotic behavior of X_n is the same as of V_n.

It is a classical observation due to Dynkin [3] that under the assumption (1.1) with $\alpha \in (0, 1)$ we have

$$Y_n/n \overset{d}{\to} \eta, \quad n \to \infty, \tag{2.3}$$

where η has density (1.3).

Let $(\eta_k)_{k \in \mathbb{N}}$ be iid copies of η. Define a random walk

$$\tilde{S}_0 = 0; \quad \tilde{S}_k = |\log \eta_1| + \ldots + |\log \eta_k|, \quad k \in \mathbb{N};$$

the corresponding renewal counting process

$$\nu_t := \#\{k \in \mathbb{N} : \tilde{S}_k \leq t\} = \sum_{k=1}^{\infty} 1_{\{\tilde{S}_k \leq t\}}, \quad t \in \mathbb{R},$$

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[1]Note that in [10] the definition of T_n is slightly different from our which results in different recursion for (V_n).

and set $W_t := \nu_{\log t} + 1_{\{t>1\}}$ for $t > 0$. Since $\nu_t = 0$ a.s. for $t \leq 0$ we have $W_t = 0$ for $t \in (0, 1]$, while for $t > 1$ the strong Markov property implies

$$W_t \stackrel{d}{=} 1 + W'_{t\eta}, \tag{2.4}$$

where $W_t \stackrel{d}{=} W'_t$ for every $t > 0$ and $(W'_t)_{t\geq 0}$ and η are independent.

Comparing recursions (2.2) and (2.4) and in view of (2.3) we may expect that the weak asymptotic behavior of X_n is the same as of W_n. We will show, assuming (1.2), that this heuristic can be made rigorous and leads to the desired result on the asymptotic of V_n.

3 Proofs

We start with a refinement of (2.3) by estimating the speed of convergence of Y_n/n to η in terms of so-called minimal L_1-distance. Let us recall its definition. Let $\mathcal{D}_1$ be the set of probability laws on $\mathbb{R}$ with finite first absolute moment. The L_1-minimal (or Wasserstein) distance on $\mathcal{D}_1$ is defined by

$$d_1(X, Y) = \inf \mathbb{E}|\widehat{X} - \widehat{Y}|, \tag{3.1}$$

where the infimum is taken over all couplings $(\widehat{X}, \widehat{Y})$ such that $X \stackrel{d}{=} \widehat{X}$ and $Y \stackrel{d}{=} \widehat{Y}$.

For ease of reference we summarize the properties of d_1 to be used in this work in the following proposition.

Proposition 3.1. *Let X, Y be random variables with finite first absolute moments. The distance d_1 has the following properties:*

(Int) *$d_1(X, Y)$ has an integral representation:*

$$d_1(X, Y) := \int_{\mathbb{R}} |\mathbb{P}\{X \leq x\} - \mathbb{P}\{Y \leq x\}| \mathrm{d}x.$$

(Rep) *$d_1(X, Y)$ has a dual representation:*

$$d_1(X, Y) = \sup_{f \in \mathcal{F}} |\mathbb{E}f(X) - \mathbb{E}f(Y)|.$$

where $\mathcal{F} := \{f \ : \ |f(x) - f(y)| \leq |x - y|\}$,

(Lin) *$d_1(cX + a, cY + a) = |c|d_1(X, Y)$ for $a, c \in \mathbb{R}$.*

(Conv) *For $X, X_n \in \mathcal{D}_1$ convergence $d_1(X_n, X) \to 0$, $n \to \infty$, is equivalent to $X_n \stackrel{d}{\to} X$ and $\mathbb{E}|X_n| \to \mathbb{E}|X|$, $n \to \infty$.*

We refer the reader to Chapter 1 in [15] for an introduction to the theory of probability metrics, in particular for the proofs of the aforementioned properties of d_1.

In view of (Conv) characterization of d_1 the next lemma is indeed a refinement of (2.3).

Proposition 3.2. *Under the assumptions of Theorem 1.1 there exists $\delta > 0$ such that*

$$d_1\left(\log \frac{Y_n}{n}, \log \eta\right) = d_1\left(\log Y_n, \log(n\eta)\right) = O(n^{-\delta}), \ \ n \to \infty.$$

Proof. The first equality follows from (Lin) property of d_1. Using (Rep) we have

$$d_1\left(\log Y_n, \log(n\eta)\right) = \sup_{f \in \mathcal{F}} \left|\mathbb{E}f(\log Y_n) - \mathbb{E}f(\log(n\eta))\right|. \tag{3.2}$$

From the distributional identity

$$Y_1 = 1, \quad Y_n \stackrel{d}{=} n1_{\{\xi \geq n\}} + Y'_{n-\xi}1_{\{\xi < n\}}, \quad n \geq 2,$$

where $Y'_k \stackrel{d}{=} Y_k$ for all $k \in \mathbb{N}$ and $(Y'_k)_{k \in \mathbb{N}}$ is independent from ξ, we infer

$$\mathbb{E}f(\log Y_n) = \mathbb{P}\{\xi \geq n\}f(\log n) + \sum_{j=1}^{n-1} p_j \mathbb{E}f(\log Y_{n-j}), \quad n \geq 2.$$

Substituting this into (3.2) and using the triangle inequality gives

$$d_1\Big(\log Y_n, \log(n\eta)\Big)$$

$$\leq \sup_{f \in \mathcal{F}} \Big| \mathbb{P}\{\xi \geq n\}f(\log n) + \sum_{j=1}^{n-1} p_j \mathbb{E}f(\log(n-j)\eta) - \mathbb{E}f(\log(n\eta)) \Big|$$

$$+ \sum_{j=1}^{n-1} p_j \sup_{f \in \mathcal{F}} \Big| \mathbb{E}f(\log Y_{n-j}) - \mathbb{E}f(\log(n-j)\eta) \Big|$$

$$= \sup_{f \in \mathcal{F}} \Big| \mathbb{P}\{\xi \geq n\}f(\log n) + \sum_{j=1}^{n-1} p_j \mathbb{E}f(\log(n-j)\eta) - \mathbb{E}f(\log(n\eta)) \Big|$$

$$+ \sum_{j=1}^{n-1} p_j d_1\Big(\log Y_{n-j}, \log(n-j)\eta\Big).$$

Let $\tilde{\xi}$ be independent of $\tilde{\eta}$ and $\tilde{\xi} \stackrel{d}{=} \xi$, $\tilde{\eta} \stackrel{d}{=} \eta$. The first term can be written as

$$\sup_{f \in \mathcal{F}} \Big| \mathbb{P}\{\xi \geq n\}f(\log n) + \sum_{j=1}^{n-1} p_j \mathbb{E}f(\log(n-j)\eta) - \mathbb{E}f(\log(n\eta)) \Big|$$

$$= d_1\Big(\log(n1_{\{\tilde{\xi} \geq n\}} + (n-\tilde{\xi})\tilde{\eta}1_{\{\tilde{\xi} < n\}}), \log(n\tilde{\eta})\Big) = d_1\Big((\log(1 - \tilde{\xi}n^{-1})\tilde{\eta})1_{\{\tilde{\xi} < n\}}, \log \tilde{\eta}\Big),$$

where we have utilized (Lin) property of d_1 in the second equality.

For every $x \geq 1$,

$$\mathbb{P}\{\xi \geq x\} = \mathbb{P}\{\xi \geq \lceil x \rceil\} = c(\lceil x \rceil)^{-\alpha} + O((\lceil x \rceil)^{-(\alpha+\varepsilon)}) = cx^{-\alpha} + O(x^{-((\alpha+\varepsilon)\wedge 1)}),$$

hence, by Lemma 4.1 with $\beta = 1$ and $x = n$, there exist $K > 0$ and $\delta \in (0, 1-\alpha)$ such that

$$d_1\Big(\log Y_n, \log(n\eta)\Big) \leq Kn^{-(\alpha+\delta)} + \sum_{j=1}^{n-1} p_j d_1\Big(\log Y_{n-j}, \log(n-j)\eta\Big).$$

Using 1-arithmetic variant of Theorem 1 in [1] (the cited result is also valid for $\alpha \in (0, 1/2]$ in view of assumption (1.4)), we obtain

$$d_1\Big(\log Y_n, \log(n\eta)\Big) = O(n^{-\delta}), \quad n \to \infty.$$

The proof is complete. □

3.1 Proof of Theorem 1.1

It is enough to prove Theorem 1.1 for V_n replaced by X_n. In view of (Conv) property of d_1, in order to prove Theorem 1.1 we need to check

$$d_1\Big(\frac{X_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \mathcal{N}(0,1)\Big) \to 0, \quad n \to \infty.$$

Using the triangle inequality yields for $n \geq 2$,

$$d_1\Big(\frac{X_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \mathcal{N}(0,1)\Big) \quad \leq \quad d_1\Big(\frac{X_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \frac{W_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}\Big)$$

$$+ d_1\Big(\frac{W_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \mathcal{N}(0,1)\Big)$$

$$= \quad d_1\Big(\frac{X_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \frac{W_n - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}\Big)$$

$$+ d_1\Big(\frac{\nu_{\log n} + 1 - \mu_a^{-1}\log n}{\sqrt{\sigma_a^2 \mu_a^{-3}\log n}}, \mathcal{N}(0,1)\Big)$$

The second term converges to zero in view of the CLT for the renewal process with finite variance (see Chapter XI.5 in [4]) as well as the convergence of first absolute moments (see Proposition A.1 in [11]). From (Lin) property of d_1 we see that it is enough to prove

$$d_1(X_n, W_n) = O(1), \quad n \to \infty. \tag{3.3}$$

Using the recursions for X_n and W_n we have, in view of (Lin) property of d_1,

$$
\begin{aligned}
t_n := d_1(X_n, W_n) &= d_1(X'_{Y_n}, W'_{n\eta}) \leq d_1(W'_{n\eta}, W'_{Y_n}) \\
&+ d_1(W'_{Y_n}, X'_{Y_n}) \leq d_1(W'_{n\eta}, W'_{Y_n}) + \mathbb{E}|\widehat{W}_{Y_n} - \widehat{X}_{Y_n}| \\
&=: c_n + \sum_{k=2}^{n} \mathbb{P}\{Y_n = k\}\mathbb{E}|\widehat{X}_k - \widehat{W}_k|,
\end{aligned}
$$

for arbitrary pairs $\{(\widehat{X}_k, \widehat{W}_k) : 2 \leq k \leq n\}$ independent of Y_n such that $\widehat{X}_k \stackrel{d}{=} X_k$, $\widehat{W}_k \stackrel{d}{=} W_k$. Passing to infimum over all such pairs in both sides of inequality leads to

$$t_n \leq c_n + \sum_{k=2}^{n} \mathbb{P}\{Y_n = k\}t_k. \tag{3.4}$$

In order to estimate c_n we proceed as follows. Let $(\hat{Y}_n, \hat{\eta})$ be a coupling of Y_n and η such that $d_1(\log Y_n, \log(n\eta)) = \mathbb{E}|\log \hat{Y}_n - \log(n\hat{\eta})|$. Let $(\hat{\nu}_t)_{t\in\mathbb{R}}$ be a copy of $(\nu_t)_{t\in\mathbb{R}}$ independent of $(\hat{Y}_n, \hat{\eta})$. We have

$$
\begin{aligned}
c_n = d_1(W'_{Y_n}, W'_{n\eta}) &= d_1(\hat{\nu}_{\log \hat{Y}_n} + 1_{\{\hat{Y}_n > 1\}}, \hat{\nu}_{\log(n\hat{\eta})} + 1_{\{n\hat{\eta} > 1\}}) \\
&\leq \mathbb{E}|\hat{\nu}_{\log \hat{Y}_n} + 1_{\{\hat{Y}_n > 1\}} - \hat{\nu}_{\log(n\hat{\eta})} - 1_{\{n\hat{\eta} > 1\}}| \\
&\leq \mathbb{E}|\hat{\nu}_{\log \hat{Y}_n} - \hat{\nu}_{\log(n\hat{\eta})}| + \mathbb{P}\{Y_n = 1\} + \mathbb{P}\{n\eta \leq 1\}
\end{aligned}
$$

where the penultimate inequality follows from the definition of d_1, since $(\hat{Y}_n, \hat{\eta}, (\hat{\nu}(t)))$ is a particular coupling. There exists $\rho > 0$ such that the last two summands are $O(n^{-\rho})$. To bound the first term, we apply the distributional subadditivity of (ν_t):

$$\nu_{x+y} - \nu_x \stackrel{d}{\leq} \nu_y, \quad x, y \in \mathbb{R},$$

which yields

$$c_n \leq \mathbb{E}\hat{\nu}_{|\log \hat{Y}_n - \log(n\hat{\eta})|} + O(n^{-\rho}). \tag{3.5}$$

Note that for every $x \geq 0$,

$$\mathbb{P}\{\tilde{S}_1 \leq x\} \leq \mathbb{E}\nu_x = \sum_{k=1}^{\infty} \mathbb{P}\{\tilde{S}_k \leq x\} \leq \sum_{k=1}^{\infty}(\mathbb{P}\{\tilde{S}_1 \leq x\})^k = \frac{\mathbb{P}\{\tilde{S}_1 \leq x\}}{\mathbb{P}\{\tilde{S}_1 > x\}},$$

hence, by the standard sandwich argument,

$$\lim_{x \downarrow 0} \frac{\mathbb{E}\nu_x}{x^\alpha} = \frac{\sin \pi\alpha}{\pi\alpha}.$$

On the other hand, from the elementary renewal theorem we have

$$\lim_{x \to \infty} \frac{\mathbb{E}\nu_x}{x} = \frac{1}{\mathbb{E}\tilde{S}_1},$$

therefore there exist constants $c_1, c_2 > 0$ such that for all $x \geq 0$,

$$\mathbb{E}\nu_x \leq c_1 x^\alpha + c_2 x. \tag{3.6}$$

Using (3.6) and (3.5) we obtain

$$\begin{aligned}
c_n &\leq c_1 \mathbb{E}|\log \hat{Y}_n - \log(n\hat{\eta})|^\alpha + c_2 \mathbb{E}|\log \hat{Y}_n - \log(n\hat{\eta})| + O(n^{-\rho}) \\
&\leq c_1 d_1^\alpha(\log Y_n, \log(n\eta)) + c_2 d_1(\log Y_n, \log(n\eta)) + O(n^{-\rho}).
\end{aligned}$$

By Lemma 3.2 we conclude $c_n = O(n^{-\rho'})$ for some $\rho' > 0$ as $n \to \infty$.

It remains to apply Lemma A.1 from [6] with $\phi_n \equiv 1$ to (3.4) to conclude that

$$t_n = O\Big(\sum_{k=1}^n \frac{k^{-\rho'}}{k} \Big) = O(1), \ \ n \to \infty.$$

The proof of Theorem 1.1 is complete.

4 Appendix

The next lemma is the main ingredient in the proof of Proposition 3.2.

Lemma 4.1. *Assume that θ is a random variable on $[1, +\infty)$ such that for some $c > 0$, $\alpha \in (0,1)$ and $\varepsilon > 0$*

$$1 - F_\theta(x) := \mathbb{P}\{\theta \geq x\} = cx^{-\alpha} + O(x^{-(\alpha+\varepsilon)}), \ \ x \to \infty. \tag{4.1}$$

Let η be a random variable with density (1.3) independent of θ. Then for every $\beta > 0$ there exists $\delta > 0$ such that

$$d_1\Big(\log((1 - \theta x^{-1})\eta)1_{\{\theta < x - \beta\}}, \log \eta \Big) = O(x^{-(\alpha+\delta)}), \ \ x \to \infty. \tag{4.2}$$

Proof. Denote the left-hand side of (4.2) by $s_\theta(x, \beta)$. In view of relations

$$s_\theta(x, \beta) = s_{c^{-1/\alpha}\theta}(c^{-1/\alpha}x, c^{-1/\alpha}\beta), \ \ x \geq 1,$$

and

$$\mathbb{P}\{c^{-1/\alpha}\theta \geq x\} = x^{-\alpha} + O(x^{-(\alpha+\varepsilon)}), \ \ x \to \infty,$$

it is enough to prove the result for $c = 1$. Fix β for the rest of the proof. Using representation (Int) from Proposition 3.1 we have

$$\begin{aligned}
s_\theta(x, \beta) &= \int_{-\infty}^0 |\mathbb{P}\{\log(1_{\{\theta \geq x - \beta\}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x - \beta\}}) \leq z\} - \mathbb{P}\{\log \eta \leq z\}|\mathrm{d}z \\
&= \int_0^1 |\mathbb{P}\{1_{\{\theta \geq x - \beta\}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x - \beta\}} \leq z\} - \mathbb{P}\{\eta \leq z\}|z^{-1}\mathrm{d}z.
\end{aligned}$$

Integrating by parts the first probability in the integrand, we obtain for $z \in [0, 1)$ and $x > 1 + \beta$,

$$\mathbb{P}\{1_{\{\theta \geq x - \beta\}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x - \beta\}} \leq z\}$$

$$= -\int_{[1, x - \beta)} \mathbb{P}\{(1 - yx^{-1})\eta \leq z\} \mathrm{d}(1 - F_\theta(y))$$

$$= -\mathbb{P}\{\eta \leq \beta^{-1} xz\}\Big(1 - F_\theta((x - \beta)-)\Big) + \mathbb{P}\{\eta \leq zx(x - 1)^{-1}\}$$

$$+ \int_{[1, x - \beta)} (1 - F_\theta(y)) \mathrm{d}_y \mathbb{P}\{(1 - yx^{-1})\eta \leq z\}.$$

Let θ_α be a random variable independent of η and with distribution

$$1 - F_{\theta_\alpha}(x) := \mathbb{P}\{\theta_\alpha \geq x\} = x^{-\alpha}, \quad x \geq 1.$$

By the same reasoning as above,

$$\mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha x^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\}$$

$$= -\int_{[1, x - \beta)} \mathbb{P}\{(1 - yx^{-1})\eta \leq z\} \mathrm{d}(1 - F_{\theta_\alpha}(y))$$

$$= -\mathbb{P}\{\eta \leq \beta^{-1} xz\}\Big(1 - F_{\theta_\alpha}((x - \beta))\Big) + \mathbb{P}\{\eta \leq zx(x - 1)^{-1}\}$$

$$+ \int_{[1, x - \beta)} (1 - F_{\theta_\alpha}(y)) \mathrm{d}_y \mathbb{P}\{(1 - yx^{-1})\eta \leq z\}.$$

Subtracting the corresponding equations and using (4.1) we have for $z \in [0, 1)$ and $x > 1 + \beta$,

$$\left| \mathbb{P}\{1_{\{\theta \geq x - \beta\}} + (1 - \theta x^{-1})\eta 1_{\{\theta < x - \beta\}} \leq z\} - \mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha x^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\} \right|$$

$$\leq K\left(\mathbb{P}\{\eta \leq \beta^{-1} xz\}(x - \beta)^{-(\alpha + \varepsilon)} + \int_{[1, x - \beta)} y^{-(\alpha + \varepsilon)} \mathrm{d}_y \mathbb{P}\{(1 - yx^{-1})\eta \leq z\} \right),$$

for some $K > 0$ which does not depend on x and z. Therefore,

$$s_\theta(x, \beta)$$

$$\leq \int_0^1 |\mathbb{P}\{1_{\{\theta_\alpha \geq x - \beta\}} + (1 - \theta_\alpha x^{-1})\eta 1_{\{\theta_\alpha < x - \beta\}} \leq z\} - \mathbb{P}\{\eta \leq z\}| z^{-1} \mathrm{d}z$$

$$+ K \int_0^1 z^{-1} \mathbb{P}\{\eta \leq \beta^{-1} xz\}(x - \beta)^{-(\alpha + \varepsilon)} \mathrm{d}z$$

$$+ K \int_0^1 z^{-1} \int_{[1, x - \beta)} y^{-(\alpha + \varepsilon)} \mathrm{d}_y \mathbb{P}\{(1 - yx^{-1})\eta \leq z\} \mathrm{d}z =: I_1(x) + I_2(x) + I_3(x).$$

Firstly we calculate $I_2(x)$ explicitly as follows:

$$
\begin{aligned}
I_2(x) &= K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^1 \mathbb{P}\{\eta \leq \beta^{-1} xz\} z^{-1} \mathrm{d}z \\
&= K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^{\beta x^{-1}} \mathbb{P}\{\eta \leq \beta^{-1} xz\} z^{-1} \mathrm{d}z + K(x - \beta)^{-(\alpha + \varepsilon)}(\log x - \log \beta) \\
&= K(x - \beta)^{-(\alpha + \varepsilon)} \int_0^1 \mathbb{P}\{\eta \leq z\} z^{-1} \mathrm{d}z + K(x - \beta)^{-(\alpha + \varepsilon)}(\log x - \log \beta) \\
&= K(x - \beta)^{-(\alpha + \varepsilon)}(\mathbb{E}|\log \eta| + \log x - \log \beta) = O(x^{-(\alpha + \varepsilon)} \log x).
\end{aligned}
$$

Pick $\varepsilon' \in (0, \varepsilon]$ such that $\alpha + \varepsilon' < 1$. The third summand $I_3(x)$ is estimated using the Fubini's theorem:

$$
\begin{aligned}
I_3(x) \quad &\leq \quad K \int_0^1 z^{-1} \int_{[1,x-\beta)} y^{-(\alpha+\varepsilon')} d_y \mathbb{P}\{(1-yx^{-1})\eta \leq z\} dz \\
&= \quad K \int_0^1 z^{-1} \int_{[1,x-\beta)} y^{-(\alpha+\varepsilon')} \mathbb{P}\{(1-\eta^{-1}z)x \in dy\} dz \\
&= \quad K \int_0^1 z^{-1} \mathbb{E}\Big((1-\eta^{-1}z)x\Big)^{-(\alpha+\varepsilon')} 1_{\{1 \leq (1-\eta^{-1}z)x \leq x-\beta\}} dz \\
&= \quad K x^{-(\alpha+\varepsilon')} \mathbb{E} \int_0^1 z^{-1}(1-\eta^{-1}z)^{-(\alpha+\varepsilon')} 1_{\{1 \leq (1-\eta^{-1}z)x \leq x-\beta\}} dz \\
&= \quad K x^{-(\alpha+\varepsilon')} \mathbb{E} \int_{\beta\eta x^{-1}}^{\eta(1-x^{-1})} z^{-1}(1-\eta^{-1}z)^{-(\alpha+\varepsilon')} dz \\
&\overset{z=\eta u}{=} \quad K x^{-(\alpha+\varepsilon')} \int_{\beta x^{-1}}^{1-x^{-1}} u^{-1}(1-u)^{-(\alpha+\varepsilon')} du = O(x^{-(\alpha+\varepsilon')} \log x).
\end{aligned}
$$

It remains to bound the first integral. To this end, note that for every $z \in [0,1)$ and $x \geq 1 + \beta$,

$$
\{1_{\{\theta_\alpha \geq x-\beta\}} + (1-\theta_\alpha x^{-1})\eta 1_{\{\theta_\alpha < x-\beta\}} \leq z\} = \{(1-\eta^{-1}z)x \leq \theta_\alpha < x - \beta\},
$$

and therefore

$$
\begin{aligned}
&\mathbb{P}\{1_{\{\theta_\alpha \geq x-\beta\}} + (1-\theta_\alpha x^{-1})\eta 1_{\{\theta_\alpha < x-\beta\}} \leq z\} \\
&= \quad \mathbb{P}\{(1-\eta^{-1}z)x \leq \theta_\alpha < x - \beta\} \\
&= \quad \mathbb{P}\{((1-\eta^{-1}z)x) \vee 1 \leq \theta_\alpha < x - \beta\} \\
&= \quad \mathbb{P}\{\eta \leq \beta^{-1}xz, ((1-\eta^{-1}z)x) \vee 1 \leq \theta_\alpha < x - \beta\} \\
&= \quad \mathbb{P}\{\eta \leq \beta^{-1}xz, ((1-\eta^{-1}z)x) \vee 1 \leq \theta_\alpha < x\} - \mathbb{P}\{\eta \leq \beta^{-1}xz\}((x-\beta)^{-\alpha} - x^{-\alpha}).
\end{aligned}
$$

Putting this into $I_1(x)$ yields

$$
\begin{aligned}
I_1(x) \quad &\leq \quad \int_0^1 |\mathbb{P}\{\eta \leq \beta^{-1}xz, ((1-\eta^{-1}z)x) \vee 1 \leq \theta_\alpha < x\} - \mathbb{P}\{\eta \leq z\}| z^{-1} dz \\
&+ \quad ((x-\beta)^{-\alpha} - x^{-\alpha}) \int_0^1 z^{-1} \mathbb{P}\{\eta \leq \beta^{-1}xz\} dz.
\end{aligned}
$$

The second term is $O(x^{-\alpha-1} \log x)$ by the same argument as was used in the estimation of $I_2(x)$. Using simple algebra we obtain that the first term is equal to

$$
\begin{aligned}
\int_0^1 \Big| &\mathbb{P}\{z < \eta \leq (x-1)^{-1}xz\} \\
&+ x^{-\alpha}\Big(\int_{((x-1)^{-1}xz)\wedge 1}^{(\beta^{-1}xz)\wedge 1} ((1-y^{-1}z)^{-\alpha})\mathbb{P}\{\eta \in dy\} - \mathbb{P}\{\eta \leq \beta^{-1}xz\}\Big)\Big| z^{-1} dz =: J(x).
\end{aligned}
$$

By the triangle inequality,

$$
\begin{aligned}
J(x) \leq \quad &\int_0^1 z^{-1} \mathbb{P}\{z < \eta \leq (x-1)^{-1}xz\} dz \\
&+ x^{-\alpha} \int_0^1 \Big|\int_{((x-1)^{-1}xz)\wedge 1}^{(\beta^{-1}xz)\wedge 1} (1-y^{-1}z)^{-\alpha}\mathbb{P}\{\eta \in dy\} - \mathbb{P}\{\eta \leq \beta^{-1}xz\}\Big| z^{-1} dz. \quad (4.3)
\end{aligned}
$$

The first summand, again by the Fubini's theorem, is calculated easily:

$$
\begin{aligned}
\int_0^1 z^{-1} \mathbb{P}\{z < \eta \le (x-1)^{-1} xz\} \mathrm{d}z &= \mathbb{E} \int_0^1 z^{-1} 1_{\{z < \eta \le (x-1)^{-1} xz\}} \mathrm{d}z \\
&= \mathbb{E} \int_0^1 z^{-1} 1_{\{x^{-1}(x-1)\eta \le z < \eta\}} \mathrm{d}z \\
&= \mathbb{E} \int_{x^{-1}(x-1)\eta}^{\eta} z^{-1} \mathrm{d}z = |\log(1 - x^{-1})| = O(x^{-1}).
\end{aligned}
$$

The inner integral in the second summand in rhs of (4.3) is equal

$$
\frac{\sin \pi\alpha}{\pi} \int_{((x-1)^{-1} xz) \wedge 1}^{(\beta^{-1} xz) \wedge 1} (y-z)^{-\alpha}(1-y)^{\alpha-1} \mathrm{d}y,
$$

and upon substitution $u := (y-z)(1-z)^{-1}$ becomes

$$
\frac{\sin \pi\alpha}{\pi} \int_{\frac{z}{(1-z)(x-1)} \wedge 1}^{\frac{(\beta^{-1} x-1)z}{1-z} \wedge 1} u^{-\alpha}(1-u)^{\alpha-1} \mathrm{d}u = \mathbb{P}\Big\{ \frac{z}{(1-z)(x-1)} \wedge 1 \le \eta \le \frac{(\beta^{-1} x-1)z}{1-z} \wedge 1 \Big\}.
$$

Since for $z \in [0,1)$ and $x > 1 + \beta$,

$$
0 \le \frac{z}{(1-z)(x-1)} \wedge 1 \le \frac{(\beta^{-1} x-1)z}{1-z} \wedge 1 \le (\beta^{-1} xz) \wedge 1,
$$

the integral in the second summand in (4.3) is

$$
\int_0^1 z^{-1} \mathbb{P}\Big\{ \eta \le \frac{z}{(1-z)(x-1)} \wedge 1 \Big\} \mathrm{d}z + \int_0^1 z^{-1} \mathbb{P}\Big\{ \frac{(\beta^{-1} x-1)z}{1-z} \wedge 1 \le \eta \le (\beta^{-1} xz) \wedge 1 \Big\} \mathrm{d}z.
$$

We will check that the second summand above is $O(x^{-1})$ as follows:

$$
\begin{aligned}
\int_0^1 z^{-1} \mathbb{P}\Big\{ \frac{(\beta^{-1} x-1)z}{1-z} &\wedge 1 \le \eta \le (\beta^{-1} xz) \wedge 1 \Big\} \mathrm{d}z \\
&= \mathbb{E} \int_0^1 z^{-1} 1_{\{\eta \beta x^{-1} \le z \le \eta(\beta^{-1} x-1+\eta)^{-1}\}} \mathrm{d}z \\
&= \mathbb{E}\Big(\log(\beta^{-1} x) - \log(\beta^{-1} x - 1 + \eta) \Big) = O(x^{-1}).
\end{aligned}
$$

The first term can be treated analogously, hence $J(x) = O(x^{-1})$. Combining all the estimates we get $s_\theta(x, \beta) = O(x^{-\alpha+\delta})$ for sufficiently small $\delta > 0$. The proof is complete. $\qquad\square$

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