

The power of choice combined with preferential attachment*

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Abstract

We prove almost sure convergence of the maximum degree in an evolving tree model combining local choice and preferential attachment. At each step in the growth of the graph, a new vertex is introduced. A fixed, finite number of possible neighbors are sampled from the existing vertices with probability proportional to degree. Of these possibilities, the new vertex attaches to the vertex from the sample that has the highest degree. The maximal degree in this model has linear or near-linear behavior. This behavior contrasts sharply with the behavior in the same choice model with uniform attachment as well as the preferential attachment model without choice. The proof is based on showing the tree has a persistent hub by comparison with the standard preferential attachment model, as well as martingale and stochastic approximation arguments.

Keywords: preferential attachment; choice; stochastic approximation.

AMS MSC 2010: 05C80.

Submitted to ECP on April 18, 2014, final version accepted on June 12, 2014.

1 Introduction

In the present work we further explore how the addition of choice affects the classic preferential attachment model (see [1, 10]), building on previous work [6, 13, 9]. The preferential attachment graph is a time-indexed sequence of graphs, inductively constructed the following way. We start with some initial graph and then on each step we add a new vertex. One of the old vertices is chosen with probability proportional to degree, and an edge is drawn between the new vertex and the chosen old vertex. Many different properties of this model have been obtained in both the math and physics literature (see [1, 10, 15, 5]).

In this work, we will study the evolution of the maximal degree of a related model. For the preferential attachment model this problem is studied in [7, 15, 16]. Letting $\Delta(n)$ be the maximum degree at time n , it is shown in [15] that $\Delta(n)n^{-1/2}$ converges almost surely to a variable with absolutely continuous distribution (see also [16] for properties of this distribution and the rate of convergence).

In [13], the authors introduce the *min-choice preferential attachment model*. This model is also built inductively, but now a new vertex chooses 2 (or d in general) existing vertices with probability proportional to degree and connects to the one with the lowest degree, breaking ties uniformly. In [13] it is shown that the maximal degree at time n

*YM gratefully acknowledges the support of the Weizmann Institute of Science, where this work was begun. EP gratefully acknowledges the support of NSF Postdoctoral Fellowship DMS-1304057.

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in such a model will be $\log \log n / \log 2 + \Theta(1)$ with high probability ($\log \log n / \log d$ in case of d choices). There, it is also conjectured that the *max-choice preferential attachment model*, where we choose the vertex with the highest degree, has maximal degree of order $n / \log n$. Subsequently, this is studied in the physics literature [9], where the analysis is expanded to show that for $d = 2$ this is indeed the case while for $d > 2$, the maximal degree has linear order.

We will give exact first-order asymptotics for the maximal degree in the max-choice model, and we will show almost sure convergence of the appropriately scaled maximal degree.

We now describe the model precisely. Define a sequence of trees $(P_m)_{m \geq 0}$ given by the following rule. Let P_1 be the one-edge tree on vertices $\{v_0, v_1\}$. Given P_m , define P_{m+1} by first adding one new vertex v_{m+1} . Let X_m^1, \dots, X_m^d , where $d \geq 2$, be i.i.d. vertices from $V(P_m)$, where $V(P)$ is the set of vertices of P , chosen with probability

$$\mathbb{P}[X_m^1 = w] = \frac{\deg w}{2m},$$

where $\deg w$ is the degree of vertex w in P_m ; note that as the graph has m edges, $\sum_w \deg w = 2m$. Finally, create a new edge between v_{m+1} and Y_m , where Y_m is whichever of X_m^1, \dots, X_m^d has highest degree. In the case of a tie, break the tie by choosing from the maximal vertices uniformly (any other tiebreaking rule works as well). We call this the *max-choice preferential attachment tree*.

Our main theorem is the following.

Theorem 1.1. *In the case $d = 2$, the maximum degree M_n of P_n has*

$$\lim_{n \rightarrow \infty} \frac{M_n \log n}{n} = 4 \text{ a.s.}$$

For $d > 2$,

$$\lim_{n \rightarrow \infty} \frac{M_n}{n} = x_* \text{ a.s.},$$

where x_* is the unique positive solution of equation $1 - (1 - x/2)^d = x$ with $x \leq 1$.

Our proof is based on the existence of a *persistent hub*, i.e. a single vertex that in some finite random time becomes the highest degree vertex for all time after. Many preferential attachment graphs are known to have persistent hubs, including the classical one (see [4]). Using the existence of a hub, instead of analyzing the maximum degree over all vertices we effectively only need to analyze the degree of just one vertex.

Proposition 1.2. *There exists random T and K that are finite almost surely so that at any time $n \geq T$, the vertex v_K has the highest degree among all vertices.*

Let L_n denote the number of vertices at time n that have maximal degree. The dynamics of M_n are given by the rule

$$M_{n+1} - M_n = \begin{cases} 1 & \text{with probability } 1 - \left(1 - \frac{M_n L_n}{2n}\right)^d, \\ 0 & \text{else.} \end{cases} \quad (1.1)$$

The effect of Proposition 1.2 is that for some $T < \infty$ random and sufficiently large, $L_n = 1$ for all $n > T$. If we were to assume that L_n were identically one, we would be considering a simple multi-choice urn.

This urn contains 2 types of balls, colored black and colored white, with the number of black balls being M_n and the number of white balls being $2n - M_n$. At every time step, d balls are sampled from the urn with replacement and then put back into the urn. If all

are white, then two white balls are added back to the urn. If at least one is black, then one white ball and one black ball are added to the urn. Such urn models with multiple samplings have appeared recently in the literature (see for example [11, 3]), although this appears to be an uncovered case.

Proof approach and organization

We start in section 3 with some initial lower-bound estimates for the maximal degree. All subsequent arguments require that the maximal degree grows quickly enough to ensure deterministic behavior takes over.

In section 4 we prove the existence of the persistent hub, which allows us to consider the degree of a single vertex instead of the maximal degree. We present an argument that follows the proof of [8] for convex preferential attachment models and consists of two steps. First, we show that the number of *possible leaders*, vertices that have maximal degree at some time, is almost surely finite; this follows on account of the maximal degree growing quickly enough that vertices added after a long time have a very small probability of ever catching up. Second, we show that any two vertices have degrees that change leadership only finitely many times. These arguments rely heavily on comparison with the preferential attachment model and the Pólya urn respectively.

In sections 5 and 6 we prove convergence of the scaled maximal degree in the cases $d = 2$ and $d > 2$ respectively, which require different analyses. From (1.1), we anticipate the maximal degree M_n of the graph evolves according to the differential equation

$$\frac{dM}{dt} = 1 - (1 - M/2t)^d.$$

Setting $u(t) = M(e^t)e^{-t}$, we get that u satisfies the autonomous differential equation

$$u' + u = 1 - (1 - u/2)^d.$$

In the case $d = 2$, this can be explicitly solved to give $M(t) = 4t/(\log t + C)$, while in the $d > 2$ case, we are led to consider critical points, which are solutions of $1 - (1 - x/2)^d = x$. When $d > 2$ there are two solutions of the equation $1 - (1 - x/2)^d = x$ in the interval $0 \leq x \leq 2$, but it only has one stable solution x_* (meaning that u' has the opposite sign of $u - x_*$ in a neighborhood of x_*).

In section 5 we prove the $d = 2$ case by considering explicit scale functions of M_n that can be guessed from the solution of the differential equation.

In section 6, we prove the $d > 2$ case, which can be formulated generally as follows. Consider a continuous function $q : [0, 1] \rightarrow [0, 1]$ and define a process $\{T(n), n \geq n_0\}$, started from point $T(n_0) = T_0, 0 < T_0 < n_0$, such that the increments $T(n+1) - T(n)$ are independent Bernoulli($q(T(n)/n$) variables conditioned on $\sigma(T_n)$. This problem has appeared many times in the stochastic approximation literature under the name of the Robbins-Monro model (see [18, 12, 2, 17]). Off the shelf techniques are applicable to this situation, but still require that we show that M_n/n does not converge to 0, which we show using martingale arguments.

2 Discussion

Theorem 1.1 allows us to complete Table 1 about the influence of choice on the maximum degree of growing random trees. In summary, for the min-choice models, the effect of choice completely overwhelms the effect of preferential attachment. On the other hand, the combined effect of preferential attachment with max-choice completely changes the structure of the graph and the order of the maximum degree (see also

Figure 1 for a simulation of these trees). In comparison, adding max-choice to the uniform attachment model does not increase the order of the maximum degree.

Theorem 1.1 along with Proposition 1.2 provide us information about the degree sequence of the graph and some structural information about the graph, but it would be nice to know more topological information about the tree. One natural topological property to consider is the diameter of the tree.

Table 1: Comparison of maximum degree at time n for max/min-choice with 2 choices versus preferential or uniform attachment.

	max-choice	no-choice	min-choice
Preferential attachment	$\frac{4n}{\log n}(1 + o(1))$	$\Theta(n^{1/2})$ ^(a)	$\frac{\log \log n}{\log 2} + \Theta(1)$ ^(b)
Uniform attachment	$O(\log n)$ ^(c)	$O(\log n)$ ^(d)	$O(\log \log n)$ ^(c)

^(a) [15] ^(b) [13] ^(c) [6] ^(d) To our knowledge, this is not claimed formally anywhere. However, getting the correct order is an elementary exercise.

In the standard preferential attachment model the diameter is known to be logarithmic [5]. It is natural to wonder if the diameter in this situation is smaller. To increase the diameter we must add an edge between a new vertex and an existing vertex of degree 1. In the max-choice model, choosing such a vertex is still not too rare; for while it is less likely to choose a degree 1 vertex than in preferential attachment, there are $\Theta(n)$ degree one vertices. Thus, degree 1 vertices are selected at each time step with some probability bounded away from 0. Conditional on choosing a vertex of degree 1, the exact choice of vertex is uniform over all possible choices. Thus we conjecture the diameter of the graph grows at a rate that is of the same order as that of preferential attachment.

The rate might be different for other rules of breaking ties. The model we study breaks ties uniformly, but in fact any tie breaking rule has the same degree sequence evolution in law. However, we anticipate it could significantly affect the structure of the graph. For example, if instead of a fair coin toss we define a function $\text{rad}(v_j) = \max_i(\text{dist}(v_i, v_j))$, and on each step we choose the vertex with the smallest value of $\text{rad}(v_i)$ among all vertices with the same degrees, we anticipate order $\log \log n$ diameter (see also [10], where such a model is considered).

We consider only graphs that are trees, but similar results should hold for classes of models with more edges. One such natural model would be to add more than one edge at each step. A second would be to flip a coin at each time step to choose between adding a new vertex or adding an edge between existing vertices. If adding a vertex, the rule would be the same as in our model, while for adding an edge there are a few natural possibilities that could affect the structure of the graph. Here is one such rule. We choose the first vertex with probability proportional to its degree (which is preferential attachment without choice), and then we choose the second vertex among all non-adjacent vertices using the max- d choice rule. Note that both these methods will only increase the average degree of the vertices of the graph.

3 *A priori* estimates

We begin with a pair of lower bounds for the growth of the maximal degree. These are needed both for the persistent hub proof and the eventual precise estimates. We will frequently use the following lemma of [8].

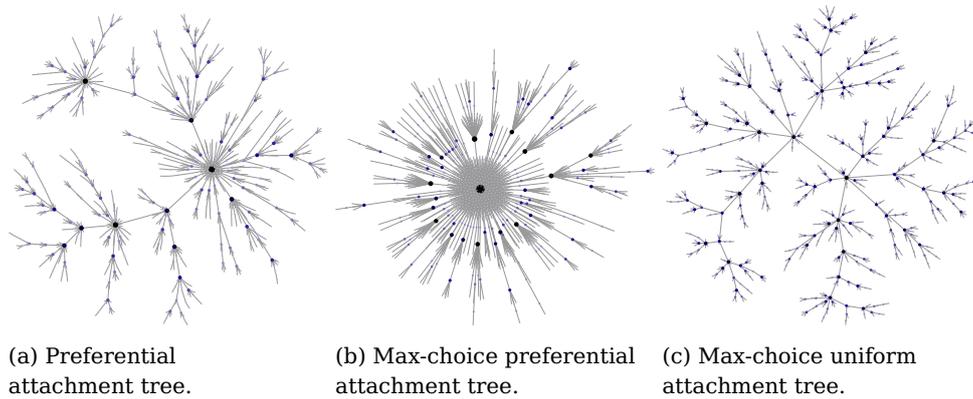


Figure 1: All simulations are with 1000 vertices and are made for $d = 2$.

Lemma 3.1. *Suppose that a sequence of positive numbers r_n satisfies*

$$r_{n+1} = r_n \left(1 + \frac{\alpha}{n+x} \right), \quad n \geq k$$

for fixed reals α, n and x satisfying $\alpha > 0, k > -x$. Then r_n/n^α has a positive limit.

This is easily checked from a direct computation.

Lemma 3.2. *With probability 1, $\inf_n M_n/n^{3/8} > 0$.*

Proof. Define $C_{n+1} = \frac{8n}{8n-3}C_n = (1 + \frac{3}{8n-3})C_n$, with $C_1 = 1$. By Lemma 3.1 we have that $C_n/n^{3/8}$ converges to a positive limit. Now, we will show that C_n/M_n is a supermartingale from which the desired conclusion follows.

Let p_n be the conditional probability that $M_{n+1} = M_n + 1$ given \mathcal{F}_n , where $\mathcal{F}_n = \sigma(P_1, P_2, \dots, P_n)$. Calculating,

$$\begin{aligned} p_n &= 1 - \left(1 - \frac{M_n L_n}{2n} \right)^d \geq 1 - \left(1 - \frac{M_n}{2n} \right)^d \\ &= 1 - \left(\frac{2n - M_n}{2n} \right)^2 = \frac{M_n}{n} - \frac{M_n^2}{4n^2} \\ &= \frac{M_n}{n} \frac{4n - M_n}{4n} \geq \frac{3M_n}{4n}. \end{aligned}$$

For $1/M_n$ we get

$$\begin{aligned} \mathbb{E} [1/M_{n+1} | \mathcal{F}_n] &= \frac{p_n}{M_n + 1} + \frac{1 - p_n}{M_n} = \frac{M_n + 1 - p_n}{M_n(M_n + 1)} \\ &= \frac{1}{M_n} \left(1 - \frac{p_n}{M_n + 1} \right) \leq \frac{1}{M_n} \left(1 - \frac{p_n}{2M_n} \right) \\ &\leq \frac{1}{M_n} \left(1 - \frac{3}{8n} \right) = \frac{1}{M_n} \frac{C_n}{C_{n+1}}, \end{aligned}$$

which shows C_n/M_n is a supermartingale. □

We will now show that with this initial argument, it is possible to improve the result again using martingale convergence.

Lemma 3.3. *For any fixed $\delta > 0$, $\liminf_{n \rightarrow \infty} M_n/n^{3/4-\delta} = \infty$ a.s.*

Proof. For each fixed $\delta > 0$ and each fixed $\epsilon > 0$, let $n_0 = n_0(\epsilon, \delta)$ be such that

$$1 - \frac{1}{1 + \epsilon n^{3/8}} \geq 1 - \frac{4\delta}{6}$$

for all $n \geq n_0$. Let ζ_ϵ be the stopping time given by

$$\zeta_\epsilon = \inf\{n > n_0 : M_n < \epsilon n^{3/8}\}.$$

From Lemma 3.2, we have that $\mathbb{P}[\zeta_\epsilon < \infty] \rightarrow 0$ as $\epsilon \rightarrow 0$. Set O_ϵ^ζ to be the event $\{\zeta_\epsilon = \infty\}$.

As in the proof of Lemma 3.2, we get that $p_n \geq \frac{3M_n}{4n}$. Then for $1/M_{n+1}$, it holds that

$$\mathbb{E}(1/M_{n+1} | \mathcal{F}_n) = \frac{1}{M_n} \left(1 - \frac{p_n}{M_n + 1}\right) \leq \frac{1}{M_n} \left(1 - \frac{3}{4n} \frac{M_n}{M_n + 1}\right).$$

For any n with $n_0 \leq n < \zeta_\epsilon$,

$$\frac{M_n}{M_n + 1} = 1 - \frac{1}{M_n + 1} \geq 1 - \frac{1}{1 + \epsilon n^{3/8}} \geq 1 - \frac{4\delta}{6}.$$

Hence for $\zeta_\epsilon > n > n_0$ we get

$$\mathbb{E}(1/M_{n+1} | \mathcal{F}_n) \leq \frac{1}{M_n} \left(1 - \frac{3/4 - \delta/2}{n}\right).$$

Define $R_{n+1} = \frac{4n}{4n-3+2\delta} R_n \geq (1 + \frac{3/4-\delta/2}{n}) R_n$, $n \geq n_0$. Then R_n/M_n is a supermartingale, and from Lemma 3.1 it follows that $R_n n^{-(3/4-\delta/2)}$ converges to a positive finite limit. Setting $A_n = R_n/M_n$, we have that by Doob's theorem $A_{n \wedge \zeta_\epsilon}$ tends to a finite limit with probability 1. Hence, conditioned on O_ϵ^ζ , we have that $M_n/n^{3/4-\delta} \rightarrow \infty$ a.s. Thus, it follows that

$$\mathbb{P}\left[\liminf_{n \rightarrow \infty} M_n/n^{3/4-\delta} = \infty\right] \geq \mathbb{P}\left[\{\liminf_{n \rightarrow \infty} M_n/n^{3/4-\delta} = \infty\} \cap O_\epsilon^\zeta\right] = \mathbb{P}[O_\epsilon^\zeta].$$

Taking $\epsilon \rightarrow 0$, we conclude the proof. □

4 Persistent hub

Our method of proof is essentially by comparison with the preferential attachment model, and we use the machinery of [8] developed for this task. First we estimate the probability that the degree of the vertex added on the $(k+1)$ -st step exceeds the degree of the vertex with highest degree at step k . For this we use the following lemma.

Lemma 4.1. *Let $\pi(k)$ be the probability that the degree of v_k becomes maximal at any future time, given \mathcal{F}_k . Then,*

$$\pi(k) \leq \frac{P(M_k)}{2^{M_k}},$$

where $P(A)$ is some polynomial of A and M_k is the maximum degree of P_k . Hence, the number of vertices that at some point in the process have maximal degree is finite almost surely.

Throughout this section, let $\deg_n v$ denote the degree of the vertex v in P_n . First we prove a comparison between the degree evolutions in the max-choice model and the standard preferential attachment model.

Lemma 4.2. *Fix $n_0 > 0$, and let v_i and v_j be any vertices from $V(P_{n_0})$. Let $(A_{n_0}, B_{n_0}) = (\deg_{n_0} v_i, \deg_{n_0} v_j)$, and let $T_n = (A_n, B_n)$ for $n \geq n_0$ denote the Pólya urn started from (A_{n_0}, B_{n_0}) , i.e. the random walk on \mathbb{Z}^2 that moves one step right or one step up with probabilities proportional to A_n and B_n respectively. The probability that there is an $n \geq n_0$ so that $\deg_n v_i = \deg_n v_j$ is bounded above by the probability that $T_n = (A_n, B_n)$ reaches the line $y = x$.*

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Proof. Consider the two-dimensional random walk $S_n = (w_n, u_n)$, where $w_n = \deg_n v_i$ and $u_n = \deg_n v_j$. Without loss of generality assume that $w_{n_0} > u_{n_0}$. We want to show that

$$\mathbb{P}[\exists n \geq n_0 : w_n = u_n] \leq \mathbb{P}[\exists n \geq n_0 : A_n = B_n].$$

To accomplish this, we will show the existence of an appropriate coupling of $(S_n)_{n \geq n_0}$ and $(T_n)_{n \geq n_0}$. Set

$$F_n = \sum_{v_k \in V(P_n)} \deg_n v_k \mathbb{1}_{\{\deg_n v_k < \deg_n v_i\}} \text{ and}$$

$$G_n = \sum_{v_k \in V(P_n)} \deg_n v_k \mathbb{1}_{\{\deg_n v_k \leq \deg_n v_j\}},$$

and let $p_n^w = \mathbb{P}[w_{n+1} = w_n + 1]$ and $p_n^u = \mathbb{P}[u_{n+1} = u_n + 1]$.

The probability that $w_n = \deg_n v_i$ increases is at least the probability that $v_i \in \{X_n^1, \dots, X_n^d\}$ and that all the other X_n^k have degree strictly less than $\deg_n v_i$. Thus

$$p_n^w \geq \left(\frac{F_n + w_n}{2n} \right)^d - \left(\frac{F_n}{2n} \right)^d.$$

Likewise, the probability that $u_n = \deg_n v_j$ increases is at most the probability that vertex $v_j \in \{X_n^1, \dots, X_n^d\}$ and $\deg_n v_j = \max_{1 \leq k \leq d} \deg_n X_n^k$. Thus

$$p_n^u \leq \left(\frac{G_n}{2n} \right)^d - \left(\frac{G_n - u_n}{2n} \right)^d.$$

So long as $w_n = \deg_n v_i > \deg_n v_j = u_n$, we have $F_n \geq G_n$. Hence

$$\begin{aligned} \frac{p_n^w}{p_n^u} &\geq \frac{(F_n + w_n)^d - (F_n)^d}{(G_n)^d - (G_n - u_n)^d} \\ &\geq \frac{(G_n + w_n)^d - (G_n)^d}{(G_n)^d - (G_n - u_n)^d}. \end{aligned}$$

Using the convexity of x^d , we have the bound $|x + y|^d \geq x^d + dx^{d-1}y$ for $x \geq 0$. Applying this to the previous inequality, we get:

$$\frac{p_n^w}{p_n^u} \geq \frac{d(G_n)^{d-1}w_n}{d(G_n)^{d-1}u_n} = \frac{w_n}{u_n}.$$

Thus,

$$\frac{p_n^w}{p_n^w + p_n^u} = \frac{1}{1 + \frac{p_n^u}{p_n^w}} \geq \frac{1}{1 + \frac{u_n}{w_n}} = \frac{w_n}{w_n + u_n}.$$

Letting $\tau_1, \tau_2, \tau_3, \dots$ be the times at which S_n moves, we have that S_{τ_n} and T_n can be coupled in such a way that both $w_{\tau_n} \geq A_n$ and $u_{\tau_n} \leq B_n$ until the first time $w_{\tau_n} = u_{\tau_n}$. Thus if at some finite time $w_n = u_n$, it must also be that there is a time $m \leq n$ at which $A_m = B_m$, completing the proof. □

The walk T_n describes the evolution of the degrees of two vertices in the preferential attachment model without choices. Hence we can apply to it some of the results from [8]. We will now use it to prove Lemma 4.1.

Proof. The vertex v_k has degree 1 at time k . Let $A_k = M_k$, $B_k = 1$, and $n_0 = k$. By Lemma 4.2, it suffices to estimate the probability that $T_n = (A_n, B_n)$ started from (A_k, B_k) reaches the line $y = x$. Corollary 15 of [8] gives the following estimate for the probability $q(M_k)$ that the walk T_n , $n > k$ moves from the point $(M_k, 1)$ to the diagonal:

$$q(M_k) \leq \frac{P(M_k)}{2^{M_k}},$$

where $P(M_k)$ is some polynomial.

By Lemma 3.2 we get that $M_n \geq Mn^{3/8}$ for some random $M > 0$ almost surely. In particular, $\pi(k)$ forms a convergent series with probability 1, and by Borel-Cantelli, the number of k for which the vertex added at the k -th step have maximal degree at some point in time is finite almost surely. \square

To complete the proof of Proposition 1.2 we now need the following lemma.

Lemma 4.3. *Consider two vertices that at some time have maximal degree. With probability 1 there are only a finite number of times when these vertices have the same degree and are maximal.*

Proof. Let v_i and v_j be two vertices that at some point have equal, maximal degree, and let m_0 be the first time that this occurs. Consider a two-dimensional random walk S with coordinates equal to $(\deg_n v_i, \deg_n v_j)$ for all time $n \geq m_0$. They have the same degree if and only if the walk is on the line $y = x$. As in Lemma 4.2, the probability that S hits the line $y = x$ when started off the line is bounded from above by the probability that T hits the line $y = x$. Hence the number of times $n \geq m_0$ that S returns to the line $y = x$ is bounded above by the number of times T returns to the line $y = x$.

It is a standard fact about the Pólya urn that if $T_n = (A_n, B_n)$ starts from a point (t, t) , then the fraction $A_n/(A_n + B_n)$ tends in law to a random variable $H(t)$ as n tends to infinity, where $H(t)$ has a beta probability distribution:

$$H(t) \sim \text{Beta}(t, t).$$

(See also Proposition 16 of [8]) Since the beta distribution is absolutely continuous, the fraction $A_n/(A_n + B_n)$ tends to an absolutely continuous probability distribution for any starting point of the process T . Thus the limit of $A_n/(A_n + B_n)$ exists almost surely, and it takes value $1/2$ with probability 0. Hence this fraction can be equal to $1/2$ only finitely many times, and so T can return to the line $y = x$ only finitely many times.

Thus, the only way that there can be infinitely many times for which $\deg_n v_i = \deg_n v_j$ is if both $\deg_n v_i$ and $\deg_n v_j$ stabilize, i.e. there is a D not depending on n and an n_1 for which $\deg_n v_i = \deg_n v_j = D$ for all $n \geq n_1$. However, in this case, these degrees are only maximal for finitely many times as the maximal degree goes to infinity by Lemma 3.2, which completes the proof. \square

Proof of Proposition 1.2. From Lemma 4.1 the number of vertices that at some point have maximal degree is finite almost surely, and from Lemma 4.3 these finitely many vertices only change leadership finitely many times almost surely. Thus, after some sufficiently long time, a single vertex remains the maximal degree vertex for all subsequent time. \square

5 The case $d = 2$

In this section, we show the limiting behavior of the maximum degree in the case $d = 2$. From Proposition 1.2 it follows that

$$\lim_{C \rightarrow \infty} \mathbb{P}[L_n = 1, \forall n \geq C] = 1.$$

Introduce events $D(C) = \{L_n = 1, \forall n \geq C\}$, and the stopping times $\eta_C = \inf_{n \geq C} \{n : L_n > 1\}$. For fixed $c > 0$ we define the following set of scale functions of M_n .

$$\begin{aligned} Q_n^c &= \exp(cn/M_n)/n \\ U_n^c &= n \exp(-cn/M_n). \end{aligned} \tag{5.1}$$

Lemma 5.1. *In the following, let $\epsilon > 0$ and $C > 0$ be fixed positive numbers.*

1. *For each $c < 4$, there is a constant $n_1 = n_1(C, c, \epsilon) \geq C$ sufficiently large so that if $\tau_\epsilon = \inf_{n > n_1} \{n : M_n < \epsilon n^{0.67}\}$ then $Q_{n \wedge \tau_\epsilon \wedge \eta_C}^c, n \geq n_1$ is a supermartingale.*
2. *For each $c > 4$, there is a constant $n_2 = n_2(C, c, \epsilon) \geq C$ sufficiently large so that if $\tau_\epsilon = \inf_{n > n_2} \{n : M_n < \epsilon n^{0.67}\}$ then $U_{n \wedge \tau_\epsilon \wedge \eta_C}^c, n \geq n_0$ is a supermartingale.*

Proof of Lemma 5.1. Since we only consider $n \leq \eta_C$ we have that $L_n = 1$ almost surely, and hence $p_n = (1 - M_n/4n)M_n/n$ for the conditional probability at the n -th step that M_n increases.

Proof of (1): We must estimate $\mathbb{E}[Q_{n+1}^c | \mathcal{F}_n]$ for $c < 4$ under the assumption that $M_n \geq \epsilon n^{0.67}$. As we wish to show this is a supermartingale, it suffices to show that there is a n_0 sufficiently large so that under these assumptions

$$\mathbb{E}\left[\frac{Q_{n+1}^c}{Q_n^c} | \mathcal{F}_n\right] \leq 1.$$

The proof follows by Taylor expansion.

$$\begin{aligned} \mathbb{E}\left[\frac{Q_{n+1}^c}{Q_n^c} | \mathcal{F}_n\right] &= \frac{n}{n+1} \left[e^{\left(\frac{c}{M_n}\right)} (1 - p_n) + p_n e^{\left(c\frac{n+1}{M_{n+1}} - \frac{cn}{M_n}\right)} \right] \\ &= 1 - \frac{1}{n} + \frac{c}{M_n} + cp_n \left(\frac{-1}{M_n} + \frac{M_n - n}{M_n(M_n + 1)} \right) + O\left(\frac{1}{M_n^2} + \frac{n^2 p_n}{M_n^4} \right). \end{aligned}$$

Noting that $p_n \leq M_n/n$ and that under our assumption, $M_n = \omega(n^{2/3})$, it follows that this error term is $o(1/n)$. Substituting in the definition of p_n , we get

$$\begin{aligned} \mathbb{E}\left[\frac{Q_{n+1}^c}{Q_n^c} | \mathcal{F}_n\right] &= 1 - \frac{1}{n} + \frac{c}{M_n} - c \left(\frac{n+1}{n(M_n+1)} \right) \left(1 - \frac{M_n}{4n} \right) + O\left(\frac{1}{n^{1.001}} \right) \\ &\leq 1 - \frac{1}{n} + \frac{c}{4n} + O\left(\frac{1}{n^{1.001}} \right). \end{aligned}$$

Note that the constant in the $O(\dots)$ term depends only on ϵ and c . Hence, when $c < 4$, we may find a constant $n_0 > C$ sufficiently large so that this is always strictly less than 1, which completes the proof.

Proof of (2) This is nearly the same calculation as was done for (1). Once more, it suffices to show that for $c > 4$,

$$\mathbb{E}\left[\frac{U_{n+1}^c}{U_n^c} | \mathcal{F}_n\right] \leq 1.$$

If we expand this expectation, we get

$$\mathbb{E}\left[\frac{U_{n+1}^c}{U_n^c} | \mathcal{F}_n\right] = \frac{n+1}{n} \left[e^{\left(\frac{-c}{M_n}\right)} (1 - p_n) + p_n e^{\left(-c\frac{n+1}{M_{n+1}} + \frac{cn}{M_n}\right)} \right].$$

The same calculus shows that we have

$$\mathbb{E}\left[\frac{U_{n+1}^c}{U_n^c} | \mathcal{F}_n\right] = 1 + \frac{1}{n} - \frac{c}{4n} + O\left(\frac{1}{n^{1.001}} \right),$$

so that when $c > 4$, the desired claim holds. □

Using the a priori estimates, we are able to use Q_n^c to prove the main theorem for $d = 2$.

Proof of Theorem 1.1. Using these supermartingales, the proof proceeds along similar lines as in Lemma 3.3. Once again set O_ϵ^τ to be the event $\{\tau_\epsilon = \infty\}$. From Lemma 3.3 we have

$$\liminf_{n \rightarrow \infty} M_n/n^{0.67} = \infty \text{ a.s.}$$

Hence, we have that

$$\lim_{\epsilon \rightarrow 0} \mathbb{1} \left\{ \inf_{n > 0} M_n/n^{0.67} \leq \epsilon \right\} = 0 \text{ a.s.}$$

Thus, $\lim_{\epsilon \rightarrow 0} \mathbb{P}[O_\epsilon^\tau] = 1$.

Recall that $D(C) = \{L_n = 1, \forall n \geq C\}$. For $c < 4$, on the event $O_\epsilon^\tau \cap D_C$, we have by positive supermartingale convergence that there is some large R_ϵ random so that

$$\sup_{n > 0} Q_n^c < R_\epsilon < \infty.$$

Hence, on this event,

$$M_n \geq \frac{cn}{\log n + \log R_\epsilon},$$

and so

$$\liminf_{n \rightarrow \infty} \frac{M_n \log n}{n} \geq c.$$

Thus we have that

$$\mathbb{P} \left[\left\{ \liminf_{n \rightarrow \infty} \frac{M_n \log n}{n} \geq c \right\} \cap O_\epsilon^\tau \cap D_C \right] = \mathbb{P} [O_\epsilon^\tau \cap D_C],$$

and so taking $\epsilon \rightarrow 0$ and $C \rightarrow \infty$ we have that

$$\liminf_{n \rightarrow \infty} \frac{M_n \log n}{n} \geq c \text{ a.s.}$$

As this holds for any $c < 4$, we conclude the desired lower bound.

The upper bound follows by the exact same machinery. On the event $O_\epsilon^\tau \cap D_C$, we have by positive supermartingale convergence that there is some large R_ϵ random so that for $c > 4$

$$\sup_{n > 0} U_n^c < R_\epsilon < \infty.$$

Hence, on this event,

$$M_n \leq \frac{cn}{\log n - \log R_\epsilon},$$

and so

$$\limsup_{n \rightarrow \infty} \frac{M_n \log n}{n} \leq c.$$

Thus we have that

$$\mathbb{P} \left[\left\{ \limsup_{n \rightarrow \infty} \frac{M_n \log n}{n} \leq c \right\} \cap O_\epsilon^\tau \cap D_C \right] = \mathbb{P} [O_\epsilon^\tau \cap D_C],$$

and so taking $\epsilon \rightarrow 0$ and $C \rightarrow \infty$ we have that

$$\limsup_{n \rightarrow \infty} \frac{M_n \log n}{n} \leq c \text{ a.s.}$$

As this holds for any $c > 4$, the proof is complete. □

6 The case $d > 2$

The case $d > 2$ requires different analysis from the case $d = 2$. Let x_* be the solution of equation $1 - (1 - x/2)^d = x$ in the interval $(0, 1)$; we will briefly argue that x_* exists and is unique. Note that by concavity of left side of the equation, there are at most 2 solutions of this equation on $(-\infty, 2]$. Further, 0 is always a solution, and for $d > 2$ the value of the derivative of the left hand side is greater than 1. As when $x = 1$, the right hand side is already greater than the left, there must be a solution on $(0, 1)$. As there are at most two solutions, x_* exists and is well-defined.

We will apply the stochastic approximation framework laid out in [17] to show that $Z_n := M_n/n \rightarrow x_*$ almost surely. A process $(Z_n)_{n \geq 0}$ adapted to the filtration $(\mathcal{F}_n)_{n \geq 0}$ is called a *stochastic approximation process* if it can be decomposed as

$$Z_{n+1} - Z_n = \frac{1}{n} (F(Z_n) + \xi_{n+1} + R_n),$$

where F is some function, $\mathbb{E}[\xi_{n+1}|\mathcal{F}_n] = 0$, and R_n is an (\mathcal{F}_n) adapted process satisfying $\sum_{n \geq 1} n^{-1}|R_n| < \infty$ almost surely.

From (1.1), we have that

$$\mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_n] = \frac{-Z_n}{n+1} + \frac{1 - (1 - \frac{M_n L_n}{2n})^d}{n+1}.$$

Set $F(x) = 1 - (1 - x/2)^d - x$. Thus, as L_n is eventually 1, we get that $\mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_n] = \frac{F(Z_n)}{n+1}$ for all n larger than some sufficiently large random time. Set

$$R_n = n\mathbb{E}[Z_{n+1} - Z_n|\mathcal{F}_n] - F(Z_n),$$

and note that we then must take $\xi_{n+1} = n(Z_{n+1} - \mathbb{E}[Z_{n+1}|\mathcal{F}_n])$.

By Proposition 1.2, there is a $T < \infty$ almost surely so that for all $n \geq T$, $L_n = 1$. As $|F(Z_n)| \leq 1$ almost surely, we have that

$$\sum_{n=1}^{\infty} \frac{|R_n|}{n} \leq \sum_{n=1}^{T-1} \frac{|R_n|}{n} + \sum_{n=T}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1} \right) = \frac{1}{T} + \sum_{n=1}^{T-1} \frac{|R_n|}{n}.$$

This is finite almost surely by the finiteness of T , and hence $(Z_n)_{n \geq 0}$ is a stochastic approximation process.

We now use the following corollary of Lemma 2.6 of [17].

Proposition 6.1. *If $\mathbb{E}[\xi_{n+1}^2|\mathcal{F}_n] \leq K$ for some $K > 0$ almost surely and if F is continuous with isolated zero set, then Z_n converges almost surely to a point in the zero set.*

As $\frac{n+1}{n}\xi_{n+1}$ is Bernoulli given \mathcal{F}_n , the second moment condition is clearly satisfied. Hence, we know that with probability 1, Z_n converges to either 0 or to x_* . Thus, we need only show that the process does not converge to 0 almost surely.

From section 5, recall the events $D(C) = \{L_n = 1, \forall n \geq C\}$, and the stopping time $\eta_C = \inf_{n \geq C} \{n : L_n > 1\}$. Given Propositions 1.2 and 6.1, it suffices to show the following.

Lemma 6.2. *Conditional on $D(C)$, for any $n_0 > C$ and $\epsilon > 0$ there is an $N < \infty$ random with $N > n_0$ so that $x_* - \epsilon < M_N/N$.*

Proof. Recall that p_n is the probability that $M_{n+1} = M_n + 1$ conditional on \mathcal{F}_n . Note that for n with $C \leq n \leq \eta_C$,

$$p_n = 1 - \left(1 - \frac{M_n}{2n}\right)^d = \frac{M_n}{2n} \left(\sum_{i=0}^{d-1} \left(1 - \frac{M_n}{2n}\right)^i \right).$$

Hence if we define the function

$$f(x) = \frac{1}{2} \sum_{i=0}^{d-1} (1 - x/2)^i,$$

then $\frac{p_n}{M_n} = \frac{1}{n} f\left(\frac{M_n}{n}\right)$. If $x \neq 0$ this function is equal to $\frac{1 - (1-x/2)^d}{x}$. Therefore x_* is the solution of equation $f(x) = 1$ in the interval $(0, 1)$. Note that for any $\epsilon > 0$ there is a $\delta > 0$ so that $f(x) > 1 + \delta$ if $0 \leq x \leq x_* - \epsilon$ and $f(x) < 1 - \delta$ if $x_* + \epsilon \leq x \leq 1$.

We will start by proving the lower bound. Assume that for $n_0, x_* - \epsilon > M_{n_0}/n_0$ (otherwise we could just put $N = n_0$). Consider the expectation

$$\begin{aligned} \mathbb{E} \left(\frac{M_n}{M_{n+1}} \mid \mathcal{F}_n \right) &= p_n \frac{M_n}{M_n + 1} + 1 - p_n = p_n \left(1 - \frac{1}{M_n + 1} \right) + 1 - p_n \\ &= 1 - \frac{p_n}{M_n} + O(M_n^{-2}) = 1 - \frac{1}{n} f\left(\frac{M_n}{n}\right) + O(M_n^{-2}). \end{aligned}$$

Thus, by the monotonicity of $f(x)$ there is a $\delta > 0$ such that

$$\mathbb{E} \left(\frac{1}{M_{n+1}} \mid \mathcal{F}_n \right) < \frac{(1 - (1 + \delta/2)/n)}{M_n},$$

provided $n \geq n_0$ for some large n_0 and $n \leq N \wedge \eta_C$. Setting $C_{n+1} = (1 + (1 + \delta)/n)C_n$, $n > n_0$, we have that $A_n = C_n/M_n$ is a supermartingale for this same range of n . By Lemma 3.1 we have that $C_n n^{-1-\delta}$ converges to a positive limit, and by Doob's theorem $A_{n \wedge \phi_1 \wedge \eta_C}$ tends to a finite limit with probability 1. Thus there is a random constant $B > 0$ so that $M_n \geq Bn^{1+\delta}$ for all $n \leq N \wedge \eta_C$. On the other hand, $M_n \leq 2n$, and so it must be that $N \wedge \eta_C < \infty$ almost surely. Thus, on the event that $\eta_C = \infty$, we have $N < \infty$. \square

Remark 6.3. *By looking at M_n/C_n , it is possible to show by an identical argument that $M_n/n < x_* + \epsilon$ infinitely often. This can be combined with an upcrossing inequality to show that M_n/n indeed converges to x_* . This argument is available in full in an earlier version of this paper on the arXiv [14].*

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Acknowledgments. The authors are grateful to Professor Itai Benjamini for proposing the model. They would also like to thank the anonymous referees for many helpful suggestions and references, especially regarding stochastic approximation.

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