Electron. Commun. Probab. **19** (2014), no. 62, 1–12.
DOI: 10.1214/ECP.v19-3249
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

A counter-example to the central limit theorem in Hilbert spaces under a strong mixing condition

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Abstract

We show that in a separable infinite dimensional Hilbert space, uniform integrability of the square of the norm of normalized partial sums of a strictly stationary sequence, together with a strong mixing condition, does not guarantee the central limit theorem.

Keywords: Central limit theorem ; Hilbert space ; mixing conditions ; strictly stationary process.
AMS MSC 2010: 60F05 ; 60G10.
Submitted to ECP on January 9, 2014, final version accepted on August 24, 2014.

1 Introduction and notations

Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and (S, d) a separable metric space. We say that the sequence of random variables $(X_n)_{n\in\mathbb{Z}}$ from Ω to S is *strictly stationary* if for all integer d and all integer k, the d- uple $(X_1, \dots, X_d)$ has the same law as $(X_{k+1}, \dots, X_{k+d})$.

Rosenblatt introduced in [18] the measure of dependence between two sub-σ-algebras $\mathcal{A}$ and $\mathcal{B}$:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup \left\{ \left| \mu(A \cap B) - \mu(A)\mu(B) \right|, A \in \mathcal{A}, B \in \mathcal{B} \right\}.$$

Another one is β-mixing, which is defined by

$$\beta(\mathcal{A}, \mathcal{B}) := \frac{1}{2} \sup \sum_{i=1}^{I} \sum_{j=1}^{J} \left| \mu(A_i \cap B_j) - \mu(A_i)\mu(B_j) \right|,$$

where the supremum is taken over the finite partitions $\{A_1, \dots, A_I\}$ and $\{B_1, \dots, B_J\}$ of Ω, which consist respectively of elements of $\mathcal{A}$ and $\mathcal{B}$. It was introduced by Volkonskii and Rozanov in [21].

In order to measure dependence of a sequence of random variables, say $X := (X_j)_{j\in\mathbb{Z}}$ (assumed strictly stationary for simplicity), we define $\mathcal{F}_m^n$ as the σ-algebra generated by the X_j for $m \leqslant j \leqslant n$, where $-\infty \leqslant m \leqslant n \leqslant +\infty$.

Then mixing coefficients are defined by

$$\alpha_X(n) := \alpha\left(\mathcal{F}_{-\infty}^0, \mathcal{F}_n^{+\infty}\right) \tag{1.1}$$

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$$\beta_X(n) := \beta\left(\mathcal{F}_{-\infty}^{0}, \mathcal{F}_{n}^{+\infty}\right), \tag{1.2}$$

which will be simply writen $\alpha(n)$ (respectively $\beta(n)$) when there is no ambiguity.

We say that the strictly stationary sequence $(X_j)_j$ is α*-mixing* (respectively β*-mixing)* if $\lim_{n\to\infty}\alpha(n)=0$ (respectively $\lim_{n\to\infty}\beta(n)=0$). Sequences which are α-mixing are also called *strong-mixing*. Notice that the inequality $2\alpha(\mathcal{A},\mathcal{B})\leqslant\beta(\mathcal{A},\mathcal{B})$ for any two sub-σ-algebras $\mathcal{A}$ and $\mathcal{B}$ implies that each β-mixing sequence is strong mixing. We refer the reader to Bradley's book [4] for further information about mixing conditions.

Let $(V,\|\cdot\|)$ be a separable normed space. We can represent a strictly stationary sequence $(X_j)_j$ by $X_j = f\circ T^j$, where $T\colon\Omega\to\Omega$ is measurable and measure preserving, that is, $\mu(T^{-1}(S))=\mu(S)$ for all $S\in\mathcal{F}$ (see [8], p.456, second paragraph).

Given an integer N, we define $S_N(f) := \sum_{j=0}^{N-1} f\circ T^j$ and $(\sigma_N(f))^2 := \mathbb{E}\left[\|S_N(f)\|^2\right]$.

When $V=\mathbb{R}^d$, $d\in\mathbb{N}^*$ it is well-known that if $\left(f\circ T^j\right)_{j\geqslant 0}$ satisfies the following assumptions:

1. $\lim_{N\to+\infty}\sigma_N(f)=+\infty$;
2. $\int f d\mu = 0$:
3. $\lim_{n\to+\infty}\alpha(n)=0$;
4. the family $\left\{\frac{\|S_N(f)\|^2}{(\sigma_N(f))^2}, N\geqslant 1\right\}$ is uniformly integrable,

then $\left(\frac{1}{\sigma_N(f)}S_N(f)\right)_{N\geqslant 1}$ converges in distribution to a Gaussian law. It was established for $d=1$ by Denker [7], Mori and Yoshihara [14] using a blocking argument. Volný [22] gave a proof for d arbitrary based on approximation by an array of independent random variables.

A natural question would be: what if we replace $\mathbb{R}^d$ by another normed space?

First, we restrict ourselves to separable normed spaces in order to avoid measurability issues of sums of random variables. Corollary 10.9. in [11] asserts that a separable Banach space B with norm $\|\cdot\|_B$ is isomorphic to a Hilbert space if and only if for all random variables X with values in B, the conditions $\mathbb{E}[\mathbf{X}]=0$ and $\mathbb{E}\left[\|\mathbf{X}\|_B^2\right]<\infty$ are necessary and sufficient for X to satisfy the central limit theorem. By "$\mathbf{X}$ satisfies the CLT", we mean that if $(\mathbf{X_j})_{j\geqslant 1}$ is a sequence of independent random variables, with the same law as X, the sequence $\left(n^{-1/2}\sum_{j=1}^{n}\mathbf{X_j}\right)_{n\geqslant 1}$ weakly converges in B. Hence we cannot expect a generalization in a class larger than separable Hilbert spaces. Such a space is necessarily isomorphic to $\mathcal{H}:=\ell^2(\mathbb{R})$, the space of square sumable sequences $(x_n)_{n\geqslant 1}$ endowed with the inner product $\langle x,y\rangle_{\mathcal{H}} := \sum_{n=1}^{+\infty}x_n y_n$. We shall denote by $\mathbf{e_n}$ the sequence whose all terms are 0, except the n-th which is 1. Bold letters denote both randoms variables taking their values in $\mathcal{H}$ and elements of this space.

General considerations about probability measures and central limit theorem in Banach spaces are contained in Araujo and Giné's book [2].

Notation 1. If $(a_n)_{n\geqslant 1}$, $(b_n)_{n\geqslant 1}$ are sequences of non- negative real numbers, $a_n\lesssim b_n$ means that $a_n\leqslant Cb_n$, where C does not depend on n. In an analogous way, we define $a_n\gtrsim b_n$. When $a_n\lesssim b_n\lesssim a_n$, we simply write $a_n\asymp b_n$.

Our main results are

Theorem A. *There exists a probability space $(\Omega, \mathcal{F}, \mu)$ such that given $0 < q < 1$, we can construct a strictly stationary sequence $\mathbf{X} = (\mathbf{f} \circ T^k) = (\mathbf{X_k})_{k \in \mathbb{N}}$ defined on Ω, taking its values in $\mathcal{H}$, such that:*

a) $\mathbb{E}[\mathbf{f}] = 0$, $\mathbb{E}[\|\mathbf{f}\|_{\mathcal{H}}^p]$ *is finite for each* p*;*

b) *the limit* $\lim_{N\to\infty} \sigma_N(\mathbf{f})$ *is infinite;*

c) *the process* $(\mathbf{X_k})_{k\in\mathbb{N}}$ *is* β*-mixing, more precisely,* $\beta_{\mathbf{X}}(j) = O\left(\frac{1}{j^q}\right)$*;*

d) *the family* $\left\{\frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1\right\}$ *is uniformly integrable;*

e) *if* $I \subset \mathbb{N}$ *is infinite, the family* $\left\{\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I\right\}$ *is not tight in* $\mathcal{H}$*; furthermore, given a sequence* $(c_N)_{N\geqslant 1}$ *of real numbers going to infinity, we have either*

- $\lim_{N\to+\infty} \frac{\sigma_N(\mathbf{f})}{c_N} = 0$, *hence* $\left(\frac{S_N(\mathbf{f})}{c_N}\right)_{N\geqslant 1}$ *converges to* $\mathbf{0}_{\mathcal{H}}$ *in distribution, or*
- $\limsup_{N\to+\infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$*, and in this case the collection* $\left\{\frac{S_N(\mathbf{f})}{c_N}, N \geqslant 1\right\}$ *is not tight.*

Theorem A'. *Let* $(b_N)_{N\geqslant 1}$ *and* $(h_N)_{N\geqslant 1}$ *be sequences of positive real numbers, with* $\lim_{N\to\infty} b_N = 0$ *and* $\lim_{N\to\infty} h_N = \infty$*. Then there exists a strictly stationary sequence* $\mathbf{X} := (\mathbf{f} \circ T^k)_{k\in\mathbb{N}} = (\mathbf{X_k})_{k\in\mathbb{N}}$ *of random variables with values in* $\mathcal{H}$ *such that A, A, A of Theorem A and the following two properties hold:*

b') *we have* $\sigma_N^2(\mathbf{f}) \lesssim N \cdot h_N$ *and* $\frac{\sigma_N^2(\mathbf{f})}{N} \to \infty$*;*

c') *the process* $(\mathbf{X_k})_{k\in\mathbb{N}}$ *is* β*-mixing, and there is an increasing sequence* $(n_k)_{k\geqslant 1}$ *of integers such that for each* k*,* $\beta_{\mathbf{X}}(n_k) \leqslant b_{n_k}$*.*

Remark 2. Theorem A shows that Denker's result does not remain valid in its full generality in the context of Hilbert space valued random variables.

Furthermore, a careful analysis of the proof of Proposition 6 shows that for the construction given in Theorem A, we have $\sigma_N^2(f) = N \cdot h(N)$ with h slowly varying in the strong sense. Theorem 1 of [12] does not remain valid in the Hilbert space setting. Indeed, the arguments given in pages 654-655 show that the conditions of Denker's theorem together with the assumption that $\sigma_N^2 = N \cdot h(N)$ with h slowly varying in the strong sense imply those of Theorem 1. These arguments are still true in the Hilbert space setting.

Remark 3. Theorem A' gives a control of the mixing coefficients on a subsequence. When $b_N := N^{-2}$ for example, the construction gives a better estimation for the considered subsequence than what we get by Theorem A.

Tone has established in [20] a central limit theorem for strictly stationary random fields with values in $\mathcal{H}$ under ρ'-mixing conditions. For sequences, these coefficients are defined by

$$\rho'(n) := \sup\left\{\frac{|\mathbb{E}[\langle \mathbf{f}, \mathbf{g}\rangle_{\mathcal{H}}] - \langle \mathbb{E}[\mathbf{f}], \mathbb{E}[\mathbf{g}]\rangle_{\mathcal{H}}|}{\|\mathbf{f}\|_{\mathbb{L}^2(\mathcal{H})} \|\mathbf{g}\|_{\mathbb{L}^2(\mathcal{H})}}\right\},$$

where the supremum is taken over all the non-zero functions $\mathbf{f}$ and $\mathbf{g}$ such that $\mathbf{f}$ and $\mathbf{g}$ are respectively $\sigma(X_j, j \in S_1)$ and $\sigma(X_j, j \in S_2)$-measurable, where S_1 and S_2 are such that $\min_{s\in S_1, t\in S_2} |s - t| \geqslant n$, while $\mathbb{L}^2(\mathcal{H})$ denote the collection of equivalence classes of random variables $\mathbf{X} \colon \Omega \to \mathcal{H}$ such that $\|\mathbf{X}\|_{\mathcal{H}}^2$ is integrable.

So "interlaced index sets" can be considered, which is not the case for α and β-mixing coefficient. Taking f and g as characteristic functions of elements of $\mathcal{F}_{-\infty}^0$ and

$\mathcal{F}_n^{+\infty}$ respectively, one can see that $\alpha(n) \leqslant \rho'(n)$, hence ρ'-mixing condition is more restrictive than α-mixing condition.

A partial generalization of the finite dimensional result was proved by Politis and Romano [15], namely, the conditions $\mathbb{E}\left\|\mathbf{X_1}\right\|_{\mathcal{H}}^{2+\delta}$ finite for some positive δ and $\sum_j \alpha_{\mathbf{X}}(j)^{\frac{\delta}{2+\delta}}$ guarantees the convergence of $n^{-1/2}\sum_{j=1}^n \mathbf{X_j}$ to a Gaussian random variable $\mathcal{N}$, whose covariance operator S satisfies

$$\mathbb{E}\left[\langle \mathcal{N}, h\rangle^2\right] = \langle Sh, h\rangle_{\mathcal{H}} = \operatorname{Var}(\langle \mathbf{X_1}, h\rangle) + 2\sum_{i=1}^{+\infty} \operatorname{Cov}\left(\langle \mathbf{X_1}, h\rangle, \langle \mathbf{X_{1+i}}, h\rangle\right).$$

Similar results were obtained by Dehling [6].

Rio's inequality [16] asserts that given two real valued random variables X and Y with finite two order moments,

$$\left|\mathbb{E}\left[XY\right] - \mathbb{E}\left[X\right]\mathbb{E}\left[Y\right]\right| \leqslant 2\int_0^{\alpha(\sigma(X),\sigma(Y))} Q_X(u)Q_Y(u)\mathrm{d}u.$$

It was extented by Merlevède et al. [13], namely, if $\mathbf{X}$ and $\mathbf{Y}$ are two random variables with values in $\mathcal{H}$, with respective quantile function $Q_{\|\mathbf{X}\|_{\mathcal{H}}}$ and $Q_{\|\mathbf{Y}\|_{\mathcal{H}}}$, then

$$\left|\mathbb{E}\left[\langle \mathbf{X}, \mathbf{Y}\rangle_{\mathcal{H}}\right] - \langle \mathbb{E}\left[\mathbf{X}\right], \mathbb{E}\left[\mathbf{Y}\right]\rangle_{\mathcal{H}}\right| \leqslant 18\int_0^{\alpha} Q_{\|\mathbf{X}\|_{\mathcal{H}}} Q_{\|\mathbf{Y}\|_{\mathcal{H}}}\mathrm{d}u,$$

where $\alpha := \alpha(\sigma(\mathbf{X}), \sigma(\mathbf{Y}))$.

From this inequality, they deduce a central limit theorem for a stationary sequence $(\mathbf{X_j})_{j\in\mathbb{Z}}$ of $\mathcal{H}$-valued zero-mean random variables satisfying

$$\int_0^1 \alpha^{-1}(u) Q^2_{\|\mathbf{X_0}\|_{\mathcal{H}}}(u)\mathrm{d}u < \infty, \tag{1.3}$$

where α^{-1} is the inverse function of $x \mapsto \alpha_{\mathbf{X}}(\lfloor x\rfloor)$.

Discussion after Corollary 1.2 in [17] proves that the later result implies Politis' one.

Relative optimality of condition (1.3) (cf. [9]) can give a finite-dimensional counter-example to the central limit theorem when this condition is not satisfied. Here, the condition of uniform integrability prevents such counter-examples.

Defining $\alpha_{2,\mathbf{X}}(n) := \sup_{i\geqslant j\geqslant n} \alpha(\mathcal{F}^0_{-\infty}, \sigma(\mathbf{X_i}, \mathbf{X_j}))$ and Q_{X_0} the right-continuous inverse of the function $t \mapsto \mu\left\{\|\mathbf{X_0}\|_{\mathcal{H}} > t\right\}$ (that is,

$Q_{\mathbf{X_0}}(u) := \inf\left\{t \in \mathbb{R}, \mu\left\{\|\mathbf{X_0}\|_{\mathcal{H}} > t\right\} \leqslant u\right\}$), Dedecker and Merlevède have shown in [5] that under the assumption

$$\sum_{k=1}^{+\infty} \int_0^{\alpha_{2,\mathbf{X}}(k)} Q^2_{\mathbf{X_0}}(u)\mathrm{d}u < \infty,$$

we can find a sequence $(\mathbf{Z_i})_{i\in\mathbb{N}}$ of Gaussian random variables with values in $\mathcal{H}$ such that almost surely,

$$\left\|\mathbf{S_n} - \sum_{i=1}^n \mathbf{Z_i}\right\|_{\mathcal{H}} = o\left(\sqrt{n\log\log n}\right).$$

2 The proof

2.1 Construction of f

In order to construct a counter-example, we shall need the following lemma, which will be proved later.

We will denote U the Koopman operator associated to T, which acts on measurable functions by $U(f)(x) := f(T(x))$.

Lemma 4. *Let $(u_k)_{k\geqslant 1}\subset(0,1)$ be a sequence of numbers. Then there exists a dynamical system $(\Omega,\mathcal{F},\mu,T)$ and a sequence of random variables $(\xi_k)_{k\geqslant 1}$ such that*

1. *for each $k\geqslant 1$, $\mu(\xi_k=1)=\mu(\xi_k=-1)=\frac{u_k}{2}$ and $\mu(\xi_k=0)=1-u_k$;*
2. *the random variables $(U^i\xi_k, k\geqslant 1, i\in\mathbb{Z})$ are mutually independent.*

Recall that $\mathbf{e_k}$ is the k-th element of the canonical orthonormal system of $\mathcal{H}=\ell^2(\mathbb{R})$. We define

$$f_k:=\sum_{i=0}^{n_k-1}U^{-i}\xi_k \text{ and } \mathbf{f}:=\sum_{k=1}^{+\infty}f_k\mathbf{e_k}, \tag{2.1}$$

where the ξ_i's are constructed using to Lemma 4 taking $u_k:=n_k^{-2}$. Conditions on the increasing sequence of integers $(n_k)_{k\geqslant 1}$ will be specified latter.

Then $\mathbf{X_k}:=\mathbf{f}\circ T^k$ is a strictly stationary sequence. Note that $\|\mathbf{f}\|_{\mathcal{H}}^2$ is an integrable random variable whenever $\sum_k\frac{1}{n_k}$ is convergent. In the sequel, the choice of n_k will guarantee this condition.

2.2 Preliminary results

We express $S_N(f_k)$ as a linear combination of independent random variables. By direct computations, we get

$$f_k=n_k\xi_k+(I-U)\sum_{i=1-n_k}^{-1}(n_k+i)U^i\xi_k, \tag{2.2}$$

hence

$$S_N(f_k)=n_k\sum_{j=0}^{N-1}U^j\xi_k+\sum_{i=1-n_k}^{-1}(n_k+i)U^i\xi_k-\sum_{i=N-n_k+1}^{N-1}(n_k+i-N)U^i\xi_k.$$

This formula can be simplified if we distinguish the cases $N\geqslant n_k$ and $n_k<N$ (we break the third sum at the index $i=0$ if necessary). This gives

$$S_N(f_k)=\sum_{j=0}^{N-1}(N-j)U^j\xi_k+\sum_{j=1-n_k}^{N-n_k}(n_k+j)U^j\xi_k + N\sum_{j=1+N-n_k}^{-1}U^j\xi_k,\quad \text{if } N<n_k, \tag{2.3}$$

$$S_N(f_k)=n_k\sum_{j=0}^{N-n_k}U^j\xi_k+\sum_{j=N-n_k+1}^{N-1}(N-j)U^j\xi_k + \sum_{j=1-n_k}^{-1}(n_k+j)U^j\xi_k,\quad \text{if } N\geqslant n_k. \tag{2.4}$$

The computation of the expectation of the square of partial sums gives

$$\sigma_N^2(f_k)=\begin{cases}\frac{1}{n_k^2}\left(2\sum_{j=1}^{N}j^2+(n_k-N-1)N^2\right) & \text{if } N<n_k,\\ \frac{1}{n_k^2}\left(n_k^2(N-n_k+1)+2\sum_{j=1}^{n_k-1}j^2\right) & \text{if } N\geqslant n_k.\end{cases} \tag{2.5}$$

Notation 5. If N is a positive integer and $(n_k)_{k\geqslant 1}$ is an increasing sequence of integers, denote by $i(N)$ the unique integer for which $n_{i(N)} \leqslant N < n_{i(N)+1}$.

Proposition 6. *Assume that $(n_k)_{k\geqslant 1}$ satisfies the condition*

$$\text{there is } p>1 \text{ such that for each } k, \quad n_{k+1} \geqslant n_k^p. \tag{C}$$

Then $\sigma_N^2(\mathbf{f}) \asymp N \cdot i(N)$.

Proof. Using (2.5), the fact that $M^3 \asymp \sum_{j=1}^M j^2$ and $\sigma_N^2(\mathbf{f}) = \sum_{k\geqslant 1} \sigma_N^2(f_k)$, we have

$$\sigma_N^2(\mathbf{f}) \geqslant \sum_{k=1}^{i(N)} \sigma_N^2(f_k) \asymp N \sum_{j=1}^{i(N)} 1 = N \cdot i(N). \tag{2.6}$$

From (2.5) in the case $n_k \geqslant N$, we deduce

$$\sum_{k\geqslant i(N)+1} \sigma_N^2(f_k) \lesssim \sum_{k\geqslant i(N)+1} \frac{N^2}{n_k} \leqslant \frac{N^2}{n_{i(N)+1}} + \sum_{k\geqslant i(N)+1} \frac{N^2}{n_k} \frac{1}{n_k^{p-1}}. \tag{2.7}$$

Since $n_{i(N)+1} \geqslant N$ and the series $\sum_{k\geqslant 1} n_k^{1-p}$ is convergent (by the ratio test), we obtain

$$\sum_{k\geqslant i(N)+1} \sigma_N^2(f_k) \lesssim N + N \sum_{k\geqslant i(N)+1} \frac{1}{n_k^{p-1}} \lesssim N. \tag{2.8}$$

Combining (2.6) and (2.8), we get

$$N \cdot i(N) \lesssim \sigma_N^2(\mathbf{f}) \lesssim \sum_{k=1}^{i(N)} \sigma_N^2(f_k) + \sum_{k\geqslant i(N)+1} \sigma_N^2(f_k) \lesssim N \cdot i(N) + N \lesssim N \cdot i(N). \tag{2.9}$$

□

Proposition 7. *Assume that $\sum_k n_k^{-a}$ is convergent for any positive real number a. Then for each integer p, $\|\mathbf{f}\|_{\mathcal{H}}$ has a finite moment of order p.*

Proof. We shall use Rosenthal's inequality (Theorem 3, [19]): there exists a constant C depending only on q such that if M is an integer, $Y_1, \dots, Y_M$ are independent real valued zero mean random variables for which $\mathbb{E}\,|Y_i|^q < \infty$ for each i, then

$$\mathbb{E}\left|\sum_{j=1}^M Y_j\right|^q \leqslant C\left(\sum_{j=1}^M \mathbb{E}\,|Y_j|^q + \left(\sum_{j=1}^M \mathbb{E}\left[Y_j^2\right]\right)^{q/2}\right). \tag{2.10}$$

If $q = 2p$ is given then we have

$$\mathbb{E}\,|f_k|^{2p} \lesssim n_k^{-1} + n_k^{-p} \lesssim n_k^{-1}. \tag{2.11}$$

□

We provide a sufficient condition for the uniform integrability of the family $\mathcal{S} := \left\{\frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1\right\}$.

Proposition 8. *If $(n_k)_{k\geqslant 1}$ satisfies* (C)*, then $\mathcal{S}$ is uniformly integrable.*

Proof. For $N \geqslant 1$, we have:

$$\frac{\|S_N(\mathbf{f})\|_{\mathcal{H}}^2}{\sigma_N^2(\mathbf{f})} = \sum_{j=1}^{i(N)-1} \frac{|S_N(f_j)|^2}{\sigma_N^2(\mathbf{f})} + \frac{\left|S_N(f_{i(N)})\right|^2}{\sigma_N^2(\mathbf{f})} + \frac{\left|S_N(f_{i(N)+1})\right|^2}{\sigma_N^2(\mathbf{f})} + \sum_{j \geqslant i(N)+2} \frac{|S_N(f_j)|^2}{\sigma_N^2(\mathbf{f})},$$

hence it is enough to prove that the families

$$\mathcal{S}_1 := \left\{ \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1 \right\},$$
$$\mathcal{S}_2 := \left\{ \frac{\left|S_N(f_{i(N)})\right|^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1 \right\} =: \{u_N, N \geqslant 1\},$$
$$\mathcal{S}_3 := \left\{ \frac{\left|S_N(f_{i(N)+1})\right|^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1 \right\} =: \{v_N, N \geqslant 1\}, \text{ and}$$
$$\mathcal{S}_4 := \left\{ \sum_{k \geqslant i(N)+2} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})}, N \geqslant 1 \right\}$$

are uniformly integrable. For $\mathcal{S}_1$ and $\mathcal{S}_4$, we shall show that these families are bounded in $\mathbb{L}^p$ for $p \in (1,2]$ as in (C).

- for $\mathcal{S}_1$: using the expression in (2.4) and (2.10) with $q := 2p > 2$, we have

$$\begin{aligned}
\mathbb{E}\left[|S_N(f_k)|^{2p}\right] &\leqslant C \left(2\sum_{j=1}^{n_k} \frac{j^{2p}}{n_k^2} + \frac{n_k^{2p}(N-n_k)}{n_k^2} \right) + C \left(2\sum_{j=1}^{n_k} \frac{j^2}{n_k^2} + \frac{(N-n_k)n_k^2}{n_k^2} \right)^p \\
&\lesssim \frac{1}{n_k^2} \left(n_k^{2p+1} + (N-n_k)n_k^{2p} \right) + \frac{1}{n_k^{2p}} \left(n_k^3 + (N-n_k)n_k^2 \right)^p \\
&= \frac{N n_k^{2p}}{n_k^2} + \frac{N^p n_k^{2p}}{n_k^{2p}} \\
&= N n_k^{2(p-1)} + N^p
\end{aligned}$$

hence

$$\left\| S_N(f_k)^2 \right\|_p \lesssim N^{1/p} n_k^{2\frac{p-1}{p}} + N,$$

which gives

$$\begin{aligned}
\left\| \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})} \right\|_p &\lesssim \frac{\sum_{k=1}^{i(N)-1} (N^{1/p} n_k^{2\frac{p-1}{p}} + N)}{\sigma_N^2(\mathbf{f})} \\
&\lesssim \frac{i(N) n_{i(N)-1}^{2\frac{p-1}{p}} + N i(N)}{\sigma_N^2(\mathbf{f})}.
\end{aligned}$$

From (2.6), we get

$$\left\| \sum_{k=1}^{i(N)-1} \frac{|S_N(f_k)|^2}{\sigma_N^2(\mathbf{f})} \right\|_p \lesssim \frac{n_{i(N)}^{2\frac{p-1}{p}}}{n_{i(N)}} + 1 = n_{i(N)}^{\frac{p-2}{p}} + 1.$$

Since $p - 2 \leqslant 0$, we obtain that $\mathcal{S}_1$ is bounded in $\mathbb{L}^p$ hence uniformly integrable.

- for $\mathcal{S}_2$: using (2.4) in the case $n_k \leqslant N$ and Proposition 6, we get

$$\|u_N\|_1 \lesssim \frac{N}{\sigma_N^2(\mathbf{f})} \lesssim \frac{1}{i(N)}. \tag{2.12}$$

  Since $\|u_N\|_1 \to 0$ and $u_N \in \mathbb{L}^1$ for each N, the family $\mathcal{S}_2$ is uniformly integrable.
- for $\mathcal{S}_3$: using (2.3) in the case $n_k > N$ and Proposition 6, we get

$$\|v_N\|_1 \lesssim \frac{N^2}{n_{i(N)+1}\sigma_N^2(\mathbf{f})} \lesssim \frac{N}{N \cdot i(N)}. \tag{2.13}$$

  Since $\|v_N\|_1 \to 0$ and $v_N \in \mathbb{L}^1$ for each N, the family $\mathcal{S}_3$ is uniformly integrable.
- for $\mathcal{S}_4$: as for $\mathcal{S}_1$, we shall show that this family is bounded in $\mathbb{L}^p$ with $p \in (1,2]$. We have, using (2.3) and (2.10)

$$\begin{aligned}\mathbb{E}\left[|S_N(f_k)|^{2p}\right] &\lesssim \frac{1}{n_k^2}(N^{2p+1} + N^{2p}(n_k - N)) + \frac{1}{n_k^{2p}}(N^3 + (n_k - N)N^2)^p \\ &= \frac{N^{2p}}{n_k} + \frac{N^{2p}}{n_k^p} \\ &\lesssim \frac{N^{2p}}{n_k}\end{aligned}$$

  as $N \leqslant n_k$. We thus get that

$$\left\|\sum_{k \geqslant i(N)+2} |S_N(f_k)|^2\right\|_p \lesssim N^2 \sum_{k \geqslant i(N)+2} \frac{1}{n_k^{1/p}}.$$

  Also, using (2.5), we have

$$\sigma_N^2(\mathbf{f}) \gtrsim N^2 \sum_{k \geqslant i(N)+1} \frac{1}{n_k}.$$

  The condition $n_{k+1} \geqslant n_k^p$ gives boundedness in $\mathbb{L}^p$ of $\mathcal{S}_4$.

This concludes the proof of A. □

Proposition 9. *Assume that $(n_k)_{k\geqslant 1}$ is such that $\mathcal{S}$ is uniformly integrable and $\sum_k n_k^{-1}$ is convergent. Then for each $I \subset \mathbb{N}$ infinite, the collection $\left\{\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I\right\}$ is not tight in $\mathcal{H}$. Its finite-dimensional distributions converge to 0 in probability.*

Furthermore, if $(c_N)_{N\geqslant 0}$ is a sequence of positive numbers going to infinity, we have either

- $\lim_{N\to+\infty} \frac{\sigma_N(\mathbf{f})}{c_N} = 0$*, hence* $\left(\frac{S_N(\mathbf{f})}{c_N}\right)_{N\geqslant 1}$ *converges to* $\mathbf{0}_{\mathcal{H}}$ *in distribution, or*
- $\limsup_{N\to+\infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$*, and in this case the sequence* $\left\{\frac{S_N(\mathbf{f})}{c_N}, N \geqslant 1\right\}$ *is not tight.*

Proof. We first prove that the finite dimensional distributions of $\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}$ converge weakly to 0.

For each $d \in \mathbb{N}$, we have $\frac{\langle S_N(\mathbf{f}), \mathbf{e_d}\rangle_{\mathcal{H}}}{\sigma_N(\mathbf{f})} \to 0$ in distribution. Indeed, we have by (2.2) that $\langle S_N(\mathbf{f}), \mathbf{e_d}\rangle_{\mathcal{H}} = n_d \sum_{i=0}^{N-1} U^i\xi_d + (I - U^N)\sum_{i=1-n_d}^{-1}(n_d + i)U^i\xi_d$. We conclude noticing that $\sigma_N(\mathbf{f})^{-1}(I - U^N)\sum_{i=1-n_d}^{-1}(n_d + i)U^i\xi_d$ goes to 0 in probability as N goes to infinity, using Proposition 6 and the estimate

$$\mathbb{E}\left(n_d \sum_{i=0}^{N-1} U^i\xi_d\right)^2 = N \lesssim \frac{\sigma_N^2(\mathbf{f})}{i(N)}$$

This can be extended replacing $\mathbf{e_d}$ by any $\mathbf{v} \in \mathcal{H}$ by an application of Theorem 4.2. in [3]. By Proposition 4.15 in [2], the only possible limit is the Dirac measure at $\mathbf{0}_{\mathcal{H}}$.

Assume that the sequence $\left\{\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \geqslant 1\right\}$ is tight. The sequence $\left(\frac{\|S_N(\mathbf{f})\|^2_{\mathcal{H}}}{\sigma^2_N(\mathbf{f})}\right)_{N \geqslant 1}$ is a uniformly integrable sequence of random variables of mean 1. A weakly convergent subsequence would go to $\mathbf{0}_{\mathcal{H}}$. According to Theorem 5.4 in [3], we should have that the limit random variable has expectation 1. This contradiction gives the result when $I = \mathbb{N} \setminus \{0\}$. Applying this reasonning to subsequences, one can see that for any infinite subset I of $\mathbb{N} \setminus \{0\}$, the family $\left\{\frac{S_N(\mathbf{f})}{\sigma_N(\mathbf{f})}, N \in I\right\}$ is not tight.

Let $(c_N)_{N \geqslant 1}$ be a sequence of positive real numbers such that $\lim_{N\to+\infty} c_N = +\infty$.

- first case: $\frac{\sigma_N(\mathbf{f})}{c_N}$ converges to 0. In this case, the sequence $\left(\frac{\|S_N(\mathbf{f})\|^2}{c^2_N}\right)_{N \geqslant 1}$ converges to 0 in $\mathbb{L}^1$, hence the sequence $\left(\frac{S_N(\mathbf{f})}{c_N}\right)_{N \geqslant 1}$ converges in distribution to $\mathbf{0}_{\mathcal{H}}$.
- second case: $\limsup_{N\to\infty} \frac{\sigma_N(\mathbf{f})}{c_N} > 0$. Hence there is some $r > 0$ and a sequence of integers $l_i \uparrow \infty$ such that for each i, $\frac{\sigma_{l_i}(\mathbf{f})}{c_{l_i}} \geqslant \frac{1}{r}$, that is, $c_{l_i} \leqslant r\sigma_{l_i}(\mathbf{f})$.

  Assume that the family $\left\{\frac{S_{l_i}(\mathbf{f})}{c_{l_i}}, i \geqslant 1\right\}$ is tight. This means that given a positive ε, one can find a compact set $K = K(\varepsilon)$ such that for each i, $\mu\left\{\frac{S_{l_i}(\mathbf{f})}{c_{l_i}} \in K\right\} > 1 - \varepsilon$. We can assume that this compact set is convex and contains 0 (we consider the closed convex hull of $K \cup \{0\}$, which is compact by Theorem 5.35 in [1]). Then we have

$$\left\{\frac{S_{l_i}(\mathbf{f})}{c_{l_i}} \in K\right\} = \left\{\frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})} \in \frac{c_{l_i}}{\sigma_{l_i}(\mathbf{f})} K\right\} \subset \left\{\frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})} \in rK\right\},$$

  and we would deduce tightness of $\left\{\frac{S_{l_i}(\mathbf{f})}{\sigma_{l_i}(\mathbf{f})}, i \geqslant 1\right\}$, which cannot happen.

Remark 10. In the second case, it may happen that the finite dimensional distributions does not converge to degenerate ones, for example with $c_N := N$.

□

2.3 Proof of Theorem A

Notice that if $n_{k+1} \geqslant n_k^p$ for some $p > 1$ and $n_1 = 2$, then $n_k \geqslant 2^{p^k}$, hence the condition of Proposition 7 is fulfilled. We get A since each f_k has expectation 0.

We denote $\lfloor x \rfloor := \sup\{k \in \mathbb{Z}, k \leqslant x\}$ the integer part of the real number x.

Proposition 11. *Let $p > 1$. With $n_k := \lfloor 2^{p^k} \rfloor$ (which satisfies (C)), we have for each positive integer l,*

$$\beta_{\mathbf{X}}(l) \lesssim \frac{1}{l^{\frac{1}{p}}}.$$

Proof. We define $\beta_k(n)$ as the n-th β-mixing coefficient of the sequence $(f_k \circ T^i)_{i \geqslant 0}$.

By Lemma 5 of [10], we have the estimate $\beta_k(0) \leqslant 4n_k^{-1}$ for each k. Using then Proposition 4 of this paper (cf. [4] for a proof), we get that $\beta_{\mathbf{X}}(n_k) \lesssim \sum_{j \geqslant k} \frac{1}{n_j}$ for each integer k. Since $p^i \geqslant i$ for i large enough,

$$\sum_{j \geqslant k} \frac{1}{n_j} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^{i+k}}} = \sum_{i=0}^{+\infty} \frac{1}{2^{p^i p^k}} \lesssim \sum_{i=0}^{+\infty} \frac{1}{2^i} \frac{1}{2^{p^k}} = \frac{2}{2^{p^k}},$$

we get

$$\beta_{\mathbf{X}}(N) \leqslant \beta_X(n_{i(N)}) \lesssim \frac{1}{n_{i(N)}} = \frac{1}{n_{i(N)+1}^{1/p}} \leqslant \frac{1}{N^{1/p}}.$$

□

This proves A. For any p, the choice $n_k := \lfloor 2^{p^k} \rfloor$ satisfies the condition of Proposition 8, which proves A. We conclude the proof by Proposition 9.

Remark 12. For each of these choices, $\sigma_N^2(\mathbf{f})$ behaves asymptotically like $N \log\log N$. Theorem A' shows that we can construct a process which satisfies the same asymptotic behavior of partial sums and has a variance close to a linear one.

A question would be: can we construct a strictly stationary sequence with all the properties of Theorem A, except A which is replaced by an assumption of linear variance?

2.4 Proof of Theorem A'

Let $(h_N)_{N \geqslant 1}$ be the sequence involved in Theorem A'. We define for an integer u the quantity $h^{-1}(u) := \inf\{j \in \mathbb{N}, h_j \geqslant u\}$.

If $(b_k)_{k \geqslant 1}$ is the given sequence (that can be assumed decreasing), we define inductively

$$n_{k+1} := \max\left\{ n_k^2, \lfloor \frac{2^k}{b_{n_k}} \rfloor, h^{-1}(k) \right\}. \tag{2.14}$$

Let N be an integer. We assume without loss of generality that the growth of the sequence $(h_N)_{N \geqslant 1}$ is slow enough in order to guarantee that there exists k such that $N = h^{-1}(k)$. We then have $i(N) \leqslant k+1 \leqslant h_N + 1$, hence using Proposition 6, we get b').

We have $n_k \geqslant 2^{2^k}$ hence by a similar argument as in the proof of Theorem A, A is satisfied.

By a similar argument as in [10], we get $\beta_{\mathbf{X}}(n_k) \leqslant b_{n_k}$, hence c') holds.

Remark 13. By (1.3), we cannot expect the relationship $\beta_{\mathbf{X}}(\cdot) \leqslant b_{\cdot}$ for the whole sequence.

Since for each k, $n_{k+1} \geqslant n_k^2$, Proposition 8 and 9 apply. This concludes the proof of Theorem A'.

Proof of Lemma 4. Let $\Omega := [0,1]^{\mathbb{N}^* \times \mathbb{Z}}$, where $[0,1]$ is endowed with Borel σ- algebra and Lebesgue measure, and Ω with the product structure.

For $(k,j) \in \mathbb{N}^* \times \mathbb{Z}$ and $S \subset [0,1]$, let $P_{k,j}(S) := \prod_{(i_1,i_2) \in \mathbb{N}^* \times \mathbb{Z}} S_{i_1,i_2}$, where $S_{i_1,i_2} = S$ if $(i_1,i_2) = (k,j)$ and $[0,1]$ otherwise. Then we define

$$A_{k,j}^{+} := P_{k,j}([0, 2^{-1}(u_k)^{-1}]),$$

$$A_{k,j}^{-} := P_{k,j}([2^{-1}(u_k)^{-1}, (u_k)^{-1}]),$$

$$A_{k,j}^{(0)} := P_{k,j}([(u_k)^{-1}, 1]),$$

the map T by $T\left((x_{k,j})_{(k,j) \in \mathbb{N}^* \times \mathbb{Z}} \right) := (x_{k,j+1})_{(k,j) \in \mathbb{N}^* \times \mathbb{Z}}$, and

$$\xi_k := \chi_{A_{k,0}^{+}} - \chi_{A_{k,0}^{-}}.$$

□

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Acknowledgments. The authors would like to thank both referees for helpful comments, and for suggesting Remark 2.