Spectral measures of powers of random matrices*

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Dedicated to the memory of Helen Murphy Tepperman.

Abstract

This paper considers the empirical spectral measure of a power of a random matrix drawn uniformly from one of the compact classical matrix groups. We give sharp bounds on the L_p-Wasserstein distances between this empirical measure and the uniform measure on the circle, which show a smooth transition in behavior when the power increases and yield rates on almost sure convergence when the dimension grows. Along the way, we prove the sharp logarithmic Sobolev inequality on the unitary group.

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1 Introduction

The eigenvalues of large random matrices drawn uniformly from the compact classical groups are of interest in a variety of fields, including statistics, number theory, and mathematical physics; see e.g. [7] for a survey. An important general phenomenon discussed at length in [7] is that the eigenvalues of an $N \times N$ random unitary matrix U, all of which lie on the circle $\mathbb{S}^1 = \{z \in \mathbb{C} : |z| = 1\}$, are typically more evenly spread out than N independently chosen uniform random points in $\mathbb{S}^1$. It was found by Rains [15] that the eigenvalues of U^N are exactly distributed as N independent random points in $\mathbb{S}^1$; similar results hold for other compact Lie groups. In subsequent work [16], Rains found that in a sense, the eigenvalues of U^m become progressively more independent as m increases from 1 to N.

In this paper we quantify in a precise way the degree of uniformity of the eigenvalues of U^m when U is drawn uniformly from any of the classical compact groups $\mathbb{U}(N)$, $\mathbb{SU}(N)$, $\mathbb{O}(N)$, $\mathbb{SO}(N)$, and $\mathbb{Sp}(2N)$. We do this by bounding, for any $p \geq 1$, the mean and tails of the L_p-Wasserstein distance W_p between the empirical spectral measure $\mu_{N,m}$ of U^m and the uniform measure ν on $\mathbb{S}^1$ (see Section 4 for the definition of W_p). In particular, we show in Theorem 11 that

$$\mathbb{E}W_p(\mu_{N,m}, \nu) \leq Cp \frac{\sqrt{m\left[\log\left(\frac{N}{m}\right) + 1\right]}}{N}. \tag{1}$$

Theorem 13 gives a subgaussian tail bound for $W_p(\mu_{N,m}, \nu)$, which is used in Corollary 14 to conclude that if $m = m(N)$, then with probability 1, for all sufficiently large N,

$$W_p(\mu_{N,m}, \nu) \leq C p \frac{\sqrt{m \log(N)}}{N^{\min\{1, 1/2 + 1/p\}}}. \tag{2}$$

In the case $m = 1$ and $1 \leq p \leq 2$, (1) and (2) are optimal up to the logarithmic factor, since the W_p-distance from the uniform measure ν to *any* probability measure supported on N points is at least cN^{-1}. When $m = N$, Rains's theorem says that $\mu_{N,N}$ is the empirical measure of N independent uniformly distributed points in $\mathbb{S}^1$, for which the estimate in (1) of order $N^{-1/2}$ is optimal (cf. [6]). We conjecture that Theorem 11 and Corollary 14, which interpolate naturally between these extreme cases, are optimal up to logarithmic factors in their entire parameter space.

In the case that $m = 1$ and $p = 1$, these results improve the authors' earlier results in [14] (where $W_1(\mu_{N,1}, \nu)$ was bounded above by $CN^{-2/3}$) to what we conjectured there was the optimal rate; the results above are completely new for $m > 1$ or $p > 1$.

The proofs of our main results rest on three foundations: the fact that the eigenvalues of uniform random matrices are determinantal point processes, Rains's representation from [16] of the eigenvalues of powers of uniform random matrices, and logarithmic Sobolev inequalities. In Section 2, we combine some remarkable properties of determinantal point processes with Rains's results to show that the number of eigenvalues of U^m contained in an arc is distributed as a sum of independent Bernoulli random variables. In Section 3, we estimate the means and variances of these sums, again using the connection with determinantal point processes. In Section 4, we first generalize the method of Dallaporta [5] to derive bounds on mean Wasserstein distances from those data and prove Theorem 11. Then by combining Rains's results with tensorizable measure concentration properties which follow from logarithmic Sobolev inequalities, we prove Theorem 13 and Corollary 14. We give full details only for the case of $\mathbb{U}(N)$, deferring to Section 5 discussion of the modifications necessary for the other groups.

In order to carry out the approach above, we needed the sharp logarithmic Sobolev inequality on the full unitary group, rather than only on $\mathbb{SU}(N)$ as in [14]. It has been noted previously (e.g. in [10, 1]) that such a result is clearly desirable, but that because the Ricci tensor of $\mathbb{U}(N)$ is degenerate, the method of proof which works for $\mathbb{SU}(N)$, $\mathbb{O}(N)$, and $\mathbb{Sp}(2N)$ breaks down. In the appendix, we prove the logarithmic Sobolev inequality on $\mathbb{U}(N)$ with a constant of optimal order.

2 A miraculous representation of the eigenvalue counting function

As discussed in the introduction, a fact about the eigenvalue distributions of matrices from the compact classical groups which we use crucially is that they are determinantal point processes. For background on determinantal point processes the reader is referred to [11]. The basic definitions will not be repeated here since all that is needed for our purposes is the combination of Propositions 1 and 5 with Proposition 2 and Lemma 6 below. The connection between eigenvalues of random matrices and determinantal point processes has been known in the case of the unitary group at least since [8]. For the other groups, the earliest reference we know of is [12]. Although the language of determinantal point processes is not used in [12], Proposition 1 below is essentially a summary of [12, Section 5.2]. We first need some terminology.

Given an eigenvalue $e^{i\theta}$, $0 \leq \theta < 2\pi$, of a unitary matrix, we refer to θ as an eigenvalue angle of the matrix. Each matrix in $\mathbb{O}(2N + 1)$ has 1 as an eigenvalue, each matrix in $\mathbb{O}^-(2N + 1)$ has -1 as an eigenvalue, and each matrix in $\mathbb{O}^-(2N + 2)$ has

both -1 and 1 as eigenvalues; we refer to all of these as trivial eigenvalues. Here $\mathbb{O}^-(N) = \{U \in \mathbb{O}(N) : \det U = -1\}$, which is considered primarily as a technical tool in order to prove our main results for $\mathbb{O}(N)$. The remaining eigenvalues of matrices in $\mathbb{O}(N)$ or $\mathbb{Sp}(2N)$ occur in complex conjugate pairs. When discussing $\mathbb{O}(N)$, $\mathbb{O}^-(N)$, or $\mathbb{Sp}(2N)$, we refer to the eigenvalue angles corresponding to the nontrivial eigenvalues in the upper half-circle as nontrivial eigenvalue angles. For $\mathbb{U}(N)$, all the eigenvalue angles are considered nontrivial.

Proposition 1. *The nontrivial eigenvalue angles of uniformly distributed random matrices in any of $\mathbb{O}(N)$, $\mathbb{O}^-(N)$, $\mathbb{U}(N)$, $\mathbb{Sp}(2N)$ are a determinantal point process, with respect to uniform measure on Λ, with kernels as follows.*

	$K_N(x,y)$	Λ
$\mathbb{O}(2N)$	$1 + \displaystyle\sum_{j=1}^{N-1} 2\cos(jx)\cos(jy)$	$[0,\pi)$
$\mathbb{O}(2N+1), \mathbb{O}^-(2N+1)$	$\displaystyle\sum_{j=0}^{N-1} 2\sin\left(\frac{(2j+1)x}{2}\right)\sin\left(\frac{(2j+1)y}{2}\right)$	$[0,\pi)$
$\mathbb{U}(N)$	$\displaystyle\sum_{j=0}^{N-1} e^{ij(x-y)}$	$[0,2\pi)$
$\mathbb{Sp}(2N), \mathbb{O}^-(2N+2)$	$\displaystyle\sum_{j=1}^{N} 2\sin(jx)\sin(jy)$	$[0,\pi)$

Proposition 1 allows us to apply the following result from [11]; see also Corollary 4.2.24 of [1].

Proposition 2. *Let $K : \Lambda \times \Lambda \to \mathbb{C}$ be a kernel on a locally compact Polish space Λ such that the corresponding integral operator $\mathcal{K} : L^2(\mu) \to L^2(\mu)$ defined by*

$$\mathcal{K}(f)(x) = \int K(x,y)f(y) \, d\mu(y)$$

is self-adjoint, nonnegative, and locally trace-class with eigenvalues in $[0,1]$. For $D \subseteq \Lambda$ measurable, let $K_D(x,y) = \mathbb{1}_D(x)K(x,y)\mathbb{1}_D(y)$ be the restriction of K to D. Suppose that D is such that K_D is trace-class; denote by $\{\lambda_k\}_{k\in\mathcal{A}}$ the eigenvalues of the corresponding operator $\mathcal{K}_D$ on $L^2(D)$ ($\mathcal{A}$ may be finite or countable) and denote by $\mathcal{N}_D$ the number of particles of the determinantal point process with kernel K which lie in D. Then

$$\mathcal{N}_D \stackrel{d}{=} \sum_{k\in\mathcal{A}} \xi_k,$$

where "$\stackrel{d}{=}$" denotes equality in distribution and the ξ_k are independent Bernoulli random variables with $\mathbb{P}[\xi_k = 1] = \lambda_k$ and $\mathbb{P}[\xi_k = 0] = 1 - \lambda_k$.

In order to treat powers of uniform random matrices, we will make use of the following elegant result of Rains. For simplicity of exposition, we will restrict attention for now to the unitary group, and discuss in Section 5 the straightforward modifications needed to treat the other classical compact groups.

Proposition 3 (Rains, [16]). *Let $m \leq N$ be fixed. If $\sim$ denotes equality of eigenvalue distributions, then*

$$\mathbb{U}(N)^m \sim \bigoplus_{0 \leq j < m} \mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right).$$

That is, if U is a uniform $N \times N$ unitary matrix, the eigenvalues of U^m are distributed as those of m independent uniform unitary matrices of sizes $\left\lfloor \frac{N}{m} \right\rfloor := \max \left\{ k \in \mathbb{N} \mid k \leq \frac{N}{m} \right\}$ and $\left\lceil \frac{N}{m} \right\rceil := \min \left\{ k \in \mathbb{N} \mid k \geq \frac{N}{m} \right\}$, such that the sum of the sizes of the matrices is N.

Corollary 4. *Let $1 \leq m \leq N$, and let $U \in \mathbb{U}(n)$ be uniformly distributed. For $\theta \in [0, 2\pi)$, denote by $\mathcal{N}_\theta^{(m)}$ the number of eigenvalue angles of U^m which lie in $[0, \theta)$. Then $\mathcal{N}_\theta^{(m)}$ is equal in distribution to a sum of independent Bernoulli random variables. Consequently, for each $t > 0$,*

$$\mathbb{P}\left[\left| \mathcal{N}_\theta^{(m)} - \mathbb{E}\mathcal{N}_\theta^{(m)} \right| > t \right] \leq 2 \exp\left(-\min\left\{ \frac{t^2}{4\sigma^2}, \frac{t}{2} \right\} \right), \tag{3}$$

where $\sigma^2 = \operatorname{Var} \mathcal{N}_\theta^{(m)}$.

Proof. By Proposition 3, $\mathcal{N}_\theta^{(m)}$ is equal to the sum of m independent random variables X_i, $1 \leq i \leq m$, which count the number of eigenvalue angles of smaller-rank uniformly distributed unitary matrices which lie in the interval. Propositions 1 and 2 together imply that each X_i is equal in distribution to a sum of independent Bernoulli random variables, which completes the proof of the first claim. The inequality (3) then follows immediately from Bernstein's inequality [19, Lemma 2.7.1]. $\qquad \square$

3   Means and variances

In order to apply (3), it is necessary to estimate the mean and variance of the eigenvalue counting function $\mathcal{N}_\theta^{(m)}$. As in the proof of Corollary 4, this reduces by Proposition 3 to considering the case $m = 1$. Asymptotics for these quantities have been stated in the literature before, e.g. in [18], but not with the uniformity in θ which is needed below, so we indicate one approach to the proofs. A different approach yielding very precise asymptotics was carried out by Rains [15] for the unitary group; we use the approach outlined below because it generalizes easily to all of the other groups and cosets.

For this purpose we again make use of the fact that the eigenvalue distributions of these random matrices are determinantal point processes. It is more convenient for the variance estimates to use here an alternative representation to the one stated in Proposition 1 (which is more convenient for verifying the hypotheses of Proposition 2 and for the mean estimates). First define

$$S_N(x) := \begin{cases} \sin\left(\frac{Nx}{2} \right) / \sin\left(\frac{x}{2} \right) & \text{if } x \neq 0, \\ N & \text{if } x = 0. \end{cases}$$

The following result essentially summarizes [12, Section 5.4]. (Note that in the unitary case, the kernels given in Propositions 1 and 5 are not actually equal, but they generate the same process).

Proposition 5. *The nontrivial eigenvalue angles of uniformly distributed random matrices in any of $\mathbb{SO}(N)$, $\mathbb{SO}^-(N)$, $\mathbb{U}(N)$, $\mathbb{Sp}(2N)$ are a determinantal point process, with respect to uniform measure on Λ, with kernels as follows.*

	$K_N(x, y)$	Λ
$\$\mathbb{O}(2N)$	$\frac{1}{2}\Big(S_{2N-1}(x-y) + S_{2N-1}(x+y)\Big)$	$[0, \pi)$
$\$\mathbb{O}(2N+1), \$\mathbb{O}^-(2N+1)$	$\frac{1}{2}\Big(S_{2N}(x-y) - S_{2N}(x+y)\Big)$	$[0, \pi)$
$\mathbb{U}(N)$	$S_N(x-y)$	$[0, 2\pi)$
$\$\mathbb{p}(2N), \$\mathbb{O}^-(2N+2)$	$\frac{1}{2}\Big(S_{2N+1}(x-y) - S_{2N+1}(x+y)\Big)$	$[0, \pi)$

The following lemma is easy to check using Proposition 2. For the details of the variance expression, see [9, Appendix B].

Lemma 6. *Let $K : I \times I \to \mathbb{R}$ be a continuous kernel on an interval I representing an orthogonal projection operator on $L^2(\mu)$, where μ is the uniform measure on I. For a subinterval $D \subseteq I$, denote by $\mathcal{N}_D$ the number of particles of the determinantal point process with kernel K which lie in D. Then*

$$\mathbb{E}\mathcal{N}_D = \int_D K(x, x)\, d\mu(x)$$

and

$$\operatorname{Var}\mathcal{N}_D = \int_D \int_{I \setminus D} K(x, y)^2\, d\mu(x)\, d\mu(y).$$

Proposition 7. *1. Let U be uniform in $\mathbb{U}(N)$. For $\theta \in [0, 2\pi)$, let $\mathcal{N}_\theta$ be the number of eigenvalues angles of U in $[0, \theta)$. Then*

$$\mathbb{E}\mathcal{N}_\theta = \frac{N\theta}{2\pi}.$$

2. Let U be uniform in one of $\$\mathbb{O}(2N)$, $\$\mathbb{O}^-(2N+2)$, $\$\mathbb{O}(2N+1)$, $\$\mathbb{O}^-(2N+1)$, or $\$\mathbb{p}(2N)$. For $\theta \in [0, \pi)$, let $\mathcal{N}_\theta$ be the number of nontrivial eigenvalue angles of U in $[0, \theta)$. Then

$$\left|\mathbb{E}\mathcal{N}_\theta - \frac{N\theta}{\pi}\right| < 1.$$

Proof. The equality for the unitary group follows from symmetry considerations, or immediately from Proposition 5 and Lemma 6.

In the case of $\$\mathbb{p}(2N)$ or $\$\mathbb{O}^-(2N+2)$, by Proposition 1 and Lemma 6,

$$\mathbb{E}\mathcal{N}_\theta = \frac{1}{\pi}\int_0^\theta \sum_{j=1}^N 2\sin^2(jx)\, dx = \frac{N\theta}{\pi} - \frac{1}{2\pi}\sum_{j=1}^N \frac{\sin(2j\theta)}{j}.$$

Define $a_0 = 0$ and $a_j = \sum_{k=1}^j \sin(2k\theta)$. Then by summation by parts,

$$\sum_{j=1}^N \frac{\sin(2j\theta)}{j} = \frac{a_N}{N} + \sum_{j=1}^{N-1} \frac{a_j}{j(j+1)}.$$

Trivially, $|a_N| \leq N$. Now observe that

$$a_j = \operatorname{Im}\left[e^{2i\theta}\sum_{k=0}^{j-1} e^{2ik\theta}\right] = \operatorname{Im}\left[e^{2i\theta}\frac{e^{2ij\theta}-1}{e^{2i\theta}-1}\right] = \operatorname{Im}\left[e^{i(j+1)\theta}\frac{\sin(j\theta)}{\sin(\theta)}\right] = \frac{\sin((j+1)\theta)\sin(j\theta)}{\sin(\theta)}.$$

Since a_j is invariant under the substitution $\theta \mapsto \pi - \theta$, it suffices to assume that $0 < \theta \leq \pi/2$. In that case $\sin(\theta) \geq 2\theta/\pi$, and so

$$\sum_{j=1}^{N-1} \frac{|a_j|}{j(j+1)} \leq \frac{\pi}{2\theta} \left[\sum_{1 \leq j \leq 1/\theta} \theta^2 + \sum_{1/\theta < j \leq N-1} \frac{1}{j(j+1)} \right] \leq \frac{\pi}{2\theta}(\theta + \theta) = \pi.$$

All together,

$$\left| \mathbb{E}\mathcal{N}_\theta - \frac{N\theta}{\pi} \right| \leq \frac{1+\pi}{2\pi}.$$

The other cases are handled similarly. $\qquad\square$

As before, we restrict attention from now on to the unitary group, deferring discussion of the other cases to Section 5.

Proposition 8. *Let U be uniform in $\mathbb{U}(N)$. For $\theta \in [0, 2\pi)$, let $\mathcal{N}_\theta$ be the number of eigenvalue angles of U in $[0, \theta)$. Then*

$$\operatorname{Var}\mathcal{N}_\theta \leq \log N + 1.$$

Proof. If $\theta \in (\pi, 2\pi)$, then $\mathcal{N}_\theta \stackrel{d}{=} N - \mathcal{N}_{2\pi-\theta}$, and so it suffices to assume that $\theta \leq \pi$. By Proposition 5 and Lemma 6,

$$\operatorname{Var}\mathcal{N}_\theta = \frac{1}{4\pi^2} \int_0^\theta \int_\theta^{2\pi} S_N(x-y)^2 \, dx \, dy = \frac{1}{4\pi^2} \int_0^\theta \int_{\theta-y}^{2\pi-y} \frac{\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz \, dy$$

$$= \frac{1}{4\pi^2} \left[\int_0^\theta \frac{z\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz + \int_\theta^{2\pi-\theta} \frac{\theta\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz + \int_{2\pi-\theta}^{2\pi} \frac{(2\pi-z)\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz \right]$$

$$= \frac{1}{2\pi^2} \left[\int_0^\theta \frac{z\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz + \int_\theta^\pi \frac{\theta\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz \right].$$

For the first integral, since $\sin\left(\frac{z}{2}\right) \geq \frac{z}{\pi}$ for all $z \in [0, \theta]$, if $\theta > \frac{1}{N}$, then

$$\int_0^\theta \frac{z\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz \leq \int_0^{\frac{1}{N}} \frac{(\pi N)^2 z}{4} \, dz + \int_{\frac{1}{N}}^\theta \frac{\pi^2}{z} \, dz = \pi^2 \left(\frac{1}{8} + \log(N) + \log(\theta) \right).$$

If $\theta \leq \frac{1}{N}$, there is no need to break up the integral and one simply has the bound $\frac{(\pi N\theta)^2}{8} \leq \frac{\pi^2}{8}$. Similarly, if $\theta < \frac{1}{N}$, then

$$\int_\theta^\pi \frac{\theta\sin^2\left(\frac{Nz}{2}\right)}{\sin^2\left(\frac{z}{2}\right)} \, dz \leq \int_\theta^{\frac{1}{N}} \frac{\theta(\pi N)^2}{4} \, dz + \int_{\frac{1}{N}}^\pi \frac{\pi^2\theta}{z^2} \, dz$$

$$= \frac{\pi^2\theta N}{4}(1 - N\theta) + \pi^2 N\theta - \pi\theta \leq \frac{5\pi^2}{4};$$

if $\theta \geq \frac{1}{N}$, there is no need to break up the integral and one simply has a bound of π^2.

All together,

$$\operatorname{Var}\mathcal{N}_\theta \leq \log(N) + \frac{11}{16}. \qquad\square$$

Corollary 9. *Let U be uniform in $\mathbb{U}(N)$ and $1 \leq m \leq N$. For $\theta \in [0, 2\pi)$, let $\mathcal{N}_\theta^{(m)}$ be the number of eigenvalue angles of U^m in $[0, \theta)$. Then*

$$\mathbb{E}\mathcal{N}_\theta^{(m)} = \frac{N\theta}{2\pi} \qquad \text{and} \qquad \operatorname{Var}\mathcal{N}_\theta^{(m)} \leq m \left(\log\left(\frac{N}{m}\right) + 1 \right).$$

Proof. By Proposition 3, $\mathcal{N}_\theta^{(m)}$ is equal in distribution to the total number of eigenvalue angles in $[0, \theta)$ of each of $U_0 \ldots, U_{m-1}$, where $U_0, \ldots, U_{m-1}$ are independent and U_j is uniform in $\mathbb{U}\left(\left\lceil \frac{N-j}{m} \right\rceil\right)$; that is,

$$\mathcal{N}_\theta^{(m)} \overset{d}{=} \sum_{j=0}^{m-1} \mathcal{N}_{j,\theta},$$

where the $\mathcal{N}_{j,\theta}$ are the independent counting functions corresponding to $U_0, \ldots, U_{m-1}$.

The bounds in the corollary are thus automatic from Propositions 7 and 8. (Note that the N/m in the variance bound, as opposed to the more obvious $\lceil N/m \rceil$, follows from the concavity of the logarithm.) $\qquad\square$

4 Wasserstein distances

In this section we prove bounds and concentration inequalities for the spectral measures of fixed powers of uniform random unitary matrices. The method generalizes the approach taken in [5] to bound the distance of the spectral measure of the Gaussian unitary ensemble from the semicircle law.

Recall that for $p \geq 1$, the L_p-Wasserstein distance between two probability measures μ and ν on $\mathbb{C}$ is defined by

$$W_p(\mu, \nu) = \left(\inf_{\pi \in \Pi(\mu, \nu)} \int |w - z|^p \, d\pi(w, z) \right)^{1/p},$$

where $\Pi(\mu, \nu)$ is the set of all probability measures on $\mathbb{C} \times \mathbb{C}$ with marginals μ and ν.

Lemma 10. *Let $1 \leq m \leq N$ and let $U \in \mathbb{U}(N)$ be uniformly distributed. Denote by $e^{i\theta_j}$, $1 \leq j \leq N$, the eigenvalues of U^m, ordered so that $0 \leq \theta_1 \leq \cdots \leq \theta_N < 2\pi$. Then for each j and $u > 0$,*

$$\mathbb{P}\left[\left| \theta_j - \frac{2\pi j}{N} \right| > \frac{4\pi}{N} u \right] \leq 4 \exp\left[-\min\left\{ \frac{u^2}{m\left(\log\left(\frac{N}{m}\right) + 1\right)}, u \right\} \right]. \tag{4}$$

Proof. For each $1 \leq j \leq N$ and $u > 0$, if $j + 2u < N$ then

$$\mathbb{P}\left[\theta_j > \frac{2\pi j}{N} + \frac{4\pi}{N} u \right] = \mathbb{P}\left[\mathcal{N}_{\frac{2\pi(j+2u)}{N}}^{(m)} < j \right] = \mathbb{P}\left[j + 2u - \mathcal{N}_{\frac{2\pi(j+2u)}{N}}^{(m)} > 2u \right]$$

$$\leq \mathbb{P}\left[\left| \mathcal{N}_{\frac{2\pi(j+2u)}{N}}^{(m)} - \mathbb{E}\mathcal{N}_{\frac{2\pi(j+2u)}{N}}^{(m)} \right| > 2u \right].$$

If $j + 2u \geq N$ then

$$\mathbb{P}\left[\theta_j > \frac{2\pi j}{N} + \frac{4\pi}{N} u \right] = \mathbb{P}[\theta_j > 2\pi] = 0,$$

and the above inequality holds trivially. The probability that $\theta_j < \frac{2\pi j}{N} - \frac{4\pi}{N} u$ is bounded in the same way. Inequality (4) now follows from Corollaries 4 and 9. $\qquad\square$

Theorem 11. *Let $\mu_{N,m}$ be the spectral measure of U^m, where $1 \leq m \leq N$ and $U \in \mathbb{U}(N)$ is uniformly distributed, and let ν denote the uniform measure on $\mathbb{S}^1$. Then for each $p \geq 1$,*

$$\mathbb{E}W_p(\mu_{N,m}, \nu) \leq Cp \frac{\sqrt{m\left[\log\left(\frac{N}{m}\right) + 1\right]}}{N},$$

where $C > 0$ is an absolute constant.

Proof. Let θ_j be as in Lemma 10. Then by Fubini's theorem,

$$
\begin{aligned}
\mathbb{E}\left|\theta_j - \frac{2\pi j}{N}\right|^p &= \int_0^\infty pt^{p-1}\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{N}\right| > t\right]\,dt \\
&= \frac{(4\pi)^p p}{N^p}\int_0^\infty u^{p-1}\mathbb{P}\left[\left|\theta_j - \frac{2\pi j}{n}\right| > \frac{4\pi}{N}u\right]\,du \\
&\le \frac{4(4\pi)^p p}{N^p}\left[\int_0^\infty u^{p-1}e^{-u^2/m[\log(N/m)+1]}\,du + \int_0^\infty u^{p-1}e^{-u}\,du\right] \\
&= \frac{4(4\pi)^p}{N^p}\left[\left(m\left[\log\left(\frac{N}{m}\right)+1\right]\right)^{p/2}\Gamma\left(\frac{p}{2}+1\right) + \Gamma(p+1)\right] \\
&\le 8\Gamma(p+1)\left(\frac{4\pi}{N}\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}\right)^p.
\end{aligned}
$$

Observe that in particular,

$$
\operatorname{Var}\theta_j \le C\frac{m\left[\log\left(\frac{N}{m}\right)+1\right]}{N^2}.
$$

Let ν_N be the measure which puts mass $\frac{1}{N}$ at each of the points $e^{2\pi ij/N}$, $1 \le j \le N$. Then

$$
\begin{aligned}
\mathbb{E}W_p(\mu_{N,m},\nu_N)^p &\le \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N\left|e^{i\theta_j} - e^{2\pi ij/N}\right|^p\right] \le \mathbb{E}\left[\frac{1}{N}\sum_{j=1}^N\left|\theta_j - \frac{2\pi j}{N}\right|^p\right] \\
&\le 8\Gamma(p+1)\left(\frac{4\pi}{N}\sqrt{m\left[\log\left(\frac{N}{m}\right)+1\right]}\right)^p.
\end{aligned}
$$

It is easy to check that $W_p(\nu_N,\nu) \le \frac{\pi}{N}$, and thus

$$
\mathbb{E}W_p(\mu_{N,m},\nu) \le \mathbb{E}W_p(\mu_{N,m},\nu_N) + \frac{\pi}{N} \le (\mathbb{E}W_p(\mu_{N,m},\nu_N)^p)^{\frac{1}{p}} + \frac{\pi}{N}.
$$

Applying Stirling's formula to bound $\Gamma(p+1)^{\frac{1}{p}}$ completes the proof. $\qquad\square$

In the case that $m = 1$ and $p \le 2$, Theorem 11 could now be combined with Corollary 2.4 and Lemma 2.5 from [14] in order to obtain a sharp concentration inequality for $W_p(\mu_{N,1},\nu)$. However, for $m > 1$ we did not prove an analogous concentration inequality for $W_p(\mu_{N,m},\nu)$ because the main tool needed to carry out the approach taken in [14], specifically, a logarithmic Sobolev inequality on the full unitary group, was not available. The appendix to this paper contains the proof of the necessary logarithmic Sobolev inequality on the unitary group (Theorem 15) and the approach to concentration taken in [14], in combination with Proposition 3, can then be carried out in the present context.

The following lemma, which generalizes part of [14, Lemma 2.3], provides the necessary Lipschitz estimates for the functions to which the concentration property will be applied.

Lemma 12. *Let $p \ge 1$. The map $A \mapsto \mu_A$ taking an $N \times N$ normal matrix to its spectral measure is Lipschitz with constant $N^{-1/\max\{p,2\}}$ with respect to W_p. Thus if ρ is any fixed probability measure on $\mathbb{C}$, the map $A \mapsto W_p(\mu_A,\rho)$ is Lipschitz with constant $N^{-1/\max\{p,2\}}$.*

Proof. If A and B are $N \times N$ normal matrices, then the Hoffman–Wielandt inequality [3, Theorem VI.4.1] states that

$$\min_{\sigma \in \Sigma_N} \sum_{j=1}^{N} \left| \lambda_j(A) - \lambda_{\sigma(j)}(B) \right|^2 \leq \|A - B\|_{HS}^2, \tag{5}$$

where $\lambda_1(A), \ldots, \lambda_N(A)$ and $\lambda_1(B), \ldots, \lambda_N(B)$ are the eigenvalues (with multiplicity, in any order) of A and B respectively, and Σ_N is the group of permutations on N letters. Defining couplings of μ_A and μ_B given by

$$\pi_\sigma = \frac{1}{N} \sum_{j=1}^{N} \delta_{(\lambda_j(A), \lambda_{\sigma(j)}(B))}$$

for $\sigma \in \Sigma_N$, it follows from (5) that

$$W_p(\mu_A, \mu_B) \leq \min_{\sigma \in \Sigma_N} \left(\frac{1}{N} \sum_{j=1}^{N} \left| \lambda_j(A) - \lambda_{\sigma(j)}(B) \right|^p \right)^{1/p}$$

$$\leq N^{-1/\max\{p,2\}} \min_{\sigma \in \Sigma_N} \left(\sum_{j=1}^{N} \left| \lambda_j(A) - \lambda_{\sigma(j)}(B) \right|^2 \right)^{1/2}$$

$$\leq N^{-1/\max\{p,2\}} \|A - B\|_{HS}. \qquad \square$$

Theorem 13. *Let $\mu_{N,m}$ be the empirical spectral measure of U^m, where $U \in \mathbb{U}(N)$ is uniformly distributed and $1 \leq m \leq N$, and let ν denote the uniform probability measure on $\mathbb{S}^1$. Then for each $t > 0$,*

$$\mathbb{P}\left[W_p(\mu_{N,m}, \nu) \geq C \frac{\sqrt{m \left[\log\left(\frac{N}{m} \right) + 1 \right]}}{N} + t \right] \leq \exp\left[-\frac{N^2 t^2}{24m} \right]$$

for $1 \leq p \leq 2$ and

$$\mathbb{P}\left[W_p(\mu_{N,m}, \nu) \geq Cp \frac{\sqrt{m \left[\log\left(\frac{N}{m} \right) + 1 \right]}}{N} + t \right] \leq \exp\left[-\frac{N^{1+2/p} t^2}{24m} \right]$$

for $p > 2$, where $C > 0$ is an absolute constant.

Proof. By Proposition 3, $\mu_{N,m}$ is equal in distribution to the spectral measure of a block-diagonal $N \times N$ random matrix $U_1 \oplus \cdots \oplus U_m$, where the U_j are independent and uniform in $\mathbb{U}\left(\left\lfloor \frac{N}{m} \right\rfloor \right)$ and $\mathbb{U}\left(\left\lceil \frac{N}{m} \right\rceil \right)$. Identify $\mu_{N,m}$ with this measure and define the function $F(U_1, \ldots, U_m) = W_p(\mu_{U_1 \oplus \cdots \oplus U_m}, \nu)$; the preceding discussion means that if $U_1, \ldots, U_m$ are independent and uniform in $\mathbb{U}\left(\left\lfloor \frac{N}{m} \right\rfloor \right)$ and $\mathbb{U}\left(\left\lceil \frac{N}{m} \right\rceil \right)$ as necessary, then $F(U_1, \ldots, U_m) \overset{d}{=} W_p(\mu_{N,m}, \nu)$.

Applying the concentration inequality in Corollary 17 of the appendix to the function F gives that

$$\mathbb{P}\left[F(U_1, \ldots, U_m) \geq \mathbb{E}F(U_1, \ldots, U_m) + t \right] \leq e^{-Nt^2/24mL^2},$$

where L is the Lipschitz constant of F, and we have used the trivial estimate $\left\lfloor \frac{N}{m} \right\rfloor \geq \frac{N}{2m}$. Inserting the estimate of $\mathbb{E}F(U_1, \ldots, U_m)$ from Theorem 11 and the Lipschitz estimates of Lemma 12 completes the proof. $\qquad \square$

Corollary 14. *Suppose that for each N, $U_N \in \mathbb{U}(N)$ is uniformly distributed and $1 \leq m_N \leq N$. Let ν denote the uniform measure on $\mathbb{S}^1$. There is an absolute constant C such that given $p \geq 1$, with probability 1, for all sufficiently large N,*

$$W_p(\mu_{N,m_N}, \nu) \leq C \frac{\sqrt{m_N \log(N)}}{N}$$

if $1 \leq p \leq 2$ and

$$W_p(\mu_{N,m_N}, \nu) \leq Cp \frac{\sqrt{m_N \log(N)}}{N^{\frac{1}{2}+\frac{1}{p}}}$$

if $p > 2$.

Proof. In Theorem 13 let $t_N = 5 \frac{\sqrt{m_N \log(N)}}{N}$ for $p \in [1,2]$ and $t_N = 5 \frac{\sqrt{m_N \log(N)}}{N^{\frac{1}{2}+\frac{1}{p}}}$ for $p > 2$, and apply the Borel–Cantelli lemma. $\square$

We observe that Corollary 14 makes no assumption about any joint distribution of the matrices $\{U_N\}_{N \in \mathbb{N}}$; in particular, they need not be independent.

As a final note, Rains's Proposition 3 above shows that, in the case $m = N$, $\mu_{N,m}$ is the empirical measure of N i.i.d. points on $\mathbb{S}^1$. By another result of Rains [15], the same is true when $m > N$. In particular, in all the above results the restriction $m \leq N$ may be removed if m is simply replaced by $\min\{m, N\}$ in the conclusion.

5 Other groups

The approach taken above can be completed in essentially the same way for $\mathbb{SO}(N)$, $\mathbb{SO}^-(N)$ and $\mathbb{Sp}(2N)$, so that all the results above hold in those cases as well, with only the precise values of constants changed.

In [16], Rains proved that the eigenvalue distributions for these groups (or rather, components, in the case of $\mathbb{SO}^-(N)$) can be decomposed similarly to the decomposition described in Proposition 3, although the decompositions are more complicated in those cases (mostly because of parity issues). The crucial fact, though, is that the decomposition is still in terms of independent copies of smaller-rank (orthogonal) groups and cosets. This allows for the representation of the eigenvalue counting function in all cases as a sum of independent Bernoulli random variables (allowing for the application of Bernstein's inequality) and as a sum of independent copies of eigenvalue counting functions for smaller-rank groups. In particular the analogue of Corollary 4 holds and it suffices to estimate the means and variances in the case $m = 1$. The analogue of Proposition 8 for the other groups can be proved similarly using Proposition 5 and Lemma 6.

With those tools and Proposition 7 on hand, the analogue of Theorem 11 can be proved in the same way, with a minor twist. One can bound as in the proof of Theorem 11 the distance between the empirical measure associated to the nontrivial eigenvalues and the uniform measure on the upper-half circle. Since the nontrivial eigenvalues occur in complex conjugate pairs and there are at most two trivial eigenvalues, one gets essentially the same bound for the distance between the empirical spectral measure and the uniform measure on the whole circle.

Finally, logarithmic Sobolev inequalities — and hence concentration results analogous to Corollary 17 — for the other groups are already known via the Bakry–Émery criterion, cf. [1, Section 4.4], so that the analogue of Theorem 13 follows as for the unitary group.

For the special unitary group $\mathbb{SU}(N)$, all the results stated above hold exactly as stated for the full unitary group, cf. the proof of [14, Lemma 2.5]. Analogous results

for the full orthogonal group $\mathbb{O}(N)$ follow from the results for $\mathbb{SO}(N)$ and $\mathbb{SO}^-(N)$ by conditioning on the determinant, cf. the proofs of Theorem 2.6 and Corollary 2.7 in [14].

Appendix: the log-Sobolev constant of the unitary group

In this section we prove a logarithmic Sobolev inequality for the unitary group with a constant of optimal order. As a consequence, we obtain a sharp concentration inequality, independent of k, for functions of k independent unitary random matrices.

Recall the following general definitions for a metric space (X, d) equipped with a Borel probability measure μ. The entropy of a measurable function $f : X \to [0, \infty)$ with respect to μ is

$$\mathrm{Ent}_\mu(f) := \int f \log(f) \, d\mu - \left(\int f \, d\mu \right) \log \left(\int f \, d\mu \right).$$

For a locally Lipschitz function $g : X \to \mathbb{R}$,

$$|\nabla g|\,(x) := \limsup_{y \to x} \frac{|g(y) - g(x)|}{d(y, x)}.$$

We say that (X, d, μ) satisfies a logarithmic Sobolev inequality (or log-Sobolev inequality for short) with constant $C > 0$ if, for every locally Lipschitz $f : X \to \mathbb{R}$,

$$\mathrm{Ent}_\mu(f^2) \le 2C \int |\nabla f|^2 \, d\mu. \tag{6}$$

Theorem 15. *The unitary group $\mathbb{U}(N)$, equipped with its uniform probability measure and the Hilbert–Schmidt metric, satisfies a logarithmic Sobolev inequality with constant $6/N$.*

If the Riemannian structure on $\mathbb{U}(N)$ is the one induced by the usual Hilbert–Schmidt inner product on matrix space, then the geodesic distance is bounded above by $\pi/2$ times the Hilbert–Schmidt distance on $\mathbb{U}(N)$ (see e.g. [4, Lemma 3.9.1]). Thus Theorem 15 implies that $\mathbb{U}(N)$ equipped with the geodesic distance also satisfies a log-Sobolev inequality, with constant $3\pi^2/2N$.

It is already known that every compact Riemannian manifold, equipped with the normalized Riemannian volume measure and geodesic distance, satisfies a log-Sobolev inequality with some finite constant [17]. For applications like those in this paper to a sequence of manifolds such as $\{\mathbb{U}(N)\}_{N=1}^\infty$, however, the order of the constant as N grows is crucial. The constant in Theorem 15 is best possible up to a constant factor; this can be seen, for example, from the fact that one can recover the sharp concentration of measure phenomenon on the sphere from Corollary 17 below.

The key to the proof of Theorem 15 is the following representation of uniform measure on the unitary group.

Lemma 16. *Let θ be uniformly distributed in $\left[0, \frac{2\pi}{N}\right]$ and let $V \in \mathbb{SU}(N)$ be uniformly distributed, with θ and V independent. Then $e^{i\theta}V$ is uniformly distributed in $\mathbb{U}(N)$.*

Proof. Let X be uniformly distributed in $[0, 1)$, K uniformly distributed in $\{0, \dots, N-1\}$, and V uniformly distributed in $\mathbb{SU}(N)$ with (X, K, V) independent. Consider

$$U = e^{2\pi i X/N} e^{2\pi i K/N} V.$$

On one hand, it is easy to see that $(X + K)$ is uniformly distributed in $[0, N]$, so that $e^{2\pi i(X+K)/N}$ is uniformly distributed on $\mathbb{S}^1$. Thus $U \stackrel{d}{=} \omega V$ for ω uniform in $\mathbb{S}^1$

and independent of V. It is then straightforward to see that $U \in \mathbb{U}(N)$ is uniformly distributed (cf. the proof of [14, Lemma 2.5]).

On the other hand, if I_N is the $N \times N$ identity matrix, then $e^{2\pi i K/N} I_N \in \mathbb{SU}(N)$. By the translation invariance of uniform measure on $\mathbb{SU}(N)$ this implies that $e^{2\pi i K/N} V \stackrel{d}{=} V$, and so $e^{2\pi i X/N} V \stackrel{d}{=} U$. $\qquad\square$

Proof of Theorem 15. First, for the interval $[0, 2\pi]$ equipped with its standard metric and uniform measure, the optimal constant in (6) for functions f with $f(0) = f(2\pi)$ is known to be 1, see e.g. [20]. This fact completes the proof — with a better constant than stated above — in the case $N = 1$, since $\mathbb{U}(1) = \mathbb{S}^1$; we assume from now on that $N \geq 2$. By reflection, the optimal constant for general locally Lipschitz functions on $[0, \pi]$ is also 1. It follows by a scaling argument that the optimal logarithmic Sobolev constant on $\left[0, \frac{\pi\sqrt{2}}{\sqrt{N}}\right)$ is $2/N$.

By the Bakry–Émery Ricci curvature criterion [2], $\mathbb{SU}(N)$ satisfies a log-Sobolev inequality with constant $2/N$ when equipped with its geodesic distance, and hence also when equipped with the Hilbert–Schmidt metric (see Section 4.4 and Appendix F of [1]). By the tensorization property of log-Sobolev inequalities in Euclidean spaces (see [13, Corollary 5.7]), the product space $\left[0, \frac{\pi\sqrt{2}}{\sqrt{N}}\right) \times \mathbb{SU}(N)$, equipped with the L_2-sum metric, satisfies a log-Sobolev inequality with constant $2/N$ as well.

Define the map $F : \left[0, \frac{\pi\sqrt{2}}{\sqrt{n}}\right) \times \mathbb{SU}(N) \to \mathbb{U}(N)$ by $F(t, V) = e^{\sqrt{2}it/\sqrt{N}}V$. By Lemma 16, the push-forward via F of the product of uniform measure on $\left[0, \frac{\pi\sqrt{2}}{\sqrt{N}}\right)$ with uniform measure on $\mathbb{SU}(N)$ is uniform measure on $\mathbb{U}(N)$. Moreover, this map is $\sqrt{3}$-Lipschitz:

$$
\begin{aligned}
\left\| e^{\sqrt{2}it_1/\sqrt{N}}V_1 - e^{\sqrt{2}it_2/\sqrt{N}}V_2 \right\|_{HS} &\leq \left\| e^{\sqrt{2}it_1/\sqrt{N}}V_1 - e^{\sqrt{2}it_1/\sqrt{N}}V_2 \right\|_{HS} \\
&\quad + \left\| e^{\sqrt{2}it_1/\sqrt{N}}V_2 - e^{\sqrt{2}it_2/\sqrt{N}}V_2 \right\|_{HS} \\
&= \|V_1 - V_2\|_{HS} + \left\| e^{\sqrt{2}it_1/\sqrt{N}}I_N - e^{\sqrt{2}it_2/\sqrt{N}}I_N \right\|_{HS} \\
&\leq \|V_1 - V_2\|_{HS} + \sqrt{2}\,|t_1 - t_2| \\
&\leq \sqrt{3}\sqrt{\|V_1 - V_2\|_{HS}^2 + |t_1 - t_2|^2}.
\end{aligned}
$$

Since the map F is $\sqrt{3}$-Lipschitz, its image $\mathbb{U}(N)$ with the (uniform) image measure satisfies a logarithmic Sobolev inequality with constant $(\sqrt{3})^2 \frac{2}{N} = \frac{6}{N}$. $\qquad\square$

Corollary 17. *Given $N_1, \ldots, N_k \in \mathbb{N}$, denote by $M = \mathbb{U}(N_1) \times \cdots \mathbb{U}(N_k)$ equipped with the L_2-sum of Hilbert–Schmidt metrics. Suppose that $F : M \to \mathbb{R}$ is L-Lipschitz, and that $\{U_j \in \mathbb{U}(N_j) : 1 \leq j \leq k\}$ are independent, uniform random unitary matrices. Then for each $t > 0$,*

$$
\mathbb{P}\left[F(U_1, \ldots, U_k) \geq \mathbb{E}F(U_1, \ldots, U_k) + t \right] \leq e^{-Nt^2/12L^2},
$$

where $N = \min\{N_1, \ldots, N_k\}$.

Proof. By Theorem 15 and the tensorization property of log-Sobolev inequalities [13, Corollary 5.7], M satisfies a log-Sobolev inequality with constant $6/N$. The stated concentration inequality then follows from the Herbst argument (see, e.g., [13], Theorem 5.3). $\qquad\square$

The lack of dependence on k is a crucial feature of the inequality in Corollary 17; unlike logarithmic Sobolev inequalities, concentration inequalities themselves do not tensorize without introducing a dependence on k.

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