Electron. Commun. Probab. **18** (2013), no. 1, 1–11.
DOI: 10.1214/ECP.v18-2525
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Anomalous heat kernel behaviour for the dynamic random conductance model

Stephen Buckley*

Abstract

We introduce the time dynamic random conductance model and consider the heat kernel for the random walk on this environment. In the case where conductances are bounded above, an example environment is presented which exhibits heat kernel decay that is asymptotically slower than in the well studied time homogeneous case - being close to $O\left(n^{-1}\right)$ as opposed to $O\left(n^{-2}\right)$. The example environment given is a modification of an environment introduced in [4]

Keywords: heat kernel; random walk in random environment; random conductances.
AMS MSC 2010: 60G50; 60J27; 05C81.
Submitted to ECP on October 28, 2011, final version accepted on November 28, 2012.

1 Introduction and results

Random walk amongst random conductances has been well studied over recent years. Formally, take a weighted graph on the square lattice $G(\omega) = \left(\mathbb{Z}^d, \mathbb{E}^d, (\omega_e)_{e \in \mathbb{E}^d}\right)$ with symmetry condition $\omega_{xy} = \omega_{yx}$ and edge weights independent and identically distributed. The random walk on $G(\omega)$ is the Markov process $(X_n)_{n \geq 0}$ with transition probabilities

$$P^\omega\left(X_n = y \,|X_{n-1} = x\right) = \frac{\omega_{xy}}{\pi(x)}, \pi(x) := \sum_{z \sim x} \omega_{xz}, \tag{1.1}$$

where we write $x \sim z$ if and only if x and z are neighbours in $\mathbb{Z}^d$. The associated heat kernel is then

$$q_n(x, y) = \frac{P^\omega\left(X_n = y \,|X_0 = x\right)}{\pi(x)}.$$

Various regimes for the conductances have been considered. If $\omega \in [a, b]$ for $0 < a < b < \infty$ almost surely, then [11] provides uniform upper and lower Gaussian bounds for q_n, with a quenched invariance principle proven in [19]. When $\omega \in [1, \infty)$, the tail behaviour of ω determines whether Brownian motion or a Fractional-Kinetics process is the correct scaling limit [3].

This paper concerns an extension to the model where $\omega \in [0, 1]$ and hence we detail known results in this case. Heat kernel decay is considered in several papers ([4], [5], [7], [8]). The following theorem is taken from [4]:

*University of Oxford, UK. E-mail: stephenpbuckley@gmail.com

Theorem 1.1 (Berger et al (2008)). *Let $d \geq 2$ and consider a collection $\omega = (\omega_b)$ of iid conductances in $[0,1]$ with $\mathbb{P}(\omega_b > 0) > p_c(d)$. For almost every $\omega \in \{0 \in \mathcal{C}_\infty\}$, there is $C = C(\omega) < \infty$ such that*

$$P_\omega^n(0,0) \leq C(\omega) \begin{cases} n^{-d/2} & d = 2,3, \\ n^{-2}\log n & d = 4, \\ n^{-2} & d \geq 5, \end{cases}$$

for all $n \geq 1$.

The same paper and [5] prove that these are the best general upper bounds available by presenting examples whose heat kernel displays corresponding lower bounds.

We will consider the dynamic random conductance model. That is, take a time inhomogeneous weighted graph

$$(G_t)_{t\in K} = \left(\mathbb{Z}^d, \mathbb{E}^d, \omega = (\omega_e(t))_{\substack{e\in\mathbb{E}^d \\ t\in K}}\right)$$

for K either $\mathbb{R}$ or $\mathbb{N}$. For any edge $e \in \mathbb{E}^d$, the edge weights $(\omega_e(t))_{t\in K}$ are taken to be iid Markov processes with unique invariant probability distribution μ. Further, take $(\omega_e(0))_{e\in\mathbb{E}^d}$ to be equal in distribution to $\mu^{\mathbb{E}^d}$. As in the time homogeneous case, the symmetry condition $\omega_{xy}(t) = \omega_{yx}(t)$ is assumed for all edges and times.

The natural extension to (1.1) would be the random walk $(X_n)_{n\geq 0}$ with transition probabilities

$$P^\omega(X_n = y \,|X_{n-1} = x) = \frac{\omega_{xy}(n-1)}{\pi_{n-1}(x)}, \pi_{n-1}(x) := \sum_{z\sim x} \omega_{xz}(n-1).$$

This walk presents several technical challenges: if one looks for an invariant measure for the walk considered on the space-time graph then the measure does not generally have a simple form and the walk is not generally reversible. Further, this invariant space-time measure is not temporally consistent - for a given environment, the projection of the invariant measure at time t onto $\mathbb{Z}^d$ will in general be different from the projection at time $s \neq t$. This restricts the tools available with which to analyze the walk.

With this in mind we introduce the variable speed random walk in continuous time. This is the Markov process $(X_t)_{t\geq 0}$ with generator at time t

$$\mathcal{L}_t f(x) = \sum_{y\sim x} \omega_{xy}(t)(f(y) - f(x)). \tag{1.2}$$

The flat measure $\pi(x) = 1$ for all $x \in \mathbb{Z}^d$, is the invariant measure for the variable speed walk with

$$\langle \mathcal{L}_t f, g\rangle_\pi = \langle f, \mathcal{L}_t g\rangle_\pi \tag{1.3}$$

for all $t \in \mathbb{R}$. Although this does not imply $P(X_t = y \,|X_s = x) = P(X_t = x \,|X_s = y)$ the fact that $\mathcal{L}_t$ is self-adjoint can be important.

One would perhaps expect that the environment evolving over time would reduce the time that the walk spends in locally anomalous regions due to the time dynamic removing these regions before the random walk can spend a large quantity of time in them. This heuristic would suggest that the heat kernel behaviour should be no worse than in the time homogeneous case. Intriguingly it is this dynamic - the disappearance of anomalous regions - that leads to a change in the heat kernel. Anomalous heat kernel decay in the static case is due to the walk becoming "trapped" close to the origin so that when the walk escapes the trap it is much closer to the origin than would normally be expected. However, the random walk must pay a price to enter and exit the trap - a price of $O\left(n^{-1}\right)$ for both entrance and exit leading to the $O\left(n^{-2}\right)$ return probabilities stated above. The key idea that we present is that it is possible to choose a dynamic environment where the traps persist for long enough to trap the walk for a good length of time - so that the walk is much closer to the origin than would be expected - but then the trap disappears leaving the walk unimpeded to return to the origin. As the walk only has to pay to enter the trap and not to exit we show that lower heat kernel bounds close to $O\left(n^{-1}\right)$ can be achieved.

The main result of the paper is the following, to be proved in Section 2.

Theorem 1.2. *Take $\kappa > \frac{1}{d}$ and $d \geq 3$. There exists a law on environments $\mathbb{P}$ with iid edge marginals that evolve in an ergodic Markov fashion with edge weights bounded by one such that for almost all $\omega \in \Omega$ there exists a constant $C(\omega) > 0$ and a sequence $(n_i(\omega))$ with $\lim_{i\to\infty} n_i = \infty$ such that*

$$P^{\omega}_{(0,0)}\left(X_{n_i} = 0\right) \geq \frac{C(\omega)\, e^{-(\log n_i)^{\kappa}}}{n_i}$$

for all i, where X_n is the variable speed continuous time random walk on ω.

The statement of the theorem is very similar to Theorem 2.2 of [4]. This is deliberate. The environment that we present is a modification of an environment presented in that paper.

Note that Theorem 1.2 implies that the dynamic heat kernel has at least three extrema: when the environment is strongly mixing the walk resembles the walk on the annealed graph and hence has heat kernel bounds of order $n^{-d/2}$; when the environment is highly persistent the walk is close to the walk on the static graph and hence the heat kernel is bounded above by order n^{-2}; in between these two cases sit the environments we have outlined with heat kernel lower bounds of order n^{-1}.

No corresponding upper bound is presented. It is natural to ask whether the walk could also enter a trap for free by being at the correct site when the trap forms. It will become clear that there is a trade-off between persistence of traps and their frequency of occurrence. As we wish the traps to be persistent so that the walk remains trapped for long time periods the traps cannot occur frequently and thus the walk is highly unlikely to be at the trap site when the trap forms. We comment further on this open problem at the end of the paper.

Dynamic random environment is not a new topic. There are several papers that prove both annealed and quenched central limit theorems under various conditions, frequently without the symmetry condition $\omega_{xy} = \omega_{yx}$ (for example [6], [2] and [16]). These papers generally show that if the walk is uniformly elliptic and the environment is well mixing in time then the rescaled process converges to non-degenerate Brownian motion. In terms of heat kernel estimates, there is not a huge amount in the literature. The Appendix of [12] proves full off-diagonal results in the case where weights are bounded away from zero and infinity. The paper [10] proves that if

$$m(x) := \sum_{y} \omega_{xy}(t)$$

is independent of t and the environment satisfies both a uniform ellipticity condition and uniform Sobolev inequality then standard off-diagonal upper bounds and on diagonal lower bounds hold. One can in fact extend this result in the spirit of [14] to prove standard on-diagonal upper bounds under assumptions of asymptotic isoperimetric dimension and ergodicity of the spatial environment over time [9], although this is not proved here. There are also recent results in the case where space is taken to be finite, these can be found in [17] and [18].

2 Heat kernel lower bounds

We take a space-time environment that only switches at discrete time points. This choice of discrete environment does lead to a somewhat peculiar hybrid pair as the walk is in continuous time. The choice does, however, make the combinatorics easier and removes issues relating to exceptional times. Take weights ω_e to be supported on $\{2^{-n} : n \geq 0\}$ and let the transition probabilities for the Markov chain $(\omega_e(n))_{n\in\mathbb{Z}}$ be:

$$\begin{aligned} K\left(1, 2^{-n}\right) &= s_n \\ K\left(2^{-n}, 1\right) &= p_n \\ K\left(2^{-n}, 2^{-n}\right) &= 1 - p_n \\ K\left(2^{-n}, 2^{-m}\right) &= 0 \text{ for } n, m \neq 0 \text{ and } m \neq n, \end{aligned}$$

where $p_n \in (0,1)$ and $\sum_{n\geq 0} s_n = 1$. Then $\omega_e(n)$ has an invariant distribution if and only if the Markov chain is positive recurrent. Hence, the invariant distribution, μ, exists if and only if

$$\sum_{n>0} \frac{s_n}{p_n} < \infty. \tag{2.1}$$

Assume this to be the case, then for $n \geq 1$ the invariant distribution satisfies

$$\mu\left(2^{-n}\right) = \mu(1) \frac{s_n}{p_n}.$$

We define our space-time environment to be $\omega = (\omega_e(n))_{\substack{e\in\mathbb{E}^d \\ n\in\mathbb{N}^{\geq 0}}}$ with ω being iid in space with $\mathbb{P}\left(\omega_e(0) = 2^{-k}\right) = \mu\left(2^{-k}\right)$. Assume for the moment that s_n and p_n are chosen such that $\mu(1) > p_c(d)$ and hence for every $t \in \mathbb{R}$, the bonds of unit conductance percolate in G_t.

The traps that we consider are of the form shown in Figure 1. They consist of the following. At time zero there is a strong spatial path (made of bonds of unit conductance) connecting the origin to a vertex x. x is connected to y by a weak bond of strength 2^{-n} and all other bonds are of lesser conductance. The trap, without necessarily the path to the origin, remains in place until some time T_n (to be chosen later) at which point ω_{xy} switches to unit conductance. At time T_n there again exists a strong spatial path to the origin.

If such a trap exists, we obtain a lower bound on $P^{\omega}_{(0,0)}(X_{T_n+1} = 0)$ by conditioning on the walk moving directly to x within one unit of time, jumping from x to y in one unit of time, not jumping from y until time T_n and then proceeding directly back to the origin within one unit of time:

$$\begin{aligned} &P^{\omega}_{(0,0)}(X_{T_n+1} = 0) \\ &\geq P^{\omega}_{(0,0)}(X_1 = x)\, P^{\omega}_{(1,x)}(X_i = y \text{ for all } i \in [2, T_n])\, P^{\omega}_{(T_n,x)}(X_{T_n+1} = 0) \\ &\geq c e^{-c_1\left(l_n^1 \log l_n^1 + l_n^2 \log l_n^2\right)} 2^{-n} c_2, \end{aligned} \tag{2.2}$$

where l_n^1 and l_n^2 are the graph distances between the origin and x at times 0 and T_n respectively. In the final line the 2^{-n} is the cost to the walk of crossing the bond $\omega_{xy}(1)$, the exponential terms come from the lower bounds on the probability of the walk moving a large distance in a short time proven in [15] (and given in more detail in the proof of Theorem 1.2 below), and we have assumed that $T_n \leq \frac{1}{2d} 2^n$ and hence the walk does not jump from y between time 2 and T_n with probability bounded independently of n. If we can take $l_n^i = O(\log n)$ and $T_n = O(2^n)$ then we would see

$$P^{\omega}_{(0,0)}\left(X_{2^n} = (2^n, 0)\right) \geq f(n)\, 2^{-n}$$

for some function f that decays slower than 2^{-n}.

In order to prove the theorem we require some combinatorial estimates on the strong paths that will connect the traps to the origin. For $t \in \mathbb{R}$ write $x \leftrightarrow y$ at time t if there exists a nearest neighbour path from x to y consisting purely of bonds of unit conductance in the graph G_t. As $\mu(1) > p_c(d)$ there exists a unique infinite cluster at each $t \in \mathbb{R}$. Call this $\mathcal{C}_\infty(t)$. The following proposition encapsulates the results required.

Proposition 2.1. *Suppose $\mu(1) > p_c(d)$. For any $x \in \mathbb{Z}^d$ and $k \geq 1$ set $B_x[k] := x + [-k, k]^d$ and define the events*

$$\begin{aligned} C_0(k, x) &= \{x \leftrightarrow \partial B_x[k] \text{ at time } 0\} \\ \text{and } C_m(k, x) &= \{x \leftrightarrow \partial B_x[k] \text{ at time } m\}. \end{aligned}$$

There exist constants $c_1, M_1 > 0$ such that for $m \geq M_1$

$$\mathbb{P}\left(C_0(k, x) \cap C_m(k, x)\right) \geq c_1$$

for all $k \geq 1$.

Further, for $t \in \mathbb{R}$ define $D_t(n)$ to be the event that every connected component contained in $B_0[n] \cap G_t$ of size at least $c_2 (\log n)^2$ is connected to $\mathcal{C}_\infty(t)$. Then there exists $c_2 > 0$ such that for almost every $\omega \in \Omega$ there exists $N(\omega) < \infty$ such that $D_0(n) \cap D_n(n)$ occurs for all $n \geq N(\omega)$.

Proof. For the first claim we use the mixing properties of the Markov chain on edge weights. We have to be careful as if $B_x[k]$ contains edges with very light conductance then these edges can take a long time to mix. To avoid this problem we delete all light edges and percolate on what remains of the box.

Let

$$\varepsilon := \frac{\mu(1) - p_c(d)}{4}$$

and take M such that

$$\sum_{n=M}^{\infty} \mu\left(2^{-n}\right) < \varepsilon.$$

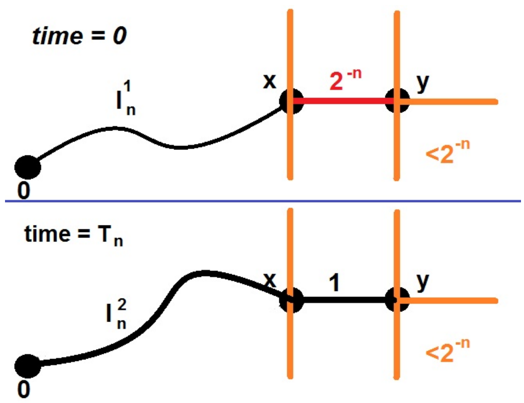

Figure 1: A space time trap

Define $E_M := \left\{e \in B_x[k] : \omega_e(0) \leq 2^{-M}\right\}$, the set of all light edges - we will throw these edges away as we cannot control their mixing properties.

Take M_1 sufficiently large so that for all $m \geq M_1$

$$\mathbb{P}\left(\omega_e(m) = 1 \,\middle|\, \omega_e(0) = 2^{-n}\right) > p_c(d) + 2\varepsilon$$

for all $n < M$. M_1 exists since the Markov chain on edge weights is irreducible, aperiodic, positive recurrent, n is bounded and $p_c(d) + 2\varepsilon < \mu(1)$ through the choice of ε.

For $i \in \{1,2\}$, take $\left(\tilde{\omega}^i_e\right)_{e\in\mathbb{Z}^d}$ to be independent bond percolation realizations with

$$\mathbb{P}\left(\tilde{\omega}^i_e = 1\right) = p_c(d) + 2\varepsilon.$$

Define $\left(\bar{\omega}^1_e(m)\right)_{e\in\mathbb{Z}^d}$, $\left(\bar{\omega}^2_e(m)\right)_{e\in\mathbb{Z}^d}$ to be independent environments with law

$$\bar{\omega}^i_e(m) = \begin{cases} 0 & \text{if } e \in E_M \\ \tilde{\omega}^i_e & \text{otherwise} \end{cases}.$$

The laws of $\omega(0)$ and $\omega(m)$ conditioned on E_M, are stochastically dominated by the conditioned laws of $\bar{\omega}^1(m)$ and $\bar{\omega}^2(m)$ respectively. Hence, on a suitably extended probability space, the event $\bar{C}_0(k,x) := \left\{x \leftrightarrow \partial B_x[k] \text{ in } \bar{\omega}^1_e(m)\right\}$ conditioned on E_M is dominated by $C_0(k,x)$ conditioned on E_M and the event $\bar{C}_m(k,x) := \{x \leftrightarrow \partial B_x[k] \text{ in } \bar{\omega}_e(m)\}$ conditioned on E_M is dominated by $C_m(k,x)$ conditioned on E_M. Hence

$$\begin{aligned}
\mathbb{P}\left(C_0(k,x) \cap C_m(k,x)\right) &= \sum_E \mathbb{P}\left(C_0(k,x) \cap C_m(k,x) \,|E_M = E\right)\mathbb{P}\left(E_M = E\right) \\
&\geq \sum_E \mathbb{P}\left(\bar{C}_0(k,x) \cap \bar{C}_m(k,x) \,|E_M = E\right)\mathbb{P}\left(E_M = E\right) \\
&= \sum_E \mathbb{P}\left(\bar{C}_m(k,x) \,|E_M = E\right)^2 \mathbb{P}\left(E_M = E\right) \\
&\geq c,
\end{aligned}$$

where the final line follows from the definition of E_M and standard percolation estimates.

The second claim is straight forward to verify via standard percolation arguments. Consider first the event $D_i(n)$ for $i \in \{0,n\}$. In either case this corresponds to static percolation and hence Theorem 8.65 of [13] gives upper bounds on $\mathbb{P}\left(D_i(n)^c\right)$ that are independent of i and summable over n. Now,

$$\begin{aligned}
\mathbb{P}\left(D_0(n) \cap D_n(n)^c\right) &= \mathbb{P}\left(D_0(n)^c \cup D_n(n)^c\right) \\
&\leq \mathbb{P}\left(D_0(n)^c\right) + \mathbb{P}\left(D_n(n)^c\right)
\end{aligned}$$

and hence is also summable over n. The Borel-Cantelli Lemma ensures that the event $D_0(n) \cap D_n(n)$ happens only finitely often with probability one. ☐

Proof of Theorem 1.2. Take $\varepsilon > 0$. For $n > 0$ set $l_n := n^{(1+(2d+1)\varepsilon)/d}$ and $T_n = 2^n$. For $n \in \mathbb{N}$ choose $s_n = c2^{-n}n^{-(1+\varepsilon)}$, $p_n = 2^{-n}$ so that $\mu\left(2^{-n}\right) = cn^{-(1+\varepsilon)}$ for $n \geq 1$ and $\mu(1) = 1 - \sum_{n\geq 1} cn^{-1+\varepsilon}$. Choosing c sufficiently small ensures that $\mu(1) > p_c(d)$.

For a fixed point $x \in \mathbb{Z}^d$, let $y = x + e_1$ and $A_n(x)$ be the event that:

- In the spatial environment at time zero, G_0, x is connected to the boundary of the spatial box of side $c_2\left(\log l_n\right)^2$ centred at x by a path of unit conductors.
- $\omega_{xy}(i) = 2^{-n}$ for $i \in [0, T_n - 1]$, $\omega_{xy}(T_n) = 1$,

- $\omega_{yz}(i) \leq 2^{-n}$ for $i \in [0, T_n]$, $z \neq x$,
- x is connected to the boundary of the spatial box of side $c_2 (\log l_n)^2$ centred at x by a path of unit conductors in the spatial environment G_{T_n}.

Proposition 2.1 holds with the events C_i modified to ensure that the paths connecting x to $\partial B_x[k]$ avoid using the edges emanating from y. Call these modified events $\tilde{C}_i$. Then

$$
\begin{aligned}
\mathbb{P}(A_n(x)) &= \mathbb{P}\left(\tilde{C}_0\left(c_2 (\log l_n)^2, x\right) \cap \tilde{C}_{T_n}\left(c_2 (\log l_n)^2, x\right)\right) \\
&\quad \times \mathbb{P}\left(\omega_{xy}(i) = 2^{-n} \text{ for } i \in [0, T_n - 1), \omega_{xy}(T_n) = 1\right) \\
&\quad \times \mathbb{P}\left(\omega_e(i) \leq 2^{-n} \text{ for } i \in [0, T_n]\right)^{2d-1} \\
&\geq c_1 \mu\left(2^{-n}\right)(1 - p_n)^{T_n} p_n \left(\sum_{i \geq n} \mu\left(2^{-i}\right)\right)^{2d-1} \\
&\geq c_4 n^{-1-2d\varepsilon},
\end{aligned}
$$

by the choice of p_n.

Taking $\mathbb{G}_n$ to be a grid of sites in $[-l_n, l_n]^d \cap \mathbb{Z}^d$ that are spaced by distance $2(\log l_n)^2$. The events $\{A_n(x) : x \in \mathbb{G}_n\}$ are independent, so using $1 - x \leq e^{-x}$ for $x \in [0, 1]$,

$$
\mathbb{P}\left(\bigcap_{x \in \mathbb{G}_n} A_n(x)^c\right) \leq \exp\left\{-c_5 \left(\frac{l_n}{(\log l_n)^2}\right)^d n^{-[1+2d\varepsilon]}\right\} \leq e^{-cn^{\varepsilon}},
$$

hence by Borel-Cantelli, the intersection occurs for only finitely many n.

By Proposition 2.1, every connected component of diameter at least $(\log l_n)^2$ in $[-l_n, l_n]^d \cap G_i$ will be connected to the largest component of $[-2l_n, 2l_n]^d \cap G_i$ for $i \in \{0, T_n\}$ and all large enough n. Now, the origin at time zero will not necessarily be connected to this largest component. Take z to be the closest point to the origin that lies on the infinite component at time zero. On the event $A_n(x)$ for n large, take l_n^1 to be the shortest path from the origin to x in G_0 that goes from 0 to z and then follows a strong path to x.

Take n_i to be a subsequence such that there is a strong path from x to 0 at time T_{n_i} of length l_n^2 with l_n^2 bounded by $c_9 l_n$. Such a subsequence (and constant c_9) exist by [1]. The results of [1] also imply that $l_n^1 \leq c_9 l_n$ for all large enough n. We take a subsequence as we then avoid the complication of the origin being surrounded by many weak bonds as such a situation would make it difficult for the walk to return to the origin.

It is shown in [15] that for a one dimensional walk the following lower bound holds: there exist constants c_i such that for any $x, y \in \mathbb{Z}^d$ and $d(x, y) \geq t \geq 1$

$$
P_x(X_t = y) \geq c_6 \exp\left(-c_7 d(x, y)\left(1 + \log d(x, y)/t\right)\right). \tag{2.3}
$$

We wish to bound the probability that the walk travels fully along a strong path of length l_n^i in a unit of time. As we can bound the probability that the walk deviates from this one dimensional path from below by $e^{-c_8 l_n^1}$, we can condition so that the walk only sees the one dimensional path and hence

$$
P^{\omega}_{(x,T_n)}\left(X_{T_n+1} = 0\right) \geq c_6 \exp\left(-c_{10} l_n^2 \left(1 + \log d l_n^2/t\right)\right) \exp\left(-c_8 l_n^2\right).
$$

Similarly for the initial strong path from z to x, with a constant dependent on the local environment around the origin replacing c_6.

Plugging this into (2.2) we obtain

$$
\begin{aligned}
P^{\omega}_{(0,0)}\left(X_{2^{n_i}+1}=0\right) &\geq C(\omega)\,2^{-n_i}\exp\left(-cn_i^{(1+(2d+1)\varepsilon)/d}\log n\right)\\
&\geq C(\omega)\,2^{-n_i}\exp\left(-c'n_i^{(1+(3d+1)\varepsilon)/d}\right),
\end{aligned}
$$

taking ε small concludes the proof. □

With this example of ω supported on $\{2^{-n} : n \in \mathbb{N}\}$ in mind we can demonstrate three distinct behaviours for the heat kernel for the dynamic random walk. Let $G_t(\omega)$ be the environment at time t for $\omega \in \Omega$. For $m > 0$ define the dynamic graph $\mathcal{G}_t^m := G_{tm}$, that is the graph speeded up so that edges flip m times per unit of time. Let X_t^m be the space-time random walk on $\mathcal{G}_t^m$ started at $(0,0)$. Theorem 1.2 proves that if $m = 1$ we have a lower bound close to $O\left(t^{-1}\right)$. As we let m tend to zero and infinity then we have two further distinct behaviours.

Proposition 2.2. *For almost all $\omega \in \Omega$ there exist a constant $C(\omega)$ such that for all $t > 0$ we have the limits:*

$$
\begin{aligned}
\frac{C(\omega)}{t^2} &\geq \lim_{m\to 0} P\left(X_t^m = 0\,|X_0^m = 0\right) \geq C(\omega)\,\frac{e^{-(\log t)^{\kappa}}}{t^2} \text{ for } d \geq 5\\
C_1 t^{-d/2} &\geq \lim_{m\to\infty} P\left(X_t^m = 0\,|X_0^m = 0\right) \geq C_2 t^{-d/2} \text{ for } d \geq 1.
\end{aligned}
$$

Proof. The first line is due to [4], as $m \to 0$ corresponds to the static case.

As m gets large the probability that the walk crosses an edge in time t tends towards the probability that the random walk crosses an edge of conductance $\mathbb{E}(\omega_e)$ in time t. Hence the limit as $m \to \infty$ corresponds to the annealed random walk. This is the simple random walk on $\mathbb{Z}^d$ with speed $2d\mathbb{E}(\omega_e)$ and hence exhibits standard on diagonal heat kernel behaviour. □

We have been slightly disingenuous when suggesting that Theorem 1.2 shows behaviour more anomalous than presented in [4] as we are considering a different model - the variable speed walk as opposed to the constant speed walk investigated in [4]. In the constant speed case the above trap is not a trap at all as the transition rates are normalized by $\sum_{y\sim x}\omega_{xy}$ so that the walk always moves at unit speed. There are thus two obvious questions: what is the behaviour of the variable speed walk in the static case and what is the behaviour of the constant speed walk in the dynamic case?

We begin with the first question. The trapping demonstrated above will lead to lower bounds close to $O\left(n^{-2}\right)$ in the static case (the n^{-2} is due to the walk now having to pay a price of $O\left(n^{-1}\right)$ to exit the trap as well as to enter). The upper bounds follow from slight modifications to the arguments of [4], with summations replaced by integrals.

The answer to the second question is presented in Proposition 2.3 below where lower bounds close to $O\left(n^{-1}\right)$ are again proven. The example that displays these lower bounds is a very similar example of space-time trapping in the constant speed case - again the environment changes at discrete time points with the walk being a continuous time Markov process.

Proposition 2.3. *Take $d \geq 3$. For any $\alpha > 0$ and $\kappa > \frac{1}{d}$, there exist non-static random space-time environments of the above form such that for almost every $\omega \in \Omega$ there exists $C(\omega) > 0$ and an increasing sequence $(n_i)_{i\geq 0}$ such that $\lim_{i\to\infty} n_i = \infty$ and for all i*

$$
P_{\omega}^{n_i}(0,0) \geq C(\omega)\,\frac{e^{-(\log n_i)^{\kappa}}}{n_i^{1+\alpha}}.
$$

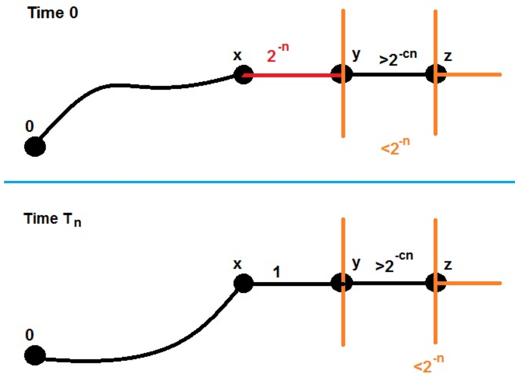

Figure 2: A space time trap for the constant speed walk

Proof. The proof is very similar to above. We will simply outline the type of traps that lead to this behaviour.

Figure 2 demonstrates the types of trap we consider around a point x. Take $y = x+e_1$ and $z = y + 2e_1$. We initially take the bond between y and z to be of weight 1, with the bond between x and y being of weight 2^{-n} and all other bonds emanating from y and z being of weight $\omega_e \leq 2^{-n}$. As time evolves all the weak bonds remain at their initial value. The strong bond will weaken but will never be weaker than 2^{-cn} for some constant c that we can take to be as small as we like. At time T_n the bond between x and y switches to unit weight. We condition on there existing strong paths between 0 and x in the spatial environments G_0 and G_{T_n}.

Take r_n to be the length of the space-time path from 0 to x at time zero and r_n' to be the length from x to 0 at time T_n. As in equation (2.2) above, if $T_n = O\left(2^{cn}\right)$ we have

$$\begin{aligned} P^{\omega}_{(0,0)}\left(X_1 = x\right) &\geq c_1 \exp\left(-c_2 r_n \left(1+\log r_n\right)\right) \\ P^{1}_{(1,x)}\left(X_1 = y\right) &\geq c_1 2^{-n} \\ P_{(2,y)}\left(\text{stay on } yz \text{ for } T_n\right) &\geq C \\ P_{r_n'}\left(x,0\right) &\geq c_1 \exp\left(-c_2 r_n' \left(1+\log r_n'\right)\right) \end{aligned}$$

and hence

$$P^{\omega}_{(0,0)}\left(X_{T_n+1} = 0\right) \geq c_2 e^{-c_3\left(r_n(1+\log r_n)+r_n'\left(1+\log r_n'\right)\right)} 2^{-n}.$$

The details are similar to the proof of Theorem 1.2. □

Note that the traps introduced in the proofs of Theorem 1.2 and Proposition 2.3 would also be traps if one considered the discrete time random walks on these environments. However, as lower bounds of the form (2.3) can never hold in discrete time, one would have to prove an exponential lower bound on the probability of moving straight from the origin to the trap and then back from the trap to the origin. One method for doing this would be to prove the existence of strong space-time paths from the origin to the trap and back again - that is, paths in space-time consisting purely of edges of unit weight that if followed would take the walk directly from the origin to the trap and vice-versa. The requisite combinatorics are beyond this paper.

We conclude by remarking on whether or not this lower bound is sharp. If the trapping strategy detailed above is the dominant strategy, then an upper bound of the same order will hold. The strategy employed above is similar to the strategy that maximizes

$\sup_y P^t(0,y)$ in the time homogeneous case - as the dominating strategy only pays to enter a trap and not to leave. Such a link between the time homogenous and time inhomogeneous case is appealing. However, one must also consider the possibility that this is not the dominating strategy in the time inhomogeneous case.

It is possible for the walk to escape from a trap for free, is it also possible for the walk to be at a trap site when the trap forms and hence enter the trap for free too? As we assume the existence of an invariant distribution for the Markov chain on edge weights, there is a trade-off between persistence of a trap and how often a trap can form. In the particular example discussed above, this is crystallized in equation (2.1). In particular this ensures that if we wish a trap to persist for time $O(n)$ then the trap is unlikely to form logarithmically close to the origin with respect to the space-time distance. This somewhat compromises the lower bound calculations of equation (2.2), as crudely bounding the probability that the walk moves from the origin to the space-time point where the trap forms by the negative exponential of the corresponding distance will lead to lower bounds that are smaller than those presented in Theorem 1.2 due to the distance being large.

Although we have found no strategy to obtaining a more anomalous lower bound than presented, we have no proof that such a bound fails to exist.

References

[1] P. Antal and A. Pisztora. On the chemical distance for supercritical Bernoulli percolation. Ann. Probab. 24 (1996), no. 2, 1036–1048. MR-1404543

[2] A. Bandyopadhyay and O. Zeitouni. Random Walk in Dynamic Markovian Random Environment. ALEA 1 (2006), 205–224. MR-2249655

[3] M.T. Barlow and J. Černý. Convergence to fractional kinetics for random walks associated with unbounded conductances. Probab. Theory Related Fields. 149 (2011), no. 3-4, 639-673. MR-2776627

[4] N. Berger, M. Biskup, C.E. Hoffman and G. Kozma. Anomalous heat kernel decay for random walk among bounded random conductances. Ann. Inst. Henri Poincaré 44 (2008), no. 2, 374–392. MR-2446329

[5] M. Biskup and O. Boukhadra. Subdiffusive heat kernel decay in four-dimensional iid random conductance models. arXiv:1010.5542v1

[6] C. Boldrighini, R. A. Minlos, and A. Pellegrinotti. Almost-sure central limit theorem for a Markov model of random walk in dynamical random environment. Probab. Theory Related Fields, 109(2):245-273, 1997. MR-1477651

[7] O. Boukhadra. Heat kernel estimates for random walk among random conductances with heavy tail. Stochastic Processes and their Applications 120 (2010), no. 2, 182-194. MR-2576886

[8] O. Boukhadra. Standard spectral dimension for the polynomial lower tail random conductances model. Electron. J. Probab. 15 (2010), no. 68, 2069–2086. MR-2745726

[9] S. Buckley. Problems in Random Walks in Random Environments. DPhil thesis, University of Oxford (2011).

[10] T. Coulhon, A. Grigor'yan and F. Zucca. The discrete integral maximum principle and its applications. Tohoku Math. J. (2) 57 (2005), no. 4, 559–587. MR-2203547

[11] T. Delmotte. Parabolic Harnack inequality and estimates of Markov chains on graphs. Rev. Mat. Iberoamericana 15 (1999) 181–232. MR-1681641

[12] G. Giacomin, S. Olla and H. Spohn. Equilibrium fluctuations for $\nabla\phi$ interface model. Ann. Probab. 29 (2001), 1138-1172. MR-1872740

[13] G.R. Grimmett. Percolation. Second edition. Grundlehren der Mathematischen Wissenschaften, 321. Springer-Verlag, Berlin, 1999. MR-1707339

[14] P. Mathieu and E. Remy. Isoperimetry and Heat Kernel Decay on Percolation Clusters. Ann. Probab. 32 (2004), no. 1, 100-128. MR-2040777

[15] M.M.H Pang. Heat kernels of graphs. J. London Math. Soc. (2) 47 (1993), no. 1, 50–64. MR-1200977

[16] F. Rassoul-Agha and T. Seppäläinen. An almost sure invariance principle for random walks in a space-time random environment. Probab. Theory Related Fields 133 (2005), no. 3, 299–314. MR-2198014

[17] L. Saloff-Coste and J. Zúñiga. Merging for time inhomogeneous finite Markov chains. I. Singular values and stability. Electron. J. Probab. 14 (2009), 1456–1494. MR-2519527

[18] L. Saloff-Coste and J. Zúñiga. Merging for time inhomogeneous finite Markov chains. II. Nash and log-Sobolev inequalities. Ann. Probab. 39 (2011), 1161–1203. MR-2789587

[19] V. Sidoravicius and A-S. Sznitman. Quenched invariance principles for walks on clusters of percolation or among random conductances. Probab. Theory Related Fields 129 (2004), no. 2, 219–244. MR-2063376

Acknowledgments. The author wishes to thank Ben Hambly for the advice and direction that lead to this work.