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On maximizing the speed of a random walk in fixed environments

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Abstract

We consider a random walk in a fixed $\mathbb{Z}$ environment composed of two point types: q-drifts (in which the probabiliy to move to the right is q, and $1-q$ to the left) and p-drifts, where $\frac{1}{2} < q < p$. We study the expected hitting time of a random walk at N given the number of p-drifts in the interval $[1, N-1]$, and find that this time is minimized asymptotically by equally spaced p-drifts.

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1 Introduction

Procaccia and Rosenthal [1] studied the maximal speed of a nearest neighbor random walk in a fixed $\mathbb{Z}$ environment, consisting of points from two types. The first type gives equal probability of moving left or right, and the second type, whose density is bounded by λ, gives probability p to move to the right and $1-p$ to the left, where $p > \frac{1}{2}$. In the finite case, the placement of a given number of p-drifts on an interval which minimizes the expected crossing time is calculated. They ask about extending their results to environments on $\mathbb{Z}$ composed of two point types: q-drifts and p-drifts, for $\frac{1}{2} < q < p \leq 1$. The goal of our work is to do so for the finite environment. See [1] for background and further related work.

Consider a nearest neighbor random walk on $0, 1, ..., N$ denoted by $\{X_n\}_{n=0}^{\infty}$ with reflection at the origin. We denote the transition law by $\omega : \{0, 1, ..., N\} \to [0, 1]$. More formally this means that for all $i \in \{0, 1, ..., N\}$:

$$\begin{aligned} P(X_{n+1} = i+1 | X_n = i) &= \omega(i) \\ P(X_{n+1} = i-1 | X_n = i) &= 1 - \omega(i) \,. \end{aligned}$$

The reflection at the origin means that $\omega(0) = 1$.

First, we prove the following proposition concerning the expected hitting time at the vertex N, in a similar way to some results in [2]:

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Proposition 1.1. *For a walk with transition law ω starting at any point $0 \le x \le N$, the hitting time $T_N = \min\{n \ge 0 | X_n = N\}$ satisfies:*

$$E_\omega^x(T_N) = N - x + 2\sum_{i=x}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k,$$

where $\rho_i = \frac{1-\omega(i)}{\omega(i)}$, and $E_\omega^x(T_N)$ stands for the expected hitting time in the environment ω starting from the vertex x. In particular:

$$E_\omega^0(T_N) = N + 2\sum_{i=1}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k.$$

The last proposition gives the following corollary:

Corollary 1.2. *The expected hitting time from 0 to N is symmetric under reflection of the environment, i.e., taking the environment $\omega' : \{0, 1, ..., N\} \to [0, 1]$ defined by:*

$$\omega'(i) = \begin{cases} \omega(N-i) & 1 \le i \le N \\ 0 & i = 0 \end{cases}$$

gives $E_{\omega'}^0(T_N) = E_\omega^0(T_N)$.

Next we turn to the case of an environment consisting of two types of vertices, q-drifts (a vertex i for which $\omega(i) = q$) and p-drifts (a vertex i for which $\omega(i) = p$), for some $\frac{1}{2} < q < p \le 1$. For a set $L \subseteq \{1, \ldots, N\}$ of size $k = |L|$ we define the environment ω_L as:

$$\forall 0 \le x \le N \; \omega_L(x) = \begin{cases} 1 & x = 0 \\ p & x \in L \\ q & x \notin L \cup \{0\} \end{cases}.$$

In [1], the exact formula for $E_\omega^0(T_N)$ was calculated for all choices of L and $q = \frac{1}{2}$, and for sufficiently large N (while keeping the drift density $\frac{k}{N}$ constant) it is approximately minimized by equaly spaced p-drifts. In this paper we extend this result for $q > \frac{1}{2}$. For given N and k, we define an environment $\omega_{\mathcal{L}(N,k)}$ in which the p-drifts are equally spaced (up to integer effects):

$$\mathcal{L}(N,k) = \left\{ \left\lfloor i \cdot \frac{N-1}{k} \right\rfloor, 1 \le i \le k \right\}$$

and prove the following theorem:

Theorem 1.3. *For every $\varepsilon > 0$ there exists n_0 such that for every $N > n_0$ and every set $L \subseteq \{1, \ldots, N\}$:*

$$\frac{E_{\omega_L}^0(T_N)}{N} > \frac{E_{\omega_{\mathcal{L}(N,k)}}^0(T_N)}{N} - \varepsilon,$$

where $k = |L|$.

Finally, we consider the set of environments $\omega_{\mathcal{L}(ak+1,k)}$ for $a \in \mathbb{N}$, and calculate $\lim\limits_{k\to\infty} \frac{E_{\omega_{\mathcal{L}(ak+1,k)}}^0(T_{ak+1})}{ak+1}$. In these calculations, as well as in the proof of Theorem 1.3, it is convinient to use the notation $\alpha = \frac{1-q}{q}$ and $\beta = \frac{1-p}{p}$.

Proposition 1.4. *Let $a \in \mathbb{N}$. Then:*

$$\lim_{k\to\infty} \frac{E^0_{\omega_{\mathcal{L}(ak+1,k)}}(T_{ak+1})}{ak+1} = 1 + \frac{2}{a} \cdot \left(\frac{\alpha^{a+1} - a\alpha^2 + (a-1)\alpha}{(1-\alpha)^2} + \frac{\beta(1-\alpha^a)^2}{(1-\alpha)^2(1-\beta\alpha^{a-1})} \right).$$

2 Proof of the main theorem

Proof of Proposition 1.1 . Let us define $v_x = E^x_\omega(T_N)$ for $0 \le x \le N$. By conditioning on the first step:

1. $v_N = 0$
2. $v_0 = v_1 + 1$
3. $v_x = \omega(x)v_{x+1} + (1-\omega(x))v_{x-1} + 1 \qquad 1 \le x \le N-1.$

To solve these equations, define $a_x = v_x - v_{x-1}$ (for $1 \le x \le N$) and $b_x = v_{x+1} - v_{x-1}$ (for $1 \le x \le N-1$). Then $\forall x \in \{1,\dots,N-1\}$:

$$\begin{aligned} b_x &= a_x + a_{x+1} \\ a_x &= \omega(x) b_x + 1 \\ a_1 &= -1. \end{aligned}$$

Thus a_x satisfies the relation $a_{x+1} = \rho_x a_x - \rho_x - 1$, whose solution is $a_x = -2\sum_{j=1}^{x-1}\prod_{k=j}^{x-1}\rho_k - 1$, and thus:

$$\begin{aligned} v_x &= \sum_{i=x+1}^{N}(v_{i-1} - v_i) + v_N \\ &= \sum_{i=x+1}^{N}(-a_i) + v_N \\ &= N - x + 2\sum_{i=x}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k. \end{aligned}$$

Finally, for $x = 0$:

$$v_0 = N + 2\sum_{i=1}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k,$$

since for $i = 0$ the inner sum is empty. □

Definition 2.1. *For $N \in \mathbb{N}$ denote:*

$$S_N = \sum_{i=1}^{N-1}\sum_{j=1}^{i}\prod_{k=j}^{i}\rho_k = \sum_{d=1}^{N-1}\sum_{j=1}^{N-d}\prod_{k=j}^{j+d-1}\rho_k.$$

In order to estimate S_N, we compare it to a similar sum on a circle. We glue the vertices 0 and $N-1$, and then sum over subintervals of the circle $\mathbb{Z}_{N-1}$, rather then summing over subinterval of the segment $[1, N-1]$.

More formally, extend ρ such that $\rho_k = \rho_{k-N+1}$ for $k \geq N$ (also setting ρ_N to be equal ρ_1). Then consider the following sum:

$$\widetilde{S}_N = \sum_{d=1}^{N-1} \sum_{j=1}^{N-1} \prod_{k=j}^{j+d-1} \rho_k.$$

Note that both S_N and $\widetilde{S}_N$ depend on the environment ω_L, so when necessary we shall use the explicit notations S_N^L and $\widetilde{S}_N^L$.

Proposition 2.2. *There exists a constant $C = C(\alpha)$ such that for every environment ω_L:*

$$\left|\widetilde{S}_N - S_N\right| \leq C(\alpha).$$

Proof. Since $\alpha = \frac{1-q}{q}$, $\beta = \frac{1-p}{p}$, and $\frac{1}{2} < q < p \leq 1$, $0 \leq \beta < \alpha < 1$, we get:

$$\begin{aligned} \left|\widetilde{S}_N - S_N\right| &= \sum_{d=1}^{N-1} \sum_{j=N-d+1}^{N-1} \prod_{k=j}^{j+d-1} \rho_k \\ &\leq \sum_{d=1}^{N-1} d\alpha^d \\ &\leq \sum_{d=1}^{\infty} d\alpha^d = C(\alpha). \end{aligned}$$

□

Definition 2.3. *Let $n_i^{(d)}$ be the number of p-drifts in the interval $[i, i+d-1]$, i.e., $n_i^{(d)} = |[i, i+d-1] \cap L|$.*

Since every drift appears in d intervals of length d, $\sum_{i=1}^{N-1} n_i^{(d)} = dk$, where $k = |L|$. In addition,

$$\begin{aligned} \widetilde{S}_N &= \sum_{d=1}^{N-1} \sum_{i=1}^{N-1} \left(\frac{\beta}{\alpha}\right)^{n_i^{(d)}} \cdot \alpha^d \\ &= \sum_{d=1}^{N-1} \sigma_d, \end{aligned}$$

where $\sigma_d = \sum_{i=1}^{N-1} \left(\frac{\beta}{\alpha}\right)^{n_i^{(d)}} \cdot \alpha^d$.

In the following claim we fix d, and see under which conditions σ_d is minimal. After fixing d, σ depends only on the vector $\mathbf{n}^{(d)} = \left(n_1^{(d)}, ..., n_{N-1}^{(d)}\right)$.

Definition 2.4. *We say that a vector $\mathbf{n} = (n_1, ..., n_{N-1}) \in \mathbb{N}^{N-1}$ is **almost constant** if there exists $a \in \mathbb{N}$ such that $n_i \in \{a, a+1\}$ for every $1 \leq i \leq N-1$.*

Claim 2.5. *Consider $\sigma_d(\mathbf{n})$ for $\mathbf{n} \in \mathbb{N}^{N-1}$, under the restriction $\sum_{i=1}^{N-1} n_i = dk$, and let $\mathbf{m} \in \mathbb{N}^{N-1}$ be an almost constant vector. Then $\mathbf{m}$ minimizes σ_d, i.e., for every $\mathbf{n} \in \mathbb{N}^{N-1}$ such that $\sum_{i=1}^{N-1} n_i = dk$, $\sigma_d(\mathbf{m}) \leq \sigma_d(\mathbf{n})$.*

Proof. For convenience, we omit d from the notation, and always assume that the domain of σ is the set of vectors in $\mathbb{N}^{N-1}$ that satisfy the restriction $\sum_{i=1}^{N-1} n_i = dk$.

We will first show that $\sigma(\mathbf{n})$ achieves its minimum for some almost constant vector $\mathbf{n}$. Secondly, we show that the value of σ on all almost constant vectors is the same, and this will complete the proof.

Let $M \subseteq \mathbb{N}^{N-1}$ be the set of vectors satisfying $\sum_{l=1}^{N-1} n_l = dk$ that minimize σ, and assume by contradiction that M doesn't contain an almost constant vector. Choose $\mathbf{m} \in M$ such that $\sum_{l=1}^{N-1} (m_l)^2$ is minimal. $\mathbf{m}$ is not almost constant, so there exist i, j for which $m_i - m_j \geq 2$, since if the difference between the maximal component of $\mathbf{m}$ and its minimal component were less than 2, it would be almost constant. Consider the vector $\mathbf{m}'$:

$$m'_l = \begin{cases} m_l & l \neq i, j \\ m_l - 1 & l = i \\ m_l + 1 & l = j \end{cases} .$$

$\mathbf{m}'$ satisfies the restriction $\sum_{l=1}^{N-1} n_l = dk$, and $\sigma(\mathbf{m}) \geq \sigma(\mathbf{m}')$:

$$\begin{aligned} \sigma(\mathbf{m}) - \sigma(\mathbf{m}') &= \sum_{t=1}^{N-1} \left(\frac{\beta}{\alpha}\right)^{m_t} \cdot \alpha^d - \sum_{t=1}^{N-1} \left(\frac{\beta}{\alpha}\right)^{m'_t} \cdot \alpha^d \\ &= \alpha^d \left(\left(\frac{\beta}{\alpha}\right)^{m_i} + \left(\frac{\beta}{\alpha}\right)^{m_j} - \left(\frac{\beta}{\alpha}\right)^{m_i - 1} - \left(\frac{\beta}{\alpha}\right)^{m_j + 1} \right) \\ &= \alpha^d \left(1 - \frac{\beta}{\alpha}\right) \left(\left(\frac{\beta}{\alpha}\right)^{m_j} - \left(\frac{\beta}{\alpha}\right)^{m_i - 1} \right) \\ &\geq 0, \end{aligned}$$

where the inequality follows from the fact that $0 \leq \frac{\beta}{\alpha} < 1$ and $m_j < m_i - 1$ from the assumption. Due to the minimality of $\sigma(\mathbf{m})$, $\sigma(\mathbf{m}')$ must also be minimal. But:

$$\begin{aligned} \sum_{l=1}^{N-1} (m_l)^2 - \sum_{l=1}^{N-1} (m'_l)^2 &= (m_i)^2 + (m_j)^2 - (m'_i)^2 - (m'_j)^2 \\ &= (m_i)^2 + (m_j)^2 - (m_i - 1)^2 - (m_j + 1)^2 \\ &= 2(m_i - m_j) - 2 \\ &\geq 2, \end{aligned}$$

which contradicts the minimality of $\sum_{l=1}^{N-1} (m_l)^2$. Therefore M must contain an almost constant vector.

Next, consider a general almost constant vector $\mathbf{n}$. Set $a = \min\{n_l : 1 \leq l \leq N-1\}$ the minimal component of $\mathbf{n}$. No component of $\mathbf{n}$ is greater then $a+1$, therefore $n_l \in \{a, a+1\}$. Defining m_0 to be the number of a's in $\mathbf{n}$ and $m_1 = N - 1 - m_0$ to be the number of $a+1$'s, we get:

$$\begin{aligned} dk &= \sum_{l=1}^{N-1} n_l \\ &= m_0 a + m_1 (a+1) \\ &= (m_0 + m_1) a + m_1 \\ &= (N-1) a + m_1. \end{aligned}$$

Since $m_1 < N-1$, there is a unique solution to the last equation for natural a, m_1. Hence, all almost constant vectors (satisfying the restriction) are the same up to re-ordering, and since $\sigma(\mathbf{n}) = \sum_{i=1}^{N-1} \left(\frac{\beta}{\alpha}\right)^{n_i} \cdot \alpha^d$, it doesn't depend on the order of the components in $\mathbf{n}$, and σ takes on the same (minimal) value for all almost constant vectors. □

Claim 2.6. *For every choice of N and k, consider the following placement $\mathcal{L}(N,k)$ of k drifts on the circle $\mathbb{Z}_{N-1}$:*

$$\mathcal{L}(N,k) = \left\{ \left\lfloor i \cdot \frac{N-1}{k} \right\rfloor \right\}_{i=1}^{k}.$$

Then, the vector $\mathbf{n}^{(d)}$ is almost constant for all d.

Proof. We calculate the number of drifts in the interval $[x, x+d-1]$. The index i_0 of the first drift inside the interval is the smallest $1 \le i_0 \le N-1$ which satisfies:

$$\left\lfloor i_0 \cdot \frac{N-1}{k} \right\rfloor \ge x.$$

That is, the smallest index satisfying $i_0 \ge x \cdot \frac{k}{N-1}$, which implies:

$$i_0 = \left\lceil x \cdot \frac{k}{N-1} \right\rceil.$$

The index i_1 of the last drift inside the interval is the greatest index satisfying:

$$\left\lfloor i_1 \cdot \frac{N-1}{k} \right\rfloor \le x+d-1.$$

This is the greatest index satisfying $i_1 \cdot \frac{N-1}{k} < x+d$, and therefore:

$$i_1 = \left\lceil (x+d) \cdot \frac{k}{N-1} \right\rceil - 1.$$

The number of drifts inside this interval therefore satisfies:

$$\begin{aligned} i_1 - i_0 + 1 &= \left\lceil (x+d) \cdot \frac{k}{N-1} \right\rceil - \left\lceil x \cdot \frac{k}{N-1} \right\rceil \\ &\ge (x+d) \cdot \frac{k}{N-1} - x \cdot \frac{k}{N-1} - 1 \\ &= \frac{dk}{N-1} - 1 \\ i_1 - i_0 + 1 &\le (x+d) \cdot \frac{k}{N-1} + 1 - x \cdot \frac{k}{N-1} \\ &= \frac{dk}{N-1} + 1. \end{aligned}$$

Consequently, for non-integer $\frac{dk}{N-1}$ the number of drifts takes on only the two values $\left\lfloor \frac{dk}{N-1} \right\rfloor, \left\lceil \frac{dk}{N-1} \right\rceil$. In the case where $\frac{dk}{N-1}$ is an integer we simply have:

$$\begin{aligned} i_1 - i_0 + 1 &= \left\lceil (x+d) \cdot \frac{k}{N-1} \right\rceil - \left\lceil x \cdot \frac{k}{N-1} \right\rceil \\ &= \frac{dk}{N-1}. \end{aligned}$$

Since this number is exactly $n_x^{(d)}$, this proves that $\mathbf{n}^{(d)}$ is an almost constant vector. □

Claim 2.7. *$\widetilde{S}_N^L$ achieves its minimum on the configuration $L = \mathcal{L}(N,k)$.*

Proof. $\widetilde{S}_N = \sum_{d=1}^{N-1} \sigma_d$, and by claims 2.5 and 2.6 each σ_d is minimized by this configuration (since $\sum_{i=1}^{N-1} n_i^{(d)} = dk$ must hold), therefore the sum is also minimized. □

Proof of Theorem 1.3. From Proposition 2.2, $0 < \widetilde{S}_N - S_N < C$. Choose $n_0 = \frac{2C}{\varepsilon}$. Then for $N > n_0$:

$$\begin{aligned} \frac{E^0_{\omega_L}(T_N)}{N} &= \frac{N + 2S_N^L}{N} \\ &= 1 + 2\frac{S_N^L}{N} \\ &> 1 + 2\frac{\widetilde{S}_N^L}{N} - \varepsilon \\ &\geq 1 + 2\frac{\widetilde{S}_N^{\mathcal{L}(N,k)}}{N} - \varepsilon \\ &\geq 1 + 2\frac{S_N^{\mathcal{L}(N,k)}}{N} - \varepsilon \\ &= \frac{E^0_{\omega_{\mathcal{L}(N,k)}}(T_N)}{N} - \varepsilon. \end{aligned}$$

where the first inequality follows from $\widetilde{S}_N - S_N < \frac{1}{2}\varepsilon N$, the second from Claim 2.7, and the last from $0 < \widetilde{S}_N - S_N$. □

Proof of Proposition 1.4. We evaluate $\lim_{k\to\infty} \frac{\widetilde{S}_{ak+1}}{ak+1}$. $\widetilde{S}_{ak+1}$ is a sum over the intervals of the circle, and we will calculate it by considering the sums over intervals containing any given number of p-drifts.

First, consider the intervals that do not contain any p-drift. In the gap between two p-drifts, there are $a-i$ intervals of length i, for every $1 \leq i \leq a-1$. Therefore, the sum for all k gaps:

$$\begin{aligned} s_0 &= k \cdot \sum_{i=1}^{a-1} (a-i)\,\alpha^i \\ &= k\frac{\alpha^{a+1} - a\alpha^2 + (a-1)\,\alpha}{(1-\alpha)^2}. \end{aligned}$$

Next, we consider the intervals that contain $0 < n < k$ p-drifts. Fixing n, there are k choices of p-drifts for such an interval. For each of them, let r be the number of q-drifts to the right of the rightmost p-drift, and s the number of q-drifts to the left of the leftmost p-drift. Then, summing over all possible values of r and s, and multiplying by k for the k different choices:

$$\begin{aligned} s_n &= k \cdot \beta^n \alpha^{(a-1)(n-1)} \cdot \sum_{r=0}^{a-1} \sum_{s=0}^{a-1} \alpha^{r+s} \\ &= k\beta^n \alpha^{(a-1)(n-1)} \cdot \frac{(1-\alpha^a)^2}{(1-\alpha)^2} \\ \sum_{n=1}^{k-1} s_n &= k\beta \frac{(1-\alpha^a)^2}{(1-\alpha)^2} \cdot \frac{1-\left(\beta\alpha^{a-1}\right)^{k-1}}{1-\beta\alpha^{a-1}}. \end{aligned}$$

For the intervals that contain all p-drifts, we first consider the intervals which do not cover the entire circle. For each of the k gaps between two adjacent p-drifts, we caculate the sum of the intervals that do not contain all points of that gap, but contain all other points of the circle. Define r and s as before, and notice that since they both count q-drifts in the same gap, and not all $a-1$ q-drifts in the gap are contained in the interval, $r+s < a-2$ must hold. Therefore:

$$\begin{aligned} s_k &= k \cdot \beta^k \alpha^{(a-1)(k-1)} \sum_{r=0}^{a-2} \sum_{s=0}^{a-r-2} \alpha^{r+s} \\ &= k\beta^k \alpha^{(a-1)(k-1)} \cdot \frac{(a\alpha - \alpha - a)\,\alpha^{a-1} + 1}{(1-\alpha)^2}. \end{aligned}$$

The last interval is the entire circle, and since it contributes to the sum S_{ak+1} an amount smaller than 1, we do not have to take it into account when calculating the limit.

Putting everything together:

$$\begin{aligned} \lim_{k\to\infty} \frac{\widetilde{S}_{ak+1}}{ak+1} &= \frac{1}{a} \lim_{k\to\infty} \frac{s_0 + \sum_{n=1}^{k-1} s_n + s_k}{k} \\ &= \frac{1}{a} \cdot \left[\frac{\alpha^{a+1} - a\alpha^2 + (a-1)\,\alpha}{(1-\alpha)^2} + \frac{\beta\,(1-\alpha^a)^2}{(1-\alpha)^2\,(1-\beta\alpha^{a-1})} + 0 \right], \end{aligned}$$

and since $\lim_{k\to\infty} \frac{\widetilde{S}_{ak+1} - S_{ak+1}}{ak+1} = 0$ from Proposition 2.2, the proof is complete. □

3 Further questions

1. Show that the optimal environment also minimizes the variance of the hitting time.
2. Can this result be extended to a random walk on $\mathbb{Z}$ with a given density of drifts (as in [1])?
3. Can similar results be found for other graphs? For example, $\mathbb{Z}_2 \times \mathbb{Z}_N$, or a binary tree.

References

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