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Some properties of generalized anticipated backward stochastic differential equations*

Ze Yang† Robert J. Elliott‡

Abstract

In this paper, after recalling the definition of generalized anticipated backward stochastic differential equations (generalized anticipated BSDEs for short) and the existence and uniqueness theorem for their solutions, we show there is a duality between them and stochastic differential delay equations. We then provide a continuous dependence property for their solutions with respect to the parameters and finally establish a comparison result for the solutions of these equations.

Keywords: Generalized anticipated BSDEs; duality; continuous dependence property; comparison theorem.
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1 Introduction

In 2009, Peng and Yang [4] introduced the following type of backward stochastic differential equations (BSDEs), called anticipated BSDEs:

$$\begin{cases} -dY_t = f(t, Y_t, Z_t, Y_{t+\delta(t)}, Z_{t+\zeta(t)})dt - Z_t dW_t, & t \in [0,T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K], \end{cases}$$

Here $\xi_\cdot \in S^2_{\mathscr{F}}(T, T+K; \mathbb{R}^m)$, $\eta_\cdot \in L^2_{\mathscr{F}}(T, T+K; \mathbb{R}^{m\times d})$ and $\delta(\cdot)$ and $\zeta(\cdot)$ are two $\mathbb{R}^+$-valued continuous functions defined on $[0,T]$ such that
(**i**) there exists a constant $K \geq 0$ such that for any $s \in [0,T]$,

$$s + \delta(s) \leq T + K; \quad s + \zeta(s) \leq T + K.$$

(**ii**) there exists a constant $L \geq 0$ such that for any $t \in [0,T]$ and nonnegative and integrable $g(\cdot)$,

$$\int_t^T g(s+\delta(s))ds \leq L \int_t^{T+K} g(s)ds; \quad \int_t^T g(s+\zeta(s))ds \leq L \int_t^{T+K} g(s)ds.$$

Further, for all $s \in [0,T]$, $f(s,\omega,y,z,\xi,\eta) : \Omega \times \mathbb{R}^m \times \mathbb{R}^{m\times d} \times L^2(\mathscr{F}_r;\mathbb{R}^m) \times L^2(\mathscr{F}_{r'};\mathbb{R}^{m\times d}) \longrightarrow L^2(\mathscr{F}_s,\mathbb{R}^m)$, where $r, r' \in [s, T+K]$, and f satisfies the following conditions:

*Supported by the NSERC and the ARC.
†Shandong University, China and University of Calgary, Canada. E-mail: yangzhezhe@gmail.com
‡University of Calgary, Canada and University of Adelaide, Australia. E-mail: relliott@ucalgary.ca

(**H1**) there exists a constant $C > 0$, such that for all $s \in [0, T]$, $y, y' \in \mathbb{R}^m$, $z, z' \in \mathbb{R}^{m\times d}$, $\xi_\cdot, \xi'_\cdot \in L^2_{\mathscr{F}}(s, T+K; \mathbb{R}^m)$, $\eta_\cdot, \eta'_\cdot \in L^2_{\mathscr{F}}(s, T+K; \mathbb{R}^{m\times d})$, $r, \bar{r} \in [s, T+K]$, we have

$$|f(s, y, z, \xi_r, \eta_{\bar{r}}) - f(s, y', z', \xi'_r, \eta'_{\bar{r}})| \leq C(|y-y'| + |z-z'| + E^{\mathscr{F}_s}[|\xi_r - \xi'_r| + |\eta_{\bar{r}} - \eta'_{\bar{r}}|]).$$

(**H2**) $E[\int_0^T |f(s, 0, 0, 0, 0)|^2 ds] < \infty$.

For these equations, [4] gives an existence and uniqueness result, and also proves some comparison theorems. Then Xu [5] gave a more general comparison theorem for anticipated BSDEs where the generators have less restrictions. In 2011 Xu [6] provided necessary and sufficient condition for the comparison theorem for multidimensional anticipated BSDEs. In 2013 Yang and Elliott [8] established a converse comparison theorem for anticipated BSDEs.

In 2006 Yang [7] generalized anticipated BSDEs as follows:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, Y, Z)ds - \int_t^T Z_s dW_s, & t \in [0, T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K]. \end{cases}$$

Here $K > 0$ is a given constant and for all $t \in [0, T]$, $f(t, Y, Z) : L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^{m\times d}) \longrightarrow L^2(\mathscr{F}_t, \mathbb{R}^m)$. It is obvious that the equations studied in [4] are a special type of this equation. In the same paper Yang deduces the existence and uniqueness theorem for solutions of the above equations. However, the notation in the above equations is not clear. In this paper we rewrite above equation as:

$$\begin{cases} Y_t = \xi_T + \int_t^T f(s, \{Y_r\}_{r\in[s,T+K]}, \{Z_r\}_{r\in[s,T+K]})ds - \int_t^T Z_s dW_s, & t \in [0, T]; \\ Y_t = \xi_t, & t \in [T, T+K]; \\ Z_t = \eta_t, & t \in [T, T+K], \end{cases}$$

where $K > 0$ is a given constant and for all $t \in [0, T]$ and f is a function defined on $L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^{m\times d})$ with values in $L^2(\mathscr{F}_t, \mathbb{R}^m)$. We call the above type of equation a generalized anticipated BSDE. In this paper, we discuss generalized anticipated BSDEs and derive a comparison theorem.

This paper is organized as follows. After a brief presentation of some known results that we shall use in Section 2, Section 3 provides some properties of generalized anticipated BSDEs, including a duality between them and stochastic differential delay equations (SDDEs), a continuous dependence property with respect to parameters for their solutions, and an important result for generalized anticipated BSDEs, the comparison theorem.

2 Preliminaries

Let $(\Omega, \mathscr{F}, P, \mathscr{F}_t,\ t \geq 0)$ be a complete stochastic basis such that $\mathscr{F}_0$ contains all P-null elements of $\mathscr{F}$ and suppose that the filtration is generated by a d-dimensional standard Brownian motion $W_\cdot = (W_t)_{t\geq 0}$. Suppose $T > 0$ is given. For all $n \in \mathbb{N}$, denote the Euclidean norm in $\mathbb{R}^n$ by $|\cdot|$. Denote:

- $L^2(\mathscr{F}_T; \mathbb{R}^m)$ ={$\mathbb{R}^m$-valued $\mathscr{F}_T$-measurable random variables such that $E[|\xi|^2] < +\infty$};
- $L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$ ={$\mathbb{R}^m$-valued and $\mathscr{F}_t$-adapted stochastic processes such that $E[\int_0^T |\varphi_t|^2 dt] < +\infty$};
- $S^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$ ={continuous processes in $L^2_{\mathscr{F}}(0, T; \mathbb{R}^m)$ such that $E[\sup_{0\leq t\leq T} |\varphi_t|^2] < +\infty$}.

If $m = 1$, we denote $L^2(\mathscr{F}_T, \mathbb{R})$ by $L^2(\mathscr{F}_T)$, $L^2_{\mathscr{F}}(0, T; \mathbb{R})$ by $L^2_{\mathscr{F}}(0, T)$ and $S^2_{\mathscr{F}}(0, T; \mathbb{R})$ by $S^2_{\mathscr{F}}(0, T)$. Then L^2 and S^2 are separable Hilbert spaces.

The following four lemmas are quoted from Peng [3]. Lemma 2.1 below is Lemma 3.1 of Peng [3]. Lemma 2.2 is Theorem 3.2 of Peng [3], and is a basic result for BSDEs: an existence and uniqueness theorem. Lemma 2.3, which is a comparison result for solutions of BSDEs, is Theorem 3.3 of Peng [3], and can also be found in El Karoui, Peng and Quenez [1].

Lemma 2.1. *For a fixed $\xi \in L^2(\mathscr{F}_T)$ and $g_0(\cdot)$ which is an $\mathscr{F}_t$-adapted process satisfying $E[(\int_0^T |g_0(t)|dt)^2] < +\infty$, there exists a unique pair of processes $(y_\cdot, z_\cdot) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^{1+d})$ satisfying the following BSDE:*

$$y_t = \xi + \int_t^T g_0(s)ds - \int_t^T z_s dW_s, \qquad t \in [0,T].$$

If $g_0(\cdot) \in L^2_{\mathscr{F}}(0,T)$, then $(y_\cdot, z_\cdot) \in S^2_{\mathscr{F}}(0,T) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. We have the following basic estimate:

$$|y_t|^2 + E^{\mathscr{F}_t}[\int_t^T (\frac{\beta}{2}|y_s|^2 + |z_s|^2)e^{\beta(s-t)}ds] \le E^{\mathscr{F}_t}[|\xi|^2 e^{\beta(T-t)}] + \frac{2}{\beta}E^{\mathscr{F}_t}[\int_t^T |g_0(s)|^2 e^{\beta(s-t)}ds]. \tag{2.1}$$

In particular,

$$|y_0|^2 + E[\int_0^T (\frac{\beta}{2}|y_s|^2 + |z_s|^2)e^{\beta s}ds] \le E[|\xi|^2 e^{\beta T}] + \frac{2}{\beta}E[\int_0^T |g_0(s)|^2 e^{\beta s}ds], \tag{2.2}$$

where $\beta > 0$ is an arbitrary constant. We also have

$$E[\sup_{0\le t\le T} |y_t|^2] \le kE[|\xi|^2 + \int_0^T |g_0(s)|^2 ds], \tag{2.3}$$

where the constant k depends only on T.

Consider the following conditions for $g = g(\omega,t,y,z) : \Omega\times[0,T]\times\mathbb{R}^m\times\mathbb{R}^{m\times d} \longrightarrow \mathbb{R}^m$:

(a) $g(\cdot,y,z)$ is an $\mathbb{R}^m$-valued and $\mathscr{F}_t$-adapted process satisfying Lipschitz condition in (y,z), i.e., there exists $\rho > 0$ such that for each $y, y' \in \mathbb{R}^m$ and $z, z' \in \mathbb{R}^{m\times d}$,

$$|g(t,y,z) - g(t,y',z')| \le \rho(|y-y'| + |z-z'|).$$

(b) $g(\cdot,0,0) \in L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$.

Lemma 2.2. *Assume that g satisfies (a) and (b). Then for any given terminal condition $\xi \in L^2(\mathscr{F}_T;\mathbb{R}^m)$, the BSDE*

$$Y_t = \xi + \int_t^T g(s,Y_s,Z_s)ds - \int_t^T Z_s dW_s, \quad 0 \le t \le T \tag{2.4}$$

has a unique solution, i.e., there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot)$ $\in S^2_{\mathscr{F}}(0,T;\mathbb{R}^m) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^{m\times d})$ satisfying equation (2.4).

Lemma 2.3. *Assume $g_j(\omega,t,y,z) : \Omega\times[0,T]\times\mathbb{R}\times\mathbb{R}^d \longrightarrow \mathbb{R}$ satisfies (a) and (b), $j = 1,2$. Let $(Y^{(1)}_\cdot, Z^{(1)}_\cdot)$ and $(Y^{(2)}_\cdot, Z^{(2)}_\cdot)$ be, respectively, the solutions of BSDEs as follows:*

$$Y_t^{(j)} = \xi^{(j)} + \int_t^T g_j(s,Y_s^{(j)},Z_s^{(j)})ds - \int_t^T Z_s^{(j)}dW_s, \quad 0 \le t \le T,$$

where $j = 1,2$. If $\xi^{(1)} \ge \xi^{(2)}$ and $g_1(t,Y_t^{(1)},\ Z_t^{(1)}) \ge g_2(t,Y_t^{(1)},Z_t^{(1)})$, a.e., a.s., then

$$Y_t^{(1)} \ge Y_t^{(2)}, \quad \text{a.e., a.s.}$$

3 Some properties of generalized anticipated BSDEs

3.1 Recalling generalized anticipated BSDEs

We first recall the basic conditions for the existence and uniqueness of solutions to generalized anticipated BSDEs in [7]. Let $K > 0$ be a given constant. Consider the following generalized anticipated BSDE:

$$\begin{cases} -dY_t = f(t, \{Y_r\}_{r\in[t,T+K]}, \{Z_r\}_{r\in[t,T+K]})dt - Z_t dW_t, & t \in [0,T]; \\ \quad Y_t = \xi_t, & t \in [T, T+K]; \\ \quad Z_t = \eta_t, & t \in [T, T+K]. \end{cases} \tag{3.1}$$

We wish to find a pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^{m\times d})$ satisfying the generalized anticipated BSDE (3.1).

Assume that for all $t \in [0,T],\ f(t, y_\cdot, z_\cdot) : L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^{m\times d}) \longrightarrow L^2(\mathscr{F}_t, \mathbb{R}^m)$, and f satisfies the conditions as follows

(**H3**) For all $t \in [0,T]$, $(y_\cdot, z_\cdot) \in L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(t, T+K; \mathbb{R}^{m\times d})$, $f(t, y_\cdot, z_\cdot)$ is $\mathscr{F}_t$-measurable.

(**H4**) There exists a constant $C > 0$ such that for all $t \in [0,T]$, $y_\cdot, y'_\cdot \in L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m)$, $z_\cdot, z'_\cdot \in L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^{m\times d})$,

$$E[\int_t^T |f(s, y_\cdot, z_\cdot) - f(s, y'_\cdot, z'_\cdot)|^2 e^{\beta s} ds] \le CE[\int_t^{T+K} (|y_s - y'_s|^2 + |z_s - z'_s|^2) e^{\beta s} ds],$$

where $\beta \ge 0$ is an arbitrary constant.

(**H4′**) There exists a constant $C' > 0$ such that for all $t \in [0,T]$, $y_\cdot, y'_\cdot \in L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m)$, $z_\cdot, z'_\cdot \in L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^{m\times d})$,

$$E[\int_t^T |f(s, y_\cdot, z_\cdot) - f(s, y'_\cdot, z'_\cdot)|^2 ds] \le C'E[\int_t^{T+K} (|y_s - y'_s|^2 + |z_s - z'_s|^2) ds].$$

(**H5**) $E[\int_0^T |f(s, 0, 0)|^2 ds] < +\infty$.

Remark 3.1.

(1) f satisfies Assumption $(H4)$ if and only if f satisfies Assumption $(H4')$.

(2) Note that $Lipschitz$ condition is stronger than Assumption $(H1)$, and Assumption $(H1)$ is stronger than Assumption $(H4)$.

Example 1. *Set $g(t, y_{t+K}) := E[y_{t+K}|\mathscr{F}_t],\ t \in [0,T],\ y_\cdot \in L^2_{\mathscr{F}}(0, T+K)$. Then g satisfies $(H1)$ but does not satisfy $Lipschitz$ condition.*

Example 2. *For any $t \in [0,T]$, $x \in [0, T+K]$, define $f(t,x) := E[\int_0^x B_r dB_r|\mathscr{F}_t]$, where $B_\cdot$ is a 1-dimensional standard Brownian motion with $B_0 = 0$. Then for any $x, y \in [0, T+K], t \in [0,T]$, we have*

$$\begin{aligned} E[\textstyle\int_t^T |f(s,x) - f(s,y)|^2 ds] &\le E[\textstyle\int_t^T E[|\int_y^x B_r dB_r|^2|\mathscr{F}_s]ds] = \int_t^T E[|\int_y^x B_r dB_r|^2]ds \\ &= \textstyle\int_t^T E[\int_y^x |B_r|^2 dr]ds \le (T+K)\int_y^x r dr = \dfrac{T+K}{2}(x-y)^2. \end{aligned}$$

That is, $(H4')$ holds. Hence $(H4)$ holds. On the other hand, for any $t \in (0,T]$, $x \in [t, T+K], y \in [0,t)$, by Itô's formula we know

$$|f(t,x) - f(t,y)| = |E[\textstyle\int_y^x B_r dB_r|\mathscr{F}_t]| = \dfrac{1}{2}|E[B_x^2 - B_y^2 - (x-y)|\mathscr{F}_t]| = \dfrac{1}{2}|B_t^2 - B_y^2 - t + y|.$$

Therefore, f does not satisfy $(H1)$.

The following is the main result of this section: the existence and uniqueness theorem for solutions of generalized anticipated BSDEs.

Theorem 3.2. *Suppose that f satisfies $(H3)$, $(H4)$ and $(H5)$. Then for arbitrary pair of given terminal conditions $\xi_{\cdot} \in S^2_{\mathscr{F}}(T, T+K; \mathbb{R}^m), \eta_{\cdot} \in L^2_{\mathscr{F}}(T, T+K; \mathbb{R}^{m\times d})$, the generalized anticipated BSDE (3.1) has a unique solution, that is, there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot, Z_\cdot) \in S^2_{\mathscr{F}}(0, T+K; \mathbb{R}^m) \times L^2_{\mathscr{F}}(0, T+K; \mathbb{R}^{m\times d})$ satisfying equation (3.1).*

The proof of theorem 3.2 is similar to the proof of theorem 4.2 in [4] or can be found in [7].

3.2 A duality between SDDEs and generalized anticipated BSDEs

El Karoui, Peng and Quenez [1] showed that there is a duality between stochastic differential equations (SDEs) and BSDEs, that is, the solutions of linear BSDEs can expressed in terms of the solutions of SDEs. In [4] Peng and Yang proved a duality between SDDEs and anticipated BSDEs. We shall investigate whether there exists a duality between SDDEs and generalized anticipated BSDEs. For the generalized anticipated BSDEs considered below, the answer is positive.

Let $\mu, \nu : \mathbb{R} \longrightarrow \mathbb{R}$ be two continuous functions. Suppose for all $s \in [t, T+K]$, $A_s : L^2_{\mathscr{F}}(t-K, s) \longrightarrow L^2(\mathscr{F}_s)$, $B_s : L^2_{\mathscr{F}}(t-K, s; \mathbb{R}^{1\times d}) \longrightarrow L^2(\mathscr{F}_s, \mathbb{R}^{1\times d})$ are defined by

$$A_s(\eta) = \mu_s \int_{t-K}^{s} \eta_r dr, \qquad B_s(\eta) = \nu_s \int_{t-K}^{s} \eta_r dr.$$

They are obviously two linear functions, that is, for all $s \in [t, T+K]$, $\eta, \eta' \in L^2_{\mathscr{F}}(t-K, s)$, $\tau, \tau' \in L^2_{\mathscr{F}}(t-K, s; \mathbb{R}^{1\times d})$, and for all α is a constant:

$$A_s(\eta+\eta') = A_s(\eta) + A_s(\eta'), \qquad \alpha A_s(\eta) = A_s(\alpha\eta),$$
$$B_s(\tau+\tau') = B_s(\tau) + B_s(\tau'), \qquad \alpha B_s(\tau) = B_s(\alpha\tau).$$

Consider the following generalized anticipated BSDE:

$$\begin{cases} -dY_t = (E^{\mathscr{F}_t}[A_t^*(\{Y_r\}_{r\in[t,T+K]}) + B_t^*(\{Z_r\}_{r\in[t,T+K]})] + l_t)dt - Z_t dW_t, & t \in [0,T]; \\ \quad Y_t = Q_t, & t \in [T, T+K]; \\ \quad Z_t = P_t, & t \in [T, T+K], \end{cases} \tag{3.2}$$

where for all $s \in [0,T]$, $\theta_\cdot \in L^2_{\mathscr{F}}(s, T+K)$, $\theta'_\cdot \in L^2_{\mathscr{F}}(s, T+K; \mathbb{R}^{1\times d})$. Then $A_s^*(\theta)$ and $B_s^*(\theta')$ are defined as follows:

$$A_s^*(\theta) = \int_s^{T+K} \mu_r \theta_r dr, \qquad B_s^*(\theta') = \int_s^{T+K} \nu_r \theta'_r dr.$$

Proposition 3.3. *Let $l_\cdot \in L^2_{\mathscr{F}}(0,T)$, $K > 0$ be a given constant and suppose there exists a constant $\gamma > 0$ such that for all $r \in [0, T+K]$, $|\mu_r| \le \gamma$ and $|\nu_r| \le \gamma$. Then for all $Q_\cdot \in S^2_{\mathscr{F}}(T, T+K)$, $P_\cdot \in L^2_{\mathscr{F}}(T, T+K; \mathbb{R}^{1\times d})$, the solution $Y_\cdot$ to the generalized anticipated BSDE (3.2) can be given by the closed formula:*

$$\begin{aligned} Y_t = & E^{\mathscr{F}_t}[\, X_T Q_T + \textstyle\int_t^T X_s l_s ds] \\ & + E^{\mathscr{F}_t}[(\textstyle\int_T^{T+K} Q_r \mu_r dr)(\int_t^T X_s ds) + (\int_T^{T+K} P_r \nu_r dr)(\int_t^T X_s ds)], \quad \text{a.e., a.s.}, \end{aligned}$$

where $\{X_s\}_{s\in[t-K,\, T+K]}$ is the solution to the following functional SDDE:

$$\begin{cases} dX_s = A_s(\{X_r\}_{r\in[t-K,s]})ds + B_s(\{X_r\}_{r\in[t-K,s]})dW_s, & s \in [t, T+K]; \\ \ X_t = 1, & \\ \ X_s = 0, & s \in [t-K, t). \end{cases}$$

Proof. There exists a unique solution to the above SDDE (see Theorem (2.1) of [2]). Apply Itô's formula to X_sY_s for $s\in[t,T]$, and take the conditional expectation under $\mathscr{F}_t$. Then

$$
\begin{aligned}
&E^{\mathscr{F}_t}[X_TY_T]-X_tY_t\\
&=E^{\mathscr{F}_t}[-\int_t^T X_sE^{\mathscr{F}_s}[A_s^*(\{Y_r\}_{r\in[s,T+K]})+B_s^*(\{Z_r\}_{r\in[s,T+K]})]ds-\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[\int_t^T Y_sA_s(\{X_r\}_{r\in[t-K,s]})ds+\int_t^T Z_sB_s(\{X_r\}_{r\in[t-K,s]})ds]\\
&=E^{\mathscr{F}_t}[-\int_t^T X_s(\int_s^{T+K}\mu_rY_rdr)ds-\int_t^T X_s(\int_s^{T+K}\nu_rZ_rdr)ds-\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[\int_t^T Y_s\mu_s(\int_{t-K}^s X_rdr)ds+\int_t^T Z_s\nu_s(\int_{t-K}^s X_rdr)ds]\\
&=E^{\mathscr{F}_t}[-\int_T^{T+K} Y_r\mu_r(\int_t^T X_sds)dr-\int_t^T Y_r\mu_r(\int_t^r X_sds)dr]\\
&\quad+E^{\mathscr{F}_t}[-\int_T^{T+K} Z_r\nu_r(\int_t^T X_sds)dr-\int_t^T Z_r\nu_r(\int_t^r X_sds)dr-\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[\int_t^T Y_s\mu_s(\int_{t-K}^s X_rdr)ds+\int_t^T Z_s\nu_s(\int_{t-K}^s X_rdr)ds]\\
&=E^{\mathscr{F}_t}[-\int_T^{T+K} Y_r\mu_r(\int_t^T X_sds)dr-\int_T^{T+K} Z_r\nu_r(\int_t^T X_sds)dr-\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[\int_t^T Y_s\mu_s(\int_{t-K}^t X_rdr)ds+\int_t^T Z_s\nu_s(\int_{t-K}^t X_rdr)ds].
\end{aligned}
$$

Since $X_t=1$ and $X_s=0,\ s\in[t-K,t)$, we deduce

$$
\begin{aligned}
Y_t&=E^{\mathscr{F}_t}[X_TQ_T+\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[\int_T^{T+K} Y_r\mu_r(\int_t^T X_sds)dr+\int_T^{T+K} Z_r\nu_r(\int_t^T X_sds)dr]\\
&=E^{\mathscr{F}_t}[X_TQ_T+\int_t^T X_sl_sds]\\
&\quad+E^{\mathscr{F}_t}[(\int_T^{T+K} Q_r\mu_rdr)(\int_t^T X_sds)+(\int_T^{T+K} P_r\nu_rdr)(\int_t^T X_sds)].
\end{aligned}
$$
□

3.3 A continuous dependence property with respect to parameters for the solutions of generalized anticipated BSDEs

The basic estimates (2.1) and (2.2) can also be applied to study the continuous dependence property of the generalized anticipated BSDEs with respect to parameters.

Proposition 3.4. *Let $(Y_\cdot^{(1)},Z_\cdot^{(1)})$ and $(Y_\cdot^{(2)},Z_\cdot^{(2)})$ be respectively the solutions to the following two generalized anticipated BSDEs:*

$$
\begin{cases}
-dY_t^{(i)}=(f(t,\{Y_r^{(i)}\}_{r\in[t,T+K]},\{Z_r^{(i)}\}_{r\in[t,T+K]})+\varphi_t^{(i)})dt-Z_t^{(i)}dW_t, & t\in[0,T];\\
Y_t^{(i)}=\xi_t^{(i)}, & t\in[T,T+K];\\
Z_t^{(i)}=\eta_t^{(i)}, & t\in[T,T+K],
\end{cases}
$$

where $i=1,2$. Assume the terminal conditions $\xi_\cdot^{(i)}$ and $\eta_\cdot^{(i)}$ are given elements in $S^2_{\mathscr{F}}(T,T+K;\mathbb{R}^m)$ and $L^2_{\mathscr{F}}(T,T+K;\mathbb{R}^{m\times d})$, respectively, $\varphi_\cdot^{(i)}$ are given processes in $L^2_{\mathscr{F}}(0,T;\mathbb{R}^m)$, where $i=1,2$. Set $(y_\cdot,z_\cdot)=(Y_\cdot^{(1)}-Y_\cdot^{(2)},Z_\cdot^{(1)}-Z_\cdot^{(2)})$, $\xi_\cdot=\xi_\cdot^{(1)}-\xi_\cdot^{(2)}$, $\eta_\cdot=\eta_\cdot^{(1)}-\eta_\cdot^{(2)}$, $\varphi_\cdot=\varphi_\cdot^{(1)}-\varphi_\cdot^{(2)}$. If f satisfies $(H3)$, $(H4)$ and $f(t,0,0)\equiv 0$ for all $t\in[0,T]$, then

$$
\begin{aligned}
&|y_t|^2+\frac{1}{2}E^{\mathscr{F}_t}[\int_t^T(|y_s|^2+|z_s|^2)e^{\beta(s-t)}ds]\\
&\le E^{\mathscr{F}_t}[|\xi_T|^2e^{\beta(T-t)}+\int_T^{T+K}(|\xi_s|^2+|\eta_s|^2)e^{\beta(s-t)}ds+\int_t^T|\varphi_s|^2e^{\beta(s-t)}ds],
\end{aligned}
\tag{3.3}
$$

where $\beta=8(1+C)$. We also have

$$
E[\sup_{0\le t\le T}|y_t|^2]\le cE[|\xi_T|^2+\int_T^{T+K}(|\xi_s|^2+|\eta_s|^2)ds+\int_0^T|\varphi_s|^2ds]. \tag{3.4}
$$

In particular, when $\varphi_s^{(1)} \equiv 0$, (*set* $\xi_\cdot^{(2)} \equiv 0,\ \eta_\cdot^{(2)} \equiv 0$,

$$E[\sup_{0\le t\le T} |Y_t^{(1)}|^2] \le cE[|\xi_T^{(1)}|^2 + \int_T^{T+K} (|\xi_s^{(1)}|^2 + |\eta_s^{(1)}|^2)ds, \tag{3.5}$$

where the constant c *depends only on the constant* C *in* $(H4)$ *and* T, K.

Proof. Since for all $t \in [0,T]$, $f(t,0,0) \equiv 0$, we obtain for any $t \in [0,T],\ A \in \mathscr{F}_t$,

$$\begin{aligned} \mathbb{I}_A Y_t^{(i)} = \quad & \mathbb{I}_A \xi_T^{(i)} + \textstyle\int_t^T (f(s, \{\mathbb{I}_A Y_r^{(i)}\}_{r\in[s,T+K]}, \{\mathbb{I}_A Z_r^{(i)}\}_{r\in[s,T+K]}) + \mathbb{I}_A \varphi_s^{(i)})ds \\ & - \textstyle\int_t^T \mathbb{I}_A Z_s^{(i)} dW_s, \end{aligned}$$

where $i = 1, 2$. From estimate (2.1) and the fact that $(a+b)^2 \le 2(a^2+b^2)$ we have

$$\begin{aligned} & \mathbb{I}_A |y_t|^2 + E^{\mathscr{F}_t}[\textstyle\int_t^T \mathbb{I}_A (\frac{\beta}{2}|y_s|^2 + |z_s|^2) e^{\beta(s-t)} ds] \\ \le\ & E^{\mathscr{F}_t}[\mathbb{I}_A |\xi_T|^2 e^{\beta(T-t)}] + \frac{4}{\beta} E^{\mathscr{F}_t}[\textstyle\int_t^T |\mathbb{I}_A \varphi_s|^2 e^{\beta(s-t)} ds] \\ & + \frac{4}{\beta} E^{\mathscr{F}_t}[\textstyle\int_t^T |f(s, \{\mathbb{I}_A Y_r^{(1)}\}_{r\in[s,T+K]}, \{\mathbb{I}_A Z_r^{(1)}\}_{r\in[s,T+K]}) \\ & \qquad\qquad - f(s, \{\mathbb{I}_A Y_r^{(2)}\}_{r\in[s,T+K]}, \{\mathbb{I}_A Z_r^{(2)}\}_{r\in[s,T+K]})|^2 e^{\beta(s-t)} ds]. \end{aligned}$$

Since $\beta = 8(1+C)$ and f satisfies $(H4)$, we take the expectation of both sides of the inequality to obtain:

$$\begin{aligned} & E[\mathbb{I}_A |y_t|^2] + E[\textstyle\int_t^T \mathbb{I}_A (\frac{\beta}{2}|y_s|^2 + |z_s|^2) e^{\beta(s-t)} ds] \\ \le\ & E[\mathbb{I}_A |\xi_T|^2 e^{\beta(T-t)}] + \frac{1}{2} E[\textstyle\int_t^T \mathbb{I}_A |\varphi_s|^2 e^{\beta(s-t)} ds]\ + \frac{1}{2} E[\textstyle\int_t^{T+K} \mathbb{I}_A (|y_s|^2 + |z_s|^2) e^{\beta(s-t)} ds]. \end{aligned}$$

Therefore,

$$\begin{aligned} & E[\mathbb{I}_A |y_t|^2] + \frac{1}{2} E[\textstyle\int_t^T \mathbb{I}_A (|y_s|^2 + |z_s|^2) e^{\beta(s-t)} ds] \\ \le\ & E[\mathbb{I}_A |\xi_T|^2 e^{\beta(T-t)}] + E[\textstyle\int_t^T \mathbb{I}_A |\varphi_s|^2 e^{\beta(s-t)} ds] + E[\textstyle\int_T^{T+K} \mathbb{I}_A (|\xi_s|^2 + |\eta_s|^2) e^{\beta(s-t)} ds]. \end{aligned}$$

By the definition of conditional expectation, we obtain (3.3) holds. From (2.3) and (3.3), we have

$$\begin{aligned} & E[\sup_{0\le t\le T} |y_t|^2] \\ \le\ & kE[|\xi_T|^2] + kE[\textstyle\int_0^T |f(s, \{Y_r^{(1)}\}_{r\in[s,T+K]}, \{Z_r^{(1)}\}_{r\in[s,T+K]}) \\ & - f(s, \{Y_r^{(2)}\}_{r\in[s,T+K]}, \{Z_r^{(2)}\}_{r\in[s,T+K]} + \varphi_s|^2 ds] \\ \le\ & kE[|\xi_T|^2] + 2kE[\textstyle\int_0^T (|f(s, \{Y_r^{(1)}\}_{r\in[s,T+K]}, \{Z_r^{(1)}\}_{r\in[s,T+K]}) \\ & - f(s, \{Y_r^{(2)}\}_{r\in[s,T+K]}, \{Z_r^{(2)}\}_{r\in[s,T+K]}|^2 +\ |\varphi_s|^2) ds] \\ \le\ & kE[|\xi_T|^2] + 2kE[\textstyle\int_0^T (|f(s, \{Y_r^{(1)}\}_{r\in[s,T+K]}, \{Z_r^{(1)}\}_{r\in[s,T+K]}) \\ & - f(s, \{Y_r^{(2)}\}_{r\in[s,T+K]}, \{Z_r^{(2)}\}_{r\in[s,T+K]}|^2 e^{\beta s} +\ |\varphi_s|^2) ds] \\ \le\ & kE[|\xi_T|^2] + 2kCE[\textstyle\int_0^{T+K} (|y_s|^2 + |z_s|^2) e^{\beta s} ds] + 2kE[\int_0^T |\varphi_s|^2 ds] \\ \le\ & kE[|\xi_T|^2] + 2kE[\textstyle\int_0^T |\varphi_s|^2 ds] + 2kCe^{\beta(T+K)} E[\int_T^{T+K} (|\xi_s|^2 + |\eta_s|^2) ds] \\ & + 4kCe^{\beta(T+K)} E[|\xi_T|^2 + \textstyle\int_0^T |\varphi_s|^2 ds + \int_T^{T+K} (|\xi_s|^2 + |\eta_s|^2) ds]. \end{aligned}$$

Since k depends only on T, it is obvious that there exists a constant c depending only on the constant C in $(H4)$ and T, K satisfying

$$E[\sup_{0\le t\le T}|y_t|^2] \le cE[|\xi_T|^2 + \int_T^{T+K}(|\xi_s|^2+|\eta_s|^2)ds + \int_0^T|\varphi_s|^2ds],$$

Therefore, (3.4) is proved. This estimate yields (3.5). □

3.4 Comparison theorem of 1-dimensional generalized anticipated BSDEs

Comparison theorems are fundamental results in the theory of BSDEs. It is natural to ask whether there is a comparison result for 1-dimensional generalized anticipated BSDEs. The answer is positive.

The following result is a comparison theorem for 1-dimensional generalized anticipated BSDEs. In this section, $m = 1$.

Theorem 3.5. *Let $(Y_\cdot^{(1)}, Z_\cdot^{(1)})$ and $(Y_\cdot^{(2)}, Z_\cdot^{(2)})$ be, respectively, the solutions to the following two 1-dimensional generalized anticipated BSDEs:*

$$\begin{cases} -dY_t^{(j)} = f_j(t, \{Y_r^{(j)}\}_{r\in[t,T+K]}, Z_t^{(j)})dt - Z_t^{(j)}dW_t, & t\in[0,T]; \\ \quad Y_t^{(j)} = \xi_t^{(j)}, & t\in[T,T+K], \end{cases}$$

where $j = 1,2$. Assume that for $j = 1,2$, f_j satisfies $(H3)$, $(H4)$ and $(H5)$, $\xi_\cdot^{(j)} \in S^2_{\mathscr{F}}(T, T+K)$, and for any $t\in[0,T]$, $z\in\mathbb{R}^d$, $f_2(t,\cdot,z)$ is increasing, i.e., $f_2(t,\theta_\cdot,z) \ge f_2(t,\theta_\cdot',z)$, if $\theta_r \ge \theta_r'$, $\theta_\cdot, \theta_\cdot' \in L^2_{\mathscr{F}}(t, T+K), r\in[t, T+K]$. If $\xi_s^{(1)} \ge \xi_s^{(2)}$, $s\in[T,T+K]$, and $f_1(t,\theta_\cdot,z) \ge f_2(t,\theta_\cdot,z)$, $t\in[0,T]$, $\theta_\cdot \in L^2_{\mathscr{F}}(t,T+K)$, $z\in\mathbb{R}^d$, then

$$Y_t^{(1)} \ge Y_t^{(2)}, \quad \textit{a.e., a.s.}$$

Proof. Set

$$\begin{cases} Y_t^{(3)} = \xi_T^{(2)} + \int_t^T f_2(s,\{Y_r^{(1)}\}_{r\in[s,T+K]}, Z_s^{(3)})ds - \int_t^T Z_s^{(3)}dW_s, & t\in[0,T]; \\ Y_t^{(3)} = \xi_t^{(2)}, & t\in[T,T+K]. \end{cases}$$

From Lemma 2.2, we know there exists a unique pair of $\mathscr{F}_t$-adapted processes $(Y_\cdot^{(3)}, Z_\cdot^{(3)}) \in S^2_{\mathscr{F}}(0,T+K)\times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ satisfying the above BSDE. Since

$$f_1(s,\{Y_r^{(1)}\}_{r\in[s,T+K]},z) \ge f_2(s,\{Y_r^{(1)}\}_{r\in[s,T+K]},z), \quad s\in[0,T],\ z\in\mathbb{R}^d,$$

by Lemma 2.3 we deduce

$$Y_t^{(1)} \ge Y_t^{(3)}, \quad \text{a.e., a.s.}$$

Set

$$\begin{cases} Y_t^{(4)} = \xi_T^{(2)} + \int_t^T f_2(s,\{Y_r^{(3)}\}_{r\in[s,T+K]}, Z_s^{(4)})ds - \int_t^T Z_s^{(4)}dW_s, & t\in[0,T]; \\ Y_t^{(4)} = \xi_t^{(2)}, & t\in[T,T+K]. \end{cases}$$

Since for all $t\in[0,T], z\in\mathbb{R}^d$, $f_2(t,\cdot,z)$ is increasing and $Y_t^{(1)} \ge Y_t^{(3)}$, a.e., a.s., from Lemma 2.3 we know

$$Y_t^{(3)} \ge Y_t^{(4)}, \ \text{a.e., a.s.}$$

For $n = 5, 6, \cdots$, consider the following classical BSDE:

$$\begin{cases} Y_t^{(n)} = \xi_T^{(2)} + \int_t^T f_2(s,\{Y_r^{(n-1)}\}_{r\in[s,T+K]}, Z_s^{(n)})ds - \int_t^T Z_s^{(n)}dW_s, & t\in[0,T]; \\ Y_t^{(n)} = \xi_t^{(2)}, & t\in[T,T+K]. \end{cases}$$

Similarly we have $Y_t^{(4)} \geq Y_t^{(5)} \geq \cdots \geq Y_t^{(n)} \geq \cdots$, a.e., a.s. We use $\|\nu(\cdot)\|_\beta$ in the proof of Theorem 3.2 as the norm in the Banach space $L^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. Set

$$\hat{Y}_s^{(n)} = Y_s^{(n)} - Y_s^{(n-1)}, \quad \hat{Z}_s^{(n)} = Z_s^{(n)} - Z_s^{(n-1)}, \qquad n \geq 5.$$

Then by the basic estimate (2.2), we have for any $n \geq 5$,

$$\begin{aligned} &E[\textstyle\int_0^T (\frac{\beta}{2}|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \\ \leq\ &\frac{2}{\beta}E[\textstyle\int_0^T |f_2(s, \{Y_r^{(n-1)}\}_{r\in[s,T+K]}, Z_s^{(n)}) - f_2(s, \{Y_r^{(n-2)}\}_{r\in[s,T+K]}, Z_s^{(n-1)})|^2 e^{\beta s}ds] \\ \leq\ &\frac{2C}{\beta}E[\textstyle\int_0^{T+K} |\hat{Y}_s^{(n-1)}|^2 e^{\beta s}ds + \int_0^T |\hat{Z}_s^{(n)}|^2 e^{\beta s}ds] = \frac{2C}{\beta}E[\int_0^T (|\hat{Y}_s^{(n-1)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds]. \end{aligned}$$

Set $\beta = 8C + 2$, then we obtain

$$E[\int_0^T ((4C+1)|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \leq \frac{1}{4}E[\int_0^T (|\hat{Y}_s^{(n-1)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds],$$

that is,

$$E[\int_0^T ((16C+4)|\hat{Y}_s^{(n)}|^2 + 3|\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \leq E[\int_0^T |\hat{Y}_s^{(n-1)}|^2 e^{\beta s}ds].$$

Therefore,

$$\begin{aligned} &E[\textstyle\int_0^T (|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \leq E[\int_0^T ((\frac{16C+4}{3})|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \\ \leq\ &\frac{1}{3}E[\textstyle\int_0^T |\hat{Y}_s^{(n-1)}|^2 e^{\beta s}ds] \leq \frac{1}{3}E[\int_0^T (|\hat{Y}_s^{(n-1)}|^2 + |\hat{Z}_s^{(n-1)}|^2)e^{\beta s}ds]. \end{aligned}$$

Hence

$$E[\int_0^T (|\hat{Y}_s^{(n)}|^2 + |\hat{Z}_s^{(n)}|^2)e^{\beta s}ds] \leq (\frac{1}{3})^{n-4}E[\int_0^T (|\hat{Y}_s^{(4)}|^2 + |\hat{Z}_s^{(4)}|^2)e^{\beta s}ds].$$

It follows that $\{Y_\cdot^{(n)}\}_{n\geq 4}$ and $\{Z_\cdot^{(n)}\}_{n\geq 4}$ are Cauchy sequences in $L^2_{\mathscr{F}}(0, T+K)$ and $L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$, respectively. Denote their limits by $Y_\cdot$ and $Z_\cdot$, respectively. Since $L^2_{\mathscr{F}}(0, T+K)$ and $L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$ are both Banach spaces, we obtain $(Y_\cdot, Z_\cdot) \in L^2_{\mathscr{F}}(0, T+K) \times L^2_{\mathscr{F}}(0,T;\mathbb{R}^d)$. Because for any $t \in [0,T]$,

$$\begin{aligned} &E[\textstyle\int_t^T |f_2(s, \{Y_r^{(n-1)}\}_{r\in[s,T+K]}, Z_s^{(n)}) - f_2(s, \{Y_r\}_{r\in[s,T+K]}, Z_s)|^2 e^{\beta s}ds] \\ \leq\ &CE[\textstyle\int_t^T (|Y_s^{(n)} - Y_s|^2 + |Z_s^{(n)} - Z_s|^2)e^{\beta s}ds] \to 0, \qquad n \to \infty \end{aligned}$$

holds, we see $(Y_\cdot, Z_\cdot)$ satisfies the following generalized anticipated BSDE

$$\begin{cases} Y_t = \xi_T^{(2)} + \int_t^T f_2(s, \{Y_r\}_{r\in[s,T+K]}, Z_s)ds - \int_t^T Z_s dW_s, & t \in [0,T]; \\ Y_t = \xi_t^{(2)}, & t \in [T, T+K]. \end{cases}$$

By Theorem 3.2 we know $Y_t = Y_t^{(2)}$, a.e., a.s. Since $Y_t^{(1)} \geq Y_t^{(3)} \geq Y_t^{(4)} \geq Y_t$, it follows immediately that

$$Y_t^{(1)} \geq Y_t^{(2)}, \quad \text{a.e., a.s.}$$

□

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