

Uniqueness for an inviscid stochastic dyadic model on a tree

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Abstract

In this paper we prove that the lack of uniqueness for solutions of the tree dyadic model of turbulence is overcome with the introduction of a suitable noise. The uniqueness is a weak probabilistic uniqueness for all l^2 -initial conditions and is proven using a technique relying on the properties of the q -matrix associated to a continuous time Markov chain.

Keywords: SPDE; shell models; dyadic model; tree dyadic model; q -matrix; fluid dynamics; Girsanov's transform; multiplicative noise.

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1 Introduction

The deterministic dyadic model on a tree

$$\begin{cases} dX_j = (c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k) dt \\ X_0(t) = 0, \end{cases} \quad (1.1)$$

was introduced as a wavelet description of Euler equations in [15] and studied in [4] as a model for energy cascade in turbulence. It can be seen as a generalization with more structure of the so called dyadic model of turbulence, studied in [7]. As we show in section 2 this deterministic model (1.1) does not have uniqueness in l^2 . The aim of this paper is to prove that we can restore uniqueness with the introduction of a suitable random noise:

$$dX_j = (c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k) dt + \sigma c_j X_j \circ dW_j - \sigma \sum_{k \in \mathcal{O}_j} c_k X_k \circ dW_k, \quad (1.2)$$

with $(W_j)_{j \in J}$ a sequence of independent Brownian motions. Let's also assume deterministic initial conditions for (1.2): $X(0) = x = (x_j)_{j \in J} \in l^2$. The main result of this paper is the weak uniqueness of solution for (1.2), proven in theorem 7.2.

This paper can be seen as a generalization to the dyadic tree model of the results proven for the classic dyadic model in [5], but the proof of uniqueness given here relies on a new, different approach (see also [8]) based on a general abstract property instead of a trick (see Section 6). The q -matrix we rely on is closely related to an infinitesimal generator, so the technique is valid for a larger class of models.

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The set J is a countable set and its elements are called nodes. We assume for the nodes a tree-like structure, where given $j \in J$, \bar{j} is the (unique) father of the node j , and $\mathcal{O}_j \subset J$ is the finite set of offsprings of j . In J we identify a special node, called root and denoted by 0. It has no parent inside J , but with slight notation abuse we will nevertheless use the symbol $\bar{0}$ when needed.

We see the nodes as eddies of different sizes, that split and transfer their kinetic energies to smaller eddies along the tree. To formalize this idea we consider the eddies as belonging to discrete levels, called *generations*, defined as follows. For all $j \in J$ we define the generation number $|j| \in \mathbb{N}$ such that $|0| = 0$ and $|k| = |j| + 1$ for all $k \in \mathcal{O}_j$.

To every eddy $j \in J$ we associate an *intensity* $X_j(t)$ at time t , such that $X_j^2(t)$ is the kinetic energy of the eddy j at time t . The relations among intensities are those given in (1.1) for the deterministic model and (1.2) for the perturbed stochastic model. The coefficients c_j are positive real numbers that represent the speed of the energy flow on the tree.

The idea of a stochastic perturbation of a deterministic model is well established in the literature, see [6] for the classical dyadic model, but also [11], [9] for different models. This stochastic dyadic model falls in the family of shell stochastic models. Deterministic shell models have been studied extensively in [12] while stochastic versions have been investigated for example in [10] and [16].

When dealing with uniqueness of solutions in stochastic shell models, the inviscid case we study is more difficult than the viscous one, since the more regular the space is, the simpler the proof and the operator associated to the viscous system regularizes, see for example [3] about GOY models, where the results are proven only in the viscous case.

In (1.2) the parameter $\sigma \neq 0$ is inserted just to stress the open problem of the zero noise limit, for $\sigma \rightarrow 0$. This has provided an interesting selection result for simple examples of linear transport equations (see [2]), but it is nontrivial in our nonlinear setting, due to the singularity that arises with the Girsanov transform, for example in (4.2).

It is worth noting that the form of the noise is unexpected: one could think that the stochastic part would mirror the deterministic one, which is not the case here, since there is a j -indexed Brownian component where we'd expect a \bar{j} one, and there is a k -indexed one instead of a j one.

One could argue that this is not the only possible choice for the random perturbation. On one hand we chose a multiplicative noise, instead of an additive one, but this is due to technical reasons (see [14]). On the other hand, there are other possible choices, for example the Brownian motion could depend on the father and not on the node itself, so that brothers would share the same Brownian motion. But the choice we made is dictated by the fact that we'd like to have a formal conservation of the energy, as we have in the deterministic case (see [4]). If we use Itô formula to calculate

$$\begin{aligned} \frac{1}{2}dX_j^2 &= X_j \circ dX_j \\ &= (c_j X_j^2 X_j - \sum_{k \in \mathcal{O}_j} c_k X_j^2 X_k)dt + \sigma c_j X_j X_j \circ dW_j - \sigma \sum_{k \in \mathcal{O}_j} c_k X_j X_k \circ dW_k, \end{aligned}$$

we can sum *formally* on the first $n + 1$ generations, taking $X_0(t) = 0$:

$$\begin{aligned} \sum_{|j|=0}^n \frac{1}{2}dX_j^2 &= - \sum_{|j|=n} \sum_{k \in \mathcal{O}_j} [c_k X_j^2 X_k dt + \sigma c_k X_j X_k \circ dW_k] \\ &= - \sum_{|j|=n} \sum_{k \in \mathcal{O}_j} c_k X_j X_k (X_j dt + \sigma \circ dW_k), \end{aligned}$$

since the series is telescoping in both the drift and the diffusion parts independently. That means we have P-a.s. the formal conservation of energy, if we define the energy as

$$\mathcal{E}_n(t) = \sum_{|j| \leq n} X_j^2(t) \quad \mathcal{E}(t) = \sum_{j \in J} X_j^2(t) = \lim_{n \rightarrow \infty} \mathcal{E}_n(t).$$

2 Non-uniqueness in the deterministic case

In [7] it has been proven that there exists examples of non uniqueness of l^2 solutions for the dyadic model if we consider solutions of the form $Y_n(t) = \frac{a_n}{t-t_0}$, called self-similar solutions, with $(a_n)_n \in l^2$. Thanks to the lifting result (Proposition 4.2 in [4]) that is enough to obtain two different solutions of the dyadic tree model, with the same initial conditions.

Following the same idea of self-similar solutions, introduced in [7] and [5] for the classic dyadic model and in section 5.1 in [4] for the tree dyadic model, we can construct a direct counterexample to uniqueness of solutions. In order to do this we need an existence result stronger than the one proven in [4].

Theorem 2.1. *For every $x \in l^2$ there exists at least one finite energy solution of (1.1), with initial conditions $X(0) = x$ and such that*

$$\sum_{j \in J} X_j^2(t) \leq \sum_{j \in J} X_j^2(s) \quad \forall 0 \leq s \leq t.$$

The proof of this theorem is classical, via Galerkin approximations, and follows that of theorem 3.3 in [4].

Now we recall the time reversing technique. We may consider the system (1.1) for $t \leq 0$: given a solution $X(t)$ of this system for $t \geq 0$, we can define $\widehat{X}(t) = -X(-t)$, which is a solution for $t \leq 0$, since

$$\begin{aligned} \frac{d}{dt} \widehat{X}_j(t) &= \frac{d}{dt} X_j(-t) = c_j X_j^2(-t) - \sum_{k \in \mathcal{O}_j} c_k X_j(-t) X_k(-t) \\ &= c_j \widehat{X}_j^2(t) - \sum_{k \in \mathcal{O}_j} c_k \widehat{X}_j(t) \widehat{X}_k(t). \end{aligned}$$

We can now consider the self similar solutions for the tree dyadic model, as introduced in [4], $X_j(t) = \frac{a_j}{t-t_0}$, defined for $t > t_0$, with $t_0 < 0$ and with $(a_j)_{j \in J} \in l^2$ such that

$$\begin{cases} a_{\bar{0}} = 0 \\ a_j + c_j a_j^2 = \sum_{k \in \mathcal{O}_j} c_k a_j a_k, \quad \forall j \in J. \end{cases}$$

We time-reverse them and we define

$$\widehat{X}_j(t) = -X_j(-t) \quad \forall j \in J \quad t < -t_0,$$

which, as we pointed out earlier, is a solution of (1.1) in $(-\infty, -t_0)$, with $-t_0 > 0$. Since

$$\lim_{t \rightarrow +\infty} |X_j(t)| = 0 \quad \text{and} \quad \lim_{t \rightarrow t_0^+} |X_j(t)| = +\infty, \quad \forall j \in J$$

we have

$$\lim_{t \rightarrow -\infty} |\widehat{X}_j(t)| = 0 \quad \text{and} \quad \lim_{t \rightarrow -t_0^-} |\widehat{X}_j(t)| = +\infty, \quad \forall j \in J.$$

Thanks to theorem 2.1 there is a solution \tilde{X} , with initial conditions $x = \hat{X}(0)$, and this solution is a finite energy one, so, in particular, doesn't blow up in $-t_0$. Yet it has the same initial conditions of \hat{X} , so we can conclude that there is no uniqueness of solutions in the deterministic case.

3 Itô formulation

Let's write the infinite dimensional system (1.2) in Itô formulation:

$$dX_j = (c_j X_j^2 - \sum_{k \in \mathcal{O}_j} c_k X_j X_k) dt + \sigma c_j X_j dW_j - \sigma \sum_{k \in \mathcal{O}_j} c_k X_k dW_k - \frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j dt. \quad (3.1)$$

We will use this formulation since it's easier to handle the calculations, while all results can also be stated in the Stratonovich formulation.

So let's now introduce the definition of weak solution. A filtered probability space $(\Omega, \mathcal{F}_t, P)$ is a probability space $(\Omega, \mathcal{F}_\infty, P)$ together with a right-continuous filtration $(\mathcal{F}_t)_{t \geq 0}$ such that \mathcal{F}_∞ is the σ -algebra generated by $\bigcup_{t \geq 0} \mathcal{F}_t$.

Definition 3.1. Given $x \in l^2$, a weak solution of (1.2) in l^2 is a filtered probability space $(\Omega, \mathcal{F}_t, P)$, a J -indexed sequence of independent Brownian motions $(W_j)_{j \in J}$ on $(\Omega, \mathcal{F}_t, P)$ and an l^2 -valued process $(X_j)_{j \in J}$ on $(\Omega, \mathcal{F}_t, P)$ with continuous adapted components X_j such that

$$X_j = x_j + \int_0^t [c_j X_j^2(s) - \sum_{k \in \mathcal{O}_j} c_k X_j(s) X_k(s)] ds + \int_0^t \sigma c_j X_j(s) dW_j(s) - \sum_{k \in \mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) - \frac{\sigma^2}{2} \int_0^t (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j(s) ds, \quad (3.2)$$

for every $j \in J$, with $c_0 = 0$ and $X_0(t) = 0$. We will denote this solution by

$$(\Omega, \mathcal{F}_t, P, W, X),$$

or simply by X .

Definition 3.2. A weak solution is an energy controlled solution if it is a solution as in Definition 3.1 and it satisfies

$$P(\sum_{j \in J} X_j^2(t) \leq \sum_{j \in J} x_j^2) = 1,$$

for all $t \geq 0$.

Theorem 3.3. There exists an energy controlled solution to (3.1) in $L^\infty(\Omega \times [0, T], l^2)$ for $(x_j) \in l^2$.

We will give a proof of this Theorem at the end of Section 7. It is a weak existence result and uses the Girsanov transform.

We'll prove in the following result that a process satisfying (3.1) satisfies (1.2) too.

Proposition 3.4. *If X is a weak solution, for every $j \in J$ the process $(X_j(t))_{t \geq 0}$ is a continuous semimartingale, so the following equalities hold:*

$$\int_0^t \sigma c_j X_{\bar{j}}(s) \circ dW_j(s) = \int_0^t \sigma c_j X_{\bar{j}}(s) dW_j(s) - \frac{\sigma^2}{2} \int_0^t c_j^2 X_j(s) ds$$

$$\int_0^t \sigma \sum_{k \in \mathcal{O}_j} c_k X_k(s) \circ dW_k(s) = \sum_{k \in \mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) + \frac{\sigma^2}{2} \int_0^t \sum_{k \in \mathcal{O}_j} c_k^2 X_j(s) ds,$$

where the Stratonovich integrals are well defined. So X satisfies the Stratonovich formulation of the problem (1.2).

Proof. We know that

$$\int_0^t \sigma c_j X_{\bar{j}}(s) \circ dW_j(s) = \int_0^t \sigma c_j X_{\bar{j}}(s) dW_j(s) + \frac{\sigma c_j}{2} [X_{\bar{j}}, W_j]_t,$$

but from (1.2) we have that the only contribution to $[X_{\bar{j}}, W_j]$ is given by the $-\sigma c_j X_j \circ dW_j$ term, so

$$\frac{\sigma c_j}{2} [X_{\bar{j}}, W_j]_t = \frac{\sigma c_j}{2} [- \int_0^t \sigma c_j X_j \circ dW_j, W_j]_t = -\frac{\sigma^2 c_j^2}{2} \int_0^t X_j ds.$$

Now if we consider the other integral, we have

$$\int_0^t \sigma \sum_{k \in \mathcal{O}_j} c_k X_k(s) \circ dW_k(s) = \sum_{k \in \mathcal{O}_j} \int_0^t \sigma c_k X_k(s) dW_k(s) + \sum_{k \in \mathcal{O}_j} \frac{\sigma c_k}{2} [X_k, W_k]_t.$$

For each X_k we get, with the same computations, that the only contribution to $[X_k, W_k]_t$ comes from the term $\sigma c_k X_j \circ dW_k$, so that we get

$$\frac{\sigma c_k}{2} [X_k, W_k]_t = \frac{\sigma c_k}{2} [\int_0^t \sigma c_k X_j \circ dW_k, W_k]_t = \frac{\sigma^2 c_k^2}{2} \int_0^t X_j ds. \quad \square$$

4 Girsanov transform

Let's consider (3.1) and rewrite it as

$$dX_j = c_j X_{\bar{j}}(X_{\bar{j}} dt + \sigma dW_j) - \sum_{k \in \mathcal{O}_j} c_k X_k (X_j dt + \sigma dW_k) - \frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j dt. \quad (4.1)$$

The idea is to isolate $X_{\bar{j}} dt + \sigma dW_j$ and prove through Girsanov's theorem that they are Brownian motions with respect to a new measure \hat{P} in $(\Omega, \mathcal{F}_\infty)$, simultaneously for every $j \in J$. This way (3.1) becomes a system of linear SDEs under the new measure \hat{P} . The infinite dimensional version of Girsanov's theorem can be found in [17] and [13].

Remark 4.1. *We can obtain the same result under Stratonovich formulation.*

Let X be an energy controlled solution: its energy $\mathcal{E}(t)$ is bounded, so we can define the process

$$M_t = -\frac{1}{\sigma} \sum_{j \in J} \int_0^t X_{\bar{j}}(s) dW_j(s) \quad (4.2)$$

which is a martingale. Its quadratic variation is

$$[M]_t = \frac{1}{\sigma^2} \int_0^t \sum_{j \in J} X_{\bar{j}}^2(s) ds.$$

Because of the same boundedness of $\mathcal{E}(t)$ stated above, by the Novikov criterion $\exp(M_t - \frac{1}{2}[M]_t)$ is a (strictly) positive martingale. We now define \widehat{P} on (Ω, \mathcal{F}_t) as

$$\begin{aligned} \frac{d\widehat{P}}{dP}\Big|_{\mathcal{F}_t} &= \exp(M_t - \frac{1}{2}[M]_t) \\ &= \exp(-\frac{1}{\sigma} \sum_{j \in J} \int_0^t X_{\bar{j}}(s) dW_j(s) - \frac{1}{2\sigma^2} \int_0^t \sum_{j \in J} X_{\bar{j}}^2(s) ds), \end{aligned} \quad (4.3)$$

for every $t \geq 0$. P and \widehat{P} are equivalent on each \mathcal{F}_t , because of the strict positivity of the exponential.

We can now prove the following:

Theorem 4.2. *If $(\Omega, \mathcal{F}_t, P, W, X)$ is an energy controlled solution of the nonlinear equation (3.1), then $(\Omega, \mathcal{F}_t, \widehat{P}, B, X)$ satisfies the linear equation*

$$dX_j = \sigma c_j X_{\bar{j}} dB_j(t) - \sigma \sum_{k \in \mathcal{O}_j} c_k X_k dB_k(t) - \frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j dt, \quad (4.4)$$

where the processes

$$B_j(t) = W_j(t) + \int_0^t \frac{1}{\sigma} X_{\bar{j}}(s) ds$$

are a sequence of independent Brownian motions on $(\Omega, \mathcal{F}_t, \widehat{P})$, with \widehat{P} defined by (4.3).

Proof. Now let's define

$$B_j(t) = W_j(t) + \int_0^t \frac{1}{\sigma} X_{\bar{j}}(s) ds.$$

Under \widehat{P} , $(B_j(t))_{j \in J, t \in [0, T]}$ is a sequence of independent Brownian motions.

Since

$$\begin{aligned} \sigma \int_0^t c_j X_{\bar{j}}(s) dB_j(s) &= \sigma \int_0^t c_j X_{\bar{j}}(s) dW_j(s) + \int_0^t c_j X_{\bar{j}}^2(s) ds \\ \sigma \int_0^t c_k X_k(s) dB_k(s) &= \sigma \int_0^t c_k X_k(s) dW_k(s) + \int_0^t c_k X_j(s) X_k(s) ds \quad k \in \mathcal{O}_j. \end{aligned}$$

Then (4.1) can be rewritten in integral form as

$$\begin{aligned} X_j(t) = x_j + \sigma \int_0^t c_j X_{\bar{j}}(s) dB_j(s) - \sigma \sum_{k \in \mathcal{O}_j} \int_0^t c_k X_k(s) dB_k(s) \\ - \frac{\sigma^2}{2} \int_0^t (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j(s) ds, \end{aligned} \quad (4.5)$$

which is a linear stochastic equation. □

Remark 4.3. *We can write our linear equation (4.4) also in Stratonovich form:*

$$dX_j = \sigma c_j X_{\bar{j}} \circ dB_j(t) - \sum_{k \in \mathcal{O}_j} \sigma c_k X_k \circ dB_k(t).$$

Remark 4.4. *If we look at (4.4) we can see that it is possible to drop the σ , considering it a part of the coefficients c_j .*

We can use Itô formula to calculate

$$\begin{aligned}
 \frac{1}{2}dX_j^2 &= X_j dX_j + \frac{1}{2}d[X_j]_t \\
 &= \sigma c_j X_j X_j dB_j - \sigma \sum_{k \in \mathcal{O}_j} c_k X_j X_k dB_k \\
 &\quad - \frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j^2 dt + \frac{\sigma^2}{2} (c_j^2 X_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 X_k^2) dt \\
 &= -\frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j^2 dt + dN_j + \frac{\sigma^2}{2} (c_j^2 X_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2 X_k^2) dt,
 \end{aligned} \tag{4.6}$$

with

$$N_j(t) = \sigma \int_0^t c_j X_j X_j dB_j - \sigma \sum_{k \in \mathcal{O}_j} \int_0^t c_k X_j X_k dB_k. \tag{4.7}$$

This equality will be useful in the following.

We now present an existence result also for system (4.4).

Proposition 4.5. *There exists a solution of (4.4) in $L^\infty(\Omega \times [0, T], l^2)$ with continuous components, with initial conditions $x \in l^2$.*

Proof. Fix $N \geq 1$ and consider the finite dimensional stochastic linear system

$$\begin{cases}
 dX_j^N = \sigma c_j X_j^N dB_j(t) - \sigma \sum_{k \in \mathcal{O}_j} c_k X_k^N dB_k(t) \\
 \quad - \frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j^N dt & j \in J, 0 \leq |j| \leq N \\
 X_k^N(t) \equiv 0 & k \in J, |k| = N + 1 \\
 X_j^N(0) = x_j & j \in J, 0 \leq |j| \leq N.
 \end{cases} \tag{4.8}$$

This system has a unique global strong solution $(X_j^N)_{j \in J}$. We can compute, using (4.6) and the definition of N_j in (4.7),

$$\begin{aligned}
 \frac{1}{2}d\left(\sum_{|j| \leq N} (X_j^N(t))^2\right) &= \sum_{|j| \leq N} \left(-\frac{\sigma^2}{2} (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) (X_j^N)^2 dt + dN_j^N\right. \\
 &\quad \left. + \frac{\sigma^2}{2} (c_j^2 (X_j^N)^2 + \sum_{k \in \mathcal{O}_j} c_k^2 (X_k^N)^2) dt\right) \\
 &= -\sum_{|j|=N} \frac{\sigma^2}{2} \sum_{k \in \mathcal{O}_j} c_k^2 (X_j^N)^2 dt \\
 &= -\frac{\sigma^2}{2} \sum_{|k|=N+1} c_k^2 (X_k^N)^2 \leq 0.
 \end{aligned}$$

Hence

$$\sum_{|j| \leq N} (X_j^N(t))^2 \leq \sum_{|j| \leq N} x_j^2 \leq \sum_{j \in J} x_j^2 \quad \widehat{P} - \text{a.s.} \quad \forall t \geq 0.$$

This implies that there exists a sequence $N_m \uparrow \infty$ such that $(X_j^{N_m})_{j \in J}$ converges weakly to some $(X_j)_{j \in J}$ in $L^2(\Omega \times [0, T], l^2)$ and also weakly star in $L^\infty(\Omega \times [0, T], l^2)$, so $(X_j)_{j \in J}$ is in $L^\infty(\Omega \times [0, T], l^2)$.

Now for every $N \in \mathbb{N}$, $(X_j^N)_{j \in J}$ is inProg, the subspace of progressively measurable processes in $L^2(\Omega \times [0, T], l^2)$. But Prog is strongly closed, hence weakly closed, so $(X_j)_{j \in J} \in \text{Prog}$.

We just have to prove that $(X_j)_{j \in J}$ solves (4.4). All the one dimensional stochastic integrals that appear in each equation in (4.5) are linear strongly continuous operators $\text{Prog} \rightarrow L^2(\Omega)$, hence weakly continuous. Then we can pass to the weak limit in (4.8). Moreover from the integral equations (4.5) we have that there is a modification of the solution which is continuous in all the components. \square

5 Closed equation for $E_{\hat{P}}[X_j^2(t)]$

Proposition 5.1. *For every energy controlled solution X of the nonlinear equation (3.1), $E_{\hat{P}}[X_j^2(t)]$ is finite for every $j \in J$ and satisfies*

$$\begin{aligned} \frac{d}{dt} E_{\hat{P}}[X_j^2(t)] = & -\sigma^2(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) E_{\hat{P}}[X_j^2(t)] \\ & + \sigma^2 c_j^2 E_{\hat{P}}[X_j^2(t)] + \sigma^2 \sum_{k \in \mathcal{O}_j} c_k^2 E_{\hat{P}}[X_k^2(t)]. \end{aligned} \quad (5.1)$$

Proof. Let $(\Omega, \mathcal{F}_t, P, W, X)$ be an energy controlled solution of the nonlinear equation (3.1), with initial condition $X \in l^2$ and let \hat{P} be the measure given by Theorem 4.2. Denote by $E_{\hat{P}}$ the expectation with respect to \hat{P} in (Ω, \mathcal{F}_t) .

Notice that

$$E_{\hat{P}}\left[\int_0^T X_j^4(t) dt\right] < \infty \quad \forall j \in J. \quad (5.2)$$

For energy controlled solutions from the definition we have that P -a.s.

$$\sum_{j \in J} X_j^4(t) \leq \max_{j \in J} X_j^2(t) \sum_{j \in J} X_j^2(t) \leq \left(\sum_{j \in J} x_j^2\right)^2,$$

because of the behavior of the energy we showed. But on every \mathcal{F}_t , $P \sim \hat{P}$, so

$$\hat{P}\left(\sum_{j \in J} X_j^4(t) \leq \left(\sum_{j \in J} x_j^2\right)^2\right) = 1,$$

and (5.2) holds.

From (5.2) it follows that $M_j(t)$ is a martingale for every $j \in J$. Moreover

$$E_{\hat{P}}\left[\sum_{j \in J} X_j^2(t)\right] < \infty,$$

since $X_j(t)$ is an energy controlled solution and the condition is invariant under the change of measure $P \leftrightarrow \hat{P}$ on \mathcal{F}_t and, in particular,

$$E_{\hat{P}}[X_j^2(t)] < \infty \quad \forall j \in J.$$

Now let's write (4.6) in integral form:

$$\begin{aligned} X_j^2(t) - x_j^2 = & -\sigma^2 \int_0^t (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) X_j^2(s) ds \\ & + 2 \int_0^t dN_j(s) + \sigma^2 \int_0^t (c_j^2 X_j^2(s) + \sum_{k \in \mathcal{O}_j} c_k^2 X_k^2(s)) ds. \end{aligned}$$

We can take the \widehat{P} expectation,

$$E_{\widehat{P}}[X_j^2(t)] - x_j^2 = -\sigma^2 \int_0^t (c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) E_{\widehat{P}}[X_j^2(s)] ds + \sigma^2 \int_0^t c_j^2 E_{\widehat{P}}[X_j^2(s)] ds + \sigma^2 \sum_{k \in \mathcal{O}_j} \int_0^t c_k^2 E_{\widehat{P}}[X_k^2(s)] ds,$$

where the N_j term vanishes, since it's a \widehat{P} -martingale. Now we can derive and the proposition is established. \square

It's worth stressing that $E_{\widehat{P}}[X_j^2(t)]$ satisfies a closed equation. Even more interesting is the fact that this is the forward equation of a continuous-time Markov chain, as we will see in the following section.

6 Associated Markov chain

We want to show and use this characterization of the second moments equation as the forward equation of a Markov chain, taking advantage of some known results in the Markov chains theory. We follow the transition functions approach to continuous times Markov chains; we don't assume any knowledge of this theory, so we will provide the basic definitions and results we need. More results can be found in the literature, see for example [1].

Definition 6.1. A non-negative function $f_{j,l}(t)$ with $j, l \in J$ and $t \geq 0$ is a transition function on J if $f_{j,l}(0) = \delta_{j,l}$,

$$\sum_{l \in J} f_{j,l}(t) \leq 1 \quad \forall j \in J, \forall t \geq 0,$$

and it satisfies the semigroup property (or Chapman-Kolmogorov equation)

$$f_{j,l}(t+s) = \sum_{h \in J} f_{j,h}(t) f_{h,l}(s) \quad \forall j, l \in J, \forall t, s \geq 0.$$

Definition 6.2. A q -matrix $Q = (q_{j,l})_{j,l \in J}$ is a square matrix such that

$$0 \leq q_{j,l} < +\infty \quad \forall j \neq l \in J, \\ \sum_{l \neq j} q_{j,l} \leq -q_{j,j} =: q_j \leq +\infty \quad \forall j \in J.$$

A q -matrix is called stable if all q_j 's are finite, and conservative if

$$q_j = \sum_{l \neq j} q_{j,l} \quad \forall j \in J.$$

If Q is a q -matrix, a Q -function is a transition function $f_{j,l}(t)$ such that $f'_{j,l}(0) = Q$.

The q -matrix shows a close resemblance to the infinitesimal generator of the transition function, but they differ, since the former doesn't determine a unique transition function, while the latter does. Still this approach can be seen as a generator approach to Markov chains in continuous times.

Now let's see these objects in our framework: let's write (5.1) in matrix form. Let Q be the infinite dimensional matrix which entries are defined as

$$q_{j,j} = -\sigma^2(c_j^2 + \sum_{k \in \mathcal{O}_j} c_k^2) \quad q_{j,\bar{j}} = \sigma^2 c_j^2 \quad q_{j,k} = \mathbb{1}_{\{k \in \mathcal{O}_j\}} \sigma^2 c_k^2 \text{ for } k \neq j, \bar{j}$$

Proposition 6.3. *The infinite matrix Q defined above is the stable and conservative q -matrix. Moreover Q is symmetric.*

Proof. It's easy to check that Q is a stable and conservative q -matrix. First of all $q_{j,j} < 0$ for all $j \in J$ and $q_{j,l} \geq 0$ for all $j \neq l$. Then

$$q_j = \sum_{l \neq j} q_{j,l} = q_{j,\bar{j}} + \sum_{k \in \mathcal{O}_j} q_{j,k} = \sigma^2 c_j^2 + \sum_{k \in \mathcal{O}_j} \sigma^2 c_k^2 = -q_{j,j}.$$

Moreover it is very easy to check that the matrix is symmetric:

$$q_{ij} = \left\{ \begin{array}{ll} \sigma^2 c_j^2 & l = \bar{j} \Leftrightarrow j \in \mathcal{O}_l \quad \sigma^2 c_j^2 \\ \sigma^2 c_l^2 & l \in \mathcal{O}_j \Leftrightarrow j = \bar{l} \quad \sigma^2 c_l^2 \end{array} \right\} = q_{lj}$$

□

Since Q is a q -matrix we can construct the process associated, as a jump and hold process on the space state, which in our case is the tree of the dyadic model. The process will wait in node j for an exponential time of parameter q_j , and then will jump to \bar{j} or $k \in \mathcal{O}_j$ with probabilities $q_{j,\bar{j}}/q_j$ and $q_{j,k}/q_j$ respectively. This process is a continuous time Markov chain that has J as a state space and also has the same skeleton as the dyadic tree model, meaning that the transition probabilities are non-zero only if one of the nodes is the father of the other one.

Given a q -matrix Q , it is naturally associated with two (systems of) differential equations:

$$\begin{aligned} y'_{jl}(t) &= \sum_{h \in J} y_{jh}(t) q_{hl} & (6.1) \\ y'_{jl}(t) &= \sum_{h \in J} q_{jh} y_{hl}(t), \end{aligned}$$

called forward and backwards Kolmogorov equations, respectively.

Lemma 6.4. *Given a stable, symmetric and conservative q -matrix Q , then the unique nonnegative solution of the forward equations (6.1) in $L^\infty([0, \infty), l^1)$, given a null initial condition $y(0) = 0$, is $y(t) = 0$.*

Proof. Let y be a generic solution, then

$$\left\{ \begin{array}{l} \frac{d}{dt} y_j(t) = \sum_{i \in J} y_i(t) q_{ij} \\ y_j(t) \geq 0 \quad j \in J \\ y_j(0) = 0 \quad j \in J \\ \sum_{j \in J} y_j(t) < +\infty. \end{array} \right. \quad (6.2)$$

We can consider for every node $\hat{y}_j = \int_0^{+\infty} e^{-t} y_j(t) dt$, the Laplace transform in 1. From the last equation of the system above, we have $\sum_j \hat{y}_j \leq M$, for some constant $M > 0$, so in particular we can consider $k \in J$ such that $\hat{y}_k \geq \hat{y}_j$, for all $j \in J$.

Now we want to show that $y'_k(t)$ is bounded: thanks to the symmetry and stability of Q we have

$$|y'_k(t)| \leq |-q_k y_k(t)| + \left| \sum_{l \neq k} y_l(t) q_{lk} \right| \leq q_k M + q_k M < +\infty.$$

We can integrate by parts

$$\begin{aligned} \hat{y}_k &= \int_0^{+\infty} e^{-t} y'_k(t) dt = \int_0^{+\infty} e^{-t} \sum_{l \in J} y_l(t) q_{lk} dt = \sum_{l \in J} \hat{y}_l q_{lk} \\ &= -\hat{y}_k q_k + \sum_{l \neq k} \hat{y}_l q_{lk} \leq \hat{y}_k (-q_k + \sum_{l \neq k} q_{kl}) = 0, \end{aligned} \quad (6.3)$$

where the last equality follows from the conservativeness of Q , and we used the stability and symmetry. Now we have $\hat{y}_k = 0$ and so all $\hat{y}_j = 0$, hence $y_j(t) = 0$ for all $j \in J$, for all $t \geq 0$. \square

7 Uniqueness

Now we can use the results of the previous section to prove the main results of this paper.

Theorem 7.1. *There is strong uniqueness for the linear system (4.4) in the class of energy controlled $L^\infty(\Omega \times [0, T], l^2)$ solutions.*

Proof. By linearity of (4.4) it is enough to prove that for null initial conditions there is no nontrivial solution. Since we have (5.1), proposition 6.3 and lemma 6.4, then $E_{\hat{P}}[X_j^2(t)] = 0$ for all j and t , hence $X = 0$ a.s. \square

Let's recall that we already proved an existence result for (4.4) with proposition 4.5.

Theorem 7.2. *There is uniqueness in law for the nonlinear system (3.1) in the class of energy controlled $L^\infty(\Omega \times [0, T], l^2)$ solutions.*

Proof. Assume that $(\Omega^{(i)}, \mathcal{F}_t^{(i)}, P^{(i)}, W^{(i)}, X^{(i)})$, $i = 1, 2$, are two solutions of (3.1) with the same initial conditions $x \in l^2$. Given $n \in \mathbb{N}$, $t_1, \dots, t_n \in [0, T]$ and a measurable and bounded function $f : (l^2)^n \rightarrow \mathbb{R}$, we want to prove that

$$E_{P^{(1)}}[f(X^{(1)}(t_1), \dots, X^{(1)}(t_n))] = E_{P^{(2)}}[f(X^{(2)}(t_1), \dots, X^{(2)}(t_n))]. \quad (7.1)$$

By theorem 4.2 and the definition of \hat{P} given in (4.3) we have that, for $i = 1, 2$,

$$\begin{aligned} E_{P^{(i)}}[f(X^{(i)}(t_1), \dots, X^{(i)}(t_n))] &= \\ &E_{\hat{P}^{(i)}}[\exp\{-M_T^{(i)} + \frac{1}{2}[M^i, M^{(i)}]_T\} f(X^{(i)}(t_1), \dots, X^{(i)}(t_n))], \end{aligned} \quad (7.2)$$

where $M^{(i)}$ is defined as in (4.2). We have proven in proposition 4.5 and theorem 7.1 that the linear system (4.4) has a unique strong solution. Thus it has uniqueness in law on $\mathcal{C}([0, T], \mathbb{R})^{\mathbb{N}}$ by Yamada-Watanabe theorem, that is under the measures $\hat{P}^{(i)}$, the processes $X^{(i)}$ have the same laws. For a detailed proof of this theorem in infinite dimension see [17].

Now we can also include $M^{(i)}$ in the system and conclude that $(X^{(i)}, M^{(i)})$ under $\hat{P}^{(i)}$ have laws independent of $i = 1, 2$, hence, through (7.2), we have (7.1). \square

We can now conclude with the proof of Theorem 3.3.

Proof of Theorem 3.3. Let $(\Omega, \mathcal{F}_t, \hat{P}, B, X)$ be the solution of (4.4) in $L^\infty(\Omega \times [0, T], l^2)$ provided by theorem 7.1. We follow the same argument as in Section 4, only from \hat{P} to P . We construct P as a measure on (Ω, \mathcal{F}_T) satisfying

$$\frac{dP}{d\hat{P}} \Big|_{\mathcal{F}_T} = \exp(\widehat{M}_T - \frac{1}{2}[\widehat{M}, \widehat{M}]_T),$$

where $\widehat{M}_t = \frac{1}{\sigma} \sum_{j \in J} \int_0^t X_j(s) dB_j(s)$. Under P the processes

$$W_j(t) = B_j(t) - \int_0^t \frac{1}{\sigma} X_j(s) ds,$$

are a sequence of independent Brownian motions. Hence $(\Omega, \mathcal{F}_t, P, W, X)$ is a solution of (3.1) and it is in L^∞ , since P and \widehat{P} are equivalent on \mathcal{F}_T . \square

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