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Avoidance Coupling

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Abstract

We examine the question of whether a collection of random walks on a graph can be coupled so that they never collide. In particular, we show that on the complete graph on n vertices, with or without loops, there is a Markovian coupling keeping apart $\Omega(n/\log n)$ random walks, taking turns to move in discrete time.

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1 Introduction

Coupling of Markov chains has proved to be a valuable tool, notably, in recent years, in proving rapid mixing. Our intent here is to isolate one very simple type of Markov chain (random walk, especially on a complete graph) and to explore one particular capability, that of avoiding collision.

As an application, one may envisage some anti-virus software moving from port to port in a computer system to check for incursions. It is natural to have such a program implement a random walk on the ports so as not to be predictable. If another program (possibly with a different purpose) also does a random walk on the ports, it may be desirable or even essential to prevent the programs from examining the same port at the same time.

If two random walks are independent, they will collide in polynomial time on any finite, connected, non-bipartite graph, even if a scheduler tries to keep them apart [4, 8]. Only if the scheduler is clairvoyant—that is, knows the entire future of each walk—is there a possibility of avoiding collision forever, and that case rests on a complex proof [5] for enormous graphs.

Coupling, on the other hand, is a much more powerful technique for keeping random walks apart. On the cycle C_n, for example (where the clairvoyant scheduler has no chance), coupling can easily keep linearly many random walks apart, simply by having them either all move clockwise, or all counter-clockwise, at the same time.

Keeping random walks apart on a complete graph K_n by coupling—especially Markovian coupling, which we define more formally below—appears to be a more difficult task. We apply a number of techniques to achieve such couplings, depending on number-theoretic properties of n. For infinitely many n there is a Markovian coupling which keeps apart $(n-1)/2$ random walkers, and for all n there is a Markovian coupling which keeps apart $\Omega(n/\log n)$ random walkers. We have essentially the same results on the looped version of the complete graph K_n^*, and in this case we also have non-Markovian couplings of linearly many walkers, for all n.

The closely related problem of coupling two Brownian motions (on various domains) so as to keep them at least some positive distance apart has been studied in some depth—see [2, 6], and [3] for a recent connection to pursuit-evasion problems.

2 Preliminaries

We refer the reader to a modern text such as [1, 7] (both of which are accessible online), for background on discrete Markov chains. All of our Markov chains are time-homogeneous and have finite state spaces.

A **coupling** of Markov chains is nothing more than an implementation of the chains on a common probability space, in such a way that each chain, viewed separately, is faithful to its transition matrix. In what follows, X_t and Y_t for $t = 0, 1, 2, \dots$ will represent simple discrete-time random walks on the loopless complete graph K_n, or its looped counterpart K_n^* (in which each vertex has a single self-loop, and the walk stays where it is with probability $1/n$ at each step). Clearly a time t for which $X_t = Y_t$ should constitute a collision, but what if $X_{t+1} = Y_t$? If in addition $Y_{t+1} = X_t$ we call such an event a "swap", otherwise a "shove".

Allowing swaps and shoves makes things easy—on K_n, for example, we could couple n walks simply by choosing, at each turn, a uniformly random derangement (or a uniformly random cycle) σ, and having the walker at i move to $\sigma(i)$.

Instead, we make the issue of swaps and shoves moot by having the walkers move alternately—in the case of two walkers, in the order $X_0, Y_0, X_1, Y_1, \dots$. Then the events $X_t = Y_t$ and $X_{t+1} = Y_t$ both constitute collisions. Multiple walkers are assumed to take turns in a fixed cyclic order, and again, a collision is deemed to occur exactly if a walker moves to a vertex currently occupied by another. We call a coupling that forbids collisions an **avoidance coupling**.

Note that a collection of random walkers who move in *continuous* time, that is, after independent exponential (mean 1) waiting times, can be coupled so as to take turns as above; if there are k walkers, we simply have walker j wait for a random time after walker $j-1$ (modulo k) has moved, where the time is distributed according to a Gamma distribution with shape parameter $1/k$ and scale parameter 1. (The sum of k independent such random variables is an exponential random variable with mean 1.) Thus, any coupling of our alternating discrete-time walkers can be applied to the continuous-time case.

A coupling is **Markovian** if it is itself a Markov chain, meaning, in the two-walker case, that X_{t+1} depends only on X_t and Y_t, while Y_{t+1} depends only on Y_t and X_{t+1}. (Brownian couplings with the analogous property are referred to as "co-adapted" in [6].) To allow the walkers to alternate, we tacitly assume that the state of the coupled chain includes the information of whether it is the first or second player's turn to move. For multiple walkers, the dependence is, similarly, only on the current locations of all the walkers, and on whose turn it is to move.

In the couplings that we construct, the individual walkers may be taken to be stationary, in the sense that their initial states X_0 and Y_0 are each uniformly distributed over the n states. However, we permit X_0 and Y_0 to be coupled in an arbitrary way. Any such coupling may be modified to make (X_0, Y_0) uniformly random over all pairs (perhaps at the expense of the Markovian property) by applying a random permutation to the states.

Note that the transformation above from discrete-time to continuous-time chains does not preserve the Markovian property. Indeed, it is rarely possible to get a Markovian avoidance coupling for continuous-time chains:

Theorem 2.1. *Let M be an irreducible, continuous-time, finite-state Markov chain. Then there is no Markovian coupling of two or more copies of M, without simultaneous transitions, that avoids all collisions.*

Proof. Suppose X and Y are copies of M that are coupled in this way. Since M is irreducible, we may fix some tour $v_0, v_1, \ldots, v_k$ of all the states that has a positive probability. Let $r_i > 0$ be the rate at which the transition $v_{i-1} \to v_i$ occurs, and consider the probability p that when started in state v_0, the single chain X follows the tour exactly and completes it in time less than ε. Then

$$p = (1 + \mathrm{o}(1)) \prod_{i=1}^{k} r_i \frac{\varepsilon^k}{k!} = \Theta(\varepsilon^k),$$

where the constants implied by the Θ notation depend on the Markov chain and the tour, but not on ε.

Next we start the coupled chain (X, Y) in state (v_0, s) for some s, and consider the probability $q(s)$ that its projection onto the first chain takes the tour and completes it in time less than ε. Since the coupled chain is collision-avoiding, it must take at least one additional step in order to move the second walker out of the way. But then at least $k + 1$ transitions must take place within time ε, thus

$$q(s) \leq \sum_{j=k+1}^{\infty} \frac{\varepsilon^j R^j}{j!} = \mathrm{O}(\varepsilon^{k+1})$$

where R is the maximum, over all states of the coupled chain, of the rate of transition out of that state.

For the coupling to be faithful, however, we must have $q(s) \geq p$ for some s. Since $p = \Theta(\varepsilon^k)$ and $q(s) = \mathrm{O}(\varepsilon^{k+1})$, this is impossible for small enough ε. □

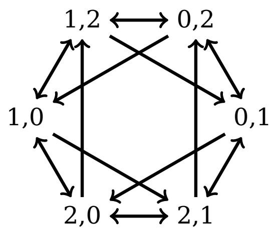
3 Two walkers on three vertices

No avoidance coupling is possible for two walkers on K_3, since there is no choice of where to move, hence no room for randomness. On the looped graph K_3^*, however, a walker stays where she is with probability $1/3$. We shall see that this is enough to permit an avoidance coupling, but not a Markovian one. In fact, we can completely analyze the more general walk on K_3^* in which a walker stays in place with some arbitrary probability s, and moves to each of the other two vertices with probability $(1-s)/2$.

Theorem 3.1. *Consider two walkers on K_3^*, each with looping probability $s \in [0, 1)$. There exists an avoidance coupling if and only if $s \geq \frac{1}{3}$, and there exists a Markovian avoidance coupling if and only if $s \geq \frac{1}{2}$.*

Proof of Theorem 3.1, non-Markovian case. We first show that an avoidance coupling exists in the case $s = 1/3$ (i.e., ordinary random walk on K_3^*). We start the coupled chain in a uniformly random pair of states (X_0, Y_0) such that $X_0 \neq Y_0$. Given X_t and Y_t, the pair (X_{t+1}, Y_{t+1}) is chosen uniformly at random among the allowed pairs, *except* (X_t, Y_t) itself (see Figure 1).

Thus, for example, if Alice is at 0 and Bob at 1, their new positions will be $(0, 2)$, $(2, 0)$, or $(2, 1)$ each with probability $1/3$. Notice that this coupling is not quite Markovian, as



$$\cdots, \quad 1,2, \quad 0,2, \quad 0,1, \quad 2,1, \quad 0,2, \quad 1,2, \quad 0,1, \quad 0,2, \quad \cdots$$

Figure 1: *Illustration of the avoidance coupling on K_3^* for $s = 1/3$. The states of Alice and Bob are naturally grouped into pairs, so that Alice and Bob are effectively jointly walking on these pairs (state diagram on top). If the sequence of pairs (bottom) is reversed, and the elements in each pair is reversed, then the law of this new sequence of pairs is the same as for the original sequence.*

Bob's move depends on Alice's *previous* position—he is not permitted to stay put when Alice has just done so.

We prove by induction on t that Y_t is uniformly random $\neq X_t$, independent of X_s and Y_s for $s < t$. We may assume $X_t = 0$ for the purpose of showing that Y_{t+1} is uniform $\neq X_{t+1}$ given X_{t+1}; then, using the induction hypothesis and the coupling definition, the triple (Y_t, X_{t+1}, Y_{t+1}) is equally likely to be any of $(1,0,2)$, $(1,2,0)$, $(1,2,1)$, $(2,0,1)$, $(2,1,0)$, $(2,1,2)$, which completes the induction.

Using this fact it easily follows that Alice's sequence is i.i.d. uniform; using the fact again, it follows that Bob's sequence is also i.i.d. uniform, as required. (We remark that this coupling is invariant under time reversal, except that Alice and Bob exchange roles.)

Turning now to the case $s \geq 1/3$, we can modify the above coupling as follows. At each round, with a suitable probability let both walkers stay in place. Otherwise they proceed to the next round. This clearly increases the probability that each walker stays in place at any step, without otherwise changing their trajectories.

Finally we must show that no avoidance coupling is possible if $s < 1/3$. Consider the event that $X_0, \ldots, X_n$ alternate between two (unspecified) states of K_3^*. This has probability $2\,(\frac{1-s}{2})^n$, since each jump has probability $(1-s)/2$. However, this event forces $Y_0 = Y_1 = \cdots = Y_{n-1}$, which has probability s^{n-1}. Thus $2\,(\frac{1-s}{2})^n \leq s^{n-1}$. Taking nth roots and letting $n \to \infty$ we find $(1-s)/2 \leq s$, so $s \geq 1/3$. □

Proof of Theorem 3.1, Markovian case. Suppose first that there is a Markovian avoidance coupling. Let p_{ab} be the probability that Alice stays at a given that it is her move, that she is at a, and that Bob is at b. Let q_{ab} be the probability that Bob stays at b, given that it is Bob's move, and again that Alice is at a and Bob at b. That these quantities may only be defined for certain pairs a, b will not interfere with our arguments.

Suppose Alice has just moved from 0 to 1. Her conditional probability of next moving back to 0 is $(1-s)/2$. Bob must have been at 2 and will stay there with probability q_{12}, after which Alice moves to 0 with probability $1-p_{12}$. We conclude that $(1-s)/2 = q_{12}(1-p_{12})$.

Similarly, suppose Bob has just moved from 0 to 2. His conditional probability of next moving back to 0 is $(1-s)/2$. Alice must have been at 1 and will stay there with probability p_{12}, after which Bob moves to 0 with probability $1-q_{12}$. So we get $(1-s)/2 = p_{12}(1-q_{12})$. Combined with the conclusion of the previous paragraph, this gives $p_{12} = q_{12}$, and similarly $p_{ab} = q_{ab}$ for all $a \neq b$.

Since the equation $p_{12}(1-p_{12}) = (1-s)/2$ has no real solutions for $s < \frac{1}{2}$, the presumed coupling cannot exist in this case.

We now demonstrate that, conversely, when $\frac{1}{2} \le s < 1$ there is a Markovian avoidance coupling for two walkers. Let p and $1-p$ be the two (possibly equal) values of x satisfying $x(1-x) = (1-s)/2$, and note that then $p^2 + (1-p)^2 = s$. Letting i' stand for $i+1 \mod 3$, put $p_{ii'} = p$, $p_{i'i} = 1-p$, and $q_{ij} = p_{ij}$. We claim that these values are the holding probabilities p_{ij}, q_{ij} (as defined earlier in the proof) of a Markovian avoidance coupling.

To show this, condition on the event that Alice is at i at time 1. We will show that, conditioned also on Bob's position at time 0, Alice's next step is to i (respectively i') with the correct probability s (respectively $(s-1)/2$); hence the probability she moves to i'' is correct also. Since the coupling is Markovian, Alice's future depends on her past only through (X_1, Y_0), so this will suffice to prove that Alice's trajectory has the correct law. By the symmetry of our construction, the same will then apply to Bob.

Suppose first that $Y_0 = i'$, that is, that Bob was at i' one move ago. Then, with all probabilities conditional on $\{X_1 = i, Y_0 = i'\}$,

$$\begin{aligned}\mathbb{P}(X_2 = i) &= \mathbb{P}(Y_1 = i')\mathbb{P}(X_2 = i \mid Y_1 = i') + \mathbb{P}(Y_1 = i'')\mathbb{P}(X_2 = i \mid Y_1 = i'') \\ &= q_{ii'}p_{ii'} + (1-q_{ii'})p_{ii''} = p^2 + (1-p)^2 = s,\end{aligned}$$

and

$$\mathbb{P}(X_2 = i') = \mathbb{P}(Y_1 = i'')\mathbb{P}(X_2 = i' \mid Y_1 = i'') = (1-q_{ii'})(1-p_{ii''}) = (1-p)p = \tfrac{1-s}{2}.$$

Observe that the coupling is invariant under replacing state i with $-i \bmod 3$, swapping $'$ and $''$, and substituting $1-p$ for p. Since the above conditional probabilities are symmetric in p and $1-p$, it follows that the distribution of X_2 conditional on $\{X_1 = i, Y_0 = i''\}$ is also correct. $\square$

4 Two walkers for composite n

Theorem 4.1. *For any composite $n = ab$, where $a, b > 1$, there exist Markovian avoidance couplings for two walkers on K_n and on K_n^*.*

Proof. We partition $[n] := \{0, 1, \ldots, n-1\}$ into b "clusters" $S_1, \ldots, S_b$ each of size a. We construct a coupling so that, when it is Alice's turn to move, she and Bob are in different clusters. (This is where we use $b > 1$.)

For the coupling on K_n, Alice's protocol is to move with probability $\frac{a(b-1)}{ab-1}$ to a random vertex in Bob's cluster (other than Bob's vertex), and move with probability $\frac{a-1}{ab-1}$ to a random vertex in her own cluster (other than her current vertex). (This is where we use $a > 1$.) Bob's protocol is to move to a random new vertex in his current cluster,

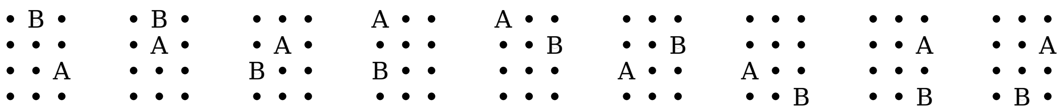


Figure 2: *Illustration of the avoidance coupling for two walkers on K_n for composite n in the case $n = 3 \times 4$. The "clusters" are the columns. Alice usually moves to Bob's cluster, but sometimes stays in her own. Bob moves to a new cluster when Alice is in his cluster, and otherwise stays in his cluster. The time-reversed process, with the roles of Alice and Bob exchanged, is equal in law to the original process.*

unless Alice is also in his cluster, in which case he moves to a uniformly random vertex in a uniformly random unoccupied cluster. (After Bob moves, he and Alice are once again in different clusters.)

The coupling for K_n^* is essentially the same, except that Alice's probability of moving to Bob's cluster is $\frac{b-1}{b}$, and when either Alice or Bob move within their own cluster, the new vertex may be the same as the current vertex.

Regardless of how Alice and Bob start, after Alice and then Bob move, Bob is at a uniformly random new vertex (for the coupling on K_n) or a uniformly random vertex (for the coupling on K_n^*). Thus Bob's walk has the correct distribution.

When the coupled walks are viewed backwards in time, the protocol is the same but with the roles of Alice and Bob reversed. Thus Alice's walk also has the correct distribution. □

Note that when $a = 2$, the above coupling has **minimum entropy**, meaning that the entropy of the coupling is equal to the entropy of a single walker. The coupling does not have minimum entropy for $a > 2$; it can, however, be modified to have minimum entropy, at the cost of giving up the Markovian property. Specifically, we can give each cluster the structure of a directed cycle and insist that when Alice moves to Bob's cluster, she chooses the site after Bob's; and if she stays in her own cluster, Bob copies Alice's movement in his own cluster. To copy Alice's movement, Bob needs to remember where Alice was on her previous turn, which is why this modified coupling is not Markovian.

5 Monotonicity

The purpose of this section is to show that existence of avoidance couplings for k walkers on K_n^* is monotone in n. We do not know whether the corresponding statement holds for the unlooped case K_n, nor if we impose the Markovian condition or the minimum entropy condition.

Theorem 5.1. *If there is an avoidance coupling of k walkers on K_n^*, then there is an avoidance coupling of k walkers on K_{n+1}^*.*

The following concept will be useful for the proof. Suppose that k walkers walk on K_2^*, taking turns in cyclic order as usual, in such a way that no two walkers are simultaneously at vertex 1 (but several walkers can be at 2), and so that the trajectory of any given walker is an i.i.d. Bernoulli sequence that is 1 with probability p at each step. We call such a coupling a **1-avoidance** coupling of $\mathrm{Bernoulli}(p)$ walkers.

Lemma 5.2. *If there is an avoidance coupling of k walkers on K_n^*, then there is a 1-avoidance coupling of k $\mathrm{Bernoulli}(1/n)$ walkers.*

Proof. Let any given Bernoulli walker be at 1 exactly when the corresponding walker on K_n^* is at vertex 1. □

Lemma 5.3. *If there is a 1-avoidance coupling of k $\mathrm{Bernoulli}(p)$ walkers, then there is a 1-avoidance coupling of k $\mathrm{Bernoulli}(q)$ walkers for all $q < p$.*

Proof. We simply thin the process of 1s. Suppose we have a $\mathrm{Bernoulli}(p)$ coupling, and take an independent process of i.i.d. coin flips, heads with probability q/p, indexed by turns (i.e. times at which any walker is allowed to move). To get the $\mathrm{Bernoulli}(q)$ coupling, take a walker to be at 1 whenever the original walker is at 1 and the corresponding coin flip is heads. □

Proof of Theorem 5.1. Suppose we have an avoidance coupling of k walkers on K_n^*. By Lemmas 5.2 and 5.3, there exists a 1-avoidance coupling of k $\mathrm{Bernoulli}(1/(n+1))$ walkers. Take such a coupling, independent of the original coupling on K_n^*. To get a coupling on K_{n+1}^*, take a given walker to be at the same vertex as the corresponding walker on K_n^*, unless the corresponding Bernoulli walker is at 1, in which case take it to be at vertex $n+1$. □

6 Linear number of walkers for special n

Theorem 6.1. *There exists a minimum-entropy Markovian avoidance coupling for k walkers on K_n^* for any $k \le 2^d$ and any $n = 2^{d+1}$ or $2^{d+1}+1$, as well as on K_n for $n = 2^{d+1}+1$.*

The avoidance coupling of 2^d walkers on $K_{2^{d+1}+1}$ is illustrated in Figure 3.

Corollary 6.2. *There exists an avoidance coupling for k walkers on K_n^* for any $k \le n/4$.*

Proof. This is immediate from Theorem 6.1 and the monotonicity result, Theorem 5.1. □

Proof of Theorem 6.1. We begin with the case of 2^d walkers, labeled $0, \dots, 2^d-1 = \omega$ on K_n^* for $n = 2^{d+1}$ or $n = 2^{d+1}+1$. Let $\varepsilon_t^{(i)}$ be independent uniform $\{\pm 1\}$ random variables. Let δ_t be independent uniform $\{0,1\}$ random variables. Let the trajectory of walker j be denoted $\{X_t^{(j)}\}$.

Let $\sum_{i=0}^{d-1} j_i 2^i$ be the binary representation of j, for $0 \le j \le \omega$. Given $X_t^{(0)}$, the positions of the other walkers are given by

$$X_t^{(j)} = X_t^{(0)} + \sum_i j_i \varepsilon_t^{(i)} 2^i$$

where all positions are understood modulo n. We then define inductively

$$X_{t+1}^{(0)} = X_t^{(\omega)} + 2^d + \delta_{t+1} \ .$$

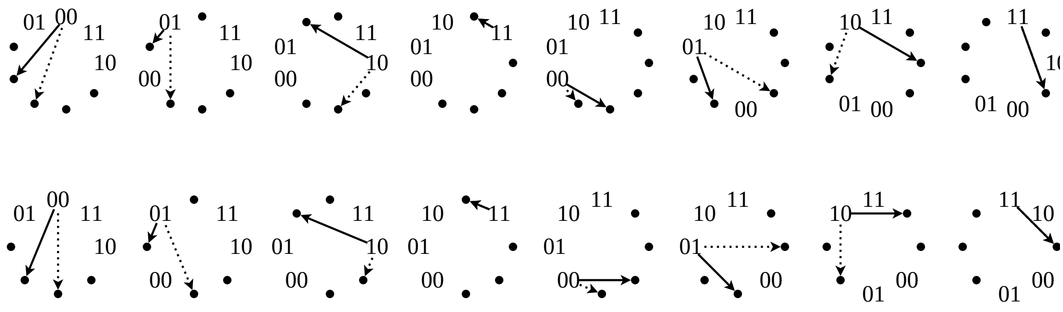


Figure 3: *Avoidance coupling of $k = 2^d$ walkers ($d = 2$) on $K_{2^{d+1}+1}$ (upper panel) and $K_{2^{d+1}}^*$ (lower panel). The walkers are naturally indexed by the vertices of a hypercube, while the states are naturally indexed by the cycle. Walker 0 and walkers 2^i each flip a coin to randomly choose from among two states, shown by the arrows, while the motions of the other walkers are determined by the motions of these walkers. In each round there are a total of $d+1$ coin flips, which is the minimum amount of randomness required for a random walk on $K_{2^{d+1}+1}$ or $K_{2^{d+1}}^*$. The avoidance coupling on $K_{2^{d+1}+1}^*$ is similar to the coupling on $K_{2^{d+1}+1}$ except that the walkers sometimes stay in place, and when they do, they stay in waves.*

We first show that this is indeed a coupling of random walkers. For all j, we have $X_t^{(\omega)} = X_t^{(j)} + \sum(1-j_i)\varepsilon_t^{(i)}2^i$ and so

$$X_{t+1}^{(j)} - X_t^{(j)} = 2^d + \delta_{t+1} + \sum_i \big[(1-j_i)\varepsilon_t^{(i)} + j_i\varepsilon_{t+1}^{(i)}\big]2^i. \tag{6.1}$$

Since $(1-j_i)\varepsilon_t^{(i)} + j_i\varepsilon_{t+1}^{(i)}$ is ± 1 we find that the sum is uniform on the odd numbers in $[-2^d, 2^d]$, and so $X_{t+1}^{(j)} - X_t^{(j)}$ is uniform on $[1, 2^{d+1}]$, as needed for the walk on $K^*_{2^{d+1}}$ or on $K_{2^{d+1}+1}$. The process $(X_t^{(j)})$ is Markov since the ε's and δ's used to define $X_{t+1}^{(j)}$ in terms of $X_t^{(j)}$ in (6.1) are disjoint from those used in any other time step. Note that the jth trajectory determines all the bits, so this is also a minimum entropy coupling.

Next we establish avoidance. Let $j < j'$ be two walkers. We have

$$\Delta := X_t^{(j')} - X_t^{(j)} = \sum_i (j'_i - j_i)\varepsilon_t^{(i)}2^i.$$

Note that $|j'_i - j_i| \le 1$, hence $|\Delta| < 2^d$, and so $\Delta = 0 \mod n$ implies $\Delta = 0$. If i_0 is the minimal index such that $j_{i_0} \neq j'_{i_0}$ then Δ is divisible by 2^{i_0} but not by 2^{i_0+1}, and so is non-zero. Thus there are no collisions within any round. Between consecutive rounds we have

$$\Delta := X_{t+1}^{(j)} - X_t^{(j')} = 2^d + \delta_{t+1} + \sum_i \big[j_i\varepsilon_{t+1}^{(i)} - (1-j'_i)\varepsilon_t^{(i)}\big]2^i. \tag{6.2}$$

Let i_1 be maximal such that $j_{i_1} \neq j'_{i_1}$. Since $j < j'$ this implies $j_{i_1} = 0$ and $j'_{i_1} = 1$. We have

$$\big|j_i\varepsilon_{t+1}^{(i)} - (1-j'_i)\varepsilon_t^{(i)}\big| \le \begin{cases} 1 & i > i_1, \\ 0 & i = i_1, \\ 2 & i < i_1. \end{cases}$$

Terms for $i > i_1$ contribute at most $2^d - 2^{i_1+1}$ in absolute value to the sum in (6.2), while terms for $i < i_1$ contribute at most $2(2^{i_1} - 1)$. Thus

$$\Delta \in [2 + \delta_{t+1}, 2^{d+1} - 2 + \delta_{t+1}]$$

and so $\Delta \neq 0 \mod n$.

To see that this coupling is Markovian, note that $X_t^{(0)}$ is determined by $X_{t-1}^{(\omega)}$ and δ_t. Similarly, $X_t^{(2^i)}$ is determined by $X_t^{(0)}$ and $\varepsilon_t^{(i)}$, and the position of any other walker $X_t^{(j)}$ (i.e., for j not a power of 2) is determined by the positions in that round of walkers with smaller index.

We can reduce the number of walkers to any value between 2 and 2^d by simply removing walkers other than 0 and ω. The Markovian property is preserved if we first remove walkers whose indices are not powers of 2.

Finally we turn to the case of k walkers on K_n^* for $n = 2^{d+1} + 1$. To do this we simply add to the coupling on K_n rounds in which all walkers rest, beginning with walker 0. For the Markovian property, we need to ensure that each walker $j \neq 0$ can detect when walker 0 has decided to rest. This is so because on K_n, given $X_t^{(\omega)}$, no vertex is a possible value for both $X_t^{(0)}$ and $X_{t+1}^{(0)}$ (otherwise $X_{t+1}^{(0)}$, which depends only on $X_t^{(\omega)}$ and δ_{t+1} but not on $X_t^{(0)}$, might stay in place). $\square$

We say that an avoidance coupling of k walkers **stays in waves** if, for some distinguished walker w, whenever w stays in place, all the other walkers do likewise at the following $k-1$ turns, while if w moves, all others do so too. (The coupling on $K^*_{2^{d+1}+1}$ in the last proof stays in waves.) Note that any Markovian avoidance coupling that stays in waves on K_n^* may be modified to obtain a Markovian avoidance coupling on K_n by removing all the looping rounds.

7 Many walkers for general n

Theorem 7.1. *There exists a Markovian avoidance coupling of k walkers on K_n^* for any $k \le n/(8\log_2 n)$, and on K_n for any $k \le n/(56\log_2 n)$.*

The constants in this theorem can easily be improved. However, as noted below, our methods will not go beyond $n/(\log_2 n)$. To prove the theorem, we make use of two lemmas which allow us to combine avoidance couplings.

Lemma 7.2. *Suppose that we have avoidance couplings of r walkers on K_m^* and of s walkers on K_n^*. Then there is an avoidance coupling of k walkers on K_{mn}^*, for any k satisfying $r+s-1 \le k \le rs$. If the given couplings are Markovian, then so is the new coupling. If the given couplings stay in waves, then so does the new coupling. If the given couplings are minimum-entropy, then the new coupling is too.*

Proof. We identify K_{mn}^* with $K_m^* \times K_n^*$, and note that if X_t and Y_t are independent random walks on K_m^* and K_n^* respectively, then (X_t, Y_t) is a random walk on K_{mn}^*. Given an avoidance coupling $\{X_t^{(i)}\}$ of r walkers on K_m^* and an independent avoidance coupling $\{Y_t^{(j)}\}$ of s walkers on K_n^*, we construct a coupling on K_{mn}^* of rs walkers with labels (i,j), for $1 \le i \le r$ and $1 \le j \le s$. The walkers move in lexicographic order. The trajectory of walker (i,j) is given by $(X_t^{(i)}, Y_t^{(j)})$, which as noted above is a random walk on K_{mn}^*. That the walkers avoid collisions follows from the product construction and the collision avoidance of the given couplings. If the given couplings are Markovian, then since the walkers on K_{mn}^* move in lexicographic order, the resulting coupling is also Markovian. It is clear that the coupling stays in waves provided both original couplings do. Finally, no randomness is required beyond that in the couplings on K_m^* and on K_n^*, so if they are minimum entropy, so is the resulting coupling.

To construct a coupling of fewer walkers, just eliminate some of the walkers, as long as walkers $(i,1)$ and $(1,j)$ (for each $1 \le i \le r$ and $1 \le j \le s$) are kept. All other trajectories are determined by those, so the Markov property is maintained. $\square$

We remark that a variant of the above construction can be used to combine an avoidance coupling of r walkers on K_m and an avoidance coupling of s walkers on K_n^* that stays in waves to produce an avoidance coupling of rs walkers on K_{mn}.

Lemma 7.3. *Suppose we have avoidance couplings for k walkers on K_m and on K_n (respectively, K_m^* and K_n^*). Then we have an avoidance coupling for k walkers on K_{m+n} (respectively, K_{m+n}^*). If the original couplings are Markovian then so is the resulting coupling.*

Proof. Partition the vertex set of K_{m+n} into two clusters U and V of sizes m and n, respectively. We will ensure that when it is the first walker's turn, all of the walkers are in the same cluster. At each of her turns, the first walker flips an appropriately biased coin to decide whether to move within her current cluster or to switch to the other cluster. If she stays in her current cluster she moves according to that cluster's coupling rules, and so do the rest of the walkers. If she switches to the other cluster, she moves to a uniformly random vertex therein. Each subsequent walker now chooses a random k-walker configuration in the new cluster (say, V) consistent with the walkers that are already in V, in accordance with the stationary distribution on configurations of the K_n (or K_n^*) coupling arising just before a move of the first walker. He then moves to his allotted space in this configuration. $\square$

Proof of Theorem 7.1. We begin with the case of K_n^*. By Theorem 6.1 and Lemma 7.2 we have a Markovian avoidance coupling for $k \le 2^d$ walkers on K_n^* where n is of the form $n = 2^{a+1}(2^{d-a+1}+1) = 2^{d+2} + 2^{a+1}$ for any $a \le d$, as well as $n = 2^{d+1}+1$, $2^{d+2}+1$, and 2^{d+1}.

For general $n > 0$, we write $n = \sum_i n_i 2^i$ where $n_i \in \{0,1\}$. We define r by

$$n = \sum_{i=0}^{d} n_i(2^{d+2} + 2^i) + r,$$

and it is clear that $2^{d+1}|r$. If $r \ge 0$, then Lemma 7.3 provides a Markovian avoidance coupling for K_n^*; this inequality indeed holds whenever

$$n \ge \sum_{i=0}^{d}(2^{d+2} + 2^i) = (d+\tfrac{3}{2})2^{d+2} - 1\ .$$

Now any $n \ge 8$ satisfies $(d+2)2^{d+2} \le n < (d+3)2^{d+3}$ for some integer $d \ge 0$. Thus the above gives a Markovian avoidance coupling for any number of walkers up to 2^d. By the first inequality, $8 \times 2^d \le n$, so $d+3 \le \log_2 n$, which combined with the second inequality gives $2^d > \frac{1}{8}n/(d+3) \ge n/(8\log_2 n)$, proving the theorem for K_n^* for $n \ge 8$. The claim of the theorem is trivial for $n < 8$.

We now turn to the case of K_n (without loops). Recall that if we have a Markovian avoidance coupling on K_n^* that stays in waves, then removing the looping rounds yields such a coupling on K_n. Fix $d \ge 1$, and let S be the set of values of n for which Markovian avoidance couplings exist on K_n^* for *every* number of walkers up to 2^{2d-1}, all of them staying in waves. By Lemma 7.3, S is closed under addition. From Theorem 6.1, we see that S contains 2^c+1 for all $c \ge 2d$. Using Theorem 6.1 and Lemma 7.2, when $a \ge 1$, $b \ge 1$, and $a + b \ge 2d+1$, there is a Markovian avoidance coupling for $n = (2^a+1)(2^b+1)$ with $x\,2^{b-1} - y$ walkers, where $1 \le x \le 2^{a-1}$ and $0 \le y < 2^{b-1}$. In particular, S contains $(2^a+1)(2^b+1)$, and specifically S contains $2^{2d+1} + 1 + 2^i + 2^{2d+1-i}$ for all $1 \le i \le d$ (and also for $i = 0$ using Theorem 6.1 and Lemma 7.3).

For any $m < 2^{d+1}$ we write $m = \sum_{i\le d} m_i 2^i$ with $m_i \in \{0,1\}$, and denote by $\widehat{m} = \sum m_i 2^{d-i}$ the number with reversed binary expansion. Then for $m \ne 0$, S contains

$$\sum_{i\le d} m_i(2^{2d+1} + 1 + 2^i + 2^{2d+1-i}) = \|m\|(2^{2d+1}+1) + m + 2^{d+1}\widehat{m},$$

where $\|m\| := \sum m_i$ denotes the Hamming weight of m. For simplicity (at the expense of the final constant) we eliminate the dependence on Hamming weight: since $\|m\| \le d+1$ and $2^{2d+1}+1 \in S$ we have

$$(d+1)(2^{2d+1}+1) + m + 2^{d+1}\widehat{m} \in S \tag{7.1}$$

(which holds also for $m = 0$). In the same way, but using $2^{2d+2} + 1 + 2^i + 2^{2d+2-i}$ instead, we find that

$$(d+1)(2^{2d+2}+1) + m + 2^{d+2}\widehat{m} \in S. \tag{7.2}$$

Write $m' = 2^{d+1}-1-m = \sum(1-m_i)2^i$, and observe that $\widehat{m'} = \widehat{m}'$. Using (7.1), together with (7.2) with m' in place of m, and adding, we get $k_0 + 2^{d+1}\widehat{m'} \in S$ where $k_0 = (3d+5)2^{2d+1} + 2d + 1$. Adding another copy of (7.1) we find that $k_1+m \in S$, where $k_1 = (d+2)2^{2d+3} - 2^{d+1} + 3d + 2$. Since the last two statements hold for all values of $m < 2^{d+1}$, we may combine them to deduce, for any $m_0, m_1 < 2^{d+1}$, that

$$k_2 + 2^{d+1}m_1 + m_0 \in S,$$

where

$$k_2 = k_0 + k_1 = (7d+13)2^{2d+1} - 2^{d+1} + 5d + 3.$$

It follows that $[k_2, k_2+2^{2d}] \subseteq S$, and since $2^{2d}+1 \in S$, any integer at least k_2 is in S. Thus for any $n \geq 7(d+2)2^{2d+1}$, there is a Markovian avoidance coupling for any number up to 2^{2d-1} walkers on K_n. Given n, choose d so that $7(d+2)2^{2d+1} \leq n < 7(d+3)2^{2d+3}$. From the second inequality we have $2^{2d-1} > \frac{n}{7\times 16(d+3)}$. From the first inequality we have $2(d+3) \leq \log_2 n - \log_2(7(d+2)) + 5 \leq \log_2 n$ (provided $d \geq 3$). But $d \geq 3$ for any $n \geq 7(3+2)2^{2\times 3+1} = 4480$. So for $n \geq 4480$ we can couple up to $\frac{n}{7\times 8\log_2 n}$ walkers on K_n.

Since there exists a Markovian avoidance coupling of 8 walkers on K_{17} and on K_{33}, such a coupling also exists on K_n for any nonzero $n = 17a + 33b$ with $a, b \geq 0$. This includes all $n > 511 = 33 \times 17 - 33 - 17$, and implies the claim for $512 \leq n \leq 4480$ (since $\frac{4480}{56\log_2 4480} < 8$). Finally, the claim is trivial for $n \leq 511$ since $\frac{n}{56\log_2 n} < 2$. ☐

We combined the number-theoretic avoidance coupling from Theorem 6.1 with the sum and product lemmas to obtain an avoidance coupling with $\Omega(n/\log n)$ walkers for any n. Given these three ingredients, this general-n construction is in a sense best possible up to constants. More precisely, we argue below that these three ingredients cannot be combined to obtain a coupling of more than $n/\|n\|$ walkers on K_n or K_n^*, where $\|n\|$ is the Hamming weight of n.

By the distributive law, any coupling that can be constructed using the sum and product lemmas 7.3 and 7.2 can be done by taking sums of products of basic constructions. Consider the product of s basic couplings of 2^{d_j} walkers on either $2^{d_j+1}+1$ or 2^{d_j+1} vertices. Note that $\|ab\| \leq \|a\|\|b\|$. The product lemma gives a coupling of 2^d walkers on K_m or K_m^*, where $d = \sum d_j$ and $m \geq 2^{d+s}$ and $\|m\| \leq 2^s$. In particular $m \geq 2^d\|m\|$. Next suppose that n is the sum of several such product terms, say $n = \sum_i m_i$, each corresponding to the same d. Then $n \geq 2^d \sum_i \|m_i\| \geq 2^d\|n\|$. In particular the number of walkers is at most $n/\|n\|$.

Thus, to improve on the $\Omega(n/\log n)$ bound for general n, more ingredients would be needed.

8 Negative result

In the negative direction, we have very little.

Theorem 8.1. *No avoidance coupling is possible for $n-1$ walkers on K_n^*, for $n \geq 4$.*

Proof. We exploit the effect that it is difficult for a walker to leave a vertex v at one step and then immediately return to v at the next step. This requires that none of the other walkers enter v in the interim. But since v is the only available vertex for a move, this means that all other walkers must remain stationary.

Let A_t^i be the event that the ith walker is in the same position at times $t-1$ and $t+1$, but in a different position at time t. Let B_t^i be the event that the ith walker is in the same position at times $t-1$ and t.

Suppose an avoidance coupling exists. From the observation in the first paragraph, the events A_t^1 and A_t^2 are disjoint, and each of them implies the event B_t^3. Since each walker individually performs a random walk, the events A_t^1 and A_t^2 have probability $(n-1)/n^2$, so that the probability of B_t^3 must be at least $2(n-1)/n^2$. But the probability of B_t^3 should be exactly $1/n$, which is less than $2(n-1)/n^2$. This gives a contradiction, as required. ☐

9 Open Problems

We have barely scratched the surface of avoidance coupling in this work; in particular we have considered only complete graphs and concentrated on discrete, alternating, Markovian couplings. Even in this limited realm, many intriguing open questions remain:

1. **Maximum number of walkers.** Is there an avoidance coupling for a linear number of walkers on the unlooped complete graph K_n for general n? Can upper bounds of the form cn for $c < 1$ be found for the maximum number of walkers that can be avoidance-coupled on K_n or K_n^*? Ditto for Markovian couplings?
2. **Monotonicity in n.** If there is an avoidance coupling for k walkers on K_n, must there necessarily be one for k walkers on K_{n+1}? Similarly in the Markovian case, for either K_n versus K_{n+1} or K_n^* versus K_{n+1}^*.
3. **Monotonicity in k.** If there is a Markovian avoidance coupling for k walkers on K_n (or K_n^*), is there one for $k-1$ walkers on the same graph? The answer is "yes" for non-Markovian couplings, since the kth walker can be imagined. The answer is "yes" for the Markovian couplings that we exhibited, but it is not clear if this holds in general.
4. **Monotonicity in loop weights.** Suppose that K_n is equipped with loops of weight w, so that a walker loops with probability $w/(w+n-1)$. If there is an avoidance coupling for k walkers on K_n with loops of weight w, must there be one with loops of weight w', where $w' > w$? The answer is "yes" for non-Markovian couplings, but what about the Markovian case? In particular, is the maximum number of Markovian avoiding walkers always at least as great on K_n^* as it is on K_n?
5. **Minimum entropy couplings.** Does existence of an avoidance coupling imply existence of a stationary avoidance coupling whose entropy equals that of a single random walk?
6. **1-avoidance.** What is the largest p for which k i.i.d. $\mathrm{Bernoulli}(p)$ sequences can be coupled, taking turns to move as usual, so that no two simultaneously take the value 1?

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References

[1] David Aldous and James Fill. *Reversible Markov Chains and Random Walks on Graphs.* 2002. Draft, http://www.stat.berkeley.edu/~aldous/RWG/book.html.

[2] Itai Benjamini, Krzysztof Burdzy, and Zhen-Qing Chen. Shy couplings. *Probab. Theory Related Fields*, 137(3-4):345–377, 2007. MR-2278461

[3] Maury Bramson, Krzysztof Burdzy, and Wilfrid S. Kendall. Shy couplings, CAT(0) spaces, and the lion and man. 2010. arXiv:1007.3199.

[4] Don Coppersmith, Prasad Tetali, and Peter Winkler. Collisions among random walks on a graph. *SIAM J. Discrete Math.*, 6(3):363–374, 1993. MR-1229691

[5] Peter Gács. Clairvoyant scheduling of random walks. *Random Structures & Algorithms*, 39(4):413–485, 2011. MR-2846299

[6] Wilfrid S. Kendall. Brownian couplings, convexity, and shy-ness. *Electron. Commun. Probab.*, 14:66–80, 2009. MR-2481667

[7] David A. Levin, Yuval Peres, and Elizabeth L. Wilmer. *Markov Chains and Mixing Times*. American Mathematical Society, Providence, RI, 2009. With a chapter by James G. Propp and David B. Wilson. `http://pages.uoregon.edu/dlevin/MARKOV/markovmixing.pdf`. MR-2466937

[8] Prasad Tetali and Peter Winkler. Simultaneous reversible Markov chains. In *Combinatorics, Paul Erdős is eighty, Vol. 1*, Bolyai Soc. Math. Stud., pages 433–451. János Bolyai Math. Soc., Budapest, 1993. MR-1249726