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Representation theorem for SPDEs via backward doubly SDEs*

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Abstract

In this paper we establish a probabilistic representation for the spatial gradient of the viscosity solution to a quasilinear parabolic stochastic partial differential equations (SPDE, for short) in the spirit of the Feynman-Kac formula, without using the derivatives of the coefficients of the corresponding backward doubly stochastic differential equations (FBDSDE, for short).

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1 Introduction

Linear backward stochastic differential equations (BSDEs, for short) have been considered by Bismut [1, 2] in the context of optimal stochastic control. However, nonlinear BSDEs and their theory have been introduced by Pardoux and Peng [15]. They have been enjoying a great interest in the last twenty year because of their connection with applied fields. For stochastic control and stochastic games (see [10]) and mathematical finance (see [6]). BSDEs also provide a probabilistic interpretation for solutions to elliptic or parabolic nonlinear partial differential equations generalizing the classical Feynman-Kac formula [16, 18]. In 1994, Pardoux and Peng [17] introduced a new class of BSDEs called backward doubly stochastic differential equations (BDSDEs, in short). Coupled with the forward SDE, we have the following: for all $s \in [0, T]$,

$$\begin{aligned} X_t &= x + \int_0^t b(r, X_r)\, dr + \int_0^t \sigma(r, X_r) dW_r, \\ Y_s &= l(X_T) + \int_s^T f(r, X_r, Y_r, Z_r)\, dr + \int_s^T g(r, X_r, Y_r)\, dB_r - \int_s^T Z_r dW_r, \end{aligned} \tag{1.1}$$

where the integral driven by the $\mathbb{R}^d$-valued process $\{B_r\}_{r\geq 0}$ is a backward Itô integral and the integral driven by the $\mathbb{R}^d$-valued process $\{W_r\}_{r\geq 0}$ is the standard forward Itô integral. The solution of such equation is the triple of measurable processes

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$(X^{t,x}, Y^{t,x}, Z^{t,x})$. They showed among other that BDSDEs are suitable tool to give a probabilistic representation for a solution of the following parabolic stochastic partial differential equations (SPDEs):

$$\begin{aligned} &du(t,x) = [\mathcal{L}u(t,x) + f(t,x,u(t,x),(\nabla u\sigma)(t,x))]\,dt + g(t,x,u(t,x))\,dB_t,\ (t,x)\in(0,T)\times\mathbb{R}^d,\\ &u(0,x) = l(x), \qquad\qquad x\in\mathbb{R}^d, \end{aligned} \tag{1.2}$$

with

$$\mathcal{L} = \frac{1}{2}\sum_{i,j=1}^{n}\sum_{l=1}^{k}\sigma_{il}\sigma_{lj}(t,x)\partial^2_{x_ix_j} + \sum_{j=1}^{n} b_j(t,x)\partial_{x_j}.$$

More precisely, assume that the functions f, g and l are smooth enough (e.g., f, g and l are both C^3 in their spatial variables), they provided an explicit expression of the random field u, the classical solution to the quasilinear SPDEs (1.2) as follows:

$$u(t,x) = Y_t^{t,x} = \mathbb{E}\left\{ l(X_T^{t,x}) + \int_t^T f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,dr + \int_t^T g(r,X_r^{t,x},Y_r^{t,x})\,dB_r|\mathcal{F}^B_{t,T}\right\}. \tag{1.3}$$

They also proved that the process Z has continuous paths, and satisfies the following identity:

$$Z_s^{t,x} = \nabla_x u(s,X_s^{t,x})\sigma(s,X_s^{t,x}), \quad s\in[t,T]. \tag{1.4}$$

The extension to the general case (f and l are only Lipchitz) is much more delicate because of difficulties to extend the notion of viscosity solutions to SPDEs. The stochastic viscosity solution for semi-linear SPDEs has been introduced firstly by Lions and Souganidis in [12]. They used the so-called "stochastic characteristic" to remove the stochastic integral from SPDE. Buckdahn and Ma [3, 4] give another approach of stochastic viscosity solution of SPDE in order to connect it to the following BDSDEs: for each $t\in[0,T]$ and $0\le s\le t$,

$$\begin{aligned} X_s^{t,x} &= x + \int_s^t b(r,X_r^{t,x})\,dr + \int_s^t \sigma(r,X_r^{t,x})dW_r \\ Y_s^{t,x} &= l(X_0^{t,x}) + \int_0^s f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})\,dr + \int_0^s g(r,X_r^{t,x},Y_r^{t,x})\,dB_r - \int_0^s Z_r^{t,x}dW_r, \end{aligned} \tag{1.5}$$

which are in fact a time reversal of that considered by Pardoux and Peng [17]. In this framework, the stochastic integral whith respect to dB is a forward Itô integral and the stochastic integral driving by dW is a backward Itô integral. By Doss-Sussman transformation, they proved that, thanks to the Blumenthal 0-1 law, $u(.,.)$, defined by (1.3) and seen as a random field, is a stochastic viscosity solution of the SPDE (1.2).

The aim of this paper is to face the representation of the spatial gradient under weak conditions. To the best of our knowledge, whenever the coefficients f and l are continuously differentiable, there has no discussion in the literature concerning the spatial gradient of the viscosity solution u, whenever it exists. In our set-up, by using arguments based on the corresponding BDSDEs (1.5), we obtain the desired result under only continuous differentiability condition on coefficients l and f. Roughly speaking, we show that the viscosity solution u to SPDE (1.2) has a continuous spatial gradient $\nabla_x u$. Moreover, the following probabilistic representation holds:

$$\nabla_x u(t,x) = \mathbb{E}\left\{ l(X_0^{t,x})N_0^t + \int_0^t f(r,X_r^{t,x},Y_r^{t,x},Z_r^{t,x})N_r^t\,dr + \int_0^t Dg(r,X_r^{t,x},Y_r^{t,x})\,dB_r|\mathcal{F}^B_t\right\}, \tag{1.6}$$

where "Dg" denotes the classical differential of the function g and N^t is some process defined on $[0,t]$, for each $t \in [0,T]$, depending only on the forward diffusion and its variational equation. Such a relation in a sense could be viewed as an extension of the nonlinear Feynman-Kac formula to stochastic PDEs, which, to our best knowledge, is new. The main significance of the formula, however, lies in that it does not depend on the derivatives of the coefficients of the BSDE, a pleasant surprise in many ways. Because of this special feature, and with the help of the identity (1.6), we hope to derive a similar representation for the martingale integrand Z, under only a Lipschitz condition on f and l. This latter representation then enables us to prove the path regularity of the process Z.

For the rest of this paper, we give all the necessary preliminaries in section 2, our main results are stated in section 3 while section 4 is devoted to its proof .

2 Preliminaries

Let $T > 0$ a fixed time horizon. Throughout this paper $\{W_t, 0 \le t \le T\}$ and $\{B_t, 0 \le t \le T\}$ will denote two independent d-dimensional Brownian motions defined on the complete probability spaces $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$ respectively. For any process $\{U_s,\ 0 \le s \le T\}$ defined on $(\Omega_i, \mathcal{F}_i, \mathbb{P}_i)$ $(i = 1, 2)$, we write $\mathcal{F}^U_{s,t} = \sigma(U_r - U_s,\ s \le r \le t)$ and $\mathcal{F}^U_t = \mathcal{F}^U_{0,t}$. Unless otherwise specified we consider

$$\Omega = \Omega_1 \times \Omega_2,\ \mathcal{F} = \mathcal{F}_1 \otimes \mathcal{F}_2 \ \text{ and } \mathbb{P} = \mathbb{P}_1 \otimes \mathbb{P}_2.$$

For each $t \in [0,T]$, we define

$$\mathbf{F}^t = \{\mathcal{F}_s = \mathcal{F}^B_s \otimes \mathcal{F}^W_{s,t} \vee \mathcal{N},\ 0 \le s \le t\}$$

where $\mathcal{N}$ is the collection of $\mathbb{P}$-null sets. Note that the collection $\mathbf{F}^t$ is neither increasing nor decreasing, it does not constitute a filtration. Next, the random variables $\xi(\omega_1)$, $\omega_1 \in \Omega_1$ and $\zeta(\omega_2)$, $\omega_2 \in \Omega_2$ are considered as random variables on Ω via the following identification:

$$\xi(\omega_1, \omega_2) = \xi(\omega_1); \quad \zeta(\omega_1, \omega_2) = \zeta(\omega_2). \tag{2.1}$$

Let E denote a generic Euclidean space (or $E_1, E_2, ..,$ if different spaces are used simultaneously); regardless of its dimension we denote $\langle ; \rangle$ the inner product and $|.|$ the norm in E. Furthermore, we use the notation $\partial_t = \frac{\partial}{\partial t},\ \partial_x = (\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, .., \frac{\partial}{\partial x_d})$ and $\partial^2 = \partial_{xx} = (\partial^2_{x_i x_j})^d_{i,j=1}$, for $(t,x) \in [0,T] \times \mathbb{R}^d$. Note that if $\psi = (\psi^1, .., \psi^d) : \mathbb{R}^d \to \mathbb{R}^d$, then $\partial_x \psi \triangleq (\partial x_j \psi^i)^d_{i,j=1}$ is a matrix. The meaning of $\partial_{xy}, \partial_{yy}, \cdot\cdot$ etc should be clear from the context.

The following spaces will be used frequently in the sequel ($\mathcal{X}$ denote a generic Banach space):

1. $L^0([0,T]; \mathcal{X})$ is the space of all measurable functions $\varphi : [0,T] \mapsto \mathcal{X}$.
2. $C([0,T]; \mathcal{X})$ is the space of all continuous functions $\varphi : [0,T] \mapsto \mathcal{X}$; further, for any $p > 0$ we denote $|\varphi|^{*,p}_{0,t} = \sup\limits_{0 \le s \le T} \|\varphi(s)\|^p_{\mathcal{X}}$ when the context is clear.
3. For any $k, n \ge 0$, $C^{k,n}([0,T] \times E; E_1)$ is the space of all E_1-valued functions $\varphi(t,e)$, $(t,e) \in [0,T] \times E$, such that they are k-times continuously differentiable in t and n-times continuously differentiable in e.
4. $C^{k,n}_b([0,T] \times E; E_1)$ is the space of those $\varphi \in C^{k,n}([0,T] \times E; E_1)$ such that all the partial derivatives are uniformly bounded.

5. For any $k,\ n,\ m \ge 0,\ C^{k,n,m}([0,T]\times E\times E';E_1)$ is the space of all E_1-valued functions $\varphi(t,e,e'),\ (t,e,e')\in[0,T]\times E\times E'$, such that they are k-times continuously differentiable in t, n-times continuously differentiable in e and m-times continuously differentiable in e'.
6. $C_b^{k,n,m}([0,T]\times E;E_1)$ is the space of those $\varphi\in C^{k,n,m}([0,T]\times E\times E';E_1)$ such that all the partial derivatives are uniformly bounded.
7. $W^{1,\infty}(E,E_1)$ is the space of all measurable functions $\psi : E\mapsto E_1$, such that for some constant $K>0$ it holds that $|\psi(x)-\psi(y)|_{E_1}\le K|x-y|_E, \forall x,y\in E$.
8. For any sub-σ-field $\mathcal{G}\subseteq\mathcal{F}_T^B$ and $0\le p<\infty,\ L^p(\mathcal{G};E)$ denote all E-valued $\mathcal{G}$-measurable random variable ξ such that $\mathbb{E}|\xi|^p<\infty$. Moreover, $\xi\in L^\infty(\mathcal{G};E)$ means it is $\mathcal{G}$-measurable and bounded.
9. For $0\le p<\infty$, $L^p(\mathbf{F},[0,T];\mathcal{X})$ is the space of all $\mathcal{X}$-valued, $\mathbf{F}$-adapted processes ξ satisfying $\mathbb{E}\left(\int_0^T\|\xi_t\|_{\mathcal{X}}^p dt\right)<\infty$. Also, $\xi\in L^\infty(\mathbf{F},[0,T];\mathbb{R}^d)$ means that the process ξ is uniformly essentially bounded in (t,ω).
10. $C(\mathbf{F},[0,T]\times E;E_1)$ is the space of E_1-valued, continuous random field $\varphi:\Omega\times[0,T]\times E$, such that for fixed $e\in E$, $\varphi(.,.,e)$ is an $\mathbf{F}$-adapted process.

To simplify notation we often write $C([0,T]\times E;E_1)=C^{0,0}([0,T]\times E;E_1)$; and if $E_1=\mathbb{R}$, then we often suppress E_1 for simplicity (e.g., $C^{k,n}([0,T]\times E;\mathbb{R})=C^{k,n}([0,T]\times E)$, $C^{k,n}(\mathbf{F},\ [0,T]\times E;\mathbb{R})=C^{k,n}(\mathbf{F},\ [0,T]\times E)$,..., etc.). Finally, unless otherwise specified, all vectors in the paper will be regarded as column vectors.

Throughout this paper we shall make use of the following standing assumptions:

($\mathbf{A1}$) The functions $\sigma\in C_b^{0,1}([0,T]\times\mathbb{R}^d;\ \mathbb{R}^{d\times d})$, $b\in C_b^{0,1}([0,T]\times\mathbb{R}^d;\mathbb{R}^d)$; and all the partial derivatives of b and σ (with respect to x) are uniformly bounded by a common constant $K>0$. Further, there exists constant $c>0$, such that

$$\xi^T\sigma(t,x)\sigma(t,x)^T\xi\ge c|\xi|^2,\ \forall\ x,\ \xi\in\mathbb{R}^d,\ t\in[0,T]. \tag{2.2}$$

($\mathbf{A2}$) The function $f\in C_b^1(\mathcal{F}^B,[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)\cap W^{1,\infty}([0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}^d)$ and $l\in W^{1,\infty}(\mathbb{R}^d)$. Furthermore, we denote the Lipschitz constants of f and l by a common one $K>0$ as in ($\mathbf{A1}$); and we assume that

$$\sup_{0\le t\le T}\{|b(t,0)|+|\sigma(t,0)|+|f(t,0,0,0)|+|l(0)|\}\le K. \tag{2.3}$$

($\mathbf{A3}$) The function $g\in C_b^{0,2,3}([0,T]\times\mathbb{R}^d\times\mathbb{R};\ \mathbb{R}^d)$.

The following results are either standard or slight variations of the well-know results in SDE and backward doubly SDE literature; we give only the statement for ready reference.

Lemma 2.1. *Suppose that* $b\in C(\mathbf{F},\ [0,T]\times\mathbb{R}^d;\ \mathbb{R}^d)\cap L^0(\mathbf{F},\ [0,T];\ W^{1,\infty}(R^d;\ \mathbb{R}^d))$, $\sigma\in C(\mathbf{F},\ [0,T]\times\mathbb{R}^d;\ \mathbb{R}^{d\times d})\cap L^0(\mathbf{F},\ [0,T];W^{1,\infty}(R^d;\ \mathbb{R}^{d\times d}))$*, with a common Lipschitz constant* $K>0$*. Suppose also that* $b(t,0)\in L^2(\mathbf{F},\ [0,T];\mathbb{R}^d)$ *and* $\sigma(t,0)\in L^2(\mathbf{F},\ [0,T];\ \mathbb{R}^{d\times d})$*. Let* X *be the unique solution of the following forward SDE*

$$X_s=x+\int_s^t b(r,X_r)\,dr+\int_s^t\sigma(r,X_r)\,dW_r. \tag{2.4}$$

Then for any $p\ge2$*, there exists a constant* $C>0$ *depending only on* p,T *and* K*, such that*

$$E(|X|_{0,t}^{*,p})\le C\left\{|x|^p+\mathbb{E}\int_0^T[|b(s,0)|^p+|\sigma(s,0)|^p]\,ds\right\} \tag{2.5}$$

Lemma 2.2. *Assume $f \in C(\mathbf{F},\ [0,T]\times\mathbb{R}\times\mathbb{R}^d)\cap L^0(\mathbf{F},\ [0,T];\ W^{1,\infty}(\mathbb{R}\times R^d))$, with a uniform Lipschitz constant $K>0$, such that $f(s,0,0)\in L^2(\mathbf{F},\ [0,T])$ and $g \in C(\mathbf{F},\ [0,T]\times\mathbb{R}\times\mathbb{R}^d;\ \mathbb{R}^d)\cap L^0(\mathbf{F},\ [0,T];\ W^{1,\infty}(\mathbb{R}\times\mathbb{R}^d;\ \mathbb{R}^d))$ with a common uniform Lipschitz constant $K>0$ with respect the first variable and the Lipschitz constant $0<\alpha<1$ which respect the second variable and such that $g(s,0,0)\in L^2(\mathbf{F},\ [0,T])$. For any $\xi \in L^2(\mathcal{F}_0;\mathbb{R})$, let (Y,Z) be the adapted solution to the BDSDE:*

$$Y_s = \xi + \int_0^s f(r,Y_r,Z_r)\,dr + \int_0^s g(r,Y_r,Z_r)\,dB_r - \int_0^s Z_r\,dW_r. \tag{2.6}$$

Then there exists a constant $C>0$ depending only on T and on the Lipschitz constants K and α, such that

$$\mathbb{E}\int_0^T |Z_s|^2 ds \leq C\mathbb{E}\left\{|\xi|^2 + \int_0^T [|f(s,0,0)|^2 + |g(s,0,0)|^2]\,ds\right\}. \tag{2.7}$$

Moreover, for all $p\geq 2$, there exists a constant $C_p>0$, such that

$$\mathbb{E}(|Y|^{*,p}_{0,t}) \leq C_p\mathbb{E}\left\{|\xi|^p + \int_0^T [|f(s,0,0)|^p + |g(s,0,0)|^p]\,ds\right\} \tag{2.8}$$

To end this section, let us consider the following variational equation that will play a important role in this paper: for $i=1,..,d$,

$$\begin{aligned}
\nabla_i X_s^{t,x} &= e_i + \int_s^t \partial_x b(r,X_r^{t,x})\nabla_i X_r^{t,x} dr + \sum_{j=1}^d \int_s^t \partial_x \sigma^j(r,X_r^{t,x})\nabla_i X_r^{t,x}\,dW_r^j,\\
\nabla_i Y_s^{t,x} &= \partial_x l(X_0^{t,x})\nabla_i X_0^{t,x}\\
&\quad + \int_0^s [\partial_x f(r,\Xi^{t,x}(r))\nabla_i X_r^{t,x} + \partial_y f(r,\Xi^{t,x}(r))\nabla_i Y_r^{t,x} + \langle \partial_z f(r,\Xi^{t,x}(r)), \nabla_i Z_r^{t,x}\rangle]dr\\
&\quad + \int_0^s [\partial_x g(r,\Theta^{t,x}(r))\nabla_i X_r^{t,x} + \partial_y g(r,\Theta^{t,x}(r))\nabla_i Y_r^{t,x}]dB_r - \int_0^s \nabla_i Z_r^{t,x}\,dW_r,
\end{aligned} \tag{2.9}$$

where $e_i = (0,...,\overset{i}{1},...,0)^T \in \mathbb{R}^d$, $\Xi^{t,x} = (\Theta^{t,x}, Z^{t,x})$, $\Theta^{t,x} = (X^{t,x}, Y^{t,x})$ and $\sigma^j(.)$ is the j-th column of the matrix $\sigma(.)$. We recall again that the superscription ^{t,x} indicates the dependence of the solution on the initial date (t,x), and will be omitted when the context is clear. We also remark that under the above assumptions,

$$\left(\nabla X^{t,x}, \nabla Y^{t,x}, \nabla Z^{t,x}\right) \in L^2(\mathbf{F};[0,T];\mathbb{R}^{d\times d}) \times C([0,T];\mathbb{R}^d) \times L^2(\mathbf{F},[0,T];\mathbb{R}^{d\times d})).$$

Further the $d\times d$-matrix-valued process $\nabla X^{t,x}$ satisfies a linear SDE and $\nabla X_t^{t,x} = I$, so that $[\nabla X_s^{t,x}]^{-1}$ exists for $s\in[0,t], \mathbb{P}$-a.s. More detail, can be found in Nualart [14] and Pardoux and Peng [16].

For a fixed $t\in[0,T]$, let consider the process M^t defined as follows

$$M_r^t = \int_r^t \sigma^{-1}(\tau, X_\tau^{t,x})\nabla X_\tau^{t,x} dW_\tau,\ 0\leq r<t\leq T, \tag{2.10}$$

which will play a key role in our representation. Clearly, for fixe $t\in[0,T]$, the process M^t is a martingale and using the Burkhölder-Davis- Gundy inequality, for any $p>1$ there exist a generic constant depending only on constants K and c appear in ($\mathbf{Al}$), the time duration T and $p>1$, such that

$$\begin{aligned}
\mathbb{E}(|M_r^t|^{2p}) &\leq C_p\left(\int_r^t |\sigma^{-1}(\tau,X_\tau^{t,x})\nabla X_\tau^{t,x}|^2 d\tau\right)^p\\
&\leq C_p(t-r)^p\mathbb{E}(|\nabla X|^{*,2p}_{s,r}) \leq C_p(t-r)^p.
\end{aligned} \tag{2.11}$$

The following lemma gives the properties of the process M^t. The proof is almost similar to the proof of Proposition 4.1 in [13].

Lemma 2.3. *For any fixed* $t \in [0,T]$ *and any* $H \in L^{p_0}(\mathbf{F}^t, [0,T];\ \mathbb{R})$, *with* $p_0 > 2$,

(i) $\mathbb{E}\left|\int_0^t \frac{1}{t-r} H_r M_r^t dr\right| < +\infty$;

(ii) *the mapping* $t \mapsto \left\{\int_0^t \frac{1}{t-r} H_r M_r^t dr\right\}(\omega)$ *is Hölder-*$([p_0-2]/[p_0(p_0+2)])$ *continuous on* $[0,T]$, $I\!P$*.a.e.* $\omega \in \Omega$;

(iii) *the mapping* $t \mapsto I\!E\left\{\int_0^t \frac{1}{t-r} H_r M_r^t dr | \mathcal{F}_t^B\right\}(\omega)$ *is continuous on* $[0,T]$, $I\!P$*.a.e.* $\omega \in \Omega$.

3 Main Results

We are now ready to state our main result which give the relation between the strategy process Z and the derivative of the random field u, (solution of the SPDE (1.2)) when the coefficients l, f and g are only continuously differentiable.

Theorem 3.1. *Assume* $(\mathbf{A1})$*-*$(\mathbf{A3})$. *Let* $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *be the adapted solution to the FBDSDE* (1.5), *and set* $u(t,x) = Y_t^{t,x}$ *the stochastic viscosity solution of SPDE* (1.2). *Then,*

(i) $\nabla_x u(t,x)$ *exists for all* $(t,x) \in [0,T] \times I\!R^d$; *and for each* (t,x), *the following representation holds:*

$$\nabla_x u(t,x) = \mathbb{E}\left\{l(X_0^{t,x})N_0^t + \int_0^t f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})N_r^t\, dr + \int_0^t Dg(r, X_r^{t,x}, Y_r^{t,x})\, dB_r | \mathcal{F}_t^B\right\} \tag{3.1}$$

where

$$N_r^t = \frac{1}{t-r} M_r^t [\nabla X_t]^{-1}, \qquad 0 \le r < t \le T.$$

Let us note that for $t \in [0,T]$, *the process* N^t *is well define. Indeed, for all* $s \in [0,t]$, ∇X_s, *the solution of the forward variational equations in* (2.9) *is invertible, thanks to the Doléans-Dade stochastic exponential formula (see, e.g.,[19]);*

(ii) *For* $\nabla_x u(t,x)$ *is continuous on* $[0,T] \times I\!R^d$;

(iii) $Z_s^{t,x} = \nabla_x u(s, X_s^{t,x})\sigma(s, X_s^{t,x}), \forall\ s \in [0,t],$ $I\!P$*-a.s.*

As a byproduct, we derive the following corollary.

Corollary 3.2. *Assume that the same conditions as in Theorem* 3.1 *hold, and let* $(X^{t,x}, Y^{t,x}, Z^{t,x})$ *be the solution of FBDSDE* (1.5). *Then, there exists a constant* $C > 0$ *depending only on* K, T, *and for any* $p \ge 1$, *a positive* $L^p(\Omega, (\mathcal{F}_s^t)_{0\le s\le t}, \mathbb{P})$*-process* $\Gamma^{t,x}$, *such that*

$$|\nabla_x u(t,x)| \le C\Gamma_t^{t,x}, \quad \forall\ (t,x) \in [0,T] \times I\!R^d, \quad I\!P - a.s.$$

Consequently, one has

$$|Z_s^{t,x}| \le C\Gamma_s^{t,x}(1 + |X_s^{t,x}|), \quad \forall s \in [0,t], \quad I\!P - a.s. \tag{3.2}$$

Furthermore, $\forall\ p > 1$, *there exists a constant* $C_p > 0$, *depending on* K, T, *and* p *such that*

$$I\!E\left\{|X^{t,x}|_{0,t}^{*,p} + |Y^{t,x}|_{0,t}^{*,p} + |Z^{t,x}|_{0,t}^{*,p}\right\} \le C_p(1 + |x|^p). \tag{3.3}$$

4 Proofs

This section is devoted to prove all results appear in this paper. To simplify presentation, we shall assume that $d = 1$. The higher dimensional case can be treated in the same way without substantial difficulty. Also, in what follows we use the simpler notation b_x, σ_x, l_x, $(f_x,\ f_y,\ f_z)$ and $(g_x,\ g_y)$ for the partial derivatives of $b,\ \sigma,\ l,\ f$ and g. Moreover, we recall:

$$\Theta = (X, Y) = (X^{t,x},\ Y^{t,x}),\ \ \Xi = (\Theta, Z) = (\Theta^{t,x}, Z^{t,x}),\ \ \nabla\Xi = (\nabla\Theta, \nabla Z) = (\nabla\Theta^{t,x}, \nabla Z^{t,x}).$$

(i) Let $(t, x) \in [0, T] \times \mathbb{R}$ be fixed. For $h \neq 0$ and $s \in [0, t]$, define:

$$\nabla X_s^h := \frac{X_s^{t,x+h} - X_s^{t,x}}{h};\ \ \nabla Y_s^h := \frac{Y_s^{t,x+h} - Y_s^{t,x}}{h};\ \ \nabla Z_s^h := \frac{Z_s^{t,x+h} - Z_s^{t,x}}{h}.$$

One can check that the processes ∇X_s^h and $(\nabla Y_s^h, \nabla Z_s^h)$ are respectively the unique solution of the following SDEs:

$$\nabla X_s^h \ = \ 1 + \int_s^t \widetilde{b}_x^h(r)\nabla X_r^h dr + \int_s^t \widetilde{\sigma}_y^h(r)\nabla X_r^h dW_r,\ 0 \le s \le t, \tag{4.1}$$

and

$$\begin{aligned}\nabla Y_s^h \ =\ & \widetilde{l}_x^h \nabla X_0^h + \int_0^s [\widetilde{f}_x^h(r)\nabla X_r^h + \widetilde{f}_y^h(r)\nabla Y_r^h + \widetilde{f}_z^h(r)\nabla Z_r^h]dr \\ & + \int_0^s [\widetilde{g}_x^h(r)\nabla X_r^h + \widetilde{g}_y^h(r)\nabla Y_r^h]B_r - \int_0^s \nabla Z_r^h dW_r,\ 0 \le s \le t,\end{aligned} \tag{4.2}$$

where

$$\begin{aligned}
\widetilde{b}_x^h(r) &= \int_0^1 b_x(r, X_r^{t,x} + \theta h \nabla X_r^h)d\theta \qquad \widetilde{b}_x^0(r) = b_x(r, X_r^{t,x}) \\
\widetilde{\sigma}_x^h(r) &= \int_0^1 \sigma_x(r, X_r^{t,x} + \theta h \nabla X_r^h)d\theta \qquad \widetilde{\sigma}_x^0(r) = \sigma_x(r, X_r^{t,x}) \\
\widetilde{l}_x^h &= \int_0^1 l_x(X_0^{t,x} + \theta h \nabla X_0^h)d\theta \qquad \widetilde{l}_x^0 = l_x(X_0^{t,x}), \\
\widetilde{\varphi}_x^h(r) &= \int_0^1 \varphi(r, \Xi_r^{t,x} + \theta h \nabla \Xi_r^h)d\theta \qquad \widetilde{\varphi}^0(r) = \varphi(r, \Xi_r^{t,x}),\ \varphi = f_x, f_y, f_z, \text{ respectively} \\
\widetilde{\psi}_x^h(r) &= \int_0^1 \psi(r, \Theta_r^{t,x} + \theta h \nabla \Theta_r^h)d\theta \qquad \widetilde{\psi}^0(r) = \psi(r, \Theta_r^{t,x}),\ \psi = g_x, g_y, \text{ respectively.}
\end{aligned}$$

Let us first state and prove this preparing lemma.

Lemma 4.1. *Assume that conditions of Theorem* 3.1 *are fulfilled. Then,*

(i)

$$\mathbb{E}|\nabla X^h - \nabla X|_{0,t}^{*,2} \to 0 \quad \text{as} \quad h \to 0.$$

(ii)

$$\mathbb{E}\left(|\nabla Y^h - \nabla Y|_{0,t}^{*,2} + \int_0^t |\nabla Z_s^h - \nabla Z_s|^2 ds\right) \to 0 \text{ as } h \to 0.$$

Proof. (i) Applying Lemma 2.1 to (4.1), there exist a constant $C>0$, independent of h, such that, for all $p\geq 2$,

$$\mathbb{E}\left(|\nabla X^h|_{0,t}^{*,p}\right)\leq C.$$

Next, according to the definition of ∇X^h, it's holds that, for all $p\geq 2$,

$$\mathbb{E}\left(|X^{t,x+h}-X^{t,x}|_{0,t}^{*,p}\right)\to 0 \qquad \text{as} \qquad h\to 0, \tag{4.3}$$

which provides that as $h\to 0$,

$$\begin{aligned} \mathbb{E}\left(\int_0^t |\widetilde{b}_x^h(r)-\widetilde{b}_x^0(r)|^p dr\right)&\to 0,\\ \mathbb{E}\int_0^t |\widetilde{\sigma}^h(r)-\widetilde{\sigma}^0(r)|^p dr &\to 0, \end{aligned} \tag{4.4}$$

On the other hand, let us recall that the process $\Delta X_s^h := \nabla X_s^h - \nabla X_s$ satisfies the following SDE:

$$\Delta X_s^h \;=\; \int_t^s [\widetilde{b}_x(r)\Delta X_r^h+\alpha^h(r)]dr+\int_t^s[\widetilde{\sigma}_x(r)\Delta X_r^h+\beta^h(r)]dW_r, \tag{4.5}$$

where

$$\begin{aligned} \alpha^h(r) &= (\widetilde{b}_x^h(r)-\widetilde{b}_x^0(r))\nabla X_r,\\ \beta^h(r) &= (\widetilde{\sigma}_x^h(r)-\widetilde{\sigma}_x^0(r))\nabla X_r. \end{aligned} \tag{4.6}$$

Once again, applying Lemma 2.1 to (4.5) together with (4.4) and (4.6), (i) is complete.
(ii) Recalling (4.2) and applying Lemma 2.2, there exits a constant $C>0$, independent of h, such that, for all $p\geq 2$,

$$\mathbb{E}\left(|\nabla Y^h|_{0,t}^{*,p}+\int_0^t |\nabla Z_s^h|^2 ds\right)\leq C.$$

Therefore, for all $p\geq 2$,

$$\mathbb{E}\left(|Y^{t,x+h}-Y^{t,x}|_{0,t}^{*,p}+\int_0^t |Z_s^{t,x+h}-Z_s^{t,x}|^2 ds\right)\to 0 \text{ as } h\to 0, \tag{4.7}$$

and

$$\begin{aligned} \mathbb{E}|\widetilde{l}_x^h-\widetilde{l}_x^0|^p &\to 0,\\ \mathbb{E}\int_0^t |\widetilde{\varphi}^h(r)-\widetilde{\varphi}^0(r)|^p dr &\to 0,\\ \mathbb{E}\int_0^t |\widetilde{\psi}^h(r)-\widetilde{\psi}^0(r)|^p dr &\to 0, \end{aligned} \tag{4.8}$$

as $h\to 0$. Hence, since $(\nabla Y,\nabla Z)$ is the solution to the (doubly backward) variational equation in (2.9), we get

$$\mathbb{E}\left(|\nabla Y^h-\nabla Y|_{0,t}^{*,2}+\int_0^t |\nabla Z_s^h-\nabla Z_s|^2 ds\right)\to 0 \text{ as } h\to 0. \tag{4.9}$$

Indeed, denoting $\Delta Y_s^h := \nabla Y_s^h - \nabla Y_s$ and $\Delta Z_s^h := \nabla Z_s^h - \nabla Z_s$, it follows from (4.2) and (2.9) that

$$\begin{aligned}\Delta Y_s^h &= \widetilde{l}_x \Delta X_0^h + (\widetilde{l}_x^h - \widetilde{l}_x^0)\nabla X_0 \\ &\quad + \int_0^s [\widetilde{f}_x(r)\Delta X_r^h + \widetilde{f}_y(r)\Delta Y_r^h + \widetilde{f}_z(r)\Delta Z_r^h + \varepsilon_1^h(r)]dr \\ &\quad + \int_0^s [\widetilde{g}_x(r)\Delta X_r^h + \widetilde{g}_y(r)\Delta Y_r^h + \epsilon_2^h(r)]dB_r \\ &\quad - \int_0^s \Delta Z_s^h dW_s, \qquad s \in [0,t],\end{aligned} \tag{4.10}$$

where

$$\begin{aligned}\varepsilon_1^h(r) &= (\widetilde{f}_x^h(r) - \widetilde{f}_x^0(r))\nabla X_r + (\widetilde{f}_y^h(r) - \widetilde{f}_y^0(r))\nabla Y_r + (\widetilde{f}_z^h(r) - \widetilde{f}_z^0(r))\nabla Z_r, \\ \epsilon_2^h(r) &= (\widetilde{g}_x^h(r) - \widetilde{g}_x^0(r))\nabla X_r + (\widetilde{g}_y^h(r) - \widetilde{g}_y^0(r))\nabla Y_r.\end{aligned} \tag{4.11}$$

Thanks to Lemma 2.2, we obtain

$$\begin{aligned}&\mathbb{E}\left\{|\Delta Y^h|_{t,T}^{*,2} + \int_0^t |\Delta Z_s^h|^2 ds\right\} \\ \le\ & C\mathbb{E}\left\{|\Delta X_0^h|^2 + |\widetilde{l}_x^h - \widetilde{l}_x^0|^2|\nabla X_0|^2 + \int_0^t (|\Delta X_r^h|^2 + |\varepsilon(r)|^2 + |\epsilon(r)|^2)dr\right\}.\end{aligned}$$

Thus, the result follows from (4.3), (4.8), (4.11) and the dominated convergence theorem. □

4.1 Proof of Theorem 3.1

(i) Let consider the σ-algebra $\mathbf{F}^t = (\mathcal{F}_s^t)_{0\le s\le t}$ defined by $\mathcal{F}_s^t = \mathcal{F}_s^B \otimes \mathcal{F}_{s,t}^W$. Then $Y^{t,x}$, $Y^{t,x+h}$, ∇Y^h and ΔY^h are all $\mathbf{F}^t$-adapted processes. In particular, since W is a Brownian motion on $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, applying the Blumenthal 0-1 law (see, e.g, [11]), $Y_t^{t,x} := u(t,x)$, $Y_t^{t,x+h} := u(t,x+h)$, $\nabla Y_t^h := \frac{1}{h}[u(t,x+h) - u(t,x)]$ and ΔY_t^h are all independent of (or a constant with respect to) $\omega_2 \in \Omega_2$. Therefore it follows from Lemma 4.1 that $\partial_x u$ exists, as the random field and is equal to $\nabla Y_t^{t,x}$, for all $(t,x) \in [0,T] \times \mathbb{R}^d$. Finally, taking the conditional expectation on the both sides of (2.9) at $s = t$, we have

$$\begin{aligned}\partial_x u(t,x) &= \mathbb{E}\Big\{l_x(X_0^{t,x})\nabla X_0^{t,x} \\ &\quad + \int_0^t [f_x(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla X_r^{t,x} + f_y(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Y_r^{t,x} + f_z(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Z_r^{t,x}]dr \\ &\quad + \int_0^t [g_x(r,\Theta_r^{t,x})\nabla X_r^{t,x} + g_y(r,\Theta_r^{t,x})\nabla Y_r^{t,x}]dB_r \mid \mathcal{F}_t^B\Big\}.\end{aligned} \tag{4.12}$$

On the other hand, it follows from Proposition 2.3 in [17] and the notation of its proof that (X,Y,Z), the solution of the FBSDEs (1.5) belong to $\mathbb{D}^{1,2}$. Moreover, one can show, for $0 < \tau < r < t$,

$$\begin{aligned}&f_x(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla X_r^{t,x} + f_y(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Y_r^{t,x} + f_z(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Z_r^{t,x} \\ =\ & D_\tau f(r,\Theta_r^{t,x}, Z_r^{t,x})\sigma^{-1}(\tau, X_\tau^{t,x})\nabla X_\tau^{t,x}.\end{aligned}$$

Since the left-hand side above is independent of τ; integrating both sides from $\tau = r < t$ to $\tau = t$ and then dividing by $t - r$, we obtain that

$$\begin{aligned}&f_x(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla X_r^{t,x} + f_y(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Y_r^{t,x} + f_z(r,\Theta_r^{t,x}, Z_r^{t,x})\nabla Z_r^{t,x} \\ =\ & \frac{1}{t-r}\int_r^t D_\tau f(r,\Theta_r^{t,x}, Z_r^{t,x})\sigma^{-1}(\tau, X_\tau^{t,x})\nabla X_\tau d\tau.\end{aligned} \tag{4.13}$$

Since σ^{-1} is bounded by (2.2), the process $\sigma^{-1}(.,X)\nabla X \in L^2(\mathbf{F}^t,[0,T])$ and therefore it belong to $Dom(\delta)$ (see section 2 in [13] combined with proof of Proposition 2.3 in [17]). Further, thanks to Lemma 2.1 and Lemma 2.2, it can be checked that

$$\left\{|f(r,\Theta_r^{t,x})|^2\int_0^t |\sigma^{-1}(\tau,X_\tau)\nabla X_\tau|^2 d\tau\right\} < +\infty,\ \forall\, r\in[0,t]. \tag{4.14}$$

Thus, by Lemma 2.5 (i) in [13] and using integration by parts, we have

$$\begin{aligned} & \int_r^t D_\tau f(r,\Theta_r^{t,x},Z_r^{t,x})\sigma^{-1}(\tau,X_\tau^{t,x})\nabla X_\tau d\tau & (4.15)\\ = \ & f(r,\Theta_r^{t,x})\int_0^t \sigma^{-1}(\tau,X_\tau)\nabla X_\tau dW_\tau \\ & -\int_r^t f(r,\Theta_r^{t,x},Z_r^{t,x})\sigma^{-1}(\tau,X_\tau^{t,x})\nabla X_\tau dW_\tau, & (4.16)\end{aligned}$$

where the second integral on the right-hand side should be understood as an anticipating stochastic integral such its conditional expectation $\mathbb{E}(.|\mathcal{F}_t^B)$ is zero. Next, plug (4.16) into the right-hand side of (4.13) and then take the conditional expectation $\mathbb{E}(.|\mathcal{F}_t^B)$ on both sides, we obtain that

$$\begin{aligned} & \mathbb{E}\left(f_x(r,\Theta_r^{t,x},Z_r^{t,x})\nabla X_r^{t,x} + f_y(r,\Theta_r^{t,x},Z_r^{t,x})\nabla Y_r^{t,x} + f_z(r,\Theta_r^{t,x},Z_r^{t,x})\nabla Z_r^{t,x}|\mathcal{F}_t^B\right)\\ = \ & \frac{1}{t-r}\mathbb{E}\left\{f(r,\Theta_r^{t,x},Z_r^{t,x})M_r^t|\mathcal{F}_t^B\right\}. \end{aligned} \tag{4.17}$$

Using similar arguments, we show that

$$\mathbb{E}\left\{l_x(X_0^{t,x})\nabla X_0^{t,x}|\mathcal{F}_t^B\right\} = \frac{1}{t}\mathbb{E}\left\{l(X_0^{t,x})M_0^t|\mathcal{F}_t^B\right\}. \tag{4.18}$$

Moreover, since g belongs in $C_b^{0,2,3}([0,T]\times\mathbb{R}^d\times\mathbb{R};\mathbb{R}^d)$, we have

$$g_x(r,\Theta_r^{t,x})\nabla X_r^{t,x} + g_y(r,\Theta_r^{t,x})\nabla Y_r^{t,x} = Dg(r,\Theta_r^{t,x}), \tag{4.19}$$

where Dg denotes the classical differential of g.
Finally, plugging (4.17), (4.18) and (4.19) into (4.12), and applying Lemma 2.3, the representation (3.1) holds and (i) is complete.
(ii) Since the variable $\partial_x u(t,x) = \nabla Y_t^{t,x}$, does not depend on ω_2, we can rewrite it as follows:

$$\partial_x u(t,x) = \mathbb{E}_2(\nabla Y_t^{t,x}), \tag{4.20}$$

where $\mathbb{E}_2$ is the expectation with respect to $\mathbb{P}_2$ introduced at the beginning of Section 2. Let $(t_i,x_i)\in[0,T]\times\mathbb{R}$, $i=1,2$. We assume without losing generality that $t_1<t_2$ and in order to simplify the notation, write for $i=1,2,\ r\in[0,t_2]$

$$\begin{aligned} \Theta^i &= (X^i,Y^i) = (X^{t_i,x_i},Y^{t_i,x_i})\\ \Xi^i &= (\Theta^i,Z^i) = (\Theta^{t_i,x_i},Z^{t_i,x_i})\\ \nabla\Xi^i &= (\nabla\Theta^i,\nabla Z^i) = (\nabla\Theta^{t_i,x_i},\nabla Z^{t_i,x_i})\\ f_x^i(r) &= \partial_x f(r,\Xi^i(r)),\ f_y^i(r) = \partial_y f(r,\Xi^i(r)),\ f_z^i(r) = \partial_z f(r,\Xi^i(r)),\\ g_x^i(r) &= \partial_x g(r,\Theta^i(r)),\ g_y^i(r) = \partial_y g(r,\Theta^i(r)),\\ l_x^i &= \partial_x l(X_0^i),\ b_y^i(r) = \partial_x b(r,X^i(r)),\ \sigma_x^i(r) = \partial_x\sigma(r,X^i(r)). \end{aligned}$$

Further, we set $\widehat{\Delta}X_r = \nabla X_r^1 - \nabla X_r^2$, $\widehat{\Delta}Y_r = \nabla Y_r^1 - \nabla Y_r^2$, $\widehat{\Delta}Z_r = \nabla Z_r^1 - \nabla Z_r^2$ and for each functions φ we denote $\widehat{\Delta}_{12}[\varphi] = \varphi^1 - \varphi^2$. Also, in the sequel $C>0$ denotes a generic constant depending only on the data and may vary line to line. Recalling (4.20) and (3.1) we have, for $\mathbb{P}_1$-almost all $\omega_1 \in \Omega_1$,

$$\begin{aligned}
&|\partial_x u(t_1,x_1) - \partial_x u(t_2,x_2)|^2 \\
\le\;& C\mathbb{E}_2\left(|l_x^1 \nabla X_0^1 - l_x^2 \nabla X_0^2|^2\right) \\
&+\mathbb{E}_2\left((t_2-t_1)\int_{t_1}^{t_2}[|f_x(r)|^2|\nabla X_r^1|^2 + |f_y(r)|^2|\nabla Y_r^1|^2 + |f_z(r)|^2|\nabla Z_r^1|^2]dr\right) \\
&+\mathbb{E}_2\left(\int_{t_1}^{t_2}[|g_x(r)|^2|\nabla X_r^1|^2 + |g_y(r)|^2|\nabla Y_r^1|^2]dr\right) \\
&+\mathbb{E}_2\left(\int_0^{t_2}[|\widehat{\Delta}_{12}[f_x\nabla X_\cdot](r)|^2 + |\widehat{\Delta}_{12}[f_x\nabla Y_\cdot](r)|^2 + |\widehat{\Delta}_{12}[f_x\nabla Z_\cdot](r)|^2]dr\right) \\
&+\mathbb{E}_2\left(\int_0^{t_2}[|\widehat{\Delta}_{12}[g_x\nabla X_\cdot](r)|^2 + |\widehat{\Delta}_{12}[g_x\nabla Y_\cdot](r)|^2]dr\right). \qquad (4.21) \\
\le\;& C\mathbb{E}_2\Big\{|\widehat{\Delta}X_0|^2 + |\nabla X_0^2|^2|\widehat{\Delta}_{12}[l_x]|^2 + (t_2-t_1)\int_{t_1}^{t_2}[|\nabla X_r^1|^2 + |\nabla Y_r^1|^2 + |\nabla Z_r^1|^2]dr \\
&+\int_{t_1}^{t_2}[|\nabla X_r^1|^2 + |\nabla Y_r^1|^2]dr + \int_0^{t_2}[|\widehat{\Delta}X_r|^2 + |\widehat{\Delta}Y_r|^2 + |\widehat{\Delta}Z_r|^2]dr \\
&+\int_0^{t_2}[|\widehat{\Delta}_{12}[f_x](r)\nabla X_r^2|^2 + |\widehat{\Delta}_{12}[f_y](r)\nabla Y_r^2|^2 + |\widehat{\Delta}_{12}[f_z](r)\nabla Z_r^2|^2]dr \\
&+\int_0^{t_2}[|\widehat{\Delta}_{12}[g_x](r)\nabla X_r^2|^2 + |\widehat{\Delta}_{12}[g_y](r)\nabla Y_r^2|^2]dr\Big\}.
\end{aligned}$$

For the estimate of the right-hand side of (4.21), we note that the process $(\widehat{\Delta}X, \widehat{\Delta}Y, \widehat{\Delta}Z)$ satisfies the following coupled FSDE-BDSDE (for $s \in [0,t_2]$):

$$\begin{aligned}
\widehat{\Delta}X_s &= (\nabla X_{t_2}^1 - 1) + \int_s^{t_2}[b_x^1(r)\widehat{\Delta}X_r + \widehat{\Delta}_{12}[b_x](r)\nabla X_r^2]dr \\
&\quad + \int_s^{t_2}[\sigma_x^1(r)\widehat{\Delta}X_r + \widehat{\Delta}_{12}[\sigma_x](r)\nabla X_r^2]\,dW_r, \qquad (4.22) \\
\widehat{\Delta}Y_s &= l_x^1\widehat{\Delta}X_0 + \Delta_{12}[l_x]\nabla X_0^2 + \int_0^s[f_x^1(r)\widehat{\Delta}X_r + f_y^1(r)\widehat{\Delta}Y_r + f_z^1(r)\widehat{\Delta}Z_r + \varepsilon_3(r)]dr \\
&\quad + \int_0^s[g_x^1(r)\widehat{\Delta}X_r + g_y^1(r)\widehat{\Delta}Y_r + \varepsilon_4(r)]dB_r - \int_0^s\widehat{\Delta}Z_r\,dW_r,
\end{aligned}$$

where

$$\varepsilon_3(r) = \widehat{\Delta}_{12}[f_x](r)\nabla X_r^2 + \widehat{\Delta}_{12}[f_y](r)\nabla Y_r^2 + \widehat{\Delta}_{12}[f_z](r)\nabla Z_r^2$$

and

$$\varepsilon_4(r) = \widehat{\Delta}_{12}[g_x](r)\nabla X_r^2 + \widehat{\Delta}_{12}[g_y](r)\nabla Y_r^2.$$

Now let $G_{t_1,t_2}(.)$ denote a generic $\mathbf{F}^W$-adapted, continuous process that is uniformly bounded and satisfies $\lim_{t_1\uparrow t_2} G_{t_1,t_2}(r) = 0$, $\forall\, r \in [0,t_2]$, $\mathbb{P}_2$-a.s. Again, we allow it to vary from line to line (i.e all $\widehat{\Delta}_{12}[\varphi](.)$, where $\varphi = b_x,\ \sigma_x,\ f_x,\ f_y,\ f_z,\ g_x,\ g_y$ can be denoted as such). Applying Lemma 2.1 and recalling the assumption on b and σ, we get,

$$\begin{aligned}
\mathbb{E}_2|\widehat{\Delta}X|_{0,t_2}^{*,2} &\le C\mathbb{E}\left\{|\nabla X_{t_2}^1 - 1|^2 + \int_0^{t_2}[|\widehat{\Delta}_{12}[b_x](s)|^2 + |\widehat{\Delta}_{12}[\sigma_x](s)|^2]|\nabla X_s^2|^2 ds\right\} \\
&\le C\mathbb{E}_2\left\{|\nabla X_{t_2}^1 - 1|^2 + \int_0^{t_2} G_{t_1,t_2}(s)|\nabla X_s^2|^2 ds\right\}. \qquad (4.23)
\end{aligned}$$

Combining (4.23) with Lemma 2.2, it holds that:

$$\begin{aligned}
&\mathbb{E}_2\left\{|\widehat{\Delta}Y|^{*,2}_{0,t_2}+\int_0^{t_2}|\widehat{\Delta}Z_s|^2ds\right\}\\
\leq\ & C\mathbb{E}_2\left\{|\widehat{\Delta}X_0|^2+|\widehat{\Delta}_{12}[l_x]|^2|\nabla X_0^2|^2\right.\\
&+\int_0^{t_2}[|\widehat{\Delta}X_r|^2+|\widehat{\Delta}_{12}[f_x](s)|^2+|\widehat{\Delta}_{12}[f_y](s)|^2+|\widehat{\Delta}_{12}[f_z](s)|^2]|\nabla\Xi^2(s)|^2ds\\
&\left.+\int_0^{t_2}[|\widehat{\Delta}X_r|^2+|\widehat{\Delta}_{12}[g_x](s)|^2+|\widehat{\Delta}_{12}[g_y](s)|^2]|\nabla\Theta^2(s)|^2ds\right\}\\
\leq\ & C\mathbb{E}_2\left\{|\nabla X^1_{t_2}-1|^2+|\widehat{\Delta}_{12}[l_x]|^2|\nabla X_0^2|^2+\int_0^{t_2}G_{t_1,t_2}(s)|\nabla\Xi^2(s)|^2ds\right\}. \qquad (4.24)
\end{aligned}$$

Plugging (4.23) and (4.24) into (4.21), we obtain that:

$$\begin{aligned}
|\partial_x u(t_1,x_1)-\partial_x u(t_2,x_2)|^4 \leq\ & C\mathbb{E}_2\left\{|\nabla X^1_{t_2}-1|^4+|\widehat{\Delta}_{12}[l_x]|^4|\nabla X_0^2|^4+(t_1-t_2)^3\int_{t_1}^{t_2}|\nabla\Theta^1(s)|^4ds\right.\\
&\left.+(t_1-t_2)\int_{t_1}^{t_2}|\Xi^1(s)|^4ds+\int_0^{t_2}G_{t_1,t_2}(s)(|\nabla\Xi^2(s)|^4+|\nabla\Theta^1(s)|^4)ds\right\}.
\end{aligned}$$

Now, for $\mathbb{P}_1$-almost all $\omega_1\in\Omega_1$, and for a fixed (t_2,x_2), by dominated convergence theorem we derive that

$$|\partial_x u(t_1,x_1)-\partial_x u(t_2,x_2)|^4\to 0 \text{ as } t_1\uparrow t_2 \text{ and } x_1\to x_2.$$

Similarly we can show that, for fixed (t_1,x_1),

$$|\partial_x u(t_1,x_1)-\partial_x u(t_2,x_2)|^4\to 0 \text{ as } t_2\uparrow t_1 \text{ and } x_2\to x_1.$$

This proves (ii).
(iii) For a continuous function φ, there exists $\{\varphi^\varepsilon\}_{\varepsilon>0}$ a family of $C^{0,\infty}$ functions such that φ^ε converges to φ uniformly on ε. Since $b,\ \sigma,\ l,\ f$ are all uniformly Lipschitz continuous, we may assume that the first order partial derivatives of $b^\varepsilon,\ \sigma^\varepsilon,\ l^\varepsilon,\ f^\varepsilon$ are all uniformly bounded, by the corresponding Lipschitz constants of b,σ,l,f uniformly in $\varepsilon>0$. Let us consider $(X^{t,x}(\varepsilon),Y^{t,x}(\varepsilon),Z^{t,x}(\varepsilon))$ the unique solution of the family of FBDSDEs

$$\begin{cases}
X_s^{t,x}(\varepsilon)=x+\int_s^t b^\varepsilon(r,X_r^{t,x}(\varepsilon))dr+\int_s^t\sigma^\varepsilon(r,X_r^{t,x}(\varepsilon))\,dW_r;\\[2ex]
Y_s^{t,x}(\varepsilon)=l^\varepsilon(X_0^{t,x}(\varepsilon))+\int_0^s f^\varepsilon(r,X_r^{t,x}(\varepsilon),Y_r^{t,x}(\varepsilon),Z_r^{t,x}(\varepsilon))dr\\
+\int_0^s g(r,X_r^{t,x}(\varepsilon),Y_r^{t,x}(\varepsilon))dB_r-\int_0^s Z_r^{t,x}(\varepsilon)\,dW_r.
\end{cases}$$

Applying Theorem 3.2 in [17], we derive that u^ε defined by $u^\varepsilon(t,x)=Y_t^{t,x}(\varepsilon)$ is the classical solution of SPDE

$$\begin{cases}
du^\varepsilon(t,x)=[\mathcal{L}^\varepsilon u^\varepsilon(t,x)+f^\varepsilon(t,x,u^\varepsilon(t,x),(\nabla u^\varepsilon\sigma^\varepsilon)(t,x))]\,dt\\[2ex]
\qquad\qquad +g(t,x,u^\varepsilon(t,x))\,dB_t,\ \ (t,x)\in(0,T)\times\mathbb{R}^d,\\[2ex]
u^\varepsilon(0,x)=l^\varepsilon(x), \qquad\qquad x\in\mathbb{R}^d.
\end{cases}$$

For any $\{x^\varepsilon\} \subset \mathbb{R}^{\mathbb{N}}$ such that $x^\varepsilon \to x$ as $\varepsilon \to 0$, define $(X^\varepsilon, Y^\varepsilon, Z^\varepsilon) = (X^{t,x^\varepsilon}(\varepsilon), Y^{t,x^\varepsilon}(\varepsilon), Z^{t,x^\varepsilon}(\varepsilon))$. Using once again [17], we have

$$Y_s^\varepsilon = u^\varepsilon(s, X_s^\varepsilon); \quad Z_s^\varepsilon = \partial_x u^\varepsilon(s, X_s^\varepsilon)\sigma^\varepsilon(s, X_s^\varepsilon), \quad \forall\, s \in [0,t], \ \mathbb{P}\text{-a.s.}$$

Moreover, since $(\nabla X^\varepsilon, \nabla Y^\varepsilon, \nabla Z^\varepsilon)$ is a unique solution of forward-backward doubly SDE

$$\begin{cases} \nabla X_s^\varepsilon = 1 + \displaystyle\int_s^t b_x^\varepsilon(r, \nabla X_r^\varepsilon)\nabla X_r^\varepsilon dr + \int_s^t \sigma_x^\varepsilon(r, \nabla X_r^\varepsilon)\nabla X_r^\varepsilon\, dW_r, \\ \nabla Y_s^\varepsilon = l_x^\varepsilon(X_0^\varepsilon)\nabla X_0^\varepsilon + \displaystyle\int_0^s [f_x^\varepsilon(r, \Xi(r)^\varepsilon)\nabla X_r^\varepsilon + f_y^\varepsilon(r, \Xi(r)^\varepsilon)\nabla Y_r^\varepsilon + f_z^\varepsilon(r, \Xi(r)^\varepsilon)\nabla Z_r^\varepsilon] dr \\ \qquad + \displaystyle\int_0^s [g_x(r, \Theta(r)^\varepsilon)\nabla X_r^\varepsilon + g_y^\varepsilon(r, \Xi(r)^\varepsilon)\nabla Y_r^\varepsilon] dB_r - \int_0^s Z_r^\varepsilon dW_r, \end{cases} \tag{4.25}$$

Lemma 2.1 and Lemma 2.2 provide that, for all $p \geq 2$

$$\mathbb{E}\left\{ |X^\varepsilon|_{0,t}^{*,p} + |Y^\varepsilon|_{0,t}^{*,p} + \int_0^t |Z_s^\varepsilon|^2 ds \right\} \leq C,$$

$$\mathbb{E}\left\{ |\nabla X^\varepsilon - \nabla X|_{0,t}^{*,p} + |\nabla Y^\varepsilon - \nabla Y|_{0,t}^{*,p} + \int_0^t |\nabla Z_s^\varepsilon - \nabla Z_s|^2 ds \right\} \to 0,$$

$$\mathbb{E}\left\{ |X^\varepsilon - X|_{0,t}^{*,p} + |Y^\varepsilon - Y|_{0,t}^{*,p} + \int_0^t |Z_s^\varepsilon - Z_s|^2 ds \right\} \to 0$$

and

$$\mathbb{E}\left\{ |\nabla X^\varepsilon - \nabla X|_{0,t}^{*,p} + |\nabla Y^\varepsilon - \nabla Y|_{0,t}^{*,p} + \int_0^t |\nabla Z_s^\varepsilon - \nabla Z_s|^2 ds \right\} \to 0,$$

as $\varepsilon \to 0$. Furthermore, passing to the limit in (4.25) and using the dominated convergence theorem one derives that

$$\partial_x u^\varepsilon(t, x^\varepsilon) \to \partial_x u(t,x), \quad as\ \ \varepsilon \to 0\ \ \mathbb{P}\text{-a.s.},$$

for each $(t,x) \in [0,T] \times \mathbb{R}$. Consequently, possibly along a subsequence, we obtain

$$Z_s = \lim_{\varepsilon \to 0} Z_s^\varepsilon = \lim_{\varepsilon \to 0} \partial u^\varepsilon(s, X_s^\varepsilon)\sigma^\varepsilon(s, X^\varepsilon) = \partial u(s, X_s)\sigma(s, X_s), \quad ds \otimes d\mathbb{P}\text{-a.e.}$$

Since for $\mathbb{P}$-a.e. ω, $\partial_x u(.,.)$ and X are both continuous, the above equalities actually holds for all $s \in [0,t]$, $\mathbb{P}$-a.s. (iii) is done and the proof is complete.

4.2 Proof of Corollary 3.2

We assume without lost of generality that $p \geq 2$. By Lemma 2.1 and Lemma 2.2, there exists a constant $C > 0$ such that

$$\mathbb{E}\left\{ |\nabla X^{t,x}|_{0,t}^{*,p} + |\nabla Y^{t,x}|_{0,t}^{*,p} + \left(\int_0^T |\nabla Z_r^{t,x}|^2 dr \right)^{p/2} \right\} \leq C.$$

Then, from the identity (3.1), we deduce immediately that $|\partial_x u(t,x)| \leq C\Gamma_t^{t,x}$, for all $(t,x) \in [0,T] \times \mathbb{R}$, where

$$\begin{aligned} \Gamma_s^{t,x} \ &= \ \mathbb{E}\Bigg(|\nabla X_0^{t,x}| + \int_0^s [|\nabla X_r^{t,x}| + |\nabla Y_r^{t,x}| + |\nabla Z_r^{t,x}|] dr \\ &\qquad + \left| \int_0^s [\nabla X_r^{t,x} + \nabla Y_r^{t,x}] dB_r \right| \mid \mathcal{F}_s^t \Bigg), \forall\, s \in [0,t]. \end{aligned}$$

Moreover, for $s \in [0,t]$, $\mathbb{E}(|\Gamma_s^{t,x}|^p) \leq C$ and hence (iii) of Theorem 3.1 implies that

$$|Z_s^{t,x}| \leq C\Gamma_s^{t,x}(1+|X_s^{t,x}|), \qquad \mathbb{P}\text{-a.s.}$$

Once again Lemma 2.1, 2.2 and (3.2) yields (3.3), for $p \geq 2$.

Remark 4.2. *Originally, this paper is devoted to extend entirely the work of Ma and Zhang [13]. More precisely, we want to establish the representation theorem for the spatial gradient $\nabla_x u$ of u, the solution of SPDE and the strategy process Z of the solution to associated BDSDE respectively as follows: for all $(t,x) \in [0,T]\times\mathbb{R}^d$, and $0 \leq s \leq t$*

$$\nabla_x u(t,x) = \mathbb{E}\left\{ l(X_0^{t,x})N_0^t + \int_0^t f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})N_r^t \, dr + \int_0^t Dg(r, X_r^{t,x}, Y_r^{t,x}) \, dB_r | \mathcal{F}_t^B \right\}$$

and

$$Z_s^{t,x} = \mathbb{E}\left\{ l(X_0^{t,x})N_0^t + \int_0^s f(r, X_r^{t,x}, Y_r^{t,x}, Z_r^{t,x})N_r^t \, dr + \int_0^s Dg(r, X_r^{t,x}, Y_r^{t,x}) \, dB_r | \mathcal{F}_s^t \right\} \sigma(s, X_s^{t,x}),.$$

Unfortunately at this stage of our work, we are limited to the first representation so that the second one becomes questionable. This is due to the real difficulty to establish the analogue of Lemma 4.1 appear in [13] which is essential to prove this representation. Indeed, since for all $t \in [0,T]$, the object $\mathbf{F} = \{\mathcal{F}_s^t = \mathcal{F}_s^B \otimes \mathcal{F}_{s,T}^W \vee \mathcal{N}, \; t \leq s \leq T\}$ is not a filtration, the optional projection method used by Ma and Zhang does not apply in our case. We hope that in our future discussion, we will find an alternative to solve this problem.

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