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Covariation representations for Hermitian Lévy process ensembles of free infinitely divisible distributions

J. Armando Domínguez-Molina* Víctor Pérez-Abreu† Alfonso Rocha-Arteaga‡

Abstract

It is known that the so-called Bercovici-Pata bijection can be explained in terms of certain Hermitian random matrix ensembles $(M_d)_{d\geq 1}$ whose asymptotic spectral distributions are free infinitely divisible. We investigate Hermitian Lévy processes with jumps of rank one associated to these random matrix ensembles introduced in [6] and [10]. A sample path approximation by covariation processes for these matrix Lévy processes is obtained. As a general result we prove that any $d \times d$ complex matrix subordinator with jumps of rank one is the quadratic variation of an $\mathbb{C}^d$-valued Lévy process. In particular, we have the corresponding result for matrix subordinators with jumps of rank one associated to the random matrix ensembles $(M_d)_{d\geq 1}$.

Keywords: Infinitely divisible random matrix, matrix subordinator, Bercovici-Pata bijection, matrix semimartingale, matrix compound Poisson..
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1 Introduction

New models of infinitely divisible random matrices have emerged in recent years from both applications and theory. On the one hand, they have been important in multivariate financial Lévy modelling where stochastic volatility models have been proposed using Lévy and Ornstein-Uhlenbeck matrix valued processes; see [3], [4], [5] and [15]. A key role in these models is played by the positive-definite matrix processes and more general matrix covariation processes.

On the other hand, in the context of free probability, Bercovici and Pata [9] introduced a bijection Λ from the set of classical infinitely divisible distributions to the set of free infinitely divisible distributions. This bijection was explained in terms of random matrix ensembles by Benaych-Georges [6] and Cabanal-Duvillard [10], providing in a more palpable way the bijection Λ and producing a new kind of infinitely divisible random matrix ensembles. Moreover, the results in [6] and [10] constitute a generalization of Wigner's result for the Gaussian Unitary Ensemble and give an alternative simple infinitely divisible random matrix model for the Marchenko-Pastur distribution, for which the Wishart and other empirical covariance matrix ensembles are not infinitely divisible.

More specifically, it is shown in [6] and [10] that for any one-dimensional infinitely divisible distribution μ there is an ensemble of Hermitian random matrices $(M_d)_{d\geq 1}$,

*Universidad Autónoma de Sinaloa, Sinaloa, México. E-mail: jadguez@uas.edu.mx
†Departamento de Probabilidad y Estadística, CIMAT, Guanajuato, México. E-mail: pabreu@cimat.mx
‡Universidad Autónoma de Sinaloa, Sinaloa, México. E-mail: arteaga@uas.edu.mx

whose empirical spectral distribution converges weakly almost surely to $\Lambda(\mu)$ as d goes to infinity. Moreover, for each $d \geq 1$, M_d has a unitary invariant matrix distribution which is also infinitely divisible in the matrix sense. From now on we call these models BGCD matrix ensembles. We consider additional facts of BGCD models in Section 3.

A problem of further interest is to understand the matrix Lévy processes $\{M_d(t)\}_{t\geq 0}$ associated to the BGCD matrix ensembles. It was pointed out in [12], [14] that the Lévy measures of these models are concentrated on rank one matrices. This means that the random matrix M_d is a realization, at time one, of a matrix valued Lévy process $\{M_d(t)\}_{t\geq 0}$ with rank one jumps $\Delta M_d(t) = M_d(t) - M_d(t-)$.

The purpose of this paper is to study the structure of a $d \times d$ Hermitian Lévy process $\{L_d(t)\}_{t\geq 0}$ with rank one jumps. It is shown in Section 4 that if L_d is a $d \times d$ complex matrix subordinator, it is the quadratic variation of an $\mathbb{C}^d$-valued Lévy process X_d, being the converse and extension of a known result in dimension one, see [11, Example 8.5]. The process X_d is constructed via its Lévy-Itô decomposition. In Section 5 we consider new realizations in terms of covariation of $\mathbb{C}^d$-valued Lévy process for matrix compound Poisson process as well as sample path approximations for Lévy processes associated to general BGCD ensembles. A new insight on Marchenko-Pastur's type results for empirical covariance matrix ensembles was recently given in [8] by considering compound Poisson models (then infinitely divisible). In this direction our results show the role of covariation of d-dimensional Lévy processes as an alternative to empirical covariance processes.

For convenience of the reader, and since the material and notation in the literature is disperse and incomplete, we include Section 2 with a review on preliminaries on complex matrix semimartingales and matrix valued Lévy processes that are used later on in this paper.

2 Preliminaries on matrix semimartingales and matrix Lévy processes

Let $\mathbb{M}_{d\times q} = \mathbb{M}_{d\times q}(\mathbb{C})$ denote the linear space of $d \times q$ matrices with complex entries with scalar product $\langle A, B\rangle = \operatorname{tr}(AB^*)$ and the Frobenius norm $\|A\| = [\operatorname{tr}(AA^*)]^{1/2}$ where tr denotes the (non normalized) trace. If $q = d$, we write $\mathbb{M}_d = \mathbb{M}_{d\times d}$. The set of Hermitian random matrices in $\mathbb{M}_d$ is denoted by $\mathbb{H}_d$. Likewise, let $\mathbb{U}_{d\times q} = \mathbb{U}_{d\times q}(\mathbb{C}) = \{U \in \mathbb{M}_{d\times q} : U^*U = \mathrm{I}_q\}$. If $q = d$, $\mathbb{U}_d = \mathbb{U}_{d\times d}$.

We denote by $\mathbb{H}_{d(1)}$ the set of matrices in $\mathbb{H}_d$ of rank one and by $\mathbb{H}_d^+$ ($\overline{\mathbb{H}}_d^+$) the set of positive (nonnegative) definite matrices in $\mathbb{H}_d$. Likewise $\mathbb{H}_{d(1)}^+ = \mathbb{H}_{d(1)} \cap \overline{\mathbb{H}}_d^+$ is the set of $d \times d$ nonnegative definite matrices of rank one. Let $\mathbb{S}(\mathbb{H}_{d(1)})$ denote the unit sphere of $\mathbb{H}_{d(1)}$.

Remark 2.1. *(a) Every $V \in \mathbb{H}_{d(1)}^+$ can be written as $V = xx^*$ where $x \in \mathbb{C}^d$. One can see that x is unique if we restrict x to the set $C_+^d = \{x = (x_1, x_2, \dots, x_d) : x_1 \geq 0, x_j \in \mathbb{C}, j = 2, ..., d\}$.*

(b) Every $V \in \mathbb{H}_{d(1)}$ can be written as $V = \lambda uu^*$ where λ is the unique nonzero eigenvalue of V and u is a unitary vector in $\mathbb{C}^d$. In this representation the $d \times d$ matrix uu^* is unique.

Covariation of complex matrix semimartingales An $\mathbb{M}_{d\times q}$-valued process $X = \{(x_{ij})(t)\}_{t\geq 0}$ is a matrix semimartingale if $x_{ij}(t)$ is a complex semimartingale for each $i = 1, ..., d, j = 1, ..., q$. Let $X = \{(x_{ij})(t)\}_{t\geq 0} \in \mathbb{M}_{d\times q}$ and $Y = \{(y_{ij})(t)\}_{t\geq 0} \in \mathbb{M}_{q\times r}$ be semimartingales. Similar to the case of matrices with real entries in [3], we define the matrix covariation of X and Y as the $\mathbb{M}_{d\times r}$-valued process $[X,Y] := \{[X,Y](t) : t \geq 0\}$ with entries

$$[X,Y]_{ij}(t) = \sum_{k=1}^{q} [x_{ik}, y_{kj}](t), \tag{2.1}$$

where $[x_{ik}, y_{kj}](t)$ is the covariation of the $\mathbb{C}$-valued semimartingales $\{x_{ik}(t)\}_{t\geq 0}$ and $\{x_{kj}(t)\}_{t\geq 0}$; see [16, pp 83]. One has the decomposition into a continuous part and a pure jump part as follows

$$[X,Y](t) = [X^c, Y^c](t) + \sum_{s\leq t} (\Delta X(s))(\Delta Y(s)), \tag{2.2}$$

where $[X^c, Y^c]_{ij}(t) := \sum_{k=1}^{q} \left[x_{ik}^c, y_{kj}^c\right](t)$. We recall that for any semimartingale x, the process x^c is the $a.s.$ unique continuous local martingale m such that $[x-m]$ is purely discontinuous.

We will use the facts that $[X] = [X, X^*]$ is a nonnegative definite $d \times d$ matrix, that $[X,Y]^\top = \left[Y^\top, X^\top\right]$ and that for any nonrandom matrices $A \in \mathbb{M}_{m\times d}, C \in \mathbb{M}_{r\times n}$ and semimartingales $X \in \mathbb{M}_{d\times q}, Y \in \mathbb{M}_{q\times r}$,

$$[AX, YC] = A[X,Y]C. \tag{2.3}$$

The natural example of a continuous semimartingale is the standard complex $d \times q$ matrix Brownian motion $B = \{B(t)\}_{t\geq 0} = \{b_{jl}(t)\}_{t\geq 0}$ consisting of independent $\mathbb{C}$-valued Brownian motions $b_{jl}(t) = \mathrm{Re}(b_{jl}(t)) + \mathrm{i}\,\mathrm{Im}(b_{jl}(t))$ where $\mathrm{Re}(b_{jl}(t)), \mathrm{Im}(b_{jl}(t))$ are independent one-dimensional Brownian motions with common variance $t/2$. Then we have $[B, B^*]_{ij}(t) = \sum_{k=1}^{q} \left[b_{ik}, \overline{b}_{jk}\right](t) = qt\delta_{ij}$ and hence the matrix quadratic variation of B is given by the $d \times d$ matrix process:

$$[B, B^*](t) = qt\mathrm{I}_d. \tag{2.4}$$

The case $q = 1$ corresponds to the $\mathbb{C}^d$-valued standard Brownian motion B. We observe this corresponds to $[B, B^*](t) = t\mathrm{I}_d$ instead of the common $2t\mathrm{I}_d$ used in the literature.

Other examples of complex matrix semimartingales are Lévy processes considered next.

Complex matrix Lévy processes An infinitely divisible random matrix M in $\mathbb{M}_{d\times q}$ is characterized by the Lévy-Khintchine representation of its Fourier transform $\mathbb{E}\mathrm{e}^{\mathrm{itr}(\Theta^* M)}$ $= \exp(\psi(\Theta))$ with Laplace exponent

$$\psi(\Theta) = \mathrm{itr}(\Theta^*\Psi\ ) - \frac{1}{2}\mathrm{tr}\left(\Theta^*\mathcal{A}\Theta^*\right) + \int_{\mathbb{M}_{d\times q}} \left(\mathrm{e}^{\mathrm{itr}(\Theta^*\xi)} - 1 - \mathrm{i}\frac{\mathrm{tr}(\Theta^*\xi)}{1+\|\xi\|^2}\right)\nu(\mathrm{d}\xi),\ \Theta \in \mathbb{M}_{d\times q}, \tag{2.5}$$

where $\mathcal{A} : \mathbb{M}_{q\times d} \to \mathbb{M}_{d\times q}$ is a positive symmetric linear operator (i.e. $\mathrm{tr}\left(\Phi^*\mathcal{A}\Phi^*\right) \geq 0$ for $\Phi \in \mathbb{M}_{d\times q}$ and $\mathrm{tr}\left(\Theta_2^*\mathcal{A}\Theta_1^*\right) = \mathrm{tr}\left(\Theta_1^*\mathcal{A}\Theta_2^*\right)$ for $\Theta_1, \Theta_2 \in \mathbb{M}_{d\times q}$), ν is a measure on $\mathbb{M}_{d\times q}$ (the Lévy measure) satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{M}_{d\times q}} (1 \wedge \|x\|^2)\nu(\mathrm{d}x) < \infty$, and $\Psi \in \mathbb{M}_{d\times q}$. The triplet $(\mathcal{A}, \nu, \Psi)$ uniquely determines the distribution of M.

Remark 2.2. *The notation $\mathcal{A}\Theta^*$ means the linear operator $\mathcal{A}$ from $\mathbb{M}_{q\times d}$ to $\mathbb{M}_{d\times q}$ acting on $\Theta^* \in \mathbb{M}_{q\times d}$. Some interesting examples of $\mathcal{A}$ and its corresponding matrix Gaussian distributions are:*

(a) $\mathcal{A}\Theta^* = \Theta$. This corresponds to a Gaussian matrix distribution invariant under left and right unitary transformations in $\mathbb{U}_d$ and $\mathbb{U}_q$, respectively.

(b) $\mathcal{A}\Theta^* = \Sigma_1\Theta\Sigma_2$ for $\Sigma_1 \in \overline{\mathbb{H}}_d^+$ and $\Sigma_2 \in \overline{\mathbb{H}}_q^+$. In this case the corresponding matrix Gaussian distribution is denoted by $\mathrm{N}_{d\times q}(0, \Sigma_1 \otimes \Sigma_2)$ and $\Sigma_1 \otimes \Sigma_2$ is called a Kronecker covariance. It holds that if N has the distribution $\mathrm{N}_{d\times q}(0, \mathrm{I}_d \otimes \mathrm{I}_q)$, then $\Sigma_1^{1/2} N \Sigma_2^{1/2}$ has distribution $\mathrm{N}_{d\times q}(0, \Sigma_1 \otimes \Sigma_2)$.

(c) When $q = d$, $\mathcal{A}\Theta^* = \mathrm{tr}(\Theta)\mathrm{I}_d$ is the covariance operator of the Gaussian random matrix $g\mathrm{I}_d$ where g is a one-dimensional random variable with a standard Gaussian distribution.

Let $\mathbb{S}_{d\times q}$ be the unit sphere of $\mathbb{M}_{d\times q}$ and let $\mathbb{M}_{d\times q}^0 = \mathbb{M}_{d\times q}\backslash\{0\}$. If ν is a Lévy measure on $\mathbb{M}_{d\times q}$, then there are a measure λ on $\mathbb{S}_{d\times q}$ with $\lambda(\mathbb{S}_{d\times q}) \geq 0$ and a measure ν_ξ for each $\xi \in \mathbb{S}_{d\times q}$ with $\nu_\xi((0,\infty)) > 0$ such that

$$\nu(E) = \int_{\mathbb{S}_{d\times q}} \lambda(\mathrm{d}\xi) \int_{(0,\infty)} 1_E(u\xi)\nu_\xi(\mathrm{d}u), \qquad E \in \mathcal{B}(\mathbb{M}_{d\times q}^0).$$

We call (λ, ν_ξ) a polar decomposition of ν. When $d = q = 1$, ν is a Lévy measure on $\mathbb{C}$ and λ is a measure in the unit sphere $\mathbb{S}_{1\times 1}$ of $\mathbb{C}$.

Any $\mathbb{M}_{d\times q}$-valued Lévy process $L = \{L(t)\}_{t\geq 0}$ with triplet $(\mathcal{A}, \nu, \Psi)$ is a semimartingale with the Lévy-Itô decomposition

$$L(t) = t\Psi + B_{\mathcal{A}}(t) + \int_{[0,t]}\int_{\|V\|\leq 1} V\widetilde{J}_L(\mathrm{d}s, \mathrm{d}V) + \int_{[0,t]}\int_{\|V\|>1} VJ_L(\mathrm{d}s, \mathrm{d}V),\ t \geq 0, \tag{2.6}$$

where:

(a) $\{B_{\mathcal{A}}(t)\}_{t\geq 0}$ is a $\mathbb{M}_{d\times q}$-valued Brownian motion with covariance $\mathcal{A}$, i.e. it is a Lévy process with continuous sample paths (a.s.) and each $B_{\mathcal{A}}(t)$ is centered Gaussian with

$$\mathbb{E}\left\{\mathrm{tr}(\Theta_1^* B_{\mathcal{A}}(t))\mathrm{tr}\left(\Theta_2^* B_{\mathcal{A}}(s)\right)\right\} = \min(s,t)\mathrm{tr}\left(\Theta_1^*\mathcal{A}\Theta_2^*\right) \text{ for each } \Theta_1, \Theta_2 \in \mathbb{M}_{d\times q},$$

(b) $J_L(\cdot,\cdot)$ is the Poisson random measure of jumps on $[0,\infty) \times \mathbb{M}_{d\times q}^0$. That is, $J_L(t, E) = \#\{0 \leq s \leq t : \Delta L_s \in E\}$, $E \in \mathbb{M}_{d\times q}^0$, with intensity measure $Leb \otimes \nu$, and independent of $\{B_{\mathcal{A}}(t)\}_{t\geq 0}$,

(c) $\widetilde{J}_L$ is the compensator measure of J_L, i.e.

$$\widetilde{J}_L(\mathrm{d}t, \mathrm{d}V) = J_L(\mathrm{d}t, \mathrm{d}V) - \mathrm{d}t\nu(\mathrm{d}V);$$

see for example [1] for the most general case of Lévy processes with values in infinite dimensional Banach spaces.

An $\mathbb{M}_{d\times q}$-valued Lévy process $L = \{L(t)\}_{t\geq 0}$ has bounded variation if and only if its Lévy-Itô decomposition takes the form

$$L(t) = t\Psi_0 + \int_{[0,t]}\int_{\mathbb{M}_{d\times q}^0} VJ_L(\mathrm{d}s, \mathrm{d}V) = t\Psi_0 + \sum_{s\leq t}\Delta L(s),\ t \geq 0, \tag{2.7}$$

where $\Psi_0 = \Psi - \int_{\|V\|\leq 1} V\nu(\mathrm{d}V)$.

The matrix quadratic variation (2.2) of L is given by the $\overline{\mathbb{H}}_d^+$-valued process

$$[L](t) = \left[B_{\mathcal{A}}, B_{\mathcal{A}}^*\right](t) + \int_{[0,t]}\int_{\mathbb{M}_{d\times q}^0} VV^* J_L(\mathrm{d}s, \mathrm{d}V) = \left[B_{\mathcal{A}}, B_{\mathcal{A}}^*\right](t) + \sum_{s\leq t}\Delta L(s)\Delta L(s)^*. \tag{2.8}$$

In Section 3 we prove a partial converse of the last result in the case $q = 1$.

Remark 2.3. *On the lines of Remark 2.2 we have the following observations for the quadratic variation of the continuous part in (2.8):*

(a) When $\mathcal{A}\Theta^* = \Theta$, $[B_{\mathcal{A}}, B_{\mathcal{A}}^*](t) = qt\mathrm{I}_d$. This follows from (2.4) since $B_{\mathcal{A}}(t)$ is a standard complex $d \times q$ matrix Brownian motion.

(b) When $\mathcal{A}\Theta^* = \Sigma_1\Theta\Sigma_2$ for $\Sigma_1 \in \overline{\mathbb{H}}_d^+$ and $\Sigma_2 \in \overline{\mathbb{H}}_q^+$, we have $B_{\mathcal{A}}(t) = \Sigma_1^{1/2}B(t)\Sigma_2^{1/2}$ where $B = \{B(t)\}_{t\geq 0}$ is a standard complex $d \times q$ matrix Brownian motion. Then, using (2.3) we have

$$[B_{\mathcal{A}}, B_{\mathcal{A}}^*](t) = \left[\Sigma_1^{1/2}B\Sigma_2^{1/2}, \Sigma_2^{1/2}B^*\Sigma_1^{1/2}\right](t) = \Sigma_1^{1/2}\left[B\Sigma_2^{1/2}, \Sigma_2^{1/2}B^*\right](t)\Sigma_1^{1/2} = t\mathrm{tr}(\Sigma_2)\Sigma_1,$$

where we have also used the easily checked fact $\left[B\Sigma_2^{1/2}, \Sigma_2^{1/2}B^*\right](t) = t\mathrm{tr}(\Sigma_2)I_d$.

(c) When $q = d$ and $\mathcal{A}\Theta^* = \mathrm{tr}(\Theta)\mathrm{I}_d$, we have $[B_{\mathcal{A}}, B_{\mathcal{A}}^*](t) = t\mathrm{I}_d$ since $B_{\mathcal{A}}(t) = b(t)\mathrm{I}_d$ where $b = \{b(t)\}_{t\geq 0}$ is a one-dimensional Brownian motion.

The extension of the notion of a real subordinator to the matrix case relies on cones. A cone K is a nonempty, closed, convex subset of $\mathbb{M}_{d\times q}$ such that if $A \in K$ and $\alpha \geq 0$ imply $\alpha A \in K$. A cone K determines a partial order in $\mathbb{M}_{d\times q}$ by defining $V_1 \leq_K V_2$ for $V_1, V_2 \in \mathbb{M}_{d\times q}$ whenever $V_2 - V_1 \in K$. A $\mathbb{M}_{d\times q}$-valued Lévy process $L = \{L(t)\}_{t\geq 0}$ is K-increasing if $L(t_1) \leq_K L(t_2)$ for every $t_1 < t_2$ almost surely. A K-increasing Lévy process with values in $\mathbb{M}_{d\times q}$ is called a matrix subordinator. It is easy to see that if $L = \{L(t)\}_{t\geq 0}$ is a Lévy process in $\mathbb{M}_{d\times q}$ then L is a subordinator if and only if L takes values in K. In this sense the matrix quadratic variation Lévy process in (2.8) with values in the cone $\overline{\mathbb{H}}_d^+$ is a matrix subordinator.

Approximation of Lévy processes The following are useful results on the sample path approximation of complex matrix Lévy processes; see [13, Th 15.17] and [17, Th. 8.7]. They follow from their corresponding real vector case by the usual identification of $\mathbb{M}_{d\times q} \to \mathbb{R}^{2dq}$ via $A \to \mathrm{vec}(A)$, $A \in \mathbb{M}_{d\times q}$ and the fact that $\mathrm{tr}\,(A^*B) = \mathrm{vec}(A)^*\mathrm{vec}(B)$, where $\mathrm{vec}(A)$ is the dq column complex vector obtained by stacking the columns of A one down each other.

Proposition 2.4. *Let L and L^n $n = 1, 2, ...$ be complex matrix Lévy processes in $\mathbb{M}_{d\times q}$ with $L^n(1) \xrightarrow{\mathcal{L}} L(1)$. Then there exist processes $\tilde{L}^n$ with the same distribution that L^n such that*

$$\sup_{0\leq s\leq t}\left|\tilde{L}^n(s) - L(s)\right| \xrightarrow{\mathrm{Pr}} 0, \quad \forall t \geq 0.$$

Proposition 2.5. *Let $M^n, n = 1, 2, ...$ be infinitely divisible random matrices in $\mathbb{M}_{d\times q}$ with triplet $(\mathcal{A}^n, \nu^n, \Psi^n)$. Let M be a random matrix in $\mathbb{M}_{d\times q}$. Then $M^n \xrightarrow{\mathcal{L}} M$ if and only if M is infinitely divisible whose triplet $(\mathcal{A}, \nu, \Psi)$ satisfies the following three conditions:*
a) If $f : \mathbb{M}_{d\times q} \to \mathbb{M}_{d\times q}$ is bounded and continuous function vanishing in a neighborhood of 0 then

$$\lim_{n\to\infty}\int_{\mathbb{M}_{d\times q}} f(\xi)\nu^n(\mathrm{d}\xi) = \int_{\mathbb{M}_{d\times q}} f(\xi)\nu(\mathrm{d}\xi).$$

b) Define the positive symmetric operator $\mathcal{A}^{n,\epsilon} : \mathbb{M}_{q\times d} \to \mathbb{M}_{d\times q}$ by

$$\mathrm{tr}\,(\Theta^*\mathcal{A}^{n,\epsilon}\Theta^*) = \mathrm{tr}\,(\Theta^*\mathcal{A}^n\Theta^*) + \int_{\|\xi\|\leq\varepsilon}|\mathrm{tr}\,(\Theta^*\xi)|^2\,\nu_n(\mathrm{d}\xi) \quad \textit{for } \Theta \in \mathbb{M}_{d\times q}.$$

Then

$$\lim_{\varepsilon\downarrow 0}\limsup_{n\to\infty}|\mathrm{tr}\,(\Theta^*\mathcal{A}^{n,\epsilon}\Theta^*) - \mathrm{tr}\,(\Theta^*\mathcal{A}\Theta^*)| = 0, \quad \textit{for } \Theta \in \mathbb{M}_{d\times q}.$$

c) $\Psi^n \to \Psi$.

3 BGCD random matrix ensembles

We now consider the matrix Lévy processes associated to the BGCD matrix ensembles $(M_d)_{d\geq 1}$ mentioned in the introduction.

When μ is the standard Gaussian distribution, M_d is a Gaussian unitary invariant random matrix, $\Lambda(\mu)$ is the semicircle distribution and $\{M_d(t)\}_{t\geq 0}$ is the Hermitian matrix valued process given by $M_d(t) = \left(1/\sqrt{d+1}\right)(B(t) + dg(t)\mathrm{I}_d)$ where $B(t)$ is a $d \times d$ Hermitian matrix Brownian motion independent of the one-dimensional Brownian motion $g(t)$; see [6, Remark 3.5].

Likewise, if μ is the Poisson distribution with parameter $\lambda > 0$, $\{M_d(t)\}_{t\geq 0}$ is the $d\times d$ matrix compound Poisson process $M_d(t) = \sum_{k=1}^{N(t)} u_k^d u_k^{d*}$ where $\{u_k^d\}_{k\geq 1}$ is a sequence of independent uniformly distributed random vectors on the unit sphere of $\mathbb{C}^d$ independent of the Poisson process $\{N(t)\}_{t\geq 0}$, and $\Lambda(\mu)$ is the Marchenko-Pastur distribution of parameter $\lambda > 0$; see [6, Remark 3.2]. We observe that in this case $\{M_d(t)\}_{t\geq 0}$ is a matrix covariation (quadratic) process rather than a covariance matrix process as in the Wishart or other empirical covariance processes.

Proposition 3.1 below collects computations in [6], [10] and [12] to summarize the Lévy triplet of a general BGCD matrix ensemble in an explicit manner. Let $\nu|_{(0,\infty)}$ and $\nu|_{(-\infty,0)}$ denote the corresponding restrictions to $(0,+\infty)$ and $(-\infty,0)$ for any Lévy measure ν, respectively.

Proposition 3.1. *Let μ be an infinitely divisible distribution in $\mathbb{R}$ with Lévy triplet (a^2,ν,ψ) and let $(M_d)_{d\geq 1}$ be a BGCD matrix ensemble for $\Lambda(\mu)$. Then, for each $d \geq 1$ M_d has the Lévy-Khintchine representation (2.5) with Lévy triplet $(\mathcal{A}_d,\nu_d,\Psi_d)$ where*

a) $\Psi_d = \psi\mathrm{I}_d$

b)

$$\mathcal{A}_d\Theta = a^2\frac{1}{d+1}(\Theta + \mathrm{tr}(\Theta)\mathrm{I}_d), \quad \Theta \in \mathbb{H}_d. \tag{3.1}$$

c)

$$\nu_d(E) = d\int_{\mathbb{S}(\mathbb{H}_{d(1)})}\int_0^\infty 1_E(rV)\,\nu_V(\mathrm{d}r)\,\Pi(\mathrm{d}V), \quad E \in \mathcal{B}(\mathbb{H}_d\backslash\{0\}), \tag{3.2}$$

where $\nu_V = \nu|_{(0,\infty)}$ or $\nu|_{(-\infty,0)}$ according to $V \geq 0$ or $V \leq 0$ and Π is a measure on $\mathbb{S}(\mathbb{H}_{d(1)})$ such that

$$\Pi(D) = \int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}\int_{\{-1,1\}} 1_D(tV)\,\lambda(\mathrm{d}t)\,\omega_d(\mathrm{d}V), \quad D \in \mathcal{B}\left(\mathbb{S}(\mathbb{H}_{d(1)})\right), \tag{3.3}$$

where λ is the spherical measure of ν and ω_d is the probability measure on $\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+$ induced by the transformation $u \to V = uu^*$, where u is a uniformly distributed column random vector in the unit sphere of $\mathbb{C}^d$.

Proof. (a) It follows from the first term in the Lévy exponent of M_d in page 635 of [10], where the notation $\Lambda_d(\mu)$ is used for the distribution of M_d. For (b), the form of the covariance operator $\mathcal{A}_d$ was implicitly computed in the first example in Section II.C of [10]. Finally, the polar decomposition of the Lévy measure (3.2) was found in [12]. $\square$

The Lévy-Itô decomposition of the Lévy process associated to the BGCD model M_d is given by

$$M_d(t) = \psi t\mathrm{I}_d + B_{\mathcal{A}_d}(t) + \int_{[0,t]}\int_{\{\|V\|\leq 1\}\cap\mathbb{H}_{d(1)}} V\widetilde{J}_d(\mathrm{d}s,\mathrm{d}V) + \int_{[0,t]}\int_{\{\|V\|>1\}\cap\mathbb{H}_{d(1)}} VJ_d(\mathrm{d}s,\mathrm{d}V), \tag{3.4}$$

where $t \geq 0$, $\mathcal{A}_d\Theta = a^2\frac{1}{d+1}(\Theta + \mathrm{tr}(\Theta)\mathrm{I}_d)$, $J_d(t,E) = \#\{0 \leq s \leq t : \Delta M_d(s) \in E\} = J_d(t, E \cap \mathbb{H}_{d(1)})$ for any measurable $E \in \mathbb{H}_d\backslash\{0\}$. Its quadratic variation is obtained by (2.8) as the matrix subordinator

$$[M_d]\,(t) = a^2 t\mathrm{I}_d + \int_{[0,t]}\int_{\mathbb{H}_{d(1)}\backslash\{0\}} VV^* J_d(\mathrm{d}s, \mathrm{d}V) = a^2 t\mathrm{I}_d + \sum_{s\leq t} \Delta M_d(s)\,(\Delta M_d(s))^*\,.$$

Remark 3.2. *It is possible to obtain BGCD models of symmetric random matrices rather than Hermitian. Indeed, slight changes in the proof of [6, Theorem 3.1] give for each $d \geq 1$, a $d \times d$ real symmetric random matrix M_d with orthogonal invariant infinitely divisible matrix distribution. The asymptotic spectral distribution of the corresponding Hermitian and symmetric ensembles is the same, similarly as the semicircle distribution is the asymptotic spectral distribution for the Gaussian Unitary Ensemble and Gaussian Orthogonal Ensemble.*

4 Bounded variation case

It is well known that the quadratic variation of a one-dimensional Lévy process is a subordinator, see [11, Example 8.5]. The following result gives a converse and a generalization to matrix subordinators with rank one jumps. The one dimensional case is given in [18, Lemma 6.5].

Theorem 4.1. *Let $L_d = \{L_d(t) : t \geq 0\}$ be a Lévy process in $\overline{\mathbb{H}}_d^+$ whose jumps are of rank one almost surely. Then there exists a Lévy process $X = \{X(t) : t \geq 0\}$ in $\mathbb{C}^d$ such that $L_d(t) = [X]\,(t)$.*

Proof. We construct X as a Lévy-Itô decomposition realization. Using (2.7), for each $d \geq 1$, L_d is an $\overline{\mathbb{H}}_d^+$-process of bounded variation with Lévy-Itô decomposition

$$L_d(t) = t\Psi_0 + \int_{[0,t]}\int_{\mathbb{H}_{d(1)}^+} V J_{L_d}(\mathrm{d}s, \mathrm{d}V),\; t \geq 0,$$

where $\Psi_0 \in \mathbb{H}_d^+$ and J_{L_d} is the Poisson random measure of L_d. Let $Leb \otimes \nu_{L_d}$ denote the intensity measure of J_{L_d}.

Consider the cone $C_+^d = \{x = (x_1, x_2, \ldots, x_d) : x_1 \geq 0,\ x_j \in \mathbb{C},\ j = 2, ..., d\}$ and let $\varphi_+ : \mathbb{R}_+ \times \mathbb{H}_{d(1)}^+ \to \mathbb{R}_+ \times C_+^d$ be defined as $\varphi_+(t, V) = (t, x)$ where $V = xx^*$ and $x \in C_+^d$. Let $\overline{\varphi}_+ : \mathbb{H}_{d(1)}^+ \to C_+^d$ be defined by $\overline{\varphi}_+(V) = x$ for $V = xx^*$ and $x \in C_+^d$. By Remark 2.1 (a), the functions φ_+ and $\overline{\varphi}_+$ are well defined.

Let us define $J(\mathrm{d}s, \mathrm{d}x) = \left(J_{L_d} \circ \varphi_+^{-1}\right)(\mathrm{d}s, \mathrm{d}x)$ the random measure induced by the transformation φ_+ which is a Poisson random measure on $\mathbb{R}_+ \times C_+^d$. Observe that $\mathbb{E}\left[J(t, F)\right] = \mathbb{E}\left[J_{L_d} \circ \varphi_+^{-1}(\{t\} \times F)\right] = t\nu_{L_d}\left(\overline{\varphi}_+(F)\right) = t\left(\nu_{L_d} \circ \overline{\varphi}_+^{-1}\right)(F)$ for $F \in \mathcal{B}(C_+^d \setminus \{0\})$. Let us denote $\nu = \nu_{L_d} \circ \overline{\varphi}_+^{-1}$ which is a Lévy measure on C_+^d since

$$\int_{C_+^d\backslash\{0\}} \left(1 \wedge |x|^2\right)\nu(\mathrm{d}x) = \int_{C_+^d\backslash\{0\}} \left(1 \wedge |x|^2\right)\nu_{L_d} \circ \overline{\varphi}_+^{-1}(\mathrm{d}x)$$

$$= \int_{C_+^d\backslash\{0\}} (1 \wedge \mathrm{tr}\,(xx^*))\,\nu_{L_d} \circ \overline{\varphi}_+^{-1}(\mathrm{d}x) = \int_{\mathbb{H}_{d(1)}^+} (1 \wedge \mathrm{tr}\,(V))\left(\nu_{L_d} \circ \overline{\varphi}_+^{-1}\right) \circ f^{-1}(\mathrm{d}V)$$

$$= \int_{\mathbb{H}_{d(1)}^+} (1 \wedge \mathrm{tr}\,(V))\,\nu_{L_d}(\mathrm{d}V) < \infty,$$

where $\left(\nu_{L_d} \circ \overline{\varphi}_+^{-1}\right) \circ f^{-1} = \nu_{L_d}$, with $f(x) = xx^*$ and we have used that $\mathrm{tr}\,(V) \leq \alpha \|V\|$ for some constant $\alpha > 0$. Thus $Leb \otimes \nu$ is the intensity measure of the Poisson random measure J.

Let us take the Lévy process in $\mathbb{C}^d$

$$X(t) = |\Psi_0|^{1/2} B_I(t) + \int_{[0,t]} \int_{\mathbb{C}^d \cap \{|x| \leq 1\}} x\widetilde{J}(\mathrm{d}s, \mathrm{d}x) + \int_{[0,t]} \int_{\mathbb{C}^d \cap \{|x| > 1\}} xJ(\mathrm{d}s, \mathrm{d}x),\ t \geq 0, \tag{4.1}$$

where B_I is a $\mathbb{C}^d$-valued standard Brownian motion with quadratic variation tI_d, (i.e. (2.4) with $q = 1$). Thus the quadratic variation of X is given by

$$[X](t) = \left[|\Psi_0|^{1/2} B_I, B_I^* |\Psi_0|^{1/2}\right](t) + \int_{[0,t]} \int_{\mathbb{C}^d \setminus \{0\}} xx^* J(\mathrm{d}s, \mathrm{d}x)$$

$$= \Psi_0 t + \int_{[0,t]} \int_{\mathbb{C}^d \setminus \{0\}} xx^* J_{L_d} \circ \varphi_+^{-1}(\mathrm{d}s, \mathrm{d}x) = \Psi_0 t + \int_{[0,t]} \int_{\mathbb{H}_{d(1)}^+} V J_{L_d} \circ \varphi_+^{-1} \circ h^{-1}(\mathrm{d}s, \mathrm{d}V)$$

$$= \Psi_0 t + \int_{[0,t]} \int_{\mathbb{H}_{d(1)}^+} V J_{L_d}(\mathrm{d}s, \mathrm{d}V) = L_d(t),$$

where $J_{L_d} \circ \varphi_+^{-1} \circ h^{-1} = J_{L_d}$, with $h(t,x) = (t, xx^*)$. □

For the general bounded variation case we have the following Wiener-Hopf type decomposition.

Theorem 4.2. *Let $L_d = \{L_d(t) : t \geq 0\}$ be a Lévy process in $\mathbb{H}_d$ of bounded variation whose jumps are of rank one almost surely. Then there exist Lévy processes $X = \{X(t) : t \geq 0\}$ and $Y = \{Y(t) : t \geq 0\}$ in $\mathbb{C}^d$ such that*

$$L_d(t) = [X](t) - [Y](t). \tag{4.2}$$

Moreover, $\{[X](t) : t \geq 0\}$ and $\{[Y](t) : t \geq 0\}$ are independent processes.

Proof. For each $d \geq 1$, L_d is an $\mathbb{H}_d$-process of bounded variation with Lévy-Itô decomposition

$$L_d(t) = \Psi t + \int_{[0,t]} \int_{\mathbb{H}_{d(1)}} V J_{L_d}(\mathrm{d}s, \mathrm{d}V),\ t \geq 0, \tag{4.3}$$

where $\Psi \in \mathbb{H}_d$ and J_{L_d} is the Poisson random measure of L_d. Let $Leb \otimes \nu_{L_d}$ denote the intensity measure of J_{L_d}.

First we prove that $L_d = L_d^1 - L_d^2$ where L_d^1 and L_d^2 are the Lévy processes in $\overline{\mathbb{H}}_d^+$ given by (4.4) and (4.5).
Every $V \in \mathbb{H}_{d(1)}$ can be written as $V = \lambda uu^*$ where λ the eigenvalue of V and u is a unitary vector in $\mathbb{C}^d$. Let us define $|V| = |\lambda| uu^*$ and $V^+ = \lambda^+ uu^*$, $V^- = \lambda^- uu^*$ where $\lambda^+ = \lambda$ if $\lambda \geq 0$ and $\lambda^- = -\lambda$ if $\lambda < 0$.

Let $\varphi_+ : \mathbb{R}_+ \times \mathbb{H}_{d(1)} \to \mathbb{R}_+ \times \mathbb{H}_{d(1)}^+$ and $\varphi_- : \mathbb{R}_+ \times \mathbb{H}_{d(1)} \to \mathbb{R}_+ \times \mathbb{H}_{d(1)}^+$ be defined as $\varphi_+(t, V) = (t, V^+)$ and $\varphi_-(t, V) = (t, V^-)$ respectively. Let $\overline{\varphi}_+ : \mathbb{H}_{d(1)} \to \mathbb{H}_{d(1)}^+$ and $\overline{\varphi}_- : \mathbb{H}_{d(1)} \to \mathbb{H}_{d(1)}^+$ be defined as $\overline{\varphi}_+(V) = V^+$ and $\overline{\varphi}_-(V) = V^-$ respectively. By Remark 2.1 (b) the functions $\varphi_+, \overline{\varphi}_+, \varphi_-$ and $\overline{\varphi}_-$ are well defined and hence $V = \overline{\varphi}_+(V) - \overline{\varphi}_-(V)$.

Let us define $J^+(\mathrm{d}s, \mathrm{d}x) = \left(J_{L_d} \circ \varphi_+^{-1}\right)(\mathrm{d}s, \mathrm{d}x)$ and $J^-(\mathrm{d}s, \mathrm{d}x) = \left(J_{L_d} \circ \varphi_-^{-1}\right)(\mathrm{d}s, \mathrm{d}x)$ the random measures induced by the transformations φ_+ and φ_- respectively, which are Poisson random measures both on $\mathbb{R}_+ \times \mathbb{H}_{d(1)}^+$. Observe that $\mathbb{E}[J^+(t, F)] = \mathbb{E}[J_{L_d} \circ \varphi_+^{-1}(\{t\} \times F)] = t\nu_{L_d}\left(\overline{\varphi}_+^{-1}(F)\right) = t\left(\nu_{L_d} \circ \overline{\varphi}_+^{-1}\right)(F)$ for $F \in \mathcal{B}\left(\mathbb{H}_{d(1)}^+\right)$ and similarly $\mathbb{E}[J^-(t, F)] = t\left(\nu_{L_d} \circ \overline{\varphi}_-^{-1}\right)(F)$. Let us denote $\nu_{L_d}^+ = \nu_{L_d} \circ \overline{\varphi}_+^{-1}$ and $\nu_{L_d}^- = \nu_{L_d} \circ \overline{\varphi}_-^{-1}$.

Note that $\nu_{L_d}^{+}$ is a Lévy measure on $\mathbb{H}_{d(1)}^{+}$ since

$$\begin{aligned}\infty &> \int_{\mathbb{H}_{d(1)}\setminus\{0\}} (1\wedge\|V\|)\,\nu_{L_d}(\mathrm{d}V) \geq \int_{\mathbb{H}_{d(1)}} \left(1\wedge\left\|\overline{\varphi}_{+}(V)\right\|\right)\nu_{L_d}(\mathrm{d}V)\\ &= \int_{\mathbb{H}_{d(1)}^{+}} (1\wedge\|W\|)\,\nu_{L_d}^{+}(\mathrm{d}W).\end{aligned}$$

Hence $Leb\otimes\nu_{L_d}^{+}$ is the intensity measure of J^{+}. Similarly, one can see that $Leb\otimes\nu_{L_d}^{-}$ is the intensity measure of J^{-}.

There exist Ψ^{+} and Ψ^{-} in $\mathbb{H}_d^{+}$ such that $\Psi=\Psi^{+}-\Psi^{-}$. Let us take the Lévy processes X and Y in $\mathbb{C}^d$

$$X(t)=\left|\Psi^{+}\right|^{1/2}B_I(t)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|\leq1\}} x\widetilde{J}^{+}(\mathrm{d}s,\mathrm{d}x)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|>1\}} xJ^{+}(\mathrm{d}s,\mathrm{d}x),\ t\geq0,$$

$$Y(t)=\left|\Psi^{-}\right|^{1/2}B_I(t)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|\leq1\}} x\widetilde{J}^{-}(\mathrm{d}s,\mathrm{d}x)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|>1\}} xJ^{-}(\mathrm{d}s,\mathrm{d}x),\ t\geq0,$$

where B_I is a $\mathbb{C}^d$-valued standard Brownian motion with quadratic variation tI_d.

Observe that

$$[X](t)=\Psi^{+}t+\int_{[0,t]}\int_{\mathbb{C}^d\setminus\{0\}} xx^{*}J_{+}(\mathrm{d}s,\mathrm{d}x)=\Psi^{+}t+\int_{[0,t]}\int_{\mathbb{H}_{d(1)}^{+}} VJ_{L_d}(\mathrm{d}s,\mathrm{d}V) \tag{4.4}$$

and

$$\begin{aligned}[Y](t)&=\Psi^{-}t+\int_{[0,t]}\int_{\mathbb{C}^d\setminus\{0\}} xx^{*}J^{-}(\mathrm{d}s,\mathrm{d}x)=\Psi^{-}t-\int_{[0,t]}\int_{\mathbb{C}^d\setminus\{0\}}(-xx^{*})\,J_{L_d}(\mathrm{d}s,\mathrm{d}x)\\ &=\Psi^{-}t-\int_{[0,t]}\int_{\mathbb{H}_{d(1)}^{-}} VJ_{L_d}(\mathrm{d}s,\mathrm{d}V),\end{aligned} \tag{4.5}$$

where $\mathbb{H}_{d(1)}^{-}$ denotes the set of negative (nonpositive) definite matrices of rank one in $\mathbb{H}_d$. The first assertion follows from (4.3). Finally, since J_{L_d} is a Poisson random measure and $\mathbb{H}_{d(1)}^{+}$ and $\mathbb{H}_{d(1)}^{-}$ are disjoint sets, from the last expressions in (4.4) and (4.5) we have that $[X]$ and $[Y]$ are independent processes, although X and Y are not. $\square$

Next we consider the matrix Lévy processes associated to the BGCD matrix ensembles $(M_d)_{d\geq1}$. We have the following two consequences of the former results.

Corollary 4.3. *Let $M_d=\{M_d(t):t\geq0\}$ be the matrix Lévy process associated to the BGCD random matrix ensembles.*

a) Let μ be the infinitely divisible distribution with triplet $(0,\nu,\psi)$ associated to M_d such that

$$\int_{|x|\leq1}(1\wedge x)\,\nu(\mathrm{d}x)<\infty,\ \ \nu((-\infty,0])=0\ \text{ and } \psi_0:=\psi-\int_{x\leq1}x\nu(\mathrm{d}x)\geq0.$$

Let us consider the Lévy-Itô decomposition of $M_d(t)$ in $\overline{\mathbb{H}}_d^{+}$

$$M_d(t)=\psi_0 tI_d+\int_{[0,t]}\int_{\mathbb{H}_{d(1)}^{+}} VJ_{M_d}(\mathrm{d}s,\mathrm{d}V).$$

Then there exists a Lévy process $X=\{X(t):t\geq0\}$ in $\mathbb{C}^d$ such that $M_d(t)=[X](t)$, where

$$X(t)=|\psi_0|^{1/2}B_I(t)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|\leq1\}} x\widetilde{J}(\mathrm{d}s,\mathrm{d}x)+\int_{[0,t]}\int_{\mathbb{C}^d\cap\{|x|>1\}} xJ(\mathrm{d}s,\mathrm{d}x),\ t\geq0,$$

B_I is a $\mathbb{C}^d$-valued standard Brownian motion with quadratic variation tI_d, and the Poisson random measure J is given by $J = J_{M_d} \circ \varphi_+^{-1}$.

b) If M_d has bounded variation then there exist Lévy processes $X = \{X(t) : t \geq 0\}$ and $Y = \{Y(t) : t \geq 0\}$ in $\mathbb{C}^d$ such that $M_d(t) = [X](t) - [Y](t)$, where $\{[X](t) : t \geq 0\}$ and $\{[Y](t) : t \geq 0\}$ are independent.

5 Covariation matrix processes approximation

We now consider approximation of general BGCD ensembles by BGCD matrix compound Poisson processes which are covariation of $\mathbb{C}^d$-valued Lévy processes.

The following results gives realizations of BGCD ensembles of compound Poisson type as the covariation of two $\mathbb{C}^d$-valued Lévy processes. Its proof is straightforward.

Proposition 5.1. *Let μ be a compound Poisson distribution on $\mathbb{R}$ with Lévy measure ν and drift $\psi \in \mathbb{R}$ and let $(M_d)_{d\geq 1}$ be the BGCD matrix ensemble for $\Lambda(\mu)$. For each $d \geq 1$, assume that*

i) $(\beta_j)_{j\geq 1}$ is a sequence of i.i.d. random variables with distribution $\nu/\nu(\mathbb{R})$.

ii) $(u_j)_{j\geq 1}$ is a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^d$.

iii) $\{N(t)\}_{t\geq 0}$ is a Poisson process with parameter one.

Assume that $(\beta_j)_{j\geq 1}$, $(u_j)_{j\geq 1}$ and $\{N(t)\}_{t\geq 0}$ are independent. Then

a) M_d has the same distribution as $M_d(1)$ where

$$M_d(t) = \psi t I_d + \sum_{j=1}^{N(t)} \beta_j u_j u_j^*, \quad t \geq 0. \tag{5.1}$$

b) $M_d(\cdot) = [X_d, Y_d](\cdot)$ where $X_d = \{X_d(t)\}_{t\geq 0}$, $Y_d = \{Y_d(t)\}_{t\geq 0}$ are the $\mathbb{C}^d$-valued Lévy processes

$$X_d(t) = \sqrt{|\psi|} B(t) + \sum_{j=1}^{N(t)} \sqrt{|\beta_j|} u_j, \quad t \geq 0, \tag{5.2}$$

$$Y_d(t) = \operatorname{sign}(\psi) \sqrt{|\psi|} B(t) + \sum_{j=1}^{N(t)} \operatorname{sign}(\beta_j) \sqrt{|\beta_j|} u_j, \quad t \geq 0, \tag{5.3}$$

and $B = \{B(t)\}_{t\geq 0}$ is a $\mathbb{C}^d$-valued standard Brownian motion independent of $(\beta_j)_{j\geq 1}$, $(u_j)_{j\geq 1}$ and $\{N(t)\}_{t\geq 0}$.

For the general case we have the following sample path approximation by covariation processes for Lévy processes generated by the BGCD matrix ensembles.

Theorem 5.2. *Let μ be an infinitely divisible distribution on $\mathbb{R}$ with triplet (a^2, ν, ψ) and let $(M_d)_{d\geq 1}$ be the corresponding BGCD matrix ensemble for $\Lambda(\mu)$. Let $d \geq 1$ fixed and assume that for $n \geq 1$*

i) $(\beta_j^n)_{j\geq 1}$ is a sequence of i.i.d. random variables with distribution $\mu^{*\frac{1}{n}}$.

ii) $(u_j^n)_{j\geq 1}$ is a sequence of i.i.d. random vectors with uniform distribution on the unit sphere of $\mathbb{C}^d$.

iii) $N^n = \{N^n(t)\}_{t\geq 0}$ is a Poisson process with parameter n.

iv) $B^n = \{B^n(t)\}_{t\geq 0}$ is a $\mathbb{C}^d$-valued standard Brownian motion.

v) $(\beta_j^n)_{j\geq 1}$, $(u_j^n)_{j\geq 1}, N^n$ and B^n are independent.

Let

$$X_d^n(t) = \sqrt{|\psi|} B^n(t) + \sum_{j=1}^{N^n(t)} \sqrt{|\beta_j^n|} u_j^n, \quad t \geq 0, \tag{5.4}$$

$$Y_d^n(t) = \operatorname{sign}(\psi)\sqrt{|\psi|}B^n(t) + \sum_{j=1}^{N^n(t)} \operatorname{sign}\left(\beta_j^n\right)\sqrt{\left|\beta_j^n\right|}u_j^n, \quad t \geq 0. \tag{5.5}$$

Then for each $d \geq 1$ *there exist* $\mathbb{M}_d$*-valued processes* $\widetilde{M}_d^n = \left\{\widetilde{M}_d^n(t)\right\}_{d\geq 1}$ *such that* $\widetilde{M}_d^n \overset{\mathcal{L}}{=} [X_d^n, Y_d^n]$*,*

$$\sup_{0<s\leq t}\left\|\widetilde{M}_d^n(s) - M_d(s)\right\| \xrightarrow[n\to\infty]{\Pr} 0, \quad \forall t \geq 0,$$

where $\{M_d(t) : t \geq 0\}$ *is the* $\mathbb{M}_d$*-valued Lévy process associated to* $(M_d)_{d\geq 1}$*.*

Proof. By the compound Poisson approximation for infinitely divisible distributions on $\mathbb{R}$ (see [17, pp 45]), we choose μ_n an infinitely divisible distribution such that $\mu_n \longrightarrow \mu$, where we take the triplet of μ_n as $(0, \nu^n, \psi^n)$, $\psi^n = \int \frac{x}{1+|x|^2}\nu^n(dx)$ and $\nu^n = n\mu^{*\frac{1}{n}}$, satisfying (see [17, Theorem 8.7]) that for every bounded continuous function f vanishing in a neighborhood of zero

$$\int_{\mathbb{R}} f(r)\,\nu^n(dr) \longrightarrow \int_{\mathbb{R}} f(r)\,\nu(dr) \text{ as } n \to \infty, \tag{5.6}$$

for each $\varepsilon > 0$

$$\int_{|r|\leq\varepsilon} r^2\nu^n(dr) \longrightarrow a^2 \text{ as } n \to \infty, \tag{5.7}$$

and $\psi^n \to \psi$.

A similar proof as for Proposition 5.1 gives

$$M_d^n(t) := [X_d^n, Y_d^{n*}](t) = \psi t \mathrm{I}_d + \sum_{j=0}^{N^n(t)} \beta_j^n u_j^n u_j^{n*},$$

which is a matrix value compound Poisson process with triplet $(\mathcal{A}_d^n, \nu_d^n, \psi_d^n)$ given by $\mathcal{A}_d^n = 0$, $\psi_d^n = \psi \mathrm{I}_d$ and

$$\nu_d^n(E) = d\int_{\mathbb{S}(\mathbb{H}_{d(1)})}\int_0^\infty 1_E(rV)\,\nu_V^n(\mathrm{d}r)\,\Pi(\mathrm{d}V), \quad E \in \mathcal{B}(\mathbb{H}_d\setminus\{0\}), \tag{5.8}$$

where $\nu_V^n = \nu^n|_{(0,\infty)}$ or $\nu^n|_{(-\infty,0)}$ according to $V \geq 0$ or $V \leq 0$ and Π is the measure on $\mathbb{S}(\mathbb{H}_{d(1)})$ in (3.3).

We will prove that $M_d^n \xrightarrow{\mathcal{L}} M_d$ by showing that the triplet $(\mathcal{A}_d^n, \nu_d^n, \psi_d^n)$ converges to the triplet $(\mathcal{A}_d, \nu_d, \psi_d)$ of the BGCD matrix ensemble in Proposition 3.1 in the sense of Proposition 2.5:

We observe that $\psi_d^n = \psi \mathrm{I}_d$ for each n.

Let $f : \mathbb{H}_{d(1)} \longrightarrow \mathbb{R}$ be a continuous bounded function vanishing in a neighborhood of zero. Using the polar decomposition (3.2) for ν_d^n we have

$$\begin{aligned}\int_{\mathbb{H}_{d(1)}} f(\xi)\,\nu_d^n(d\xi) &= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})}\int_0^\infty f(rV)\,\nu_V^n(dr)\,\Pi(dV)\\ &= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}\int_{\{-1,1\}}\int_0^\infty f(trV)\,\nu_V^n(dr)\,\lambda^n(dt)\,\omega_d(dV).\end{aligned} \tag{5.9}$$

For $V \in \mathbb{S}(\mathbb{H}_{d(1)}) \cap \overline{\mathbb{H}}_d^+$ fixed,

$$\begin{aligned}\int_{\{-1,1\}}\int_0^\infty f(trV)\,\nu_V^n(dr)\,\lambda^n(dt) = \lambda^n(\{1\})&\int_0^\infty f(rV)\,\nu^n(dr)\\ &+ \lambda^n(\{-1\})\int_{-\infty}^0 f(rV)\,\nu^n(dr).\end{aligned}$$

As a function of r, $f(rV)$ is a real valued continuous bounded function vanishing in a neighborhood of zero, hence using (5.6)

$$\lambda^n(\{1\})\int_0^\infty f(rV)\,\nu^n(dr)\longrightarrow\lambda(\{1\})\int_0^\infty f(rV)\,\nu(dr)$$

and

$$\lambda^n(\{-1\})\int_{-\infty}^0 f(rV)\,\nu^n(dr)\longrightarrow\lambda(\{-1\})\int_{-\infty}^0 f(rV)\,\nu(dr).$$

Then from (5.9)

$$\begin{aligned}\int_{\mathbb{H}_{d(1)}} f(\xi)\,\nu_d^n(d\xi)&\longrightarrow d\int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}\int_{\{-1,1\}}\int_0^\infty f(trV)\,\nu_V(dr)\,\lambda(dt)\,\omega_d(dV)\\&= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})}\int_0^\infty f(rV)\,\nu_d(dr)\,\Pi(dV)=\int_{\mathbb{H}_{d(1)}} f(\xi)\,\nu_d(d\xi).\end{aligned}$$

Next, we verify the convergence of the Gaussian part.

Let us define, for each $\varepsilon>0$ and $n\geq 1$, the operator $\mathcal{A}^{n,\varepsilon}:\mathbb{H}_d\longrightarrow\mathbb{H}_d$ by

$$\operatorname{tr}(\Theta\mathcal{A}^{n,\varepsilon}\Theta)=\int_{\|\xi\|\leq\varepsilon}|\operatorname{tr}(\Theta\xi)|^2\,\nu_d^n(d\xi).$$

From (5.8) we get

$$\begin{aligned}&\int_{\|\xi\|\leq\varepsilon}|\operatorname{tr}(\Theta\xi)|^2\,\nu_d^n(d\xi)=d\int_{\mathbb{S}(\mathbb{H}_{d(1)})}\int_0^\infty \mathbf{1}_{\{\|rV\|\leq\varepsilon\}}(rV)\,|\operatorname{tr}(r\Theta V)|^2\,\nu_V^n(dr)\,\Pi(dV)\\&= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}\int_{\{-1,1\}}\int_0^\infty \mathbf{1}_{\{r\leq\varepsilon\}}(rtV)\,r^2\,|\operatorname{tr}(\Theta V)|^2\,\nu_V^n(dr)\,\lambda(dt)\,\omega_d(dV)\\&= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}\int_{\mathbb{R}} \mathbf{1}_{\{r\leq\varepsilon\}}(rV)\,r^2\,|\operatorname{tr}(\Theta V)|^2\,\nu^n(dr)\,\omega_d(dV)\\&= d\int_{\mathbb{S}(\mathbb{H}_{d(1)})\cap\overline{\mathbb{H}}_d^+}|\operatorname{tr}(\Theta V)|^2\int_{|r|\leq\varepsilon} r^2\nu^n(dr)\,\omega_d(dV).\end{aligned}$$

Then using (5.7),

$$\int_{\|\xi\|\leq\varepsilon}|\operatorname{tr}(\Theta\xi)|^2\,\nu_d^n(d\xi)\longrightarrow da^2E_u\,|\operatorname{tr}(\Theta uu^*)|^2,$$

where u is a uniformly distributed column random vector in the unit sphere of $\mathbb{C}^d$. Finally

$$da^2E_u\,|\operatorname{tr}(\Theta uu^*)|^2=\frac{a^2}{d+1}\left(\operatorname{tr}\left(\Theta^2\right)+(\operatorname{tr}(\Theta))^2\right)=\operatorname{tr}(\Theta^*\mathcal{A}_d\Theta^*),\tag{5.10}$$

where $\mathcal{A}_d$ is as in (3.1) and the first equality in (5.10) follows from page 637 in [10]. Thus $M_d^n\xrightarrow{\mathcal{L}}M_d$ and the conclusion follows from Proposition 2.4. □

6 Final remarks

1. For the present work we do not have a specific financial application in mind. However, infinitely divisible nonnegative definite matrix processes with rank one jumps as characterized in Theorem 4.1, might be useful in the study of multivariate high-frequency data using realized covariation, where matrix covariation processes appear; see for example [2]. Moreover, it seems interesting to explore the construction of financial oriented matrix Lévy based models as in [4] for the specific case of rank one jumps matrix process of bounded variation.

2. In the direction of free probability, it is well known that the so-called Hermitian Brownian motion matrix ensemble $\{B_d(t) : t \geq 0\}$, $d \geq 1$, is a realization of the free Brownian motion. It is an open question if the matrix Lévy processes from BGCD models $\{M_d(t) : t \geq 0\}$, $d \geq 1$, are realizations of free Lévy processes. A first step in this direction would be to prove that the increments of a BGCD ensemble become free independent. A second step, more related to our work, would be to have an insight of the implication of the rank one condition of the matrix Lévy BGCD process in Corollary 4.3 as realization of a positive free Lévy process. These two problems are the subjects of current research of one of the coauthors.
3. In [7] a new Bercovici-Pata bijection for certain free convolution $\boxplus_c$ is established and a $d \times d'$ random matrix model for this bijection which is very close to the one given by the BGCD random matrix model is established. It can be seen that the Lévy measures of these rectangular BGCD random matrices are supported in the subset of $d \times d'$ complex matrices of rank one, in a similar way as done in [12] for the BGCD case. It would be of interest to have the analogue results on bounded variation of Section 4 for the Lévy processes associated to these rectangular BGCD random matrices, considering an appropriate nonnegative definite notion for rectangular matrices.

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