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A scaling proof for Walsh's Brownian motion extended arc-sine law

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Abstract

We present a new proof of the extended arc-sine law related to Walsh's Brownian motion, known also as Brownian spider. The main argument mimics the scaling property used previously, in particular by D. Williams [12], in the 1-dimensional Brownian case, which can be generalized to the multivariate case. A discussion concerning the time spent positive by a skew Bessel process is also presented.

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1 Introduction

$a)$ Recently, some renewed interest has been shown (see e.g. [9]) in the study of the law of the vector

$$\overrightarrow{A_1} = \left(\int_0^1 1_{(W_s \in I_i)} ds; \; i = 1, 2, \ldots, n \right) \; ,$$

where (W_s) denotes a Walsh Brownian motion, also called Brownian spider (see [10] for Walsh's lyrical description) living on $I = \bigcup_{i=1}^n I_i$, the union of n half-lines of the plane, meeting at 0.

For the sake of simplicity, we assume $p_1 = p_2 = \ldots = p_n = 1/n$, i.e.: when returning to 0, Walsh's Brownian motion chooses, loosely speaking, its "new" ray in a uniform way. In fact, excursion theory and/or the computation of the semi-group of Walsh's Brownian motion (see [1]) allow to define the process rigorously.

Since $(d(0, W_s); s \geq 0)$, for d the Euclidian distance, is a reflecting Brownian motion, we denote by $(L_t, t \geq 0)$ the unique continuous increasing process such that:
$(d(0, W_s) - L_s; s \geq 0)$ is a $\mathcal{W}_s = \sigma \{W_u, u \leq s\}$ Brownian motion.
Let

$$\overrightarrow{A_t} = \left(A_t^{(1)}, A_t^{(2)}, \ldots, A_t^{(n)} \right)$$

denote the random vector of the times spent in the different rays. In Section 2 we will state and prove our main Theorem concerning the distribution of $\overrightarrow{A_t}$ for a fixed time.

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Section 3 deals with the general case of stable variables, First, we recall some known results and then we state and prove our main Theorem. Finally, Section 4 is devoted to some remarks and comments.

$b)$ **Reminder on the arc-sine law:**
A random variable A follows the arc-sine law if it admits the density:

$$\frac{1}{\pi\sqrt{x(1-x)}}\, 1_{[0,1)}(x). \tag{1.1}$$

Some well known representations of an arc-sine variable are the following:

$$A \overset{(law)}{=} \frac{N^2}{N^2+\hat{N}^2} \overset{(law)}{=} \cos^2(U) \overset{(law)}{=} \frac{T}{T+\hat{T}} \overset{(law)}{=} \frac{1}{1+C^2}, \tag{1.2}$$

where $N, \hat{N} \sim \mathcal{N}(0,1)$ and are independent, U is uniform on $[0, 2\pi]$, T and $\hat{T}$ stand for two iid stable (1/2) unilateral variables, and C is a standard Cauchy variable.
With $(B_t, t \geq 0)$ denoting a real Brownian motion, two well known examples of arc-sine distributed variables are:

$$g_1 = \sup\{t < 1 : B_t = 0\}, \quad \text{and} \quad A_1^+ = \int_0^1 ds\; 1_{(B_s>0)}\ ,$$

a result that is due to Paul Lévy (see e.g. [6, 7, 13]).

$c)$ This point gives some motivation for Section 3. From (1.2), one could think that more general studies of the time spent positive by diffusions may bring 2 independent gamma variables (this because N^2 and $\hat{N}^2$ are distributed like two independent gamma variables of parameter 1/2), or 2 independent stable (μ) variables. It turns out that it is the second case which seems to occur more naturally. We devote Section 3 to this case.

2 Main result

Our aim is to prove the following:

Theorem 2.1. *The random vectors* $\overrightarrow{A_T}/T$ *for:*

(i) $T = t$; *(ii)* $T = \alpha_s^{(j)} = \inf\{t : A_t^{(j)} > s\}$; *(iii)* $T = \tau_l$, *the inverse local times,*

have the same distribution. In particular, it is specified by the iid stable (1/2) subordinators:

$$\left(\left(A_{\tau_l}^{(j)}, l \geq 0\right); 1 \leq j \leq n\right).$$

Hence:

$$\overrightarrow{A_1} \overset{(law)}{=} \frac{\overrightarrow{A_{\tau_1}}}{\tau_1}\ , \tag{2.1}$$

which yields that:

$$\overrightarrow{A_1} \overset{(law)}{=} \left(\frac{T_j}{\sum_{i=1}^n T_i}\ ;\ j \leq n\right)\ , \tag{2.2}$$

where T_j *are iid, stable (1/2) variables.*

The law of the right-hand side of (2.1) is easily computed, and consequently so is its left-hand side. We refer the reader to [2] for explicit expressions of this law, which for $n = 2$ reduces to the classical arc-sine law.

Proof. $a)$ Clearly, (ii) plays a kind of "bridge" between (i) and (iii).
$b)$ We shall work with $\left(\alpha_s^{(1)}, s \geq 0\right)$, the inverse of $\left(A_t^{(1)}, t \geq 0\right)$. It is more convenient to use the notation $\left(\alpha_s^{(+)}, s \geq 0\right)$ for $\left(\alpha_s^{(1)}, s \geq 0\right)$. We then follow the main steps of [13] (Section 3.4, p. 42), which themselves are inspired by Williams [12]; see also Watanabe (Proposition 1 in [11]) and Mc Kean [8].
$\left(A_t^{(j)}\right)$ denotes the time spent in I_j, for any $j \neq 1$. Since

$$\begin{cases} A^{(j)}_{\alpha_1^{(+)}} = A^{(j)}_{\tau(L_{\alpha_1^{(+)}})} \overset{(law)}{=} (L_{\alpha_1^{(+)}})^2 A^{(j)}_{\tau_1} \ , \\ \alpha_1^{(+)} = 1 + \sum_j A^{(j)}_{\alpha_1^{(+)}} \ , \\ \quad \text{and} \\ \text{for every} \ \ u, t \geq 0, \ \ \left(L^2_{\alpha_u^{(+)}} < t\right) = \left(u < A^{(1)}_{\tau_{\sqrt{t}}}\right) \ , \end{cases}$$

and invoking the scaling property, we can write jointly for all j's:

$$\begin{aligned} \left(A^{(j)}_{\alpha_1^{(+)}}, L^2_{\alpha_1^{(+)}}, \alpha_1^{(+)}\right) & \overset{(law)}{=} \left(L^2_{\alpha_1^{(+)}} A^{(j)}_{\tau_1}, L^2_{\alpha_1^{(+)}}, 1 + \sum_j L^2_{\alpha_1^{(+)}} A^{(j)}_{\tau_1}\right) \\ & \overset{(law)}{=} \left(\frac{A^{(j)}_{\tau_1}}{A^{(1)}_{\tau_1}}, \frac{1}{A^{(1)}_{\tau_1}}, \frac{\tau_1}{A^{(1)}_{\tau_1}}\right). \end{aligned} \tag{2.3}$$

Dividing now both sides by $\alpha_1^{(+)}$ and remarking that: $\alpha_1^{(+)} A^{(1)}_{\tau_1} = \tau_1$, we deduce:

$$\frac{1}{\alpha_1^{(+)}} \left(A^{(j)}_{\alpha_1^{(+)}}, L^2_{\alpha_1^{(+)}}\right) \overset{(law)}{=} \frac{1}{\tau_1} \left(A^{(j)}_{\tau_1}, 1\right). \tag{2.4}$$

With the help of the scaling Lemma below, we obtain:

$$\begin{aligned} E\left[1_{(W_1 \in I_1)} f(\overrightarrow{A_1}, L_1^2)\right] & = E\left[\frac{1}{\alpha_1^{(+)}} f\left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}}, \frac{L^2_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}\right)\right] \\ & \overset{\text{from (2.3)}}{=} E\left[\frac{A^{(1)}_{\tau_1}}{\tau_1} f\left(\frac{\overrightarrow{A_{\tau_1}}}{\tau_1}, \frac{1}{\tau_1}\right)\right]. \end{aligned} \tag{2.5}$$

I_1 may be replaced by I_m, for any $m \in \{2, \ldots, n\}$. Adding the m quantities found in (2.5) and remarking that:

$$\tau_1 = \sum_{i=1}^{n} A^{(i)}_{\tau_1} \ , \tag{2.6}$$

we get:

$$E\left[f(\overrightarrow{A_1}, L_1^2)\right] = E\left[f\left(\frac{\overrightarrow{A_{\tau_1}}}{\tau_1}, \frac{1}{\tau_1}\right)\right].$$

which proves (2.1). Note that from (2.4), the latter also equals:

$$E\left[f\left(\frac{\overrightarrow{A_{\alpha_1^{(+)}}}}{\alpha_1^{(+)}},\frac{L^2_{\alpha_1^{(+)}}}{\alpha_1^{(+)}}\right)\right].$$

Equality in law (2.2) follows now easily. Indeed, we denote by $\boldsymbol{\nu}$ the Itô measure of the Brownian spider, and we have:

$$\boldsymbol{\nu}=\frac{1}{n}\sum_{j=1}^{n}\nu_j\ , \tag{2.7}$$

where ν_j is the canonical image of $\mathbf{n}$, the standard Itô measure of the space of the excursions of the standard Brownian motion, on the space of the excursions on I_j. Hence, with $\lambda_j,\ j=1,\ldots,n$ denoting positive constants:

$$\begin{aligned} E\left[\exp\left(-\sum_{j=1}^{n}\lambda_j A^{(j)}_{\tau_1}\right)\right] &= \exp\left(-\frac{1}{n}\sum_{j=1}^{n}\int\nu_j(d\varepsilon_j)(1-e^{-\lambda_j\nu_j})\right)\\ &= \exp\left(-\frac{1}{n}\sum_{j=1}^{n}\sqrt{2\lambda_j}\right)\ , \end{aligned}$$

thus:

$$\overrightarrow{A_{\tau_1}}=\left(A^{(j)}_{\tau_1}\ ;\ j\le n\right)\overset{(law)}{=}\left(\frac{1}{n^2}T_j\ ;\ j\le n\right).$$

The latter, using (2.6) yields:

$$\overrightarrow{A_1}=\frac{\overrightarrow{A_{\tau_1}}}{\tau_1}=\frac{\overrightarrow{A_{\tau_1}}}{\sum_{i=1}^{n}A^{(i)}_{\tau_1}}\overset{(law)}{=}\left(\frac{T_j}{n^2\sum_{i=1}^{n}n^{-2}T_i}\ ;j\le n\right),$$

finishes the proof. □

It now remains to state the scaling Lemma which played a role in (2.5), and which we lift from [13] (Corollary 1, p. 40) in a "reduced" form.

Lemma 2.2. (Scaling Lemma) *Let $U_t=\int_0^t ds\theta_s$, with the pair (W,θ) satisfying:*

$$\left(W_{ct},\theta_{ct};t\ge 0\right)\overset{(law)}{=}\left(\sqrt{c}W_t,\theta_t;t\ge 0\right). \tag{2.8}$$

Then,

$$E\left[F\left(W_u,u\le 1\right)\theta_1\right]=E\left[\frac{1}{\alpha_1}F\left(\frac{1}{\sqrt{\alpha_1}}W_{v\alpha_1},v\le 1\right)\right], \tag{2.9}$$

where $\alpha_t=\inf\{s:U_s>t\}$.

3 Stable subordinators

3.1 Reminder and preliminaries on stable variables

In this Section, we consider S_μ and S'_μ two independent stable variables with exponent $\mu\in(0,1)$, i.e. for every $\lambda\ge 0$, the Laplace transform of S_μ is given by:

$$E[\exp(-\lambda S_\mu)]=\exp(-\lambda^\mu). \tag{3.1}$$

Concerning the law of S_μ, there is no simple expression for its density (except for the case $\mu = 1/2$; see e.g. Exercise 4.20 in [3]). However, we have that, for every $s < 1$ (see e.g. [15] or Exercise 4.19 in [3]):

$$E[(S_\mu)^{\mu s}] = \frac{\Gamma(1-s)}{\Gamma(1-\mu s)} \ . \tag{3.2}$$

We consider now the random variable of the ratio of two μ-stable variables:

$$X = \frac{S_\mu}{S'_\mu} \ . \tag{3.3}$$

Following e.g. Exercise 4.23 in [3], we have respectively the following formulas for the Stieltjes and the Mellin transforms of X:

$$E\left[\frac{1}{1+sX}\right] = \frac{1}{1+s^\mu} \ , \ s \geq 0 \ , \tag{3.4}$$

$$E\left[X^s\right] = \frac{\sin(\pi s)}{\mu \sin(\frac{\pi s}{\mu})}, \ 0 < s < \mu \ . \tag{3.5}$$

Moreover, the density of the random variable X^μ is given by (see e.g. [14, 5] or Exercise 4.23 in [3]):

$$P(X^\mu \in dy) = \frac{\sin(\pi\mu)}{\pi\mu} \frac{dy}{y^2 + 2y\cos(\pi\mu) + 1} \ , \ y \geq 0, \tag{3.6}$$

or equivalently:

$$\left(\frac{S_\mu}{S'_\mu}\right)^\mu = (C_\mu | C_\mu > 0), \tag{3.7}$$

where, with C denoting a standard Cauchy variable and U a uniform variable in $[\ 0, 2\pi\ )$,

$$C_\mu = \sin(\pi\mu)C - \cos(\pi\mu) \overset{(law)}{=} \frac{\sin(\pi\mu - U)}{U} \ .$$

3.2 The case of 2 stable variables

We turn now our study to the random variable:

$$A = \frac{S'_\mu}{S'_\mu + S_\mu} = \frac{1}{1+X}, \tag{3.8}$$

Theorem 3.1. *The density function of the random variable A is given by:*

$$P\left(A \in dz\right) = \frac{\sin(\pi\mu)}{\pi} \frac{dz}{z(1-z)\left[\left(\frac{1-z}{z}\right)^\mu + \left(\frac{z}{1-z}\right)^\mu + 2\cos(\pi\mu)\right]} \ , \ \ z \in [0,1]. \tag{3.9}$$

Proof. Identity (3.8) is equivalent to:

$$X = \frac{1}{A} - 1 \ .$$

Hence, (3.4) yields:

$$E\left[\frac{1}{1+sX}\right] = E\left[\frac{A}{(1-s)A + s}\right] = \frac{1}{1+s^\mu} \ .$$

We consider now a test function f and invoking the density (3.6) we have ($\nu = \frac{1}{\mu} > 1$):

$$E\left[f\left(\frac{1}{1+X}\right)\right] = \frac{\sin(\pi\mu)}{\pi\mu}\int_0^\infty \frac{dy}{y^2+2y\cos(\pi\mu)+1}\; f\left(\frac{1}{1+y^\nu}\right).$$

Changing the variables $z = \frac{1}{1+y^\nu}$, we deduce:

$$E\left[f\left(A\right)\right] = \frac{\sin(\pi\mu)}{\pi}\int_0^1 \frac{dz(1-z)^{\mu-1}}{z^{\mu+1}}\; f\left(z\right)\Delta(z),$$

where:

$$\begin{aligned}\Delta(z) &= \frac{1}{(z^{-1}-1)^{2\mu}+2(z^{-1}-1)^{\mu}\cos(\pi\mu)+1}\\ &= \frac{z^{2\mu}}{(1-z)^{2\mu}+2(1-z)^{\mu}z^{\mu}\cos(\pi\mu)+z^{2\mu}}\ ,\end{aligned}$$

and (3.9) follows easily. □

In Figure 1, we have plotted the density function g of A, for several values of μ.

Remark 3.2. *Similar discussions have been made in [4] in the framework of a skew Bessel process with dimension $2-2\alpha$ and skewness parameter p. Formula (3.9) is a particular case of formula in [4] for the density of the time spent positive (called $f_{p,\alpha}$ in [4]).*

3.3 The case of many stable (1/2) variables

In this Subsection, we consider again n iid stable (1/2) variables, i.e.: $T_1,\dots,T_n$, and we will study the distribution of:

$$A_1^{(1)} = \frac{T_1}{T_1+\ldots+T_n}\ . \tag{3.10}$$

The following Theorem answers to an open question (and even in a more general sense) stated at the end of [9].

Theorem 3.3. *The density function of the random variable $A_1^{(1)}$ is given by:*

$$P\left(A_1^{(1)}\in dz\right) = \frac{1}{\pi}\frac{dz}{\sqrt{z}\sqrt{1-z}\left[(n-1)z+\frac{1}{n-1}\left(1-z\right)\right]}\ ,\quad z\in[0,1]. \tag{3.11}$$

Proof. We first remark that, with C denoting a standard Cauchy variable, using e.g. (1.2):

$$A_1^{(1)} \overset{(law)}{=} \frac{T_1}{T_1+(n-1)^2T_2} \overset{(law)}{=} \frac{1}{1+(n-1)^2C^2}\ . \tag{3.12}$$

Hence, with f standing again for a test function, and invoking the density of a standard Cauchy variable, that is: for every $x\in\mathbb{R}$, $g(x)=\frac{1}{\pi(1+x^2)}$ we have:

$$\begin{aligned}E\left[f\left(A_1^{(1)}\right)\right] &= E\left[f\left(\frac{1}{1+(n-1)^2C^2}\right)\right]\\ &= \frac{1}{\pi}\int_{-\infty}^\infty \frac{dx}{1+x^2}\; f\left(\frac{1}{1+(n-1)^2x^2}\right)\\ &\overset{x^2=y}{=} \frac{2}{\pi}\int_0^\infty \frac{dy}{2\sqrt{y}(1+y)}\; f\left(\frac{1}{1+(n-1)^2y}\right)\end{aligned}$$

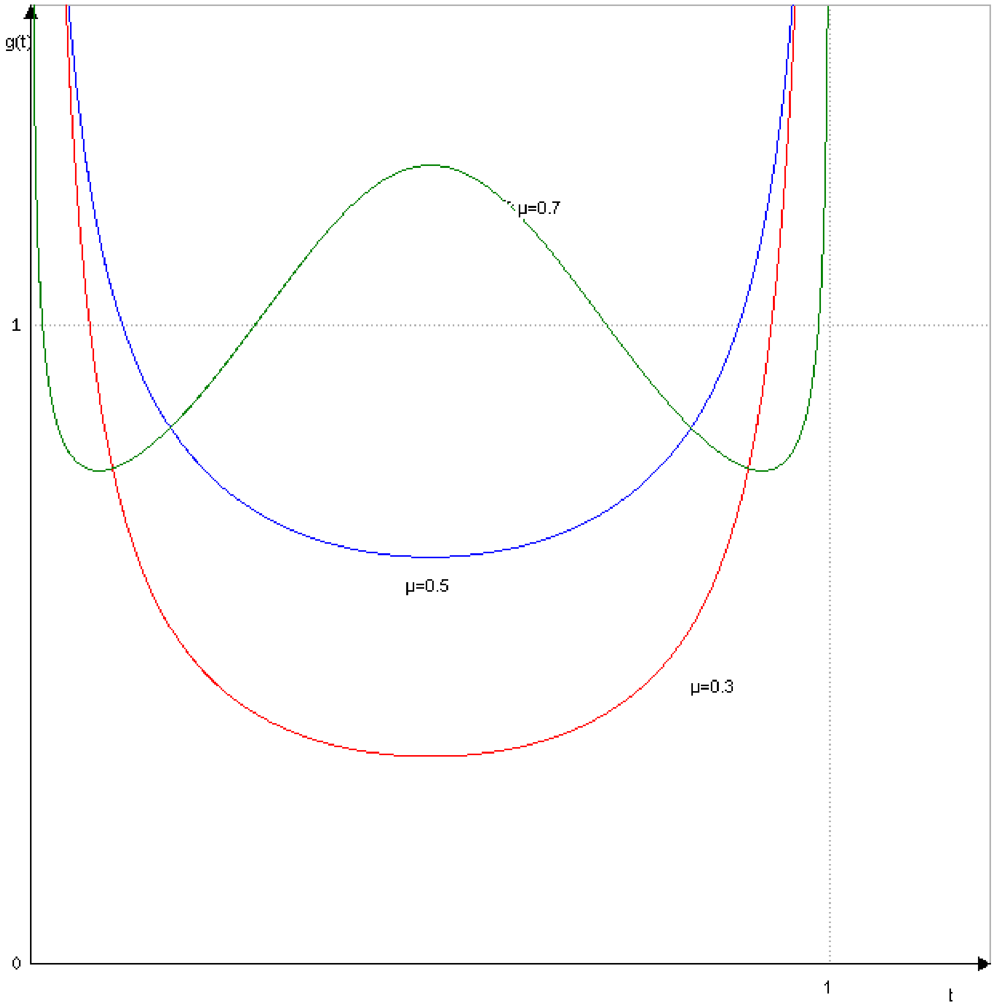

Figure 1: The density function g of A, for several values of μ.

Changing the variables $z = \frac{1}{1+(n-1)^2 y}$, we deduce:

$$E\left[f\left(A_1^{(1)}\right)\right] = \frac{1}{\pi}\int_0^1 \frac{dz}{(n-1)^2 z^2}\ \frac{(n-1)\sqrt{z}}{\sqrt{z-1}\left(1+\frac{1}{(n-1)^2}\left(\frac{1}{z}-1\right)\right)}\ f(z),$$

and (3.11) follows easily. □

Figure 2 presents the plot of the density function h of $A_1^{(1)}$, for several values of n.

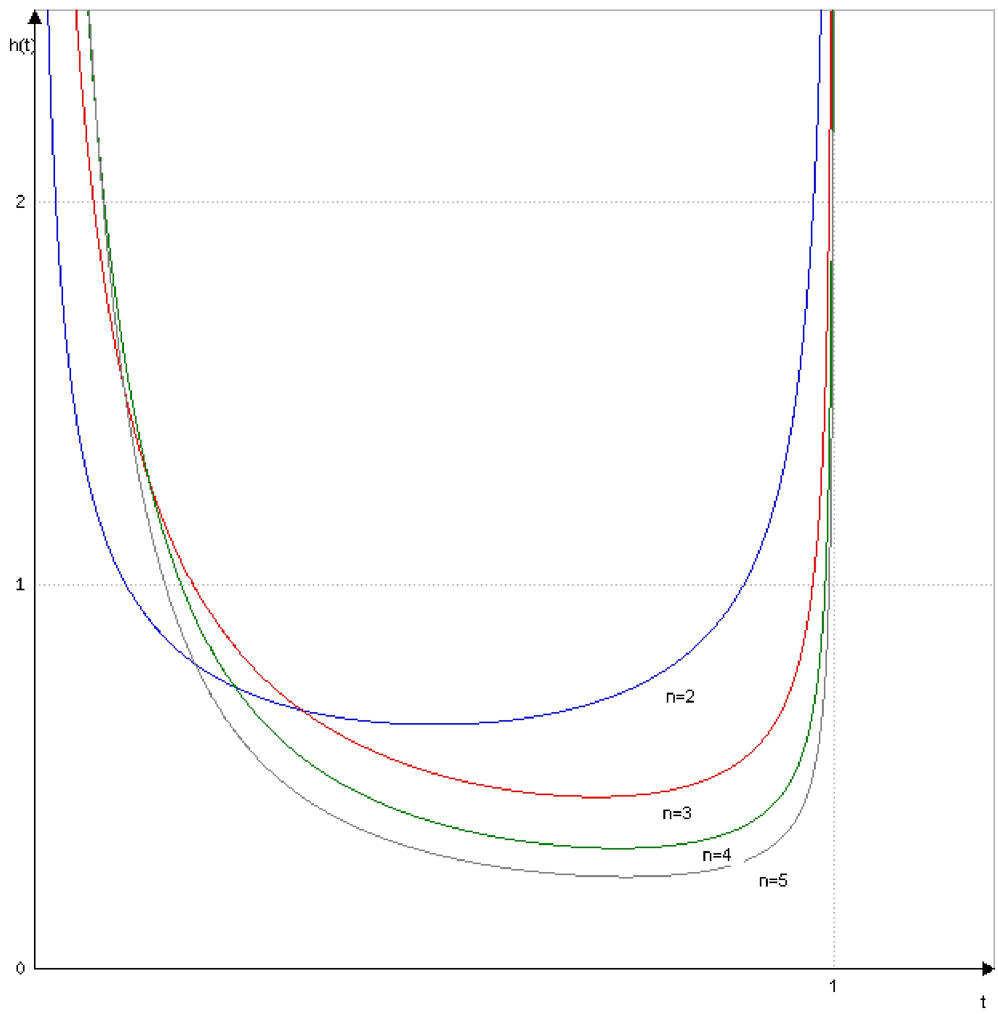

Figure 2: The density function h of $A_1^{(1)}$, for several values of n.

Corollary 3.4. *The following convergence in law holds:*

$$n^2 A_1^{(1)}(n) \underset{n\to\infty}{\overset{(law)}{\longrightarrow}} C^2\ . \tag{3.13}$$

Proof. It follows from Theorem 3.3 by simply remarking that $C \overset{(law)}{=} C^{-1}$. Hence:

$$n^2 A_1^{(1)}(n) = \frac{n^2}{1+(n-1)^2 C^2} = \frac{1}{\frac{1}{n^2}+\left(\frac{n-1}{n}\right)^2 C^2} \overset{n\to\infty}{\longrightarrow} \frac{1}{C^2} \overset{(law)}{=} C^2.$$

□

4 Conclusion and comments

We end up this article with some comments: usually, a scaling argument is "one-dimensional", as it involves a time-change. Exceptionally (or so it seems to the authors), here we could apply a scaling argument in a multivariate framework. We insist that the scaling Lemma plays a key role in our proof. The curious reader should also look at the totally different proof of this Theorem in [2], which mixes excursion theory and the Feynman-Kac method.

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