

A note on linearization methods and dynamic programming principles for stochastic discontinuous control problems

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Abstract

Using the linear programming approach to stochastic control introduced in [6] and [10], we provide a semigroup property for some set of probability measures leading to dynamic programming principles for stochastic control problems. An abstract principle is provided for general bounded costs. Linearized versions are obtained under further (semi)continuity assumptions.

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1 Introduction

Linear programming tools have been efficiently used to deal with stochastic control problems (see [3], [4], [11], [12], [13], [14] and references therein). An approach relying mainly on Hamilton-Jacobi(-Bellman) equations has been developed in [9] for deterministic control systems. This approach has been generalized to controlled Brownian diffusions (cf. [6] for infinite horizon, discounted control problems and [10] for Mayer and optimal stopping problems). The control processes and the associated solutions (and, eventually stopping times) can be embedded into a space of probability measures satisfying a convenient condition. This condition is given in terms of the coefficient functions. Using Hamilton-Jacobi-Bellman techniques, it is proven in [6] and [10] that minimizing continuous cost functionals with respect to the new set of constraints leads to the same value. These formulations turn out to provide the generalized solution of the (discontinuous) Hamilton-Jacobi-Bellman equation. For further details, the reader is referred to [10] and references therein.

For regular cost functionals, the dynamic programming principle has been extensively studied (e.g. [7], [15]). In general, dynamic programming principles are rather difficult to prove if the regularity of the value function is not known a priori. An alternative is to use a weak formulation where the value function is replaced by a test function (cf. [5]). This short paper aims at giving another approach to the dynamic programming principles based on linear programming techniques (cf. [6], [10]).

We begin by briefly recalling the linearization techniques from [10] for Lipschitz-continuous cost functionals in section 2.1. As a by-product, this allows to characterize

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the set of constraints as the closed convex hull of occupational measures associated to control processes. We give the linear programming formulation of the value in the case of lower and upper semicontinuous cost functionals and specify the connections between classical value function, primal and dual values (in section 2.2). Using the characterization of the set of constraints, we prove a semigroup property (section 3.1). This property follows naturally from the structure of the sets of constraints. We derive a dynamic programming principle for general bounded cost functionals in section 3.2. In the upper semicontinuous (or convex, lower semicontinuous) setting, we provide a further linearized inequality. This becomes an equality providing a linearized programming principle if the cost functionals are continuous.

2 Linear formulations for finite horizon stochastic control problems

We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space endowed with a filtration $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ satisfying the usual assumptions and W be a standard, d -dimensional Brownian motion with respect to this filtration. We denote by $T > 0$ a finite time horizon and we let U be a compact metric space. We consider the stochastic control system

$$\begin{cases} dX_s^{t,x,u} = b(s, X_s^{t,x,u}, u_s) ds + \sigma(s, X_s^{t,x,u}, u_s) dW_s, \text{ for all } s \in [t, T], \\ X_t^{t,x,u} = x \in \mathbb{R}^N, \end{cases} \quad (2.1)$$

where $t \in [0, T]$. Throughout the paper, we use the following standard assumption on the coefficient functions $b : \mathbb{R}^{N+1} \times U \rightarrow \mathbb{R}^N$ and $\sigma : \mathbb{R}^{N+1} \times U \rightarrow \mathbb{R}^{N \times d}$:

$$\begin{cases} \text{(i) the functions } b \text{ and } \sigma \text{ are bounded and uniformly continuous on } \mathbb{R}^{N+1} \times U, \\ \text{(ii) there exists a real constant } c > 0 \text{ such that} \\ |b(t, x, u) - b(s, y, u)| + |\sigma(t, x, u) - \sigma(s, y, u)| \leq c(|x - y| + |t - s|^{\delta_0}), \end{cases} \quad (2.2)$$

for all $(s, t, x, y, u) \in \mathbb{R}^2 \times \mathbb{R}^{2N} \times U$, and for some positive $\delta_0 > 0$. We recall that an admissible control process is any \mathbb{F} -progressively measurable process with values in the compact metric space U . We let \mathcal{U} denote the class of all admissible control processes on $[0, T] \times \Omega$. Under the assumption (2.2), for every $(t, x) \in [0, T] \times \mathbb{R}^N$ and every admissible control process $u \in \mathcal{U}$, there exists a unique solution to (2.1) starting from (t, x) denoted by $X^{t,x,u}$.

2.1 Lipschitz continuous cost functionals

In this subsection, we recall the basic tools that allow to identify the primal and dual linear formulations associated to (finite horizon) stochastic control problems. The results can be found in [10] (see also [6] for the infinite time horizon).

To any $(t, x) \in [0, T] \times \mathbb{R}^N$ and any $u \in \mathcal{U}$, we associate the (expectation of the) occupational measures

$$\begin{aligned} \gamma_{t,T,x,u}^1(A \times B \times C) &= \frac{1}{T-t} \mathbb{E} \left[\int_t^T 1_{A \times B \times C}(s, X_s^{t,x,u}, u_s) ds \right], \\ \gamma_{t,T,x,u}^2(D) &= \mathbb{E} [1_D(X_T^{t,x,u})], \end{aligned}$$

for all Borel subsets $A \times B \times C \times D \subset [t, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N$. Also, we can define $\gamma_{T,T,x,u}^1 = \delta_{T,x,u_T}$, $\gamma_{T,T,x,u}^2 = \delta_x$. We denote by

$$\Gamma(t, T, x) = \{ \gamma_{t,T,x,u} = (\gamma_{t,T,x,u}^1, \gamma_{t,T,x,u}^2) \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : u \in \mathcal{U} \}.$$

Here, $\mathcal{P}(\mathcal{X})$ stands for the set of probability measures on the metric space \mathcal{X} . Due to the assumption (2.2), there exists a positive constant C_0 such that, for every $T > 0$, every $(t, x) \in [0, T] \times \mathbb{R}^N$ and every $u \in \mathcal{U}$, one has

$$\sup_{s \in [t, T]} \mathbb{E} \left[|X_s^{t,x,u}|^2 \right] \leq e^{C_0(T-t)} \left(|x|^2 + C_0(T-t) \right). \tag{2.3}$$

Therefore,

$$\begin{cases} \int_{\mathbb{R}^N} |y|^2 \gamma_{t,T,x,u}^1([t, T], dy, U) \leq e^{C_0(T-t)} \left(|x|^2 + C_0(T-t) \right), \\ \int_{\mathbb{R}^N} |y|^2 \gamma_{t,T,x,u}^2(dy) \leq e^{C_0(T-t)} \left(|x|^2 + C_0(T-t) \right). \end{cases} \tag{2.4}$$

We define

$$\Theta(t, T, x) = \left\{ \begin{array}{l} \gamma \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : \forall \phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N), \\ \int_{[t, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N} \begin{bmatrix} (T-t) \mathcal{L}^v \phi(s, y) \\ + \phi(t, x) - \phi(T, z) \end{bmatrix} \gamma^1(ds, dy, dv) \gamma^2(dz) = 0, \end{array} \right\} \tag{2.5}$$

where

$$\mathcal{L}^v \phi(s, y) = \frac{1}{2} Tr[(\sigma \sigma^*)(s, y, v) D^2 \phi(s, y)] + \langle b(s, y, v), D \phi(s, y) \rangle + \partial_t \phi(s, y),$$

for all $(s, y) \in [0, T] \times \mathbb{R}^N$, $v \in U$ and all $\phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N)$. Itô's formula applied to test functions $\phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N)$ yields

$$\Gamma(t, T, x) \subset \Theta(t, T, x).$$

Moreover, the set $\Theta(t, T, x)$ is convex and a closed subset of $\mathcal{P}([t, T] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$. For further details, the reader is referred to [10].

Let us suppose that $f : \mathbb{R} \times \mathbb{R}^N \times U \rightarrow \mathbb{R}$, $g : \mathbb{R}^N \rightarrow \mathbb{R}$ are bounded and uniformly continuous such that

$$|f(t, x, u) - f(s, y, u)| + |g(x) - g(y)| \leq c(|x - y| + |t - s|), \tag{2.6}$$

for all $(s, t, x, y, u) \in \mathbb{R}^2 \times \mathbb{R}^{2N} \times U$, and for some positive $c > 0$. We introduce the usual value function

$$\begin{aligned} V(t, x) &= \inf_{u \in \mathcal{U}} \mathbb{E} \left[\int_t^T f(s, X_s^{t,x,u}, u_s) ds + g(X_T^{t,x,u}) \right] \\ &= \inf_{\gamma \in \Gamma(t, T, x)} \left((T-t) \int_{[t, T] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds, dy, du) + \int_{\mathbb{R}^N} g(y) \gamma^2(dy) \right); \end{aligned} \tag{2.7}$$

and the primal linearized value function

$$\Lambda(t, x) = \inf_{\gamma \in \Theta(t, T, x)} \left((T-t) \int_{[t, T] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds, dy, du) + \int_{\mathbb{R}^N} g(y) \gamma^2(dy) \right), \tag{2.8}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$. We also consider the dual value function

$$\mu^*(t, x) = \sup \left\{ \begin{array}{l} \mu \in \mathbb{R} : \exists \phi \in C_b^{1,2}([0, T] \times \mathbb{R}^N) \\ \text{s.t. } \forall (s, y, v, z) \in [t, T] \times \mathbb{R}^N \times U \times \mathbb{R}^N, \\ \mu \leq (T-t) (\mathcal{L}^v \phi(s, y) + f(s, y, v)) + g(z) - \phi(T, z) + \phi(t, x), \end{array} \right\} \tag{2.9}$$

for all $(t, x) \in [0, T] \times \mathbb{R}^N$. The following result is a slight generalization of theorem 4 in [10]. The proof is very similar and will be omitted.

Theorem 2.1. (see [10], theorem 4). Under the assumptions (2.2) and (2.6),

$$V = \Lambda = \mu^*.$$

Since this result holds true for arbitrary (regular) functions f and g , a standard separation argument yields:

Corollary 2.2. The set of constraints $\Theta(t, T, x)$ is the closed, convex hull of $\Gamma(t, T, x)$:

$$\Theta(t, T, x) = \overline{\text{co}}\Gamma(t, T, x). \tag{2.10}$$

The closure is taken with respect to the usual (narrow) convergence of probability measures.

Remark 2.3. 1. Due to the inequality (2.4), Prohorov's theorem yields that $\text{co}\Gamma(t, T, x)$ is relatively compact and, thus, $\Theta(t, T, x)$ is compact. Moreover,

$$\begin{cases} \int_{\mathbb{R}^N} |y|^2 \gamma^1([t, T], dy, U) \leq e^{C_0(T-t)} (|x|^2 + C_0(T-t)), \\ \int_{\mathbb{R}^N} |y|^2 \gamma^2(dy) \leq e^{C_0(T-t)} (|x|^2 + C_0(T-t)), \end{cases} \tag{2.11}$$

for all $\gamma = (\gamma^1, \gamma^2) \in \Theta(t, T, x)$.

2. In fact, similar estimates can be obtained for any $p \geq 2$. The set $\Theta(t, T, x)$ can then be defined by taking test functions whose derivatives have at most quadratic growth ($C_2^{1,2}$). Moreover, we can impose a bounded moment of order $2 + \delta > 2$. In this case, the set of constraints $\Theta(t, T, x)$ is also the closed, convex hull of $\Gamma(t, T, x)$ with respect to the Wasserstein distance W_2 .

2.2 Semicontinuous cost functionals

In the case when f and g are only semicontinuous, one can still define V , Λ and μ^* . One can also (formally) consider the associated Hamilton-Jacobi-Bellman equation

$$\{-\partial_t V(t, x) + H(t, x, DV(t, x), D^2V(t, x))\} = 0, \tag{2.12}$$

for all $(t, x) \in (0, T) \times \mathbb{R}^N$, and $V(T, \cdot) = g(\cdot)$ on \mathbb{R}^N , where the Hamiltonian is given by

$$H(t, x, p, A) = \sup_{u \in U} \left\{ -\frac{1}{2} \text{Tr}(\sigma \sigma^*(t, x, u) A) - \langle b(t, x, u), p \rangle - f(t, x, u) \right\}, \tag{2.13}$$

for all $(t, x, p, A) \in \mathbb{R} \times \mathbb{R}^N \times \mathbb{R}^N \times \mathcal{S}^N$. We recall that \mathcal{S}^N stands for the family of symmetric $N \times N$ -type matrix. The following theorem states the connection between these various functions and the equation (2.12). Proofs for these results can be found in section 4 of [10]. They are based on inf/sup-convolutions and the compactness of $\Theta(t, T, x)$.

Theorem 2.4. If the functions f and g are bounded and lower semicontinuous, then the primal and dual value functions coincide $\Lambda = \mu^*$. The common value is the smallest bounded l.s.c. viscosity supersolution of (2.12).

If the functions f and g are bounded and upper semicontinuous, then V coincides with Λ . The common value is the largest bounded u.s.c. viscosity subsolution of (2.12).

In the l.s.c. case, the assertions are given by theorem 7 in [10]. The u.s.c. case is covered by theorem 8 in [10].

Remark 2.5. If the functions f and g are bounded and lower semicontinuous (but not continuous), V and Λ might not coincide. Under usual convexity assumptions, Λ coincides with the weak formulation of V (i.e.

$$\Lambda(t, x) = \inf_{\pi = (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, u)} \mathbb{E}^\pi \left[\int_t^T f(s, X_s^{t,x,u}, u_s) ds + g(X_T^{t,x,u}) \right].$$

For further details, the reader is referred to proposition 12 and example 13 in [10]. If the functions f and g are bounded and upper semicontinuous, then V and μ^* may not coincide (cf. example 9 in [10]).

Remark 2.6. In the case of control problems with semicontinuous cost and governed by a deterministic equation

$$x'(t) = f(t, x(t), u(t)),$$

the standard assumption is convexity of the dynamics. In this case, the value function is also semicontinuous (see for instance [2, 8]). Moreover, if the cost is continuous, the value function associated to the previous dynamics coincides with the value function associated to the convexified differential inclusion

$$x'(t) \in F(t, x(t)), \quad F(t, x) = \overline{\text{co}}(f(t, x, u) : u \in U).$$

because of Fillipov's theorem (see for instance [1]). However, when the dynamics is not convex and the cost is l.s.c. the value functions associated to the previous dynamics might not coincide. Consequently, in order to obtain the existence of optimal trajectories/controls and to characterize the l.s.c. value function as the generalized solution (smallest bounded l.s.c. viscosity supersolution) to the associated Hamilton-Jacobi system, one would have to consider the convexified differential inclusion. This definition is similar to (2.8).

In the stochastic framework, one can reason in a similar way, by taking the weak limit of solutions (i.e. the closed convex hull of occupational measures). This set turns out to be Θ introduced in our paper. We emphasize that the actual procedure is somewhat reversed: we consider the explicit set and deduce that it is the closed convex hull of occupational measures. It is our opinion that this is the natural method allowing to obtain good properties of the value function whenever the cost is semicontinuous (or just bounded).

3 A linear formulation for the dynamic programming principle

3.1 A semigroup property

We fix $x_0 \in \mathbb{R}^N$. Let us consider $t_1, t_2 \geq 0$ such that $0 < t_1 + t_2 \leq T$, where $T > 0$ is a fixed terminal time. By analogy to $\Theta(t, T, x)$ for $x \in \mathbb{R}^N$, we define sets of constraints starting from measures. If $\gamma \in \mathcal{P}([0, t_1] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$, we define the set

$$\Theta(t_1, t_1 + t_2, \gamma) = \left\{ \begin{array}{l} \eta = (\eta^1, \eta^2) \in \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) : \\ \forall \phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N), \\ \int_{\mathbb{R}^N} \phi(t_1, x) \gamma^2(dx) + \int_{[t_1, t_1 + t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta^1(ds dy du) \\ - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z) \eta^2(dz) = 0, \\ \int_{\mathbb{R}^N} |y|^2 \eta^1([t_1, t_1 + t_2], dy, U) \leq e^{C_0(t_1 + t_2)} (|x_0|^2 + C_0(t_1 + t_2)), \\ \int_{\mathbb{R}^N} |z|^2 \eta^2(dz) \leq e^{C_0(t_1 + t_2)} (|x_0|^2 + C_0(t_1 + t_2)). \end{array} \right.$$

The reader is invited to notice that x can be identified with $\gamma_{t,t,x,u}$, for all $u \in \mathcal{U}$.

Proposition 3.1. *For every $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$ and every $\gamma = (\gamma^1, \gamma^2) \in \Theta(0, t_1, x_0)$, the set $\Theta(t_1, t_1 + t_2, \gamma)$ is nonempty, convex and compact with respect to the usual (weak *) convergence of probability measures.*

Proof. We only need to prove that $\Theta(t_1, t_1 + t_2, \gamma)$ is nonempty. Convexity and closedness is obvious from the definition, while the second-order moment inequalities guarantee relative compactness (due to Prohorov's theorem). We consider an arbitrary test function $\phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$. We begin by assuming that $\gamma = \gamma_{0,t_1,x_0,u}$, for some admissible control process $u \in \mathcal{U}$. We define a couple of probability measures $(\eta^1, \eta^2) \in \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ by setting

$$\eta^1(dsdydu) = \frac{t_1 + t_2}{t_2} 1_{[t_1, t_1 + t_2]}(s) \gamma_{0,t_1+t_2,x_0,u}^1(dsdydu), \quad \eta^2 = \gamma_{0,t_1+t_2,x_0,u}^2.$$

One has

$$\begin{aligned} & \int_{\mathbb{R}^N} \phi(t_1, x) \gamma^2(dx) + \int_{[t_1, t_1 + t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta^1(dsdydu) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z) \eta^2(dz) \\ &= \mathbb{E} \left[\phi(t_1, X_{t_1}^{0,x_0,u}) + \int_{t_1}^{t_1+t_2} \mathcal{L}^{u_s} \phi(s, X_s^{0,x_0,u}) ds - \phi(t_1 + t_2, X_{t_1+t_2}^{0,x_0,u}) \right] = 0 \end{aligned} \quad (3.1)$$

Therefore, combining (3.1) with (2.3), the assertion of our proposition holds true for occupational couples.

To prove the result for general couples $\gamma = (\gamma^1, \gamma^2) \in \Theta(0, t_1, x_0)$, we will use Corollary 2.2. Whenever $\gamma = (\gamma^1, \gamma^2) \in \Theta(0, t_1, x_0)$, there exists a family of convex combinations

$$\left(\sum_{i=1}^{k_n} \alpha_n^i \gamma_{0,t_1,x_0,u_n^i} \right)_{n \geq 0}$$

converging to γ . One defines

$$\eta_n^1 = \frac{t_1 + t_2}{t_2} \sum_{i=1}^{k_n} \alpha_n^i \gamma_{0,t_1+t_2,x_0,u_n^i}^1, \quad \eta_n^2 = \sum_{i=1}^{k_n} \alpha_n^i \gamma_{0,t_1+t_2,x_0,u_n^i}^2,$$

for $n \geq 0$. Inequality (2.3) (also (2.4)) yields the existence of a subsequence converging to some $\eta = (\eta^1, \eta^2) \in \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$. One notices that $\eta \in \Theta(t_1, t_1 + t_2, \gamma)$. \square

Definition 3.2. *Whenever $\gamma \in \Theta(0, t_1, x_0)$ and $\eta \in \Theta(t_1, t_1 + t_2, \gamma)$, we define $\eta \circ \gamma \in \mathcal{P}([0, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ by setting:*

$$\begin{aligned} (\eta \circ \gamma)^1(A, dydu) &= \frac{t_1}{t_1+t_2} \gamma^1(A \cap [0, t_1], dydu) + \frac{t_2}{t_1+t_2} \eta^1(A \cap [t_1, t_1 + t_2], dydu), \\ (\eta \circ \gamma)^2 &= \eta^2, \end{aligned}$$

for all Borel sets $A \subset [0, t_1 + t_2]$.

We introduce the following notations:

$$\begin{cases} \Theta(t_1, t_1 + t_2, \Theta(0, t_1, x_0)) = \bigcup_{\gamma \in \Theta(0, t_1, x_0)} \Theta(t_1, t_1 + t_2, \gamma), \\ \Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \{\eta \circ \gamma : \gamma \in \Theta(0, t_1, x_0), \eta \in \Theta(t_1, t_1 + t_2, \gamma)\}, \\ \Theta(t_1, t_1 + t_2, \cdot) \circ \Gamma(0, t_1, x_0) = \{\eta \circ \gamma : \gamma \in \Gamma(0, t_1, x_0), \eta \in \Theta(t_1, t_1 + t_2, \gamma)\} \end{cases}$$

Remark 3.3. Whenever $\gamma \in \Theta(0, t_1, x_0)$ and $\eta \in \Theta(t_1, t_1 + t_2, \gamma)$, it is clear that $\eta \circ \gamma \in \Theta(0, t_1 + t_2, x_0)$. Indeed, if $\phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$, one gets

$$\begin{aligned} \phi(0, x_0) + \int_{[0, t_1] \times \mathbb{R}^N \times U} t_1 \mathcal{L}^u \phi(s, y) \gamma^1(ds dy du) - \int_{\mathbb{R}^N} \phi(t_1, z) \gamma^2(dz) = 0, \text{ and} \\ \int_{\mathbb{R}^N} \phi(t_1, z) \gamma^2(dz) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta^1(ds dy du) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z) \eta^2(dz) = 0. \end{aligned}$$

The conclusion follows by summing the two equalities.

In fact, we can prove that the converse also holds true.

Proposition 3.4. 1. We have the following semigroup property

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) = \Theta(0, t_1 + t_2, x_0),$$

for all $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$.

2. $\Theta(t_1, t_1 + t_2, \cdot) \circ \Gamma(0, t_1, x_0) \supset \Gamma(0, t_1 + t_2, x_0)$, for all $t_1, t_2 \geq 0$ such that $t_1 + t_2 \leq T$.

Proof. We only prove the inclusion

$$\Theta(t_1, t_1 + t_2, \cdot) \circ \Theta(0, t_1, x_0) \supset \Theta(0, t_1 + t_2, x_0).$$

Let us suppose that $\bar{\gamma} = (\bar{\gamma}^1, \bar{\gamma}^2) \in \Theta(0, t_1 + t_2, x_0)$ is an occupational measure associated to $(x_0, u) \in \mathbb{R}^N \times \mathcal{U}$. We define

$$\begin{cases} \gamma^1(A, dy du) = \gamma_{0, t_1, x_0, u}^1(A, dy du) = \frac{t_1+t_2}{t_1} \bar{\gamma}^1(A, dy du), \gamma^2 = \gamma_{0, t_1, x_0, u}^2, \\ \eta^1(A', dy du) = \frac{t_1+t_2}{t_2} \gamma_{0, t_1+t_2, x_0, u}^1(A', dy du) = \frac{t_1+t_2}{t_2} \bar{\gamma}^1(A', dy du), \\ \eta^2 = \gamma_{0, t_1+t_2, x_0, u}^2 = \bar{\gamma}^2, \end{cases}$$

for all Borel sets $A \subset [0, t_1]$, $A' \subset [t_1, t_1 + t_2]$. It is clear that

$$\gamma = (\gamma^1, \gamma^2) \in \Theta(0, t_1, x_0), \eta = (\eta^1, \eta^2) \in \Theta(t_1, t_1 + t_2, \gamma) \text{ and } \bar{\gamma} = \eta \circ \gamma.$$

The conclusion in the general case follows by recalling that $\Theta = \overline{co}(\Gamma)$ by construction similar to proposition 3.1 and due to the compactness arguments. \square

The following propositions give different ways of constructing elements of $\Theta(t_1, t_1 + t_2, \gamma)$.

Proposition 3.5. If $\eta : \mathbb{R}^N \rightarrow \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ is measurable and such that $\eta(z) = (\eta^1(z), \eta^2(z)) \in \Theta(t_1, t_1 + t_2, z)$ for all z , then

$$\bar{\eta} = \left(\int_{\mathbb{R}^N} \eta^1(z) (ds dy du) \gamma^2(dz), \int_{\mathbb{R}^N} \eta^2(z) (dy) \gamma^2(dz) \right) \in \Theta(t_1, t_1 + t_2, \gamma).$$

In particular, whenever $\pi = (B_n)_n$ is a partition of \mathbb{R}^N into Borel sets, the family

$$\begin{aligned} \Gamma_\pi(t_1, t_1 + t_2, \gamma) = \\ \left\{ \left(\int_{\mathbb{R}^N} \eta_{u^n}^1(z) \gamma^2(dz), \int_{\mathbb{R}^N} \eta_{u^n}^2(z) \gamma^2(dz) \right) : (u^n) \subset \mathcal{U} \right\}, \end{aligned}$$

with

$$\eta_{u^n}^1(z) = \sum_n 1_{B_n}(z) \gamma_{t_1, t_1+t_2, z, u^n}^1, \eta_{u^n}^2(z) = \sum_n 1_{B_n}(z) \gamma_{t_1, t_1+t_2, z, u^n}^2,$$

is a subset of $\Theta(t_1, t_1 + t_2, \gamma)$.

Proof. Indeed, if $\phi \in C_b^{1,2}(\mathbb{R}_+ \times \mathbb{R}^N)$, for every $z \in \mathbb{R}^N$,

$$\phi(t_1, z) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta^1(z) (ds dy du) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, y) \eta^2(z) (dy) = 0.$$

Then, integrating with respect to γ^2 , one gets

$$\int_{\mathbb{R}^N} \phi(t_1, z) \gamma^2(dz) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \bar{\eta}^1(ds dy du) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, y) \bar{\eta}^2(dy) = 0.$$

Also, due to (2.11), we have

$$\begin{aligned} \int_{\mathbb{R}^N} |y|^2 \bar{\eta}^2(dy) &= \int_{\mathbb{R}^N} \left(\int_{\mathbb{R}^N} |y|^2 \eta^2(z) (dy) \right) \gamma^2(dz) \leq \int_{\mathbb{R}^N} e^{C_0 t_2} (|z|^2 + C_0 t_2) \gamma^2(dz) \\ &\leq e^{C_0 t_2} \left[C_0 t_2 + e^{C_0 t_1} (|x_0|^2 + C_0 t_1) \right] \leq e^{C_0(t_1+t_2)} (|x_0|^2 + C_0(t_1 + t_2)). \end{aligned}$$

Similar estimates hold true for $\int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} |y|^2 \bar{\eta}^1(ds dy du)$ and the first assertion follows.

Whenever $u \in \mathcal{U}$, standard estimates yield that the function $z \mapsto \gamma_{t_1, t_1+t_2, z, u}$ is continuous (thus measurable). The last assertion follows from 1. \square

Proposition 3.6. For $\varepsilon > 0$, we consider a partition (B_n^ε) of \mathbb{R}^N and a family $(x_n^\varepsilon)_n \subset \mathbb{R}^N$ such that $|x - x_n^\varepsilon| \leq \varepsilon$, for all $x \in B_n^\varepsilon$. If $\gamma_n \in \Theta(t_1, t_1 + t_2, x_n^\varepsilon)$, we introduce

$$\eta^{\varepsilon,1} = \sum_n \gamma^2(B_n^\varepsilon) \gamma_n^1 \left(= \int_{\mathbb{R}^N} \sum_n \gamma_n^1 1_{B_n^\varepsilon}(z) \gamma^2(dz) \right), \quad \eta^{\varepsilon,2} = \sum_n \gamma^2(B_n^\varepsilon) \gamma_n^2, \quad \eta^\varepsilon = (\eta^{\varepsilon,1}, \eta^{\varepsilon,2}).$$

Then the family $(\eta^\varepsilon)_\varepsilon \subset \mathcal{P}([t_1, t_1 + t_2] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$ is relatively compact and any limit η is in $\Theta(t_1, t_1 + t_2, \gamma)$.

Proof. First, let us notice that, whenever $x \in B_n^\varepsilon$,

$$|x_n^\varepsilon|^2 \leq (|x| + \varepsilon)^2 \leq (1 + \varepsilon) |x|^2 + (\varepsilon + \varepsilon^2).$$

Due to (2.11), one has

$$\begin{aligned} \int_{\mathbb{R}^N} |y|^2 \eta^{\varepsilon,2}(dy) &\leq \int_{\mathbb{R}^N} \sum_n 1_{B_n^\varepsilon}(z) e^{C_0 t_2} (|x_n^\varepsilon|^2 + C_0 t_2) \gamma^2(dz) \\ &\leq \int_{\mathbb{R}^N} e^{C_0 t_2} (|z|^2 (1 + \varepsilon) + (\varepsilon + \varepsilon^2) + C_0 t_2) \gamma^2(dz) \\ &\leq (1 + \varepsilon) e^{C_0(t_1+t_2)} (|x_0|^2 + \varepsilon + C_0(t_1 + t_2)). \end{aligned}$$

Similar estimates hold true for $\eta^{\varepsilon,1}$. Prohorov's theorem yields the relative compactness. Moreover, if η is a limit as $\varepsilon \rightarrow 0$ of some subsequence,

$$\int_{\mathbb{R}^N} |y|^2 \eta^2(dy) \leq e^{C_0(t_1+t_2)} (|x_0|^2 + C_0(t_1 + t_2)).$$

Furthermore, if $\phi \in C_b^{1,2}$,

$$\begin{aligned} & \left| \int_{\mathbb{R}^N} \phi(t_1, z) \gamma^2(dz) + \int_{[t_1, t_1+t_2] \times \mathbb{R}^N \times U} t_2 \mathcal{L}^u \phi(s, y) \eta^{\varepsilon,1}(dsdydu) - \int_{\mathbb{R}^N} \phi(t_1 + t_2, z) \eta^{\varepsilon,2}(dz) \right| \\ &= \left| \int_{\mathbb{R}^N} \left[\phi(t_1, z) - \sum_n 1_{B_n^\varepsilon}(z) \phi(t_1, x_n^\varepsilon) \right] \gamma^2(dz) \right| \leq c\varepsilon, \end{aligned}$$

for some constant c depending only on ϕ . We conclude that $\eta \in \Theta(t_1, t_1 + t_2, \gamma)$. \square

3.2 Dynamic programming principles

Let us now return to the primal value function introduced by (2.8). If $\gamma = (\gamma^1, \gamma^2) \in \Theta(t_0, t_0, x_0)$, then $\gamma^2 = \delta_{x_0}$ and it seems natural to impose

$$\Lambda(t_0, \gamma) = \Lambda(t_0, x_0).$$

If $t_0 < t \leq T$ and $\gamma \in \Theta(t_0, t, x_0)$ we extend Λ to initial measures by setting

$$\Lambda(t, \gamma) = \inf_{\eta \in \Theta(t, T, \gamma)} \left((T-t) \int_{[t, T] \times \mathbb{R}^N \times U} f(s, y, u) \eta^1(dsdydu) + \int_{\mathbb{R}^N} g(z) \eta^2(dz) \right). \quad (3.2)$$

Proposition 3.7. *If f and g are l.s.c. and bounded, then the restriction of the functional $\gamma \mapsto \Lambda(t, \gamma)$ to $\Theta(t_0, t, x_0)$ is convex, for all $t_0 < t \leq T$ and all $x_0 \in \mathbb{R}^N$.*

Proof. We introduce the functional $\chi : \mathcal{P}([t, T] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N) \rightarrow \mathbb{R}$ given by

$$\chi(\eta) = (T-t) \int_{[t, T] \times \mathbb{R}^N \times U} f(s, y, u) \eta^1(dsdydu) + \int_{\mathbb{R}^N} g(z) \eta^2(dz),$$

for all $\eta \in \mathcal{P}([t, T] \times \mathbb{R}^N \times U) \times \mathcal{P}(\mathbb{R}^N)$. Then

$$\Lambda(t, \gamma) = \inf_{\eta \in \Theta(t, T, \gamma)} \chi(\eta),$$

for all $\gamma \in \Theta(t_0, t, x_0)$. If f and g are l.s.c., χ is l.s.c. Whenever $\gamma \in \Theta(t_0, t, x_0)$, using the compactness of $\Theta(t, T, \gamma)$, one gets the existence of some $\eta \in \Theta(t, T, \gamma)$ such that

$$\Lambda(t, \gamma) = \chi(\eta).$$

If $\gamma_1, \gamma_2 \in \Theta(t_0, t, x_0)$ and $\alpha \in [0, 1]$, then there exist $\eta_i \in \Theta(t, T, \gamma_i)$ such that $\Lambda(t, \gamma_i) = \chi(\eta_i)$. Moreover, $\alpha\eta_1 + (1-\alpha)\eta_2 \in \Theta(t, T, \alpha\gamma_1 + (1-\alpha)\gamma_2)$. This implies that the functional $\gamma \mapsto \Lambda(t, \gamma)$ is convex. \square

We can now state and prove the dynamic programming principle(s) for the value function Λ . In the bounded case, an abstract DPP is obtained. Under further assumptions, the DPP has a linear form. We only have an inequality if the cost functions are semicontinuous and one gets a linear DPP in the uniformly continuous framework. This is just a first step in studying the equations that would characterize the discontinuous value function.

Theorem 3.8. *(Dynamic Programming Principles)*

1. Let us suppose that the functions f and g are bounded. Then, the following equality holds true

$$\Lambda(t_0, x_0) = \inf_{\gamma \in \Theta(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds dy du) + \Lambda(t, \gamma) \right), \quad (3.3)$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (t_0, T)$.

2. If f and g are bounded and u.s.c, then

$$\Lambda(t_0, x_0) \geq \inf_{\gamma \in \Theta(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds dy du) + \int_{\mathbb{R}^N} \Lambda(t, y) \gamma^2(dy) \right).$$

The same holds true if $\{(\sigma\sigma^*(t, x, u), b(t, x, u), f(t, x, u)) : u \in U\}$ is convex and f and g are bounded and l.s.c.

3. If f and g are bounded and Lipschitz-continuous,

$$\Lambda(t_0, x_0) = \inf_{\gamma \in \Theta(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds dy du) + \int_{\mathbb{R}^N} \Lambda(t, y) \gamma^2(dy) \right),$$

for all $x_0 \in \mathbb{R}^N$ and all $t \in (t_0, T)$.

Remark 3.9. If f and g are bounded and Lipschitz-continuous, Λ coincides with the classical value function V (cf. Theorem 4 in [10]), being, therefore, uniformly continuous. The method can be extended to uniformly continuous cost functions.

Proof. Let us fix $x_0 \in \mathbb{R}^N$ and $t \in (0, T]$. To simplify notation, we assume that $t_0 = 0$.

1. Proposition 3.4 and definition (3.2) yield

$$\begin{aligned} \Lambda(0, x_0) &= \inf_{\substack{\gamma \in \Theta(0, t, x_0) \\ \eta \in \Theta(t, T, \gamma)}} \left(T \int_{[0, T] \times \mathbb{R}^N \times U} f(s, y, u) (\eta \circ \gamma)^1(ds dy du) + \int_{\mathbb{R}^N} g(z) (\eta \circ \gamma)^2(dz) \right) \\ &= \inf_{\gamma \in \Theta(0, t, x_0)} \left(t \int_{[0, t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(ds dy du) + \Lambda(t, \gamma) \right). \end{aligned}$$

2. In the upper semicontinuous case, due to [10], proposition 11,

$$\Lambda(s, x) = V^w(s, x), \quad (3.4)$$

for all $(s, x) \in [0, T] \times \mathbb{R}^N$. Here, V^w is the weak value function and it is defined on the family of weak control processes $\mathcal{U}^w = \left\{ \pi : \pi = \left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P}, W, u \right) \right\}$. It is known by the classical optimality principles (see, for example [15], chapter 4, lemma 3.2 and theorem 3.3) that

$$V^w(0, x_0) \geq \inf_{\pi \in \mathcal{U}^w} \mathbb{E}^\pi \left[\int_0^t f(s, X_s^{0, x_0, u}, u_s) ds + V^w(t, X_t^{0, x_0, u}) \right]. \quad (3.5)$$

To every weak control π we can associate, as before, an occupational couple

$$\gamma_{0, t, x_0, \pi}^1(A \times B \times C) = \frac{1}{t} \mathbb{E}^\pi \left[\int_0^t 1_{A \times B \times C}(s, X_s^{0, x_0, u}, u_s) ds \right],$$

$$\gamma_{0,t,x_0,\pi}^2(D) = \mathbb{P}^\pi \left(X_t^{0,x_0,u} \in D \right)$$

for all Borel subsets $A \times B \times C \times D \subset [0, t] \times \mathbb{R}^N \times U \times \mathbb{R}^N$. As in the strong case, $\gamma_{0,t,x_0,\pi} \in \Theta(0, t, x_0)$. With this notation, the inequality (3.5) gives

$$V^w(0, x_0) \geq \inf_{\gamma \in \Theta(0,t,x_0)} \left(t \int_{[0,t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(dsdydu) + \int_{\mathbb{R}^N} V^w(t, y) \gamma^2(dy) \right),$$

and, we conclude using (3.4). In the l.s.c. case, one uses [10], proposition 12.

3. Let us fix for the moment $\gamma \in \Theta(0, t, x_0)$. For $\varepsilon > 0$, we consider a partition (B_n^ε) of \mathbb{R}^N and a family $(x_n^\varepsilon)_n \subset \mathbb{R}^N$ such that $|x - x_n^\varepsilon| \leq \varepsilon$, for all $x \in B_n^\varepsilon$. We denote by ω the continuity modulus of $\Lambda(t, \cdot)$. Then

$$\Lambda(t, z) + \omega(\varepsilon) \geq \sum_n 1_{B_n^\varepsilon}(z) \Lambda(t, x_n^\varepsilon), \tag{3.6}$$

for all $z \in \mathbb{R}^N$. For every n , by compactness of $\Theta(t, T, x_n^\varepsilon)$, there exists an optimal measure $\gamma_n \in \Theta(t, T, x_n^\varepsilon)$ such that

$$\Lambda(t, x_n^\varepsilon) = (T - t) \int_{[t,T] \times \mathbb{R}^N \times U} f(s, y, u) \gamma_n^1(dsdydu) + \int_{\mathbb{R}^N} g(y) \gamma_n^2(dy).$$

We introduce

$$\eta^{\varepsilon,1} = \sum_n \gamma^2(B_n^\varepsilon) \gamma_n^1, \quad \eta^{\varepsilon,2} = \sum_n \gamma^2(B_n^\varepsilon) \gamma_n^2, \quad \eta^\varepsilon = (\eta^{\varepsilon,1}, \eta^{\varepsilon,2}).$$

Due to (3.6), one has

$$(T - t) \int_{[t,T] \times \mathbb{R}^N \times U} f(s, y, u) \eta^{\varepsilon,1}(dsdydu) + \int_{\mathbb{R}^N} g(y) \eta^{\varepsilon,2}(dy) \leq \int_{\mathbb{R}^N} \Lambda(t, z) \gamma^2(dz) + \omega(\varepsilon). \tag{3.7}$$

Due to proposition 3.6, $(\eta^\varepsilon)_{\varepsilon>0}$ is relatively compact and there exists some subsequence, still denoted (η^ε) converging to some $\eta \in \Theta(t, T, \gamma)$. Proposition 3.4 yields $\eta \circ \gamma \in \Theta(0, t, x_0)$. Passing to the limit in (3.7) and recalling that f and g are continuous, one has

$$\begin{aligned} \Lambda(0, x_0) &\leq T \int_{[0,T] \times \mathbb{R}^N \times U} f(s, y, u) (\eta \circ \gamma)^1(dsdydu) + \int_{\mathbb{R}^N} g(y) (\eta \circ \gamma)^2(dy) \\ &= t \int_{[0,t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(dsdydu) + (T - t) \int_{[t,T] \times \mathbb{R}^N \times U} f(s, y, u) \eta^1(dsdydu) \\ &\quad + \int_{\mathbb{R}^N} g(y) \eta^2(dy) \leq t \int_{[0,t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(dsdydu) + \int_{\mathbb{R}^N} \Lambda(t, z) \gamma^2(dz). \end{aligned}$$

Taking the infimum over $\gamma \in \Theta(0, t, x_0)$ we finally get

$$\Lambda(0, x_0) \leq \inf_{\gamma \in \Theta(0,t,x_0)} \left(t \int_{[0,t] \times \mathbb{R}^N \times U} f(s, y, u) \gamma^1(dsdydu) + \int_{\mathbb{R}^N} \Lambda(t, z) \gamma^2(dz) \right).$$

The proof is completed by 2. □

In the deterministic framework ($\sigma = 0$), if γ is an occupational couple associated to x_0 and $u \in \mathcal{U}$, then, $\Theta(t, T, \gamma) = \Theta(t, T, x_t^{t_0, x_0, u})$ and, by definition,

$$\int_{\mathbb{R}^N} \Lambda(t, x) \gamma^2(dx) = \Lambda(t, x_t^{t_0, x_0, u}) = \Lambda(t, \gamma).$$

Thus, in general,

$$\Lambda(t_0, x_0) \leq \inf_{\gamma \in \Gamma(t_0, t, x_0)} \left((t - t_0) \int_{[t_0, t] \times \mathbb{R}^N \times \mathcal{U}} g(s, y, u) \gamma^1(ds dy du) + \int_{\mathbb{R}^N} \Lambda(t, x) \gamma^2(dx) \right).$$

It follows from the second assertion of the previous theorem that we have equality for u.s.c. costs and the proof of 3 is much simpler.

References

- [1] J.P. Aubin and A. Cellina. *Differential inclusions. Set-valued maps and viability theory*. Springer-Verlag, Berlin, 1984. MR-0755330
- [2] E.N. Barron and R. Jensen. Semicontinuous viscosity solutions for Hamilton-Jacobi equations with convex Hamiltonians. *Commun. Partial Differ. Equations*, 15(12):1713–1742, 1990. MR-1080619
- [3] A.G. Bhatt and V.S. Borkar. Occupation measures for controlled Markov processes: Characterization and optimality. *Ann. of Probability*, 24:1531–1562, 1996. MR-1411505
- [4] V. Borkar and V. Gaitsgory. Averaging of singularly perturbed controlled stochastic differential equations. *Appl. Math. Optimization*, 56(2):169–209, 2007. MR-2352935
- [5] B. Bouchard and N. Touzi. Weak dynamic programming principle for viscosity solutions. *to appear in SIAM J. Control Optim.* MR-2806570
- [6] R. Buckdahn, D. Goreac, and M. Quincampoix. Stochastic optimal control and linear programming approach. *Appl. Math. Optimization*, 63(2):257–276, 2011. MR-2772196
- [7] W. H. Fleming and H. M. Soner. *Controlled Markov processes and viscosity solutions*, volume 25 of *Applications of Mathematics (New York)*. Springer-Verlag, New York, 2006. MR-2179357
- [8] H. Frankowska. Lower semicontinuous solutions of Hamilton-Jacobi-Bellman equations. *SIAM J. Control Optimization*, 31(1):257–272, 1993. MR-1200233
- [9] V. Gaitsgory and M. Quincampoix. Linear programming approach to deterministic infinite horizon optimal control problems with discounting. *SIAM J. Control Optimization*, 48(4):2480–2512, 2009. MR-2556353
- [10] D. Goreac and O.S. Serea. Mayer and optimal stopping stochastic control problems with discontinuous cost. *Journal of Mathematical Analysis and Applications*, 380(1):327–342, 2011. MR-2786205
- [11] T.G. Kurtz and R.H. Stockbridge. Existence of Markov controls and characterization of optimal Markov control. *SIAM J. Control Optim.*, 36(2):609–653, 1998. MR-1616514
- [12] J.B. Lasserre, D. Henrion, C. Prieur, and Trélat. E. Nonlinear optimal control via occupational measures and LMI-Relaxations. *SIAM J. Control Optim.*, 47(4):1643–1666, 2008. MR-2421324
- [13] R.H. Stockbridge. Time-average control of a martingale problem. Existence of a stationary solution. *Ann. of Probability*, 18:190–205, 1990. MR-1043943
- [14] G.G. Yin and Q. Zhang. *Continuous-Time Markov Chains and Applications. A singular Perturbation Approach*. Springer-Verlag, New York, 1997. MR-1488963
- [15] J. Yong and X.Y. Zhou. *Stochastic Controls. Hamiltonian Systems and HJB Equations*. Springer-Verlag, New York, 1999. MR-1696772