Electron. Commun. Probab. **17** (2012), no. 25, 1–7.
DOI: 10.1214/ECP.v17-1834
ISSN: 1083-589X

ELECTRONIC COMMUNICATIONS in PROBABILITY

Testing the finiteness of the support of a distribution: a statistical look at Tsirelson's equation

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Abstract

We consider the following statistical problem: based on an i.i.d. sample of size n of integer valued random variables with common law μ, is it possible to test whether or not the support of μ is finite as n goes to infinity? This question is in particular connected to a simple case of Tsirelson's equation, for which it is natural to distinguish between two main configurations, the first one leading only to laws with finite support, and the second one including laws with infinite support. We show that it is in fact not possible to discriminate between the two situations, even using a very weak notion of statistical test.

Keywords: Hypothesis testing; Tsirelson's equation.
AMS MSC 2010: 62F03; 62F05.
Submitted to ECP on February 25, 2012, final version accepted on July 4, 2012.

1 Introduction

In this paper, we consider an i.i.d. sample of integer valued random variables with common law μ and address the following problem: is it possible to build an asymptotic test for the finiteness (or for the non finiteness) of the support of μ as the sample size goes to infinity? This question is in fact motivated by Tsirelson's equation in discrete time, see [1, 2]. Let $(\nu_k)_{k\in-\mathbb{N}}$ be a sequence of probability laws on the torus $\mathbb{T}=\mathbb{R}/\mathbb{Z}$. A process $(\eta_k)_{k\in-\mathbb{N}}$ taking values in $\mathbb{T}$ is a weak solution of Tsirelson's equation associated to $(\nu_k)_{k\in-\mathbb{N}}$ if for all $k\in-\mathbb{N}$, the random variable $\xi_k=\eta_k-\eta_{k-1}$ has law ν_k and is independent of $\mathcal{F}^{\eta}_{k-1}=\sigma(\eta_n, n\leq k-1)$. If moreover for all $k\in-\mathbb{N}$, $\mathcal{F}^{\eta}_k=\mathcal{F}^{\xi}_k$, then the process $(\eta_k)_{k\in-\mathbb{N}}$ is said to be a strong solution. It is proved by Yor in [6] that there always exists a weak solution to the equation: if U and ζ_k, $k\in-\mathbb{N}$, are independent random variables taking values in $\mathbb{T}$ such that U is uniform and, for all k, ζ_k has distribution ν_k, then the process $(\eta^*_k)_{k\in-\mathbb{N}}$ defined by

$$\eta^*_0=U,\quad \eta^*_{-n}=U-\zeta_0-\cdots-\zeta_{-n+1}\quad \text{for all } n\geq 1, \tag{1.1}$$

is a weak solution of Tsirelson's equation associated to $(\nu_k)_{k\in-\mathbb{N}}$.

In this work we consider only the simple case where $\nu_k=\nu$ for all k. It can be shown that in terms of existence and unicity of strong or weak solutions to the equation, depending on the law ν, three very different situations have to be considered, see [6] for details:

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- *Case 1*: If ν is a Dirac measure: *existence of a strong solution, no uniqueness in law.*
- *Case 2*: If the support of ν contains at least two points and there exists an integer $p \geq 2$ and $x \in \mathbb{T}$ such that it is included in $x + \{k/p,\ k = 0, \dots, p-1\}$: *no strong solution, no uniqueness in law.*
- *Case 3*: If ν is a law which does not satisfy the conditions of Case 1 or Case 2: *no strong solution, uniqueness in law.*

In [4, 5], Tsirelson's equation is seen as a cosmological caricature: η_{k-1} represents the state of the universe at time $k-1$ and ξ_k the action of the evolution process at time k. In particular, the authors raise the issue of the "initial magma" that is the description of the σ-algebra

$$\mathcal{F}^{\eta}_{-\infty} = \bigcap_{n\geq 1} \sigma\big(\eta_k, k \leq -n\big).$$

It turns out that it is necessarily a trivial σ-algebra in Case 3.

Our goal here is to give a statistical look at this cosmological caricature. More precisely, based on the historical observations of the state of the universe $\eta_0, \eta_{-1}, \dots, \eta_{-n}$, can we (at least partially) decide whether we are in Case 1, Case 2 or Case 3 ? Of course we can already remark that under Case 1, the ξ_k are constant and under Case 2 the $\rho_k = \xi_k - \xi_{k-1}$ are all rational. Therefore, if we observe two different values for the ξ_k in the sample, we can conclude that we are not in Case 1 and if we observe one irrational ρ_k, we can discard Case 2. Consequently, if there are at least two different ξ_k and one irrational ρ_k, we can conclude that we are in Case 3. However, in practice, we have only access to rounded values so it does not really make sense to build a procedure based on the fact that some quantity is irrational or not.

In order to rigorously answer the question, we use the notion of statistical test. We first consider an auxiliary problem, which is interesting on its own, namely the possibility of building a consistent test for the finiteness (or for the non finiteness) of the support of an integer valued distribution μ. based on an i.i.d. sample $(X_1, \dots, X_n)$ with law μ. We prove that designing such a test procedure is in fact impossible, even using a very weak notion of consistent test. Then we show that a corollary of this result is the impossibility to build a consistent test in order to separate on the one hand the union of Case 1 and Case 2 and on the other hand Case 3.

The paper is organized as follows. The framework we use in order to test for the finiteness (or for the non finiteness) of the support of a distribution based on i.i.d. data is detailed in Section 2. In Section 3, we suggest an a priori natural functional in order to build such a test. The impossibility of building a consistent test is stated in Section 4 together with the connection with Tsirelson's equation. The proofs are relegated to Section 5.

2 Testing framework

In this section we consider an i.i.d. sample $(X_1, \dots, X_n)$ of integer valued random variables with common law μ. Our goal is to investigate the possibility of testing for the finiteness (or for the non finiteness) of the support of μ as n goes to infinity. We refer to [3] for a detailed presentation of the notion of asymptotic statistical test. In the following, we write $\mathbb{P}_\mu$ for the probability measure on the canonical space $\mathbb{N}^{\mathbb{N}^*}$ under which the canonical process $(X_i)_{i\geq 1}$ is a family of i.i.d. random variables with law μ.

Moreover, we denote by $\mathcal{P}_f$ the set of laws on $\mathbb{N}$ with finite support and by $\mathcal{P}_\infty$ the set of laws on $\mathbb{N}$ with infinite support. To fix ideas, let us consider in this section the null hypothesis of the finiteness of the support against the alternative of the non finiteness of it (the following definitions can be adapted in an obvious way if the null and alternative hypotheses are switched).

2.1 Uniform test

In order to discriminate between the two situations, the first idea is to try to build a consistent test with asymptotic uniform level α, with α given in $[0, 1)$. This means designing a rejection area W_n in $\mathbb{N}^n$ such that

$$\limsup_n \sup_{\mu \in \mathcal{P}_f} \mathbb{P}_\mu \big[(X_1, \dots, X_n) \in W_n\big] \le \alpha \tag{2.1}$$

and for all $\mu \in \mathcal{P}_\infty$,

$$\lim_n \mathbb{P}_\mu \big[(X_1, \dots, X_n) \in W_n\big] = 1.$$

This is clearly impossible. Indeed, for fixed n, any law $\mu_1^{\otimes n}$ with $\mu_1 \in \mathcal{P}_f$ can be approached with arbitrary accuracy (in total variation norm for example) by a law $\mu_2^{\otimes n}$ with $\mu_2 \in \mathcal{P}_\infty$.

2.2 Pointwise test

We now consider a much weaker notion of test: we replace (2.1) by

$$\sup_{\mu \in \mathcal{P}_f} \limsup_n \mathbb{P}_\mu \big[(X_1, \dots, X_n) \in W_n\big] \le \alpha.$$

The statistical meaning of this notion of test is of course very arguable but our point of view is the following: we want to show that it is impossible to build a consistent test with given level α even in this very weak setting.

Remark 2.1. *If we do not consider all distributions with finite support but only distributions with support in $[0, N]$ for some known N, a test statistics can of course be easily designed. Indeed, it is then enough to consider the statistics max$(X_1, \dots, X_n) \wedge (N+1)$: when X_1 follows $\mu_1 \in \mathcal{P}_f$ it converges to the right endpoint of μ_1 which is smaller than N, when X_1 follows $\mu_2 \in \mathcal{P}_\infty$ it converges to $N+1$.*

3 Candidate test statistics for pointwise test

We want to investigate potential test statistics for discriminating between our two situations. Since the laws in $\mathcal{P}_f$ and $\mathcal{P}_\infty$ are different because they have different support, a natural idea is to design a test statistics based on the empirical support. For example, we can split the sample of size $2n$, into two parts and compare the maximum over each subsample, that is we consider

$$S_n = \max_{1 \le i \le n} X_i, \quad \widetilde{S}_n = \max_{n+1 \le i \le 2n} X_i, \quad \text{and} \quad T_{2n} = \mathbf{1}_{\{S_n = \widetilde{S}_n\}}.$$

If μ has finite support, it is clear that for n large enough, the maxima of the two subsamples coincide and in particular T_{2n} converges in probability to 1. In the infinite support case, one may expect an opposite behavior. In fact, we have the following result whose proof is given in Section 5.

Proposition 3.1. *If μ has infinite support, then T_{2n} does not converge in $\mathbb{P}_\mu$-probability to* 1.

Therefore for any μ_1 with finite support and μ_2 with infinite support the statistics T_{2n} shows different asymptotic behaviors under μ_1 and μ_2. However, this is not enough to build a consistent test since we need to investigate the asymptotic probability that T_{2n} is different from 1. In fact, the first part of Theorem 4.1 in the next section implies that there exist a distribution μ_2 with infinite support and a subsequence of T_{2n} which converges to 1 in probability under $\mathbb{P}_{\mu_2}$.

4 Impossibility of testing

The following theorem shows that it is not possible to build a test for the finiteness (or for the non finiteness) of the support. Indeed, it states that if one finds a set in which an i.i.d. sample of size n from any law with finite support is very unlikely to be, there is also one distribution with infinite support so that an i.i.d. sample of it is very unlikely to be in this set, and conversely.

We write $\mathbb{E}_\mu$ for the expectation with respect to $\mathbb{P}_\mu$. We have the following result.

Theorem 4.1. *Let $\alpha \in [0,1)$, let $\varphi : \mathbb{N}^* \to \mathbb{N}^*$ be an increasing map and let $A_n : \mathbb{N}^{\varphi(n)} \to [0,1]$, $n \geq 1$, be a sequence of measurable functions.*

1. *Assume that for any distribution μ_1 on $\mathbb{N}$ with finite support,*
$$\limsup_n \, \mathbb{E}_{\mu_1}\left[A_n(X_1,\ldots,X_{\varphi(n)})\right] \leq \alpha.$$
Then, there exists a distribution μ_2 on $\mathbb{N}$ with infinite support such that
$$\liminf_n \, \mathbb{E}_{\mu_2}\left[A_n(X_1,\ldots,X_{\varphi(n)})\right] \leq \alpha.$$
2. *Assume that for any distribution μ_2 on $\mathbb{N}$ with infinite support,*
$$\limsup_n \, \mathbb{E}_{\mu_2}\left[A_n(X_1,\ldots,X_{\varphi(n)})\right] \leq \alpha.$$
Then, for all $\alpha' \in (\alpha,1)$ there exists a distribution μ_1 on $\mathbb{N}$ with finite support such that
$$\liminf_n \, \mathbb{E}_{\mu_1}\left[A_n(X_1,\ldots,X_{\varphi(n)})\right] \leq \alpha'.$$

Remark 4.2. *In Theorem 4.1, we do not only consider the usual testing framework where the A_n are indicators of a set and $\varphi(n) = n$. Indeed, we allow for randomized test procedures and subsequences in the sample size. This is slightly more general and will be useful for the proof of Corollary 4.3.*

We now come back to Tsirelson's equation and state the result showing the impossibility of testing the hypothesis that ν belongs to the union of Case 1 and Case 2 against the one that ν belongs to Case 3, and conversely. Denote by $\mathbb{P}^*_\nu$ the law on $\mathbb{T}^{-\mathbb{N}}$ of the "uniform solution" $(\eta^*_k)_{k\in-\mathbb{N}}$, given by (1.1) (when $\nu_k = \nu$ for all k) and by $(\eta_k)_{k\in-\mathbb{N}}$ the canonical process on $\mathbb{T}^{-\mathbb{N}}$. We have the following corollary.

Corollary 4.3. *Let $\alpha \in [0,1)$, let $\varphi : \mathbb{N} \to \mathbb{N}$ be an increasing map and let $B_n \subset \mathbb{T}^{\varphi(n)+1}$ be a sequence of measurable sets.*

1. *Assume that for any distribution ν_1 on $\mathbb{T}$ belonging to Case 1 or Case 2 of Section 1,*
$$\limsup_n \mathbb{P}^*_{\nu_1}\left[(\eta_0, \dots, \eta_{-\varphi(n)}) \in B_n\right] \leq \alpha.$$
Then, there exists a distribution ν_2 belonging to Case 3 such that
$$\liminf_n \mathbb{P}^*_{\nu_2}\left[(\eta_0, \dots, \eta_{-\varphi(n)}) \in B_n\right] \leq \alpha.$$

2. *Assume that for any distribution ν_2 on $\mathbb{T}$ belonging to Case 3 of Section 1,*
$$\limsup_n \mathbb{P}^*_{\nu_2}\left[(\eta_0, \dots, \eta_{-\varphi(n)}) \in B_n\right] \leq \alpha.$$
Then, for all $\alpha' \in (\alpha, 1)$ there exists a distribution ν_1 belonging to Case 1 or Case 2 of Section 1 such that
$$\liminf_n \mathbb{P}^*_{\nu_1}\left[(\eta_0, \dots, \eta_{-\varphi(n)}) \in B_n\right] \leq \alpha'.$$

5 Proofs

5.1 Proof of Proposition 3.1

Let μ be a probability measure on $\mathbb{N}$ and assume that $\mathbb{P}_\mu[T_{2n} = 1] \to 1$. Since S_n and $\widetilde{S}_n$ are independent with the same law, it implies that there exists a sequence of integers k_n such that $\mathbb{P}_\mu[S_n = k_n] \to 1$.

Let us show that k_n is a bounded sequence, which will imply that the support of μ is finite. If k_n is not bounded, then there exist a subsequence n_ℓ such that $k_{1+n_\ell} > k_{n_\ell}$ for all ℓ and $k_{n_\ell} \to \infty$. Therefore
$$\mathbb{P}_\mu[S_{n_\ell} = k_{n_\ell}] \to 1$$
and
$$\mathbb{P}_\mu[S_{1+n_\ell} = k_{n_\ell}] \to 0$$
because $P\big(S_{1+n_\ell} = k_{1+n_\ell}\big) \to 1$ and $k_{1+n_\ell} \neq k_{n_\ell}$. Moreover, we have that
$$\mathbb{P}_\mu[S_{1+n_\ell} = k_{n_\ell}] - \mathbb{P}_\mu[S_{n_\ell} = k_{n_\ell}]$$
is equal to
$$\mathbb{P}_\mu[S_{n_\ell} = k_{n_\ell}]\mu([0, k_{n_\ell}]) + \mathbb{P}_\mu[S_{n_\ell} < k_{n_\ell}]\mu(k_{n_\ell}) - \mathbb{P}_\mu[S_{n_\ell} = k_{n_\ell}].$$
This absolute value of this last quantity is smaller than $\mu([k_{n_\ell}, +\infty))$ which goes to zero as n goes to infinity. This shows the contradiction.

5.2 Proof of Theorem 4.1

5.2.1 Proof of Part 1 in Theorem 4.1

We recursively define a sequence of distributions μ_2^n, $n \geq 1$, and a sequence of integers $\psi(n)$, $n \geq 0$.
– At rank $n = 1$, we consider μ_2^1 the Dirac measure at point 0, $\psi(0) = 1$ and $\psi(1) = 1$.
– At rank $n > 1$, we define μ_2^n and $\psi(n)$. The law μ_2^n is the distribution with discrete support $\{0, \dots, n-1\}$ defined by
$$\mu_2^n(k) = \frac{c_n}{(\varphi \circ \psi(k))^2}, \quad k \in \{0, \dots, n-1\},$$

with c_n such that
$$\sum_{k=0}^{n-1} \frac{c_n}{(\varphi \circ \psi(k))^2} = 1.$$
The integer $\psi(n)$ is taken so that
$$\psi(n) > \max(\psi(n-1), n^2) \tag{5.1}$$
and for all $m \geq \psi(n)$,
$$\mathbb{E}_{\mu_2^n}\left[A_m(X_1, \ldots, X_{\varphi(m)})\right] \leq \alpha + \frac{1}{n}.$$
Note that finding such a $\psi(n)$ is always possible thanks to the assumption on the sequence $(A_m)_m$. In particular, from (5.1), we have that the sequence $\psi(n)$ is increasing and satisfies
$$\sum_{k=0}^{+\infty} \frac{1}{(\varphi \circ \psi(k))^2} < \infty.$$
Now define the distribution μ_2 on $\mathbb{N}$ with infinite support by
$$\mu_2(k) = \frac{c}{(\varphi \circ \psi(k))^2},\ k \in \mathbb{N},$$
with c such that
$$\sum_{k=0}^{+\infty} \frac{c}{(\varphi \circ \psi(k))^2} = 1.$$
We have that
$$\mathbb{E}_{\mu_2}\left[A_{\psi(n)}(X_1, \ldots, X_{\varphi\circ\psi(n)})\right]$$
is smaller than
$$\begin{aligned}
&\mathbb{E}_{\mu_2}\Big[A_{\psi(n)}(X_1, \ldots, X_{\varphi\circ\psi(n)}) \mid \bigcap_{i=1}^{\varphi\circ\psi(n)} \{X_i \leq n-1\}\Big] + \mathbb{P}_{\mu_2}\Big[\bigcup_{i=1}^{\varphi\circ\psi(n)} \{X_i \geq n\}\Big] \\
&= \mathbb{E}_{\mu_2^n}\left[A_{\psi(n)}(X_1, \ldots, X_{\varphi\circ\psi(n)})\right] + \mathbb{P}_{\mu_2}\Big[\bigcup_{i=1}^{\varphi\circ\psi(n)} \{X_i \geq n\}\Big] \\
&\leq \alpha + \frac{1}{n} + \varphi\circ\psi(n) \sum_{k=n}^{+\infty} \frac{1}{(\varphi \circ \psi(k))^2}.
\end{aligned}$$
Using the fact that $\psi(n)$ is increasing and the inequality $\psi(n) > n^2$, we obtain
$$\alpha + \frac{1}{n} + \varphi\circ\psi(n) \sum_{k=n}^{+\infty} \frac{1}{(\varphi \circ \psi(k))^2} \leq \alpha + \frac{1}{n} + \sum_{k=n}^{+\infty} \frac{1}{\varphi \circ \psi(k)} \leq \alpha + \frac{1}{n} + \sum_{k=n}^{+\infty} \frac{1}{k^2}.$$
This quantity goes to α as n goes to infinity, which gives the result.

5.2.2 Proof of Part 2 in Theorem 4.1

Let $\alpha' \in (\alpha, 1)$. Assume that for any distribution μ_1 on $\mathbb{N}$ with finite support
$$\liminf_n\ \mathbb{E}_{\mu_1}\left[A_n(X_1, \ldots, X_{\varphi(n)})\right] > \alpha'.$$
This last inequality is equivalent to
$$\limsup_n\ \mathbb{E}_{\mu_1}\left[1 - A_n(X_1, \ldots, X_{\varphi(n)})\right] < 1 - \alpha'.$$

According to point 1. of Theorem 4.1, there exists a distribution μ_2 on $\mathbb{N}$ with infinite support such that

$$\liminf_n \mathbb{E}_{\mu_2}\big[1 - A_n(X_1, \dots, X_{\varphi(n)})\big] \leq 1 - \alpha',$$

that is

$$\limsup_n \mathbb{E}_{\mu_2}\big[A_n(X_1, \dots, X_{\varphi(n)})\big] \geq \alpha'$$

which gives the contradiction since $\alpha' > \alpha$.

5.3 Proof of Corollary 4.3

Let f be an injection from $\mathbb{N}$ into $\mathbb{T} \cap \mathbb{Q}/\mathbb{Z}$. If μ is a probability measure on $\mathbb{N}$, denote by $f(\mu) = \mu \circ f^{-1}$ the image of μ by f. On the one hand, one has:
- if μ has finite support, $f(\mu)$ belongs to Case 1 or Case 2.
- if μ has infinite support, $f(\mu)$ belongs to Case 3 since $f(\mu)$ has infinite support.
On the other hand, because of the definition of $\mathbb{P}^*_\nu$, one has

$$\mathbb{P}^*_{f(\mu)}\big[(\eta_0, \dots, \eta_{-\varphi(n)}) \in B_n\big] = \mathbb{E}_\mu\big[A_n(X_1, \dots, X_{\varphi(n)})\big]$$

where $A_n(X_1, \dots, X_{\varphi(n)})$ is equal to

$$\int_0^1 \mathbf{1}_{B_n}\big(u, u - f(X_1), u - f(X_1) - f(X_2), \dots, u - f(X_1) - f(X_2) - \dots - f(X_{\varphi(n)})\big)\, du.$$

Using these two facts, together with Theorem 4.1, the proof of Corollary 4.3 is easily completed.

Acknowledgments. We are very grateful to Kouji Yano and Marc Yor for introducing us to the statistical question linked to Tsirelson's equation.

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