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Martingale approach to subexponential asymptotics for random walks*

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Abstract

Consider the random walk $S_n = \xi_1 + \cdots + \xi_n$ with independent and identically distributed increments and negative mean $\mathbf{E}\xi = -m < 0$. Let $M = \sup_{0\le i} S_i$ be the supremum of the random walk. In this note we present derivation of asymptotics for $\mathbf{P}(M > x), x \to \infty$ for long-tailed distributions. This derivation is based on the martingale arguments and does not require any prior knowledge of the theory of long-tailed distributions. In addition the same approach allows to obtain asymptotics for $\mathbf{P}(M_\tau > x)$, where $M_\tau = \max_{0\le i<\tau} S_i$ and $\tau = \min\{n \ge 1 : S_n \le 0\}$.

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1 Introduction, statement of results and discussion

Let $\xi, \xi_1, \xi_2, \ldots$ be independent random variables with a common distribution function F and negative mean, i.e., $\mathbf{E}\xi = -a < 0$. Let S_n denote the random walk with the increments ξ_k, that is,

$$S_0 = 0, \quad S_n = \xi_1 + \xi_2 + \cdots + \xi_n, \ \ n \ge 1.$$

It follows from the assumption $\mathbf{E}\xi < 0$ that the total maximum $M := \sup_{n\ge 0} S_n$ is finite almost surely. The asymptotic behaviour of $\mathbf{P}(M > x)$ has been considered by many authors. The first results are due to Cramer and Lundberg: if there exists $h_0 > 0$ such that $\mathbf{E}e^{h_0\xi} = 1$ and $\mathbf{E}\xi e^{h_0\xi} < \infty$ then

$$\mathbf{P}(M > x) \sim c_0 e^{-h_0 x} \quad \text{as } x \to \infty \tag{1.1}$$

for some $c_0 \in (0, 1)$ and, furthermore,

$$\mathbf{P}(M > x) \le e^{-h_0 x} \quad \text{for all } x > 0. \tag{1.2}$$

The proof of these statements is based on the following observation: The assumption $\mathbf{E}e^{h_0\xi} = 1$ implies that the sequence $e^{h_0 S_n}$ is a martingale. Applying the Doob inequality

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we obtain immediately (1.2). The same martingale property allows one to make an exponential change of measure, which is used in the proof of (1.1).

If the distribution of ξ is long-tailed, i.e., $\mathbf{E}e^{h\xi} = \infty$ for all $h > 0$, then one can investigate $\mathbf{P}(M > x)$ under some additional regularity restrictions on the tail function $\overline{F}(x) := 1 - F(x)$. One of the most popular regularity assumption is the so-called subexponentiality of the distribution tails.

Definition 1.1. *The distribution function F on $\mathbf{R}_+$ is called subexponential if*

$$\int_0^x \overline{F}(x-y)dF(y) \sim 2\overline{F}(x) \quad \text{as } x \to \infty.$$

The following result is known in the literature as Veraverbecke's theorem: Let F_I be defined by the tail $\overline{F_I}(x) := \min\left(1, \int_x^\infty \overline{F}(y)dy\right)$, $x > 0$. If F_I is subexponential then

$$\mathbf{P}(M > x) \sim \frac{1}{a}\overline{F_I}(x) \quad \text{as } x \to \infty. \tag{1.3}$$

We next turn to the maximum of the positive excursion of the random walk. Let

$$\tau := \inf\{n \geq 1 : S_n \leq 0\}$$

and

$$M_\tau := \max_{0 \leq n < \tau} S_n.$$

If the Cramer-Lundberg condition holds then one can derive the asymptotics for the quantity $\mathbf{P}(M_\tau > x)$ from that for the total maximum M. This way has been suggested first by Iglehart [10]. Namely, it follows from the Markov property that

$$\mathbf{P}(M > x) = \mathbf{P}(M_\tau > x) + \int_{-\infty}^0 \mathbf{P}(M > x-y)\mathbf{P}(S_\tau \in dy, M_\tau \leq x).$$

Thus,

$$\mathbf{P}(M_\tau > x) = \mathbf{P}(M > x)\left(1 - \int_{-\infty}^0 \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)}\mathbf{P}(S_\tau \in dy, M_\tau \leq x)\right).$$

Noting that (1.1) yields

$$\lim_{x\to\infty} \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)} = e^{h_0 y} \quad \text{for every } y < 0,$$

and applying the dominated convergence, we obtain

$$\int_{-\infty}^0 \frac{\mathbf{P}(M > x-y)}{\mathbf{P}(M > x)}\mathbf{P}(S_\tau \in dy, M_\tau \leq x) \sim \mathbf{E}e^{h_0 S_\tau}.$$

As a result we get

$$\mathbf{P}(M_\tau > x) \sim \left(1 - \mathbf{E}e^{h_0 S_\tau}\right)\mathbf{P}(M > x) \sim \left(1 - \mathbf{E}e^{h_0 S_\tau}\right)c_0 e^{-h_0 x}. \tag{1.4}$$

It turns out that Iglehart's approach can not be applied to heavy-tailed random walks without further restrictions on the distribution of ξ. Here one has to assume that F is *strong subexponential*. This class of distribution functions was introduced by Klüppelberg [11].

Definition 1.2. *The distribution function F on $\mathbf{R}$ belongs to the class $\mathcal{S}^*$ if*

$$\int_0^x \overline{F}(x-y)\overline{F}(y)dy \quad \sim \quad 2a_+\overline{F}(x) \quad \text{as } x\to\infty, \tag{1.5}$$

where $a_+ = \int_0^\infty \overline{F}(y)dy \in (0,\infty)$.

Denisov [3] adopted Iglehart's reduction from M_τ to M to the class of strong subexponential distributions: If $F \in \mathcal{S}^*$ then

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau\overline{F}(x), \quad x\to\infty. \tag{1.6}$$

The asymptotics (1.6) were found first by Asmussen [1] for $F \in \mathcal{S}^*$ and by Heath, Resnick and Samorodnitsky [9] for regularly varying F. An extension of this result to the general stopping time can be found in Foss and Zachary [8], and in Foss, Palmowsky and Zachary [7]. These extensions rely on (1.6).

The *main purpose* of the present note is to give alternative proofs of (1.3) and (1.6) using martingale techniques.

In order to state our main result we introduce some notation. For any $y>0$ let

$$\mu_y := \min\{n\ge 0 : S_n > y\}.$$

The latter stopping time is naturally connected with the supremum since

$$\mathbf{P}(M>x) = \mathbf{P}(\mu_x < \infty).$$

Let

$$\overline{F}_s(x) := \int_x^\infty \overline{F}(u)du$$

and

$$G_c(x) = \begin{cases} \overline{F}_s(x), & \text{if } x\ge 0 \\ c, & \text{if } x<0 \end{cases}. \tag{1.7}$$

Define also

$$\widehat{G}_c(x) := \min\{G_c(x), c\}. \tag{1.8}$$

Theorem 1.3. *Assume that F is long-tailed. For any $\varepsilon>0$ there exists $R>0$ such that the stopped sequence*

$$\widehat{G}_{a+\varepsilon}(x - S_{n\wedge\mu_{x-R}}) \quad \text{is a submartingale.} \tag{1.9}$$

Assume in addition that $F\in\mathcal{S}^*$. For any $\varepsilon>0$ there exists $R>0$ such that the stopped sequence

$$\widehat{G}_{a-\varepsilon}(x - S_{n\wedge\mu_{x-R}}) \quad \text{is a supermartingale.} \tag{1.10}$$

Having constructed super- and submartingale we can obtain subexponential asymptotics for M and M_τ by applying the optional stopping theorem.

Corollary 1.4. *For any long-tailed distribution function F with negative mean,*

$$\liminf_{x\to\infty} \frac{\mathbf{P}(M>x)}{\overline{F}_s(x)} \ge \frac{1}{a}. \tag{1.11}$$

Assume in addition that $F\in\mathcal{S}^*$. Then,

$$\mathbf{P}(M>x) \sim \frac{1}{a}\overline{F}_s(x), \quad x\to\infty.$$

To the best of our knowledge, all existing in the literature proofs of the Veraverbecke theorem are based on representations via geometric sums. More precisely, $\mathbf{P}(M > x)$ can be estimated by $\sum_{n=1}^{\infty}(1-p)p^n\mathbf{P}(Y_1 + Y_2 + \ldots + Y_n > x)$, where $p \in (0,1)$ and Y_i are independent identically distributed random variables with $\mathbf{P}(Y_1 > x) \sim \overline{F_s}(x)$. In order to obtain (1.3) from that geometric sum one uses the following two properties of subexponential distributions:

(a) $\mathbf{P}((Y_1 + Y_2 + \ldots + Y_n > x)) \sim n\mathbf{P}(Y_1 > x)$ for every fixed k,
(b) For every $\varepsilon > 0$ there exists $C(\varepsilon) < \infty$ such that

$$\mathbf{P}((Y_1 + Y_2 + \ldots + Y_n > x)) \le C(\varepsilon)(1+\varepsilon)^n\mathbf{P}(Y_1 > x).$$

A recent elegant proof based on (a) and (b) can be found in [13]. Our proof does not use any property of F besides (1.5).

Unfortunately, our method does not allow us to derive (1.3) for the whole class of subexponential distributions. The condition $F_I \in \mathcal{S}$ and $F \in \mathcal{S}^*$ are close but do not coincide, see Section 6 in [4]. But we can apply the same construction to M_τ and, as it was shown in [8], the strong subexponentiality is necessary and sufficient for asymptotics (1.6) to hold.

Corollary 1.5. *Let $F \in \mathcal{S}^*$. Then*

$$\mathbf{P}(M_\tau > x) \sim \mathbf{E}\tau\overline{F}(x). \tag{1.12}$$

It is worth mentioning that, in contrast to all previous proofs, our approach to (1.12) is direct, i.e., it does not use any knowledge on the asymptotic behaviour of M.

One of the important advantages of the martingale approach is the possibility to obtain non-asymptotic inequalities for $\mathbf{P}(M > x)$ and $\mathbf{P}(M_\tau > x)$. For example, it follows from (1.9) that for every $\varepsilon > 0$ there exists $R > 0$ such that (see the proof of Corollary 1.4)

$$\mathbf{P}(M > x) \ge \frac{\overline{F_s}(x+R)}{a+\varepsilon}, \quad x > 0. \tag{1.13}$$

Using a supermartingale property of $G_{a-\varepsilon}$ we obtain the following upper bound

$$\mathbf{P}(M > x) \le \frac{\overline{F_s}(x-R')}{a-\varepsilon}, \quad x > R'. \tag{1.14}$$

Of course, in order to apply these inequalities, one has to know how to compute R and R' for given values of ε. And we believe that one can do it rather easy for certain subclasses of $\mathcal{S}^*$, e.g., for regularly varying or Weibull tails.

Foss, Korshunov and Zachary [6] have shown that the inequality

$$\mathbf{P}(M > x) \ge \frac{\overline{F_s}(x)}{a + \overline{F_s}(x)}, \quad x > 0$$

holds without any restriction on the distribution function F, see Theorem 5.1 in [6]. This bound is better than (1.13). It's proof is based on the fact, that the distribution of M is the stationary distribution of the Lindley recursion $W_{n+1} = (W_n + \xi_{n+1})^+$. This property of M can be written as follows: Let ξ' a copy of ξ, which is independent of M. Then $\mathcal{L}(M) = \mathcal{L}((M+\xi'))$. This can be seen as a martingale property: Define $\pi(x) := \mathbf{P}(M > x)$. Then the sequence $\pi(x - S_{n\wedge\mu_x})$ is a martingale.

Using (1.10), one gets for all $x > R'$ the inequality

$$\mathbf{P}(M_\tau > x) \le \frac{\overline{F_s}(x-R') - \mathbf{E}\overline{F_s}(x-R'-S_\tau)}{a-\varepsilon}.$$

And an upper estimate for the difference in the nominator is easy to get:

$$\overline{F_s}(x-R') - \mathbf{E}\overline{F_s}(x-R'-S_\tau) = \mathbf{E}\left[\int_{x-R'}^{x-R'-S_\tau} \overline{F}(z)dz\right] \le \overline{F}(x-R')\mathbf{E}[-S_\tau].$$

Applying the Wald identity, we obtain

$$\mathbf{P}(M_\tau > x) \le \frac{a}{a-\varepsilon}\mathbf{E}\tau\overline{F}(x-R'). \tag{1.15}$$

A lower bound is not as obvious. Here we can conclude from (1.9) that

$$\mathbf{P}(M_\tau > x) \ge \frac{\overline{F_s}(x+R) - \mathbf{E}\overline{F_s}(x+R-S_\tau)}{a+\varepsilon}. \tag{1.16}$$

Thus one needs an appropriate estimate for the difference in the nominator.

Martingale approach has been used also by Kugler and Wachtel [12] in deriving upper bounds for $\mathbf{P}(M > x)$ and $\mathbf{P}(M_{\tau_z} > x)$, where $\tau_z := \min\{k : S_n \le -z\}$ under the assumption that some power moments of ξ are finite. Their strategy is completely different: They truncate the summands ξ_i in order to construct an exponential supermartingale for the random walk with truncated increments.

2 Proofs.

2.1 Proof of Theorem 1.3.

Fix $\varepsilon > 0$. To prove the submartingale property we need to show that

$$\mathbf{E}\widehat{G}_{a+\varepsilon}(x-y-\xi) \ge \widehat{G}_{a+\varepsilon}(x-y) \tag{2.1}$$

for all $y \le x-R$.

Put, for brevity, $t := x-y \ge R$. By the definition (1.7),

$$\begin{aligned}\mathbf{E}\widehat{G}_{a+\varepsilon}(t-\xi) &= (a+\varepsilon)\mathbf{P}(\xi > t-r_c) + \int_{-\infty}^{t-r_c} F(dz)\overline{F_s}(t-z)\\ &= (a+\varepsilon)\overline{F}(t-r_c) + \left(\int_0^{t-r_c} + \int_{-\infty}^0\right) F(dz)\overline{F_s}(t-z),\end{aligned}$$

where $r_c := \min\{x \ge 0 : \overline{F_s}(x) \le c\}$. Integrating the first integral by parts, we obtain

$$\int_0^{t-r_c} F(dz)\overline{F_s}(t-z) = \overline{F}(0)\overline{F_s}(t-r_c) - \overline{F}(t-r_c)\overline{F_s}(0) + \int_0^{t-r_c} dz\overline{F}(z)\overline{F}(t-z).$$

Integrating the second integral by parts, we obtain

$$\int_{-\infty}^0 F(dz)\overline{F_s}(t-z) = F(0)\overline{F_s}(t) - \int_{-\infty}^0 dz\overline{F}(t-z)F(z).$$

Combining the above inequalities, we get

$$\begin{aligned}\mathbf{E}\widehat{G}_{a+\varepsilon}(t-\xi) &= (a+\varepsilon)\overline{F}(t-r_c) - \overline{F}(t-r_c)\overline{F_s}(0) + \overline{F}(0)\overline{F_s}(t-r_c)\\ &\quad + \int_0^{t-r_c} dz\overline{F}(z)\overline{F}(t-z) + F(0)\overline{F_s}(t) - \int_{-\infty}^0 dz\overline{F}(t-z)F(z).\end{aligned} \tag{2.2}$$

It is clear that

$$\int_{-\infty}^0 dz\overline{F}(t-z)F(z) \le \overline{F}(t)\int_{-\infty}^0 dzF(z) = a_-\overline{F}(t).$$

Further,

$$\overline{F}(0)\overline{F_s}(t-r_c)+F(0)\overline{F_s}(t)=\overline{F_s}(t)+\overline{F}(0)\int_{t-r_c}^{t}\overline{F}(z)dz$$

and

$$\int_0^{t-r_c}dz\overline{F}(z)\overline{F}(t-z)=\int_0^{t}dz\overline{F}(z)\overline{F}(t-z)-\int_{t-r_c}^{t}dz\overline{F}(z)\overline{F}(t-z).$$

Now, put $a_+:=\overline{F_s}(0), a_-:=\int_{-\infty}^0 dzF(z)$ and note that $a=a_--a_+$. Consequently,

$$\begin{aligned}\mathbf{E}\widehat{G}_{a+\varepsilon}(t-\xi)\geq\ &\overline{F_s}(t)+(a+\varepsilon)\overline{F}(t-r_c)-a_+\overline{F}(t-r_c)-a_-\overline{F}(t)\\&+2\int_0^{t/2}dz\overline{F}(z)\overline{F}(t-z)+\int_{t-r_c}^{t}\overline{F}(z)\left(\overline{F}(0)-\overline{F}(t-z)\right)dz\\ \geq\ &\overline{F_s}(t)+(-2a_++\varepsilon)\overline{F}(t-r_c)+2\overline{F}(t)\int_0^{t/2}dz\overline{F}(z).\end{aligned}$$

Now, taking R_1 sufficiently large, we can ensure that

$$2\int_0^{t/2}\overline{F}(z)dz\geq 2a_+-\frac{\varepsilon}{2}\quad\text{for all } t\geq R_1.$$

Furthermore, we can choose R_2 so large that

$$\frac{\overline{F}(t-r_c)-\overline{F}(t)}{\overline{F}(t)}\leq\frac{\varepsilon}{4a_+}.$$

As a result, for $t>\max\{R_1,R_2\}$ we have

$$\mathbf{E}\widehat{G}_{a+\varepsilon}(t-\xi)\geq\overline{F_s}(t)$$

This proves (1.9).

To prove the supermartingale property it sufficient to show that

$$\mathbf{E}G_{a-\varepsilon}(x-y-\xi)\leq G_{a-\varepsilon}(x-y)\tag{2.3}$$

for all $y\leq x-R$. Using (2.2) with $r_c=0$, we obtain

$$\begin{aligned}\mathbf{E}G_{a-\varepsilon}(t-\xi)=\ &G_{a-\varepsilon}(t)+(a-\varepsilon-a_+)\overline{F}(t)\\&+\int_0^t dz\overline{F}(z)\overline{F}(t-z)-\int_{-\infty}^0 dz\overline{F}(t-z)F(z).\end{aligned}$$

According to the definition of $\mathcal{S}^*$ there exists R_1 such that

$$\int_0^t dz\overline{F}(z)\overline{F}(t-z)\leq(2a_++\varepsilon/2)\overline{F}(t)$$

for all $t\geq R_1$. Furthermore, since F is long-tailed, we have

$$\lim_{t\to\infty}\frac{1}{\overline{F}(t)}\int_{-\infty}^0 dz\overline{F}(t-z)F(z)=\int_{-\infty}^0 dzF(z)=a_-.$$

Therefore, there exists R_2 such that

$$\int_{-\infty}^0 dz\overline{F}(t-z)F(z)\geq(a_-+\varepsilon/2)\overline{F}(t),\quad t\geq R_2.$$

This immediately implies (1.10) with $R=\max\{R_1,R_2\}$.

2.2 Proof of Corollary 1.4.

Fix $\varepsilon > 0$ and pick R such that

$$Y_n = \widehat{G}_{a+\varepsilon}(x - S_{n\wedge\mu_{x-R}})$$

is a submartingale. Then,

$$\begin{aligned}\overline{F_s}(x) = \widehat{G}_{a+\varepsilon}(x) &= \mathbf{E}Y_0 \le \mathbf{E}Y_\infty \\ &= \mathbf{E}\left[\widehat{G}_{a+\varepsilon}(x - S_{\mu_{x-R}}), \mu_{x-R} < \infty\right] \\ &\le (a+\varepsilon)\mathbf{P}(\mu_{x-R} < \infty).\end{aligned}$$

Hence,

$$\mathbf{P}(M > x) = \mathbf{P}(\mu_x < \infty) \ge \frac{1}{a+\varepsilon}\overline{F_s}(x+R).$$

Letting x to infinity we obtain,

$$\liminf_{x\to\infty} \frac{\mathbf{P}(M > x)}{\overline{F_s}(x)} \ge a + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary the lower bound in (1.11) holds.

To prove the corresponding upper bound fix $\varepsilon > 0$ and pick R such that the process $Y_n = G_{a-\varepsilon}(x - S_{n\wedge\mu_{x-R}})$ is a supermartingale. Then,

$$\begin{aligned}\overline{F_s}(x) = G_{a-\varepsilon}(x) = \mathbf{E}Y_0 &\ge \mathbf{E}Y_\infty \\ &= (a-\varepsilon)\mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{E}\left[\overline{F_s}(x - S_{\mu_{x-R}}); \mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x-R, x]\right] \\ &\ge (a-\varepsilon)\mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\quad + \overline{F_s}(R)\mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x-R, x]).\end{aligned} \tag{2.4}$$

Let $r > 0$ be a number which we pick later. Then,

$$\begin{aligned}\mathbf{P}(M > x + r) &\le \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\qquad + \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x-R, x], M > x + r) \\ &\le \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\qquad + \mathbf{P}(M > r)\mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x-R, x]),\end{aligned}$$

where we use the strong Markov property. Now pick sufficiently large r such that $\mathbf{P}(M > r) \le \overline{F_s}(R)/(a-\varepsilon)$. Then,

$$\begin{aligned}\mathbf{P}(M > x + r) &\le \mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} > x) \\ &\qquad + \frac{\overline{F_s}(R)}{a-\varepsilon}\mathbf{P}(\mu_{x-R} < \infty, S_{\mu_{x-R}} \in (x-R, x]).\end{aligned}$$

Combining this with (2.4), we get

$$\mathbf{P}(M > x + r) \le \frac{\overline{F_s}(x)}{a-\varepsilon}.$$

Letting x to infinity we obtain,

$$\limsup_{x\to\infty} \frac{\mathbf{P}(M > x)}{\overline{F_s}(x)} \le \frac{1}{a-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary the upper bound holds.

2.3 Proof of Corollary 1.5

We start with a lower bound. Fix $\varepsilon > 0$ and pick R such that $Y_n = \widehat{G}_{a+\varepsilon}(x - S_{n\wedge\mu_{x-R}})$ is a submartingale. Then,

$$\begin{aligned}\overline{F_s}(x) = \widehat{G}_{a+\varepsilon}(x) = \mathbf{E}Y_0 \le \mathbf{E}Y_\tau \\ \le (a+\varepsilon)\mathbf{P}(\mu_{x-R} < \tau) + \mathbf{E}\overline{F_s}(x - S_\tau).\end{aligned}$$

Hence,

$$\mathbf{P}(M_\tau > x + R) = \mathbf{P}(\mu_{x+R} < \tau) \ge \frac{1}{a+\varepsilon}\left(\overline{F_s}(x) - \mathbf{E}\overline{F_s}(x - S_\tau)\right).$$

Now

$$\begin{aligned}\overline{F_s}(x) - \mathbf{E}\overline{F_s}(x - S_\tau) &= \int_0^\infty \mathbf{P}(S_\tau \in -dt)\left(\overline{F_s}(x) - \overline{F_s}(x+t)\right) \\ &\sim |\mathbf{E}S_\tau|\overline{F}(x), \quad x \to \infty. \qquad (2.5)\end{aligned}$$

By the Wald's identity $|\mathbf{E}S_\tau| = a\mathbf{E}\tau$. Therefore,

$$\liminf_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \ge \frac{a\mathbf{E}\tau}{a+\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary we obtain the lower bound

$$\liminf_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \ge \mathbf{E}\tau.$$

To show the upper bound fix $\varepsilon > 0$ and pick R such that $Y_n = G_{a-\varepsilon}(x - S_{n\wedge\mu_{x-R}})$ is a supermartingale. Then,

$$\begin{aligned}\overline{F_s}(x) = G_{a-\varepsilon}(0) &= \mathbf{E}Y_0 \ge \mathbf{E}Y_\tau \\ &= (a-\varepsilon)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{E}\left[\overline{F_s}(x - S_{\mu_{x-R}}); \mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x-R, x]\right] \\ &\quad + \mathbf{E}\overline{F_s}(x - S_\tau) \\ &\ge (a-\varepsilon)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) + \mathbf{E}\overline{F_s}(x - S_\tau) \\ &\quad + \overline{F_s}(R)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x-R, x]).\end{aligned}$$

Similarly to the corresponding argument in the proof of Corollary 1.4,

$$\begin{aligned}\mathbf{P}(M_\tau > x + r) &\le \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x-R, x], M_\tau > x + r) \\ &\le \mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} > x) \\ &\quad + \mathbf{P}(M > r)\mathbf{P}(\mu_{x-R} < \tau, S_{\mu_{x-R}} \in (x-R, x]),\end{aligned}$$

Consequently,

$$\mathbf{P}(M_\tau > x + r) \le \frac{1}{a+\varepsilon}\left(\overline{F_s}(x) - \mathbf{E}\overline{F_s}(x - S_\tau)\right).$$

Now, we can apply (2.5) and obtain

$$\limsup_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \le \frac{a\mathbf{E}\tau}{a-\varepsilon}.$$

Since $\varepsilon > 0$ is arbitrary we obtain the upper bound

$$\limsup_{x\to\infty} \frac{\mathbf{P}(M_\tau > x)}{\overline{F}(x)} \le \mathbf{E}\tau.$$

□

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References

[1] Asmussen, S. (1998). Subexponential asymptotics for stochastic processes: extremal behaviour, stationary distributions and first passage probabilities. *Ann. Appl. Probab.* **8** 354–374. MR-1624933

[2] Asmussen, S. (2003). *Applied probability and queues.* 2nd ed., Springer-Verlag, New York. MR-1978607

[3] Denisov D. (2005). A note on the asymptotics for the maximum on a random time interval of a random walk. *Markov Processes and related fields* **11** 165–169. MR-2133349

[4] Denisov, D., Foss S. and Korshuov, D. (2004). *Queueing Systems* **46** No 1-2, 15–33. MR-2072274

[5] Feller, W. (1971).*An Introduction to Probability Theory and Its Applications.* Vol. 2, 2nd ed., Wiley, New York. MR-0270403

[6] Foss, S., Korshunov, D., and Zachary, S. (2011). *An Introduction to Heavy-Tailed and Subexponetial Distributions.* Springer Science+Business Media. MR-2810144

[7] Foss, S., Palmowsky, Z. and Zachary, S. (2003). The probability of exceeding a high boundary on a random time interval for a heavy-tailed random walk. *Ann. Appl. Probab.* **15** 1936–1957. MR-2152249

[8] Foss, S. and Zachary, S. (2003). The maximum on a random time interval of a random walk with long-tailed increments and negative drift. *Ann. Appl. Probab.* **13** 37–53. MR-1951993

[9] Heath, D., Resnick, S. and Samorodnitsky, G. (1997). Patterns of buffer overflow in a class of queues with long memory in the input stream. *Ann. Appl. Probab.* **7** 1021–1057. MR-1484796

[10] Iglehart, D.L. (1972). Extreme values in GI/GI/1 queue. *Ann. Math. Statist.* **43** 627–635. MR-0305498

[11] Klüppelberg, C. (1988). Subexponential distributions and integrated tails. *J. Appl. Probab.* **25** 132–141. MR-0929511

[12] Kugler, J. and Wachtel,V. Upper bounds for the maximum of a random walk with negative drift. ArXiv Preprint: 1107.5400.

[13] Zachary, S. (2004). A Note on Veraverbeke's Theorem . *Queueing Systems* **46** No 1-2, 9–14. MR-2072273