

Influence of the initial condition in equilibrium last-passage percolation models

Eric A. Cator* Leandro P. R. Pimentel† Marcio W. A. Souza‡

Abstract

In this paper we consider an equilibrium last-passage percolation model on an environment given by a compound two-dimensional Poisson process. We prove an L^2 -formula relating the initial measure with the last-passage percolation time. This formula turns out to be a useful tool to analyze the fluctuations of the last-passage times along non-characteristic directions.

Keywords: Last passage percolation; interacting particle system; Hammersley process; equilibrium measures.

AMS MSC 2010: Primary 60C05 ; 60K35, Secondary 60F05.

Submitted to ECP on February 23, 2011, final version accepted on September 10, 2011.

Supersedes arXiv:1102.4737v2.

1 Introduction and the main result

1.1 The last-passage percolation model

Let $\mathbf{P} \subseteq \mathbb{R}^2$ be a two-dimensional Poisson random set of intensity one. On each point $\mathbf{p} \in \mathbf{P}$ we put a random positive weight $\omega_{\mathbf{p}}$ and we assume that $\{\omega_{\mathbf{p}} : \mathbf{p} \in \mathbf{P}\}$ is a collection of i.i.d. random variables, distributed according to a distribution function F , which are also independent of \mathbf{P} . Throughout this paper we will make the following assumption on the distribution function F of the weights:

$$\int_0^\infty e^{ax} dF(x) < +\infty, \text{ for some } a > 0. \quad (1.1)$$

This condition was used in [7] to prove the existence of invariant measures for the Hammersley's interacting fluid process we will introduce below. For each $\mathbf{p}, \mathbf{q} \in \mathbb{R}^2$, with $\mathbf{p} < \mathbf{q}$ (inequality in each coordinate, $\mathbf{p} \neq \mathbf{q}$), let $\Pi(\mathbf{p}, \mathbf{q})$ denote the set of all increasing (or up-right) paths, consisting of points in \mathbf{P} , from \mathbf{p} to \mathbf{q} , where we exclude all points that share (at least) one coordinate with \mathbf{p} . So we consider the points in the rectangle $]\mathbf{p}, \mathbf{q}]$, where we leave out the south and the west side of the rectangle. The last-passage time between $\mathbf{p} \leq \mathbf{q}$ is defined by

$$L(\mathbf{p}, \mathbf{q}) := \max_{\varpi \in \Pi(\mathbf{p}, \mathbf{q})} \left\{ \sum_{\mathbf{p}' \in \varpi} \omega_{\mathbf{p}'} \right\}.$$

*Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands
E-mail: E.A.Cator@tudelft.nl

†Federal University of Rio de Janeiro, Postal Code 68530, 21941-909 Rio de Janeiro-RJ, Brazil
E-mail: leandro@im.ufrj.br

‡University of São Paulo, Postal Code 66281, 05311-970 São Paulo-SP, Brazil
E-mail: msouza@ime.usp.br

When F is the Dirac distribution concentrated on 1 (each point has weight 1 and we will denote this F by δ_1), then we refer to this model as the classical Hammersley model [1, 9].

A crucial result is the following shape theorem (see Theorem 1.1 in [8], p.164): set $\mathbf{0} = (0, 0)$, $\mathbf{n} = (n, n)$,

$$\gamma = \gamma(F) = \sup_{n \geq 1} \frac{\mathbb{E}(L(\mathbf{0}, \mathbf{n}))}{n} > 0 \quad \text{and} \quad f(x, t) := \gamma\sqrt{xt}. \tag{1.2}$$

Then $\gamma(F) < \infty$ and for all $x, t > 0$,

$$\lim_{r \rightarrow \infty} \frac{L(\mathbf{0}, (rx, rt))}{r} = \lim_{r \rightarrow \infty} \frac{\mathbb{E}L(\mathbf{0}, (rx, rt))}{r} = f(x, t). \tag{1.3}$$

1.2 The interacting fluid system formulation

It is well known that the classical Hammersley model has a representation as an interacting particle system [1, 9]. The general model has a similar description, although a better name might be an interacting fluid system. We start by restricting the compound Poisson process $\{\omega_{\mathbf{p}} : \mathbf{p} \in \mathbf{P}\}$ to $\mathbb{R} \times \mathbb{R}_+$. To each measure ν on \mathbb{R} we associate a non-decreasing process $\nu(\cdot)$ defined by

$$\nu(x) = \begin{cases} \nu((0, x]) & \text{for } x \geq 0 \\ -\nu((x, 0]) & \text{for } x < 0. \end{cases}$$

Let \mathcal{N} be the set of all positive, locally finite measures ν such that

$$\liminf_{y \rightarrow -\infty} \frac{\nu(y)}{y} > 0.$$

We need this condition to define the evolution of the process, since otherwise all mass will be pulled to minus infinity. The Hammersley interacting fluid system $(M_t^\nu : t \geq 0)$ will be defined as a Markov process with values in \mathcal{N} , as was done in [7]. Its evolution is defined as follows: if there is a Poisson point with weight ω at a point (x_0, t) , then $M_t^\nu(\{x_0\}) = M_{t-}^\nu(\{x_0\}) + \omega$, and for $x > x_0$,

$$M_t^\nu((x_0, x]) = (M_{t-}^\nu((x_0, x]) - \omega)_+.$$

Here, M_{t-}^ν is the “mass distribution” of the fluid at time t if the Poisson point at (x_0, t) would be removed. To the left of x_0 the measure does not change. In words, the Poisson point at (x_0, t) moves a total mass ω to the left, to the point x_0 , taking the mass from the first available fluid to the right of x_0 . (See Figure 1 for a visualization, in case of atomic measures, of the process inside a space-time box.)

In this paper we follow the Aldous and Diaconis [1] graphical representation in the last-passage model (compare to the result in the classical case, found in their paper): For each $\nu \in \mathcal{N}$, $x \in \mathbb{R}$ and $t \geq 0$ let

$$L_\nu(x, t) := \sup_{z \leq x} \{\nu(z) + L((z, 0), (x, t))\}. \tag{1.4}$$

The measure M_t^ν defined by

$$M_t^\nu((x, y]) := L_\nu(y, t) - L_\nu(x, t) \quad \text{for } x < y,$$

defines a Markov process on \mathcal{N} and it evolves according to the Hammersley interacting fluid system [7].

We now make the following important observation for a random initial condition ν , which basically follows from translation invariance.

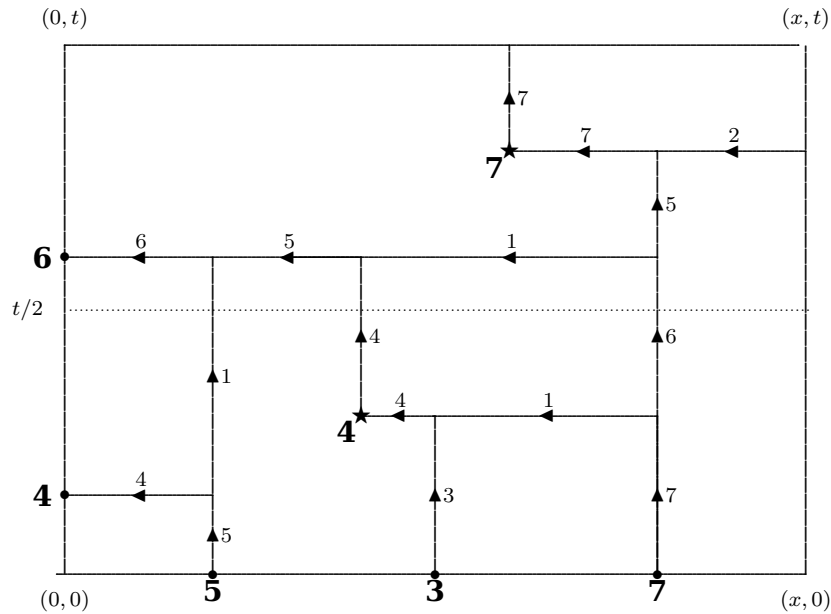


Figure 1: In this picture, restricted to $[0, x]$, the measure ν consists of three atoms of weight 5, 3 and 7. The Poisson process, restricted to $[0, x] \times [0, t]$, has two points with weights 4 and 7. The measure $M_{t/2}^\nu$ consists of three atoms of weight 1, 4 and 6, while at time t , it consists of one atom with weight 7. A total weight of $4 + 6$ has left the box due to Poisson points to the left of the box, while a total weight of 2 has entered.

Theorem 1.1. *Suppose $\nu \in \mathcal{N}$ is a random initial measure on \mathbb{R} independent of the Poisson process in $\mathbb{R} \times \mathbb{R}^+$, whose distribution is translation invariant. For any speed $V \in \mathbb{R}$ and any $x \in \mathbb{R}$, we have*

$$L_\nu(Vt, t) \stackrel{\mathcal{D}}{=} L_\nu(x, t) - \nu(x - Vt).$$

The relevance of this result is most clear when we consider equilibrium measures of the Hammersley's interacting fluid process. Assume that we have a probability measure defined on \mathcal{N} and consider $\nu \in \mathcal{N}$ as a realization of this probability measure. We say that ν is time invariant for the Hammersley interacting fluid process (in law) if

$$M_t^\nu \stackrel{\mathcal{D}}{=} M_0^\nu = \nu \quad \text{for all } t \geq 0.$$

In this case, we also say that the underlying probability measure on \mathcal{N} is an equilibrium measure. It is known that there is only one family of ergodic equilibrium measures for the Hammersley interacting fluid system [7]. Let us denote it by $\{\nu_\lambda : \lambda > 0\}$, where

$$\lambda := \mathbb{E}\nu_\lambda(1). \tag{1.5}$$

For simple notation, put $L_\lambda := L_{\nu_\lambda}$. The main result of this paper is the following formula:

Corollary 1.2. Recall (1.2) and (1.5), and let

$$V_\lambda := \left(\frac{\gamma}{2\lambda}\right)^2 \quad \text{and} \quad \psi_\lambda := \frac{\gamma^2}{2\lambda}. \quad (1.6)$$

Here, V_λ is the characteristic speed corresponding to L_λ and ψ_λ is the growth rate of $L_\lambda(V_\lambda t, t)$. Then

$$\mathbb{E}(\{L_\lambda(x, t) - [\nu_\lambda(x - V_\lambda t) + \psi_\lambda t]\}^2) = \text{Var}(L_\lambda(V_\lambda t, t)). \quad (1.7)$$

1.3 A central limit theorem for the classical model

To illustrate the importance of (1.7), let us restrict ourselves to the classical Hammersley model. In this set-up, the equilibrium measures are one-dimensional Poisson processes of intensity λ , and $\gamma = \gamma(\delta_1) = 2$. Thus,

$$V_\lambda := \frac{1}{\lambda^2} \quad \text{and} \quad \psi_\lambda := \frac{2}{\lambda}.$$

Cator and Groeneboom [6] proved that the variance of L_λ grows sub-linearly along the characteristic speed λ^{-2} . Together with Corollary 1.2, this implies

Corollary 1.3. Let $(z_t)_{t \geq 0}$ be a deterministic path. Then

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}(\{L_\lambda(z_t, t) - [\nu_\lambda(z_t - \lambda^{-2}t) + 2\lambda^{-1}t]\}^2)}{t} = \lim_{t \rightarrow \infty} \frac{\text{Var}(L_\lambda(\lambda^{-2}t, t))}{t} = 0. \quad (1.8)$$

Proof of Corollary 1.3: Formula (1.7), applied to the classical model, gives us

$$\mathbb{E}(\{L_\lambda(z_t, t) - [\nu_\lambda(z_t - \lambda^{-2}t) + 2\lambda^{-1}t]\}^2) = \text{Var}(L_\lambda(\lambda^{-2}t, t)).$$

On the other hand, [6] shows that

$$\lim_{t \rightarrow \infty} \frac{\text{Var}(L_\lambda(\lambda^{-2}t, t))}{t} = 0,$$

which proves (1.8). □

Corollary 1.4. Let $(z_t)_{t \geq 0}$ be a deterministic path such that

$$\lim_{t \rightarrow \infty} \frac{z_t}{t} = a.$$

Then

$$\lim_{t \rightarrow \infty} \frac{\text{Var}(L_\lambda(z_t, t))}{t} = \sigma^2 := \left|a\lambda - \frac{1}{\lambda}\right|. \quad (1.9)$$

Furthermore, if $a \neq \lambda^{-2}$ then

$$\lim_{t \rightarrow \infty} \mathbb{P}(L_\lambda(z_t, t) \leq \lambda z_t + \frac{t}{\lambda} + (\sigma\sqrt{t})u) = \mathbb{P}(N \leq u), \quad (1.10)$$

where N is a standard Gaussian random variable.

Proof of Corollary 1.4: Corollary 1.3 shows that

$$\lim_{t \rightarrow \infty} \frac{L_\lambda(z_t, t) - [\nu_\lambda(z_t - \lambda^{-2}t) + 2\lambda^{-1}t]}{\sqrt{t}} = 0,$$

in the \mathbb{L}^2 sense. Since ν_λ is a one-dimensional Poisson process of intensity λ , this implies (1.9) and (1.10). □

Remark 1.5. Cator and Groeneboom [6] proved that $\sqrt{\text{Var}(L_\lambda(\lambda^{-2}t, t))}$ is of order $t^{1/3}$, which gives us the same order for the \mathbb{L}^2 -distance between

$$L_\lambda(z_t, t) \quad \text{and} \quad [\nu_\lambda(x - \lambda^{-2}t) + 2\lambda^{-1}t].$$

Remark 1.6. The central limit theorem for L_λ (along any direction) was proved by Baik and Rains [5]. Their method was based on very particular combinatorial properties of the classical model that do not seem to hold for the general set-up. Our approach reveals the strong relationship with the initial configuration.

Remark 1.7. In the general set-up, Corollary 1.2 implies: If the variance of L_λ along the characteristic speed V_λ is sub-linear, and the equilibrium measure has Gaussian fluctuations, then L_λ will also have Gaussian fluctuations along non-characteristic directions.

Remark 1.8. For the classical Hammersley process an important formula for the variance of $L_\lambda(x, t)$ was derived in [6], Theorem 2.1:

$$\text{Var}(L_\lambda(x, t)) = -\lambda x + \frac{t}{\lambda} + 2\lambda \mathbb{E}(x - X_\lambda(t))_+,$$

where $X_\lambda(t)$ is the position at time t of a second class particle starting at zero. This formula was pivotal in deriving the cube-root behavior of L_λ in [6], and later corresponding formulas were used to prove cube-root behavior for TASEP [3] and for ASEP [4]. However, this formula does not directly show the relationship with the initial configuration. Also, there seems to be no direct way to deduce (1.7) from this formula, even if we reformulate it, as was done in Equation (3.6) of [6], in terms of the exit-point of the longest path from $(0, 0)$ to (x, t) , which is the right-most z for which the supremum in (1.4) is attained.

2 Proof of Theorem 1.1 and Corollary 1.2

Recall that

$$L_\nu(x, t) = \sup_{z \leq x} \{\nu(z) + L((z, 0), (x, t))\}.$$

Clearly, $L((z, 0), (Vt, t)) \stackrel{\mathcal{D}}{=} L((z + x - Vt, 0), (x, t))$. By assumption, ν has a translation invariant distribution, independent of L . This implies that

$$\{z \mapsto \nu(z)\} \stackrel{\mathcal{D}}{=} \{z \mapsto \nu(z + x - Vt) - \nu(x - Vt)\},$$

and

$$\begin{aligned} L_\nu(Vt, t) &\stackrel{\mathcal{D}}{=} \sup_{z \leq Vt} \{\nu(z + x - Vt) - \nu(x - Vt) + L((z + x - Vt, 0), (x, t))\} \\ &= \sup_{z \leq x} \{\nu(z) + L((z, 0), (x, t))\} - \nu(x - Vt) \\ &= L_\nu(x, t) - \nu(x - Vt). \end{aligned}$$

This proves Theorem 1.1. □

Corollary 1.2 now follows from results in [7]: there it is shown that for any speed V , the stationarity of L_λ leads to

$$\mathbb{E}L_\lambda(Vt, t) = V\lambda t + \frac{1}{4}\gamma^2 t/\lambda.$$

This follows from the fact that the Hammersley fluid process has intensity λ on the bottom side of the rectangle between $(0, 0)$ and (x, t) , and intensity $\gamma^2/(4\lambda)$ on the left side (this refers to the expected mass of the fluid leaving the interval $[0, x]$ through 0 per time unit). When we define the characteristic speed $V_\lambda = \gamma^2/(4\lambda^2)$, then

$$\mathbb{E}L_\lambda(V_\lambda t, t) = \psi_\lambda t.$$

This together with Theorem 1.1 immediately shows (1.7). \square

3 The lattice last-passage percolation model

In the lattice last-passage percolation model one considers i.i.d. weights $\{\omega_{\mathbf{p}} : \mathbf{p} \in \mathbb{Z}^2\}$, distributed according to a distribution function F . For $F(x) = 1 - e^{-x}$ (exponential weights), we have a similar shape theorem as (1.3) with limit shape given by

$$f(x, t) = (\sqrt{x} + \sqrt{t})^2.$$

We know from [3] that the invariant measures are given by

$$\nu_\rho((x, y]) \stackrel{\mathcal{D}}{=} \sum_{z=x+1}^y X_z,$$

where $\{X_z : z \in \mathbb{Z}\}$ is a collection of i.i.d. exponential random variables with parameter ρ . The analog to formula (1.7) is

$$\mathbb{E}(\{L_\rho(x, t) - [\nu_\rho(x - \lfloor V_\rho t \rfloor) + \psi_\rho t]\}^2) = \text{Var}(L_\rho(\lfloor V_\rho t \rfloor, t)),$$

where

$$V_\rho := \frac{\rho^2}{(1-\rho)^2} \quad \text{and} \quad \psi_\rho := \frac{1}{(1-\rho)^2}.$$

Together with the cube-root asymptotics [3], this implies that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}(\{L_\rho(z_t, t) - [\nu_\rho(z_t - V_\rho t) + \psi_\rho t]\}^2)}{t} = 0.$$

Therefore, if

$$\lim_{t \rightarrow \infty} \frac{z_t}{t} = a$$

then

$$\lim_{t \rightarrow \infty} \frac{\text{Var}(L_\rho(z_t, t))}{t} = \sigma^2 := \frac{|a(1-\rho)^2 - \rho^2|}{\rho^2(1-\rho)^2},$$

and if $a \neq V_\rho$ then

$$\lim_{t \rightarrow \infty} \mathbb{P}\left(L_\rho(z_t, t) \leq \frac{z_t}{\rho} + \frac{t}{1-\rho} + (\sigma\sqrt{t})u\right) = \mathbb{P}(N \leq u),$$

where N is a standard Gaussian random variable.

Remark 3.1. *Ferrari and Fontes [10] determined the dependence on the initial condition for the totally asymmetric exclusion process, which is isomorphic to the lattice last-passage percolation model with exponential weights. The method developed in this paper resembles the ideas in their paper. Balázs [2] used a different method to get a generalization of the Ferrari-Fontes result for certain types of deposition models. It is not clear to us whether our methods would work for these more general deposition models.*

Remark 3.2. *In the general lattice model, the shape theorem (1.3) holds. However, not much is known about the limit shape f . If this function would not be strictly curved (we know it is convex, so this would mean that there are “flat” pieces), then the methods used in [8] to prove the existence and uniqueness of semi-infinite geodesics in a fixed direction do not apply, and we are not able to prove the existence of equilibrium measures.*

References

- [1] D. Aldous and P. Diaconis, Hammersley’s interacting particle process and longest increasing subsequences, *Probab. Theory Related Fields* **103** (1995), no. 2, 199–213. MR-1355056
- [2] M. Balázs, Growth fluctuations in a class of deposition models, *Ann. Inst. H. Poincaré Probab. Statist.* **39** (2003), no. 4, 639–685. MR-1983174
- [3] M. Balázs, E. Cator and T. Seppäläinen, Cube root fluctuations for the corner growth model associated to the exclusion process, *Electron. J. Probab.* **11** (2006), no. 42, 1094–1132 (electronic). MR-2268539
- [4] M. Balázs and T. Seppäläinen, Fluctuation bounds for the asymmetric simple exclusion process, *ALEA Lat. Am. J. Probab. Math. Stat.* **6** (2009), 1–24. MR-2485877
- [5] J. Baik and E. M. Rains, Limiting distributions for a polynuclear growth model with external sources, *J. Statist. Phys.* **100** (2000), no. 3-4, 523–541. MR-1788477
- [6] E. Cator and P. Groeneboom, Second class particles and cube root asymptotics for Hammersley’s process, *Ann. Probab.* **34** (2006), no. 4, 1273–1295. MR-2257647
- [7] E. Cator and L.P.R. Pimentel, Busemman functions and equilibrium measures in last-passage percolation models., To appear in *Probab. Theory Relat. Fields*. Available from *PTRF Online First* (2011).
- [8] E. Cator and L.P.R. Pimentel, A shape theorem and semi-infinite geodesics for the Hammersley model with random weights, *ALEA Lat. Am. J. Probab. Math. Stat.* **8** (2011), 163–175.
- [9] J. M. Hammersley, A few seedlings of research, in *Proceedings of the Sixth Berkeley Symposium on Mathematical Statistics and Probability (Univ. California, Berkeley, Calif., 1970/1971), Vol. I: Theory of statistics*, 345–394, Univ. California Press, Berkeley, CA. MR-0405665
- [10] P. A. Ferrari and L. R. G. Fontes, Current fluctuations for the asymmetric simple exclusion process, *Ann. Probab.* **22** (1994), no. 2, 820–832. MR-1288133