CORRELATION INEQUALITIES
FOR EDGE-REINFORCED RANDOM WALK

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Abstract

We prove correlation inequalities for linearly edge-reinforced random walk. These correlation inequalities concern the first entry tree, i.e. the tree of edges used to enter any vertex for the first time. They also involve the asymptotic fraction of time spent on particular edges. Basic ingredients are known FKG-type inequalities and known negative associations for determinantal processes.

1 Introduction and results

The model. Let $\mathcal{G} = (V, E)$ be a finite undirected connected graph with vertex set V and edge set $E \subseteq \{\{u, v\} : u, v \in V, u \neq v\}$. Linearly edge-reinforced random walk on $\mathcal{G}$ with initial weights $a = (a_e)_{e \in E} \in (0, \infty)^E$ and starting vertex $v_0 \in V$ is a nearest-neighbor random walk $(X_t)_{t \in \mathbb{N}_0}$ on $\mathcal{G}$ defined as follows: We realize X_t as projection to the t-th coordinate on the space $\Omega \subseteq V^{\mathbb{N}_0}$ of nearest-neighbor paths on $\mathcal{G}$. Every edge $e \in E$ is given a weight $w_t(e)$ depending on time t. Initially the edge weights are given by the initial weights $a = (a_e)_{e \in E}$:

$$w_0(e) = a_e \quad \text{for all } e \in E. \tag{1.1}$$

In each discrete time-step, the reinforced random walker jumps from its current position to a neighboring vertex with probability proportional to the weight of the connecting edge. Each time an edge is traversed, its weight is increased by 1. More formally, the edge weights are given by

$$w_t(e) = a_e + \sum_{s=0}^{t-1} \mathbf{1}_{\{\{X_s, X_{s+1}\}=e\}} \tag{1.2}$$

for $t \in \mathbb{N}_0$ and $e \in E$. The law $P_a := P_{v_0,a}^{\mathcal{G}}$ of the edge-reinforced random walk is specified by the following requirements:

$$P_a[X_0 = v_0] = 1, \tag{1.3}$$

$$P_a[X_{t+1} = v|X_0, X_1, \dots, X_t] = \frac{w_t(\{X_t, v\})}{\sum_{e \ni X_t} w_t(e)} \mathbf{1}_{\{\{X_t, v\} \in E\}}. \tag{1.4}$$

First entry tree and asymptotic ratios of visits. Let $\mathcal{S}$ denote the set of all subtrees of $\mathcal{G}$, endowed with the discrete topology, and let $\mathcal{T} \subseteq \mathcal{S}$ denote the set of spanning trees of $\mathcal{G}$. For $t \in \mathbb{N}$ and $v \in \{X_1, X_2, \dots, X_t\} \setminus \{X_0\}$, let $e_t^{\text{first entry}}(v)$ denote the edge the random walk path $(X_0, X_1, \dots, X_t)$ uses to enter the vertex v for the first time. Set

$$T_t^{\text{first entry}} = \{e_t^{\text{first entry}}(v) : v \in \{X_1, X_2, \dots, X_t\} \setminus \{X_0\}\}. \tag{1.5}$$

Then, $T_t^{\text{first entry}} \in \mathcal{S}$. Since linearly edge-reinforced random walk on a finite graph $\mathcal{G}$ visits every vertex almost surely (see e.g. Proposition 1 in [KR00]), $T_t^{\text{first entry}} \in \mathcal{T}$ for all t large enough almost surely.

For $t \in \mathbb{N}_0$, denote by $k_t(e)$ the number of crossings of edge e by the random walker up to time t:

$$k_t(e) = |\{s \in \{0, 1, \dots, t-1\} : \{X_s, X_{s+1}\} = e\}|. \tag{1.6}$$

Let $W := (0, \infty)^E$ denote the set of positive edge weights on $\mathcal{G}$. For any given reference edge $e_0 \in E$, we introduce

$$W_{e_0} := \{(x_e)_{e \in E} \in W : x_{e_0} = 1\}. \tag{1.7}$$

For $x = (x_e)_{e \in E} \in W$ and $v \in V$, we set

$$x_v := \sum_{e \ni v} x_e. \tag{1.8}$$

Similarly we use the notation $a_v = \sum_{e \ni v} a_e$ for the initial weights $a = (a_e)_{e \in E} \in W$. Abbreviating $\mathbb{R}_+ = (0, \infty)$, we introduce the map

$$\phi_{v_0, a} : \mathcal{T} \times W \to \mathbb{R}_+,$$
$$(T, x) \mapsto \frac{\prod_{e \in E} x_e^{a_e - 1}}{x_{v_0}^{a_{v_0}/2} \prod_{v \in V \setminus \{v_0\}} x_v^{(a_v + 1)/2}} \frac{\prod_{e \in T} x_e}{\sqrt{\sum_{S \in \mathcal{T}} \prod_{e \in S} x_e}}. \tag{1.9}$$

As explained below, it plays an important role in the asymptotic description of edge-reinforced random walks.

Let dT denote the counting measure on $\mathcal{T}$. The Lebesgue measure supported on W_{e_0} is denoted by

$$\rho_{e_0}(dx) := \delta_1(dx_{e_0}) \times \prod_{e \in E \setminus \{e_0\}} dx_e, \quad x \in W. \tag{1.10}$$

By Lemma 9.1 in [MR08], $\phi_{v_0,a}(T, x) \, dT \, \rho_{e_0}(dx)$ is a finite measure. Let $1/c(v_0, a)$ denote its total mass. We remark that the normalizing constant $c(v_0, a)$ is known explicitly and given by the following formula:

$$c(v_0, a) = \frac{\Gamma(a_{v_0}/2) \prod_{v \in V \setminus \{v_0\}} \Gamma((a_v + 1)/2)}{\prod_{e \in E} \Gamma(a_e)} \frac{2^{1 - |V| + \sum_{e \in E} a_e}}{\pi^{(|V| - 1)/2}}; \tag{1.11}$$

see Lemma 9.1 in [MR08]. Note that $c(v_0, a)$ does not depend on the choice of the reference edge e_0.

We show the following:

Theorem 1.1 (First entry tree and asymptotic ratios of visits) *The random vector*

$$\left(T_t^{\text{first entry}}, \left(k_t(e)/k_t(e_0) \right)_{e \in E} \right) \tag{1.12}$$

converges almost surely as $t \to \infty$ to a limit $(T^{\text{first entry}}, x^{\text{arv}})$, where "arv" is our abbreviation for "asymptotic ratio of visits". The joint law of $T^{\text{first entry}}$ and x^{arv} is denoted by $\mathbb{P}_{v_0, a, e_0}$. It is supported on $\mathcal{T} \times W_{e_0}$ and equals

$$\mathbb{P}_{v_0, a, e_0}(dT \, dx) = c(v_0, a) \phi_{v_0, a}(T, x) \, dT \rho_{e_0}(dx). \tag{1.13}$$

For any spanning tree $T \in \mathcal{T}$, one has

$$P_a \left[T^{\text{first entry}} = T \mid x^{\text{arv}} \right] = \text{tree}(x^{\text{arv}})^{-1} \prod_{e \in T} x_e^{\text{arv}} \quad P_a\text{-a.s.} \tag{1.14}$$

with the normalizing constant

$$\text{tree}(x) = \sum_{T \in \mathcal{T}} \prod_{e \in T} x_e, \quad x \in W. \tag{1.15}$$

Of course, x^{arv} depends also on the choice of the reference edge e_0. However, we suppress this in the notation as e_0 is fixed.

We always use the version described by formula (1.14) of the conditional distribution of the first entry tree $T^{\text{first entry}}$ given the asymptotic ratios of visits x^{arv}. For other conditional distributions derived from it we use the corresponding versions.

Correlation inequalities. Let $\mathscr{P}(E)$ denote the power set of E. We call a function $f : \mathscr{P}(E) \to \mathbb{R}$ increasing if $f(A) \leq f(B)$ holds for all $A \subseteq B$ in $\mathscr{P}(E)$. For $F \subseteq E$, let $\mathscr{F}_F := \sigma(T \cap F)$ denote the σ-field generated by $T \cap F$, where here T denotes the canonical process (i.e. the identity map) on $\mathscr{P}(E)$. From now on, E_a denotes expectation with respect to P_a.

The following theorem claims negative associations for the first entry tree given the asymptotic ratios of visits.

Theorem 1.2 (Correlation inequality given the asymptotic ratios of visits)
Let $F, G \subseteq E$ be disjoint sets of edges. Let $f, g : \mathscr{P}(E) \to \mathbb{R}$ be increasing functions, measurable with respect to $\mathscr{F}_F$ and $\mathscr{F}_G$, respectively. Then, the following correlation inequality holds:

$$E_a[f(T^{\text{first entry}})g(T^{\text{first entry}}) \mid x^{\text{arv}}] \leq E_a[f(T^{\text{first entry}}) \mid x^{\text{arv}}] E_a[g(T^{\text{first entry}}) \mid x^{\text{arv}}]. \tag{1.16}$$

For $F \subseteq E$ and $x \in W$, we abbreviate $x_F := (x_e)_{e \in F}$. We write $x_F \leq y_F$ if $x_e \leq y_e$ for all $e \in F$. We say that a function $f : W \to \mathbb{R}$ is increasing on $F \subseteq E$ if for all $x, y \in W$ with $x_F \leq y_F$ and $x_{E \setminus F} = y_{E \setminus F}$ one has $f(x) \leq f(y)$. A function which is increasing on E is simply called increasing. The following theorem shows positive correlations for increasing functions of the asymptotic ratios of visits conditional on the first entry tree.

Theorem 1.3 (FKG inequality given the first entry tree) *Let $e_0 \in E$ be fixed and let $F \subseteq E$. For any bounded measurable functions $f, g : W \to \mathbb{R}$ which are increasing on $E \setminus F$, one has*

$$E_a[f(x^{\mathrm{arv}})g(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}}]$$
$$\geq E_a[f(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}}] E_a[g(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}}]. \tag{1.17}$$

Here is an application of Theorem 1.3. If one increases the initial weights of some edges, then, conditionally on the asymptotic ratio of visits of these edges and on the first entry tree, the asymptotic ratio of visits of the other edges decreases stochastically. More formally:

Corollary 1.4 *Let $F \subseteq E$. Let $a = (a_e)_{e \in E}$ and $b = (b_e)_{e \in E} \in W$ be such that $a \leq b$ and $a_{E \setminus F} = b_{E \setminus F}$. Then, for any e_0 and any bounded measurable function $f : W \to \mathbb{R}$ which is increasing on $E \setminus F$, one has*

$$E_a[f(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}}] \geq E_b[f(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}}]. \tag{1.18}$$

The next theorem roughly states the following: conditionally on the first entry tree and on the values x_F of the asymptotic ratios of visits of some edges, the asymptotic ratios of visits of the other edges increase stochastically as the values x_F increase.

Theorem 1.5 *Let $e_0 \in E$ be fixed and let $F \subseteq E$ with $e_0 \notin F$. Let $f : W \to \mathbb{R}$ be increasing. Then, for any $x_F \leq y_F$ in $(0, \infty)^F$, one has*

$$E_a[f(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}} = x_F] \leq E_a[f(x^{\mathrm{arv}}) \,|\, T^{\mathrm{first\ entry}}, x_F^{\mathrm{arv}} = y_F]. \tag{1.19}$$

Here, conditioning on $x_F^{\mathrm{arv}} = x_F$ makes only sense for $e_0 \notin F$, as $x_{e_0}^{\mathrm{arv}} = 1$ holds P_a-almost surely.

A fundamental ingredient of the proofs is on the one hand a theorem on negative associations for determinantal processes proved by Lyons [Lyo03]. On the other hand, the proofs rely on FKG-type inequalities as summarized in Keane and den Hollander [dHK86]; see also Preston [Pre74].

2   Proofs

2.1   First entry tree and asymptotic ratios of visits

For the proof of Theorem 1.1, we need to review a representation of the edge-reinforced random walk as a random walk in a random environment. For $v_0 \in V$ and $x \in W$, we denote by $Q_{v_0, x}$ the law of the nearest-neighbor Markovian random walk on $\mathcal{G}$ which starts in v_0 and jumps along edge e with probability proportional to the weight x_e:

$$Q_{v_0, x}[X_0 = v_0] = 1, \tag{2.1}$$

$$Q_{v_0, x}[X_{t+1} = v | X_0, X_1, \ldots, X_t] = \frac{x_{\{X_t, v\}}}{x_{X_t}} \mathbb{1}_{\{\{X_t, v\} \in E\}} \tag{2.2}$$

for all $t \in \mathbb{N}_0$, $v \in V$.

The edge-reinforced random walk on $\mathcal{G}$ is a mixture of the Markov chains $Q_{v_0, x}$, $x \in W$. Formally, for every fixed $e_0 \in E$ and all events $B \subseteq \Omega$, one has

$$P_a[(X_t)_{t \in \mathbb{N}_0} \in B] = \int_{\mathcal{T} \times W_{e_0}} Q_{v_0, x}[(X_t)_{t \in \mathbb{N}_0} \in B] \, \mathbb{P}_{v_0, a, e_0}(dT \, dx). \tag{2.3}$$

This result can be found in Lemma 9.1 in [MR08].

Proof of Theorem 1.1. By Theorem 1 of [KR00], $(k_t(e)/t)_{e \in E}$ converges almost surely as $t \to \infty$ to a limit which is almost surely strictly positive. As an increasing sequence of subsets of E, $(T_t^{\text{first entry}})_{t \in \mathbb{N}_0}$ is eventually constant. Hence, the random vector in (1.12) converges almost surely as well. The limiting distribution is supported on $\mathscr{T} \times W_{e_0}$.

Observe that

$$P_a[T^{\text{first entry}} = T, x^{\text{arv}} \in A] = E_a[1_{\{x^{\text{arv}} \in A\}} P_a[T^{\text{first entry}} = T | x^{\text{arv}}]] \tag{2.4}$$

for all measurable sets $A \subseteq W$. The next statement uses that for all $x \in W_{e_0}$, $x^{\text{arv}} = x$ holds $Q_{v_0,x}$-almost surely. Since by (2.3), the edge-reinforced random walk is a mixture of reversible Markov chains, $P_a[\cdot|x^{\text{arv}}]$ equals $Q_{v_0,x^{\text{arv}}}$. We claim that for all $x \in W$, one has

$$Q_{v_0,x}[T^{\text{first entry}} = T] = \text{tree}(x)^{-1} \prod_{e \in T} x_e, \tag{2.5}$$

and thus (1.14) holds. The distribution of x^{arv} equals

$$c(v_0, a) \sum_{T \in \mathscr{T}} \phi_{v_0,a}(T, x) \rho_{e_0}(dx), \tag{2.6}$$

as follows from Theorem 1 of [KR00]; see also formula (9.1) of [MR08]. If we prove (2.5), then the description (1.13) for the joint law of $T^{\text{first entry}}$ and x^{arv} for the edge-reinforced random walk follows from (2.4).

It remains to prove (2.5). For $x \in W$, let $\tilde{x} = (\tilde{x}_v)_{v \in V}$ with $\tilde{x}_v = x_v / \sum_{u \in V} x_u$ be the normalization of the weights x_v of the vertices to a probability function. The corresponding distribution is stationary with respect to the transition law, defined in (2.2), underlying the Markov chains $Q_{v,x}$ for any $v \in V$. We consider the associated two-sided infinite stationary Markov chain $(X_t)_{t \in \mathbb{Z}}$, realized as projections on $V^{\mathbb{Z}}$, with the same transition kernel and stationary distribution given by $\tilde{x}$. Its law is denoted by $Q_{\tilde{x},x}$. We denote the time shift by $\theta : V^{\mathbb{Z}} \to V^{\mathbb{Z}}$, $(v_t)_{t \in \mathbb{Z}} \mapsto (v_{t+1})_{t \in \mathbb{Z}}$.

For every vertex $v \in \{X_t : t \le 0\} \setminus \{X_0\}$ let $e^{\text{last exit}}(v)$ denote the edge the random walk path $(X_t)_{t \le 0}$ uses to leave the vertex v for the last time before time 0. Let $T^{\text{last exit}} = \{e^{\text{last exit}}(v) : v \in \{X_t : t \le 0\} \setminus \{X_0\}\}$ denote the last exit tree generated by $(X_t)_{t \le 0}$; it is $Q_{\tilde{x},x}$-almost surely a spanning tree. We observe the following

$$Q_{v_0,x}[T^{\text{first entry}} = T] = Q_{\tilde{x},x}[T^{\text{first entry}} = T | X_0 = v_0]. \tag{2.7}$$

By reversibility, $(T^{\text{first entry}}, X_0)$ and $(T^{\text{last exit}}, X_0)$ have the same distribution with respect to $Q_{\tilde{x},x}$. Thus,

$$Q_{\tilde{x},x}[T^{\text{first entry}} = T | X_0 = v_0] = Q_{\tilde{x},x}[T^{\text{last exit}} = T | X_0 = v_0]. \tag{2.8}$$

It is well known that $((T^{\text{last exit}}, X_0) \circ \theta^t)_{t \in \mathbb{Z}}$ is a stationary irreducible Markov chain on $\mathscr{T} \times V$ with respect to $Q_{\tilde{x},x}$ with stationary measure

$$\mu_x(\{(T, v)\}) = \left(\prod_{u \in V} x_u \right)^{-1} \cdot x_v \cdot \left(\prod_{e \in T} x_e \right), \quad T \in \mathscr{T}, v \in V. \tag{2.9}$$

This quantity can be interpreted as follows. Let $\vec{T}_v$ denote the directed tree (viewed as a set of directed edges (u, u')) obtained from the undirected tree T by directing all edges $\{u, u'\}$ towards

v, i.e. u' is closer to v in T than u. Note that for every vertex $u \neq v$ there is a unique edge (u, u') leaving u in $\vec{T}_v$. Then,

$$\mu_x(\{(T, v)\}) = \Big(\prod_{u \in V \setminus \{v\}} x_u \Big)^{-1} \cdot \Big(\prod_{e \in T} x_e \Big) = \prod_{(u, u') \in \vec{T}_v} p(u, u'), \tag{2.10}$$

where $p(u, u') = x_{\{u, u'\}} / x_u$ equals the transition probability from u to u'.

The stationary measure is unique up to scaling. In particular, with respect to $Q_{\tilde{x}, x}$, the joint distribution of $T^{\text{last exit}}$ and X_0 equals $c_1(x) \mu_x$ with a normalizing constant $c_1(x)$. Now, $c_1(x) \mu_x = \mu_x^1 \times \mu_x^2$ is the product of the two distributions

$$\mu_x^1(\{T\}) = \text{tree}(x)^{-1} \prod_{e \in T} x_e, \ (T \in \mathcal{T}), \quad \mu_x^2(\{v\}) = \Big(\sum_{u \in V} x_u \Big)^{-1} x_v, \ (v \in V). \tag{2.11}$$

It follows that $T^{\text{last exit}}$ and X_0 are independent under $Q_{\tilde{x}, x}$ with laws μ_x^1 and μ_x^2, respectively. We conclude

$$Q_{\tilde{x}, x}[T^{\text{last exit}} = T | X_0 = v_0] = \text{tree}(x)^{-1} \prod_{e \in T} x_e. \tag{2.12}$$

Combining (2.7), (2.8), and (2.12), the claim (2.5) follows. ∎

2.2 Correlation inequality given the asymptotic ratios of visits

Just for bookkeeping reasons, we assign to each edge $e \in E$ a counting direction. Thus, $\mathcal{G}$ is viewed as a directed graph. Let $\Sigma = (\sigma_{v, e})_{v \in V, e \in E}$ denote the signed incidence matrix of $\mathcal{G}$: $\sigma_{v, e} = 1$ if e is an ingoing edge into v, $\sigma_{v, e} = -1$ if e is an outgoing edge from v, and $\sigma_{v, e} = 0$ otherwise. For $x \in W$, we introduce the matrix $D_x := \text{diag}(\sqrt{x_e}, e \in E) \in \mathbb{R}^{E \times E}$, and set $\Sigma_x := \Sigma D_x$. Let $H_x \subseteq \mathbb{R}^E$ denote the range of the transpose $\Sigma_x^\top$ of Σ_x, given by $H_x = \{\Sigma_x^\top y : y \in \mathbb{R}^V\}$. Let $r := \dim H_x$ denote its dimension. Note that $r = \text{rank}(\Sigma) = |V| - 1$ as $x_e > 0$ for all $e \in E$ and the graph $\mathcal{G}$ is connected.

We endow $\mathbb{R}^E$ with the standard Euclidean scalar product $\langle \cdot, \cdot \rangle$. For $e \in E$, the e-th standard unit vector in $\mathbb{R}^E$ is denoted by ε_e. Consider the r-th exterior power $\wedge^r \mathbb{R}^E$. It inherits a scalar product, also denoted by $\langle \cdot, \cdot \rangle$, given by the bilinear extension of

$$\langle \wedge_{i=1}^r u_i, \wedge_{j=1}^r v_j \rangle = \det(\langle u_i, v_j \rangle)_{1 \le i, j \le r}, \tag{2.13}$$

$u_1, \ldots, u_r, v_1, \ldots, v_r \in \mathbb{R}^E$. An orthonormal basis of $\wedge^r \mathbb{R}^E$ is given by $(\varepsilon_S)_{S \subseteq E, |S| = r}$, where $\varepsilon_S := \wedge_{e \in S} \varepsilon_e$, with respect to any prescribed ordering of S.

As H_x is r-dimensional, the linear subspace $\wedge^r H_x$ of $\wedge^r \mathbb{R}^E$ is one-dimensional. Let $\xi_{H_x} \in \wedge^r H_x$ denote a unit vector, $\|\xi_{H_x}\| = 1$; it is unique up to a sign. Then,

$$\sum_{S \subseteq E, |S| = r} \langle \varepsilon_S, \xi_{H_x} \rangle^2 = \|\xi_{H_x}\|^2 = 1. \tag{2.14}$$

Hence, the following definition makes sense:

Definition 2.1 *We define the probability measure P^{H_x} on the sample space $\mathscr{P}(E)$, the power set of E, by*

$$P^{H_x}(\{S\}) = \begin{cases} \langle \varepsilon_S, \xi_{H_x} \rangle^2 & \text{if } S \subseteq E \text{ with } |S| = r, \\ 0 & \text{else.} \end{cases} \tag{2.15}$$

As an illustrative example, consider the triangle graph with vertex set $V = \{1, 2, 3\}$ and set of directed edges $\vec{E} = \{e, f, g\}$, where $e = (1, 2)$, $f = (2, 3)$, and $g = (3, 1)$. The canonical unit vectors in $\mathbb{R}^V$ and $\mathbb{R}^{\vec{E}}$ are denoted by ε_1, ε_2, ε_3 and ε_e, ε_f, ε_g, respectively. In this example, H_x is the two-dimensional space spanned by $\sqrt{x_f}\varepsilon_f - \sqrt{x_e}\varepsilon_e$ and $\sqrt{x_g}\varepsilon_g - \sqrt{x_f}\varepsilon_f$, and

$$\pm\xi_{H_x} = \frac{(\sqrt{x_f}\varepsilon_f - \sqrt{x_e}\varepsilon_e) \wedge (\sqrt{x_g}\varepsilon_g - \sqrt{x_f}\varepsilon_f)}{\|(\sqrt{x_f}\varepsilon_f - \sqrt{x_e}\varepsilon_e) \wedge (\sqrt{x_g}\varepsilon_g - \sqrt{x_f}\varepsilon_f)\|}$$

$$= \frac{\sqrt{x_f x_g}\varepsilon_f \wedge \varepsilon_g + \sqrt{x_e x_f}\varepsilon_e \wedge \varepsilon_f - \sqrt{x_e x_g}\varepsilon_e \wedge \varepsilon_g}{\sqrt{\text{tree}(x)}}, \tag{2.16}$$

where $\text{tree}(x) = x_e x_f + x_e x_g + x_f x_g$.

The following lemma is implicitly contained in Lyons [Lyo03], using his Remark 5.6. However, in order to make things more explicit, we briefly sketch below the main steps to get it.

Lemma 2.2 P^{H_x} *equals the conditional distribution of* $T^{\text{first entry}}$ *given* $x^{\text{arv}} = x$:

$$P^{H_x}(\{T\}) = P_a[T^{\text{first entry}} = T \mid x^{\text{arv}} = x] = \text{tree}(x)^{-1} \prod_{e \in T} x_e \, 1_{\{T \in \mathcal{T}\}} \tag{2.17}$$

for $T \subseteq E$ *with* $\text{tree}(x)$ *defined in (1.15).*

For any matrix A, we denote by $\wedge A$ the corresponding linear map on exterior powers, which is the linear extension of $\wedge A(\wedge_{i=1}^r u_i) = \wedge_{i=1}^r (Au_i)$ for vectors $u_1, \ldots, u_r$.

Lemma 2.3 *(a)* $\wedge \Sigma_x(\varepsilon_S) = 0$ *if* $S \subseteq E$ *with* $|S| = r$ *and* $S \notin \mathcal{T}$.

(b) For all $S, T \in \mathcal{T}$, *one has*

$$\frac{\wedge \Sigma_x(\varepsilon_S)}{\prod_{e \in S} \sqrt{x_e}} = \pm \frac{\wedge \Sigma_x(\varepsilon_T)}{\prod_{e \in T} \sqrt{x_e}} \neq 0. \tag{2.18}$$

In the above example, for $S = \{e, f\}$ and $T = \{f, g\}$, one has

$$\wedge \Sigma_x(\varepsilon_S) = \Sigma_x \varepsilon_e \wedge \Sigma_x \varepsilon_f = \sqrt{x_e}(\varepsilon_2 - \varepsilon_1) \wedge \sqrt{x_f}(\varepsilon_3 - \varepsilon_2)$$

$$= \sqrt{x_e x_f}(\varepsilon_1 \wedge \varepsilon_2 - \varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_3) \tag{2.19}$$

and similarly $\wedge \Sigma_x(\varepsilon_T) = \sqrt{x_f x_g}(\varepsilon_1 \wedge \varepsilon_2 - \varepsilon_1 \wedge \varepsilon_3 + \varepsilon_2 \wedge \varepsilon_3)$.

Proof of Lemma 2.3. Since $\Sigma_x \varepsilon_e = \sqrt{x_e}\Sigma\varepsilon_e$, we have $\wedge \Sigma_x(\varepsilon_S) = (\prod_{e \in S} \sqrt{x_e}) \cdot \wedge \Sigma(\varepsilon_S)$; thus the lemma follows immediately from its special case all $x_e = 1$. In this special case, one can see it as follows:

(a) As $|S| = r$ and S is not a spanning tree, S contains a cycle C. Summing up $\Sigma\varepsilon_e$ over $e \in C$ with appropriate signs, one gets zero. This shows that $\Sigma\varepsilon_e$, $e \in S$, are linearly dependent. As a consequence, the wedge product $\wedge \Sigma(\varepsilon_S)$ of these vectors vanishes.

(b) For all $S, T \in \mathcal{T}$, the collections of vectors $B_S = (\Sigma\varepsilon_e)_{e \in S}$ and $B_T = (\Sigma\varepsilon_e)_{e \in T}$ are bases of the same r-dimensional subspace of $\mathbb{R}^V$, namely of range $\Sigma = \{(y_v)_{v \in V} : \sum_{v \in V} y_v = 0\}$. The claim follows from the observation that the base change matrix transforming B_S to B_T has determinant ± 1.

■

Proof of Lemma 2.2. Since the graph $\mathcal{G}$ is connected, the null space $\ker \Sigma_x^\top = \{y \in \mathbb{R}^V : \Sigma_x^\top y = 0\}$ contains only the vectors $y \in \mathbb{R}^V$ with constant entries. Thus, using range $\Sigma_x = \text{range} \, \Sigma = \{y = (y_v)_{v \in V} \in \mathbb{R}^V : \sum_{v \in V} y_v = 0\}$, we get $\ker \Sigma_x^\top \cap \text{range} \, \Sigma_x = \{0\}$. Combining this with Lemma 2.3, it follows that there is a constant $c_x \neq 0$ not depending on $T \in \mathcal{T}$ such that for all $T \in \mathcal{T}$ one has

$$\xi_{H_x} = \pm c_x \cdot \wedge \Sigma_x^\top \left(\frac{\wedge \Sigma_x(\varepsilon_T)}{\prod_{e \in T} \sqrt{x_e}} \right). \tag{2.20}$$

For $S \subseteq E$ with $|S| = r$ and $T \in \mathcal{T}$, we get

$$\langle \varepsilon_S, \xi_{H_x} \rangle \prod_{e \in T} \sqrt{x_e} = \pm c_x \langle \varepsilon_S, \wedge (\Sigma_x^\top \Sigma_x)(\varepsilon_T) \rangle = \pm c_x \langle \wedge \Sigma_x(\varepsilon_S), \wedge \Sigma_x(\varepsilon_T) \rangle. \tag{2.21}$$

If $S \notin \mathcal{T}$, then $\wedge \Sigma_x(\varepsilon_S) = 0$ by Lemma 2.3(a). This shows that the support of P^{H_x} is contained in $\mathcal{T}$.

On the other hand, if $S \in \mathcal{T}$, we may choose $T = S$ in the representation (2.20) of ξ_{H_x}. Using (2.21), this yields

$$\langle \varepsilon_S, \xi_{H_x} \rangle = \pm c_x \frac{\|\wedge \Sigma_x(\varepsilon_S)\|^2}{\prod_{e \in S} \sqrt{x_e}} = \pm c_x \left(\frac{\|\wedge \Sigma_x(\varepsilon_S)\|}{\prod_{e \in S} \sqrt{x_e}} \right)^2 \prod_{e \in S} \sqrt{x_e}. \tag{2.22}$$

The squared bracket on the right hand side does not depend on S by Lemma 2.3(b). Consequently, we have

$$\langle \varepsilon_S, \xi_{H_x} \rangle^2 = c_x' \prod_{e \in S} x_e \tag{2.23}$$

with a constant $c_x' > 0$ independent of $S \in \mathcal{T}$. Since P^{H_x} is a probability measure, it follows that $c_x' = 1/\text{tree}(x)$. ■

Proof of Theorem 1.2. Let $f, g : \mathscr{P}(E) \to \mathbb{R}$ fulfill the hypotheses in Theorem 1.2. Theorem 6.5 of Lyons [Lyo03] specialized to the r-dimensional subspace H_x of $\mathbb{R}^E$ claims that

$$E^{H_x}[fg] \leq E^{H_x}[f] E^{H_x}[g], \tag{2.24}$$

where E^{H_x} denotes expectation with respect to P^{H_x}. In view of Lemma 2.2 this proves Theorem 1.2. ■

2.3   FKG inequality given the first entry tree

Let G be any finite set. For $x = (x_e)_{e \in G} \in \mathbb{R}^G$ and $y = (y_e)_{e \in G} \in \mathbb{R}^G$, we define $x \vee y = (x_e \vee y_e)_{e \in G}$ and $x \wedge y = (x_e \wedge y_e)_{e \in G}$. Recall that $\mathbb{R}_+ = (0, \infty)$.

Definition 2.4 *Let $\varphi, \psi : \mathbb{R}_+^G \to \mathbb{R}_+$. We say that φ satisfies the FKG assumption if*

$$\varphi(x \vee y) \varphi(x \wedge y) \geq \varphi(x) \varphi(y) \quad \text{holds for all } x, y \in \mathbb{R}_+^G. \tag{2.25}$$

We say that φ and ψ satisfy the Holley assumption if

$$\varphi(x \vee y) \psi(x \wedge y) \geq \varphi(x) \psi(y) \quad \text{holds for all } x, y \in \mathbb{R}_+^G. \tag{2.26}$$

The names for these two assumptions are motivated by the following well-known results:

Lemma 2.5

1. **(FKG inequality, see page 181 in Keane and den Hollander [dHK86])**. *Assume that $\varphi : \mathbb{R}_+^G \to \mathbb{R}_+$ is a probability density satisfying the FKG assumption. Then, the following FKG inequality holds for all bounded increasing functions $f, g : \mathbb{R}_+^G \to \mathbb{R}$:*

$$\int_{\mathbb{R}_+^G} f(x)g(x)\varphi(x)\,dx \geq \int_{\mathbb{R}_+^G} f(x)\varphi(x)\,dx \int_{\mathbb{R}_+^G} g(x)\varphi(x)\,dx. \qquad (2.27)$$

2. **(Holley inequality, see Theorem 3 in Preston [Pre74])**. *Assume that $\varphi, \psi : \mathbb{R}_+^G \to \mathbb{R}_+$ are probability densities satisfying the Holley assumption. Then, the following Holley inequality holds for any bounded increasing function $f : \mathbb{R}_+^G \to \mathbb{R}$:*

$$\int_{\mathbb{R}_+^G} f(x)\varphi(x)\,dx \geq \int_{\mathbb{R}_+^G} f(x)\psi(x)\,dx. \qquad (2.28)$$

For a discussion and generalizations of these inequalities, see also [dHK86].
We recall the tree term $\mathrm{tree}(x)$, $x \in W$, introduced in formula (1.15).

Lemma 2.6 *The function $1/\mathrm{tree} : W \to \mathbb{R}_+$ satisfies the FKG assumption.*

Proof. We prove this lemma with the help of Theorem 1.2. Let $x, y \in W$ be fixed. We introduce $x' = (x'_e)_{e \in E}$, $x'_e := x_e/(x_e \wedge y_e)$ and similarly $y' = (y'_e)_{e \in E}$, $y'_e := y_e/(x_e \wedge y_e)$. Furthermore, we define $f, g : \mathscr{P}(E) \to \mathbb{R}_+$, $f(A) = \prod_{e \in A} x'_e$, $g(A) = \prod_{e \in A} y'_e$. Note that $x', y' \geq 1$ holds. As a consequence, f and g are increasing functions. Set $F = \{e \in E : x_e > y_e\}$ and $G = \{e \in E : x_e < y_e\}$. Obviously, F and G are disjoint, and f restricted to $E \setminus F$ and g restricted to $E \setminus G$ equal 1. As a consequence, f and g are measurable with respect to $\mathscr{F}_F$ and $\mathscr{F}_G$, respectively. Thus, we can apply Theorem 1.2. Using the explicit form (1.14) of the conditional law of $T^{\text{first entry}}$ given $x^{\text{arv}} = x \wedge y$, its claim (1.16) reads as follows

$$\sum_{T \in \mathscr{T}} f(T)g(T)\frac{\prod_{e \in T}(x_e \wedge y_e)}{\mathrm{tree}(x \wedge y)} \leq \sum_{T \in \mathscr{T}} f(T)\frac{\prod_{e \in T}(x_e \wedge y_e)}{\mathrm{tree}(x \wedge y)} \sum_{S \in \mathscr{T}} g(S)\frac{\prod_{e \in S}(x_e \wedge y_e)}{\mathrm{tree}(x \wedge y)}. \qquad (2.29)$$

Now, the following hold:

$$\sum_{T \in \mathscr{T}} f(T)g(T)\prod_{e \in T}(x_e \wedge y_e) = \mathrm{tree}(x \vee y) \qquad (2.30)$$

and

$$\sum_{T \in \mathscr{T}} f(T)\prod_{e \in T}(x_e \wedge y_e) \sum_{S \in \mathscr{T}} g(S)\prod_{e \in S}(x_e \wedge y_e) = \mathrm{tree}(x)\,\mathrm{tree}(y). \qquad (2.31)$$

Thus, (2.29) can be rewritten as

$$\mathrm{tree}(x \wedge y)\,\mathrm{tree}(x \vee y) \leq \mathrm{tree}(x)\,\mathrm{tree}(y). \qquad (2.32)$$

This yields the claim. ∎

Lemma 2.7 *For any $T \in \mathcal{T}$, the function $x \mapsto \phi_{v_0,a}(T,x)$ satisfies the FKG assumption.*

Proof. It is enough to show that $x \mapsto \phi_{v_0,a}(T,x)$ can be written as a product of finitely many non-negative functions which satisfy the FKG assumption.

Lemma 2.6 implies that $1/\sqrt{\text{tree}}$ satisfies the FKG assumption. Clearly, for any $e \in E$ and any $\alpha \in \mathbb{R}$, $x \mapsto x_e^\alpha$ satisfies the FKG assumption with equality. It remains to show that for any $v \in V$ the map $x \mapsto 1/x_v$ satisfies the FKG assumption. We need to show $x_v y_v \geq (x \wedge y)_v (x \vee y)_v$ for $x, y \in \mathbb{R}_+^E$. Let $E_v = \{e \in E : v \in E\}$ denote the set of edges adjacent to v. We set

$$a = \sum_{e \in E_v} x_e 1_{\{x_e \leq y_e\}}, \qquad\qquad A = \sum_{e \in E_v} y_e 1_{\{x_e \leq y_e\}}, \qquad (2.33)$$

$$b = \sum_{e \in E_v} y_e 1_{\{x_e > y_e\}}, \qquad\qquad B = \sum_{e \in E_v} x_e 1_{\{x_e > y_e\}}. \qquad (2.34)$$

In particular, we have $A \geq a$, $B \geq b$, and $x_v = a+B$, $y_v = A+b$, $(x \wedge y)_v = a+b$, $(x \vee y)_v = A+B$. This yields the claim as follows: $x_v y_v - (x \wedge y)_v (x \vee y)_v$ equals

$$(a+B)(A+b) - (a+b)(A+B) = (A-a)(B-b) \geq 0. \qquad (2.35)$$

Thus, $1/x_v^\alpha$ satisfies the FKG assumption as well for any $\alpha > 0$. ∎

Proof of Theorem 1.3. Let $F \subseteq E$, $T \in \mathcal{T}$, and $y \in W$. The conditional density of $x_{E \setminus F}^{\text{arv}}$ given $T^{\text{first entry}} = T$ and $x_F^{\text{arv}} = y_F$ is given by $\mathbb{R}_+^{E \setminus F} \ni x \mapsto \phi_{v_0,a}(T,x,y_F)/Z(T,y_F)$ with a normalizing constant $Z(T,y_F)$. Since $W \ni x \mapsto \phi_{v_0,a}(T,x)$ satisfies the FKG assumption, so does the function $\mathbb{R}_+^{E \setminus F} \ni x \mapsto \phi_{v_0,a}(T,x,y_F)$. Hence, the claim follows from Lemma 2.5(a). ∎

Proof of Corollary 1.4. For $e \in E$, we introduce

$$\xi_e = \xi_e(x) := \frac{x_e}{\sqrt{\prod_{v \in e} x_v}}; \qquad (2.36)$$

the product in the denominator is taken over the two vertices incident to e.

Take $T \in \mathcal{T}$ and $y_F \in \mathbb{R}_+^F$. Recall the explicit form (1.13) of the joint law $\mathbb{P}_{v_0,a,e_0}$ of $(T^{\text{first entry}}, x^{\text{arv}})$ with respect to P_a. Note that in the explicit form (1.9) of the function $\phi_{v_0,a}$, the initial weights a appear only as a factor $\prod_{e \in E} \xi_e^{a_e}$. As a consequence, we have the following density of $\mathbb{P}_{v_0,a,e_0}$ with respect to $\mathbb{P}_{v_0,b,e_0}$:

$$\frac{d\mathbb{P}_{v_0,a,e_0}}{d\mathbb{P}_{v_0,b,e_0}}(T,x) = Z_{a,b}^{-1} \prod_{e \in E} \xi_e(x)^{a_e - b_e} = Z_{a,b}^{-1} \prod_{e \in F} \xi_e(x)^{a_e - b_e}, \quad (T,x) \in \mathcal{T} \times W_{e_0}, \qquad (2.37)$$

with a normalizing constant $Z_{a,b} > 0$. It follows that

$$E_a[f(x^{\text{arv}}) \mid T^{\text{first entry}} = T, x_F^{\text{arv}} = y_F]$$
$$= \frac{E_b\left[f(x^{\text{arv}}) \prod_{e \in F} \xi_e(x^{\text{arv}})^{a_e - b_e} \mid T^{\text{first entry}} = T, x_F^{\text{arv}} = y_F\right]}{E_b\left[\prod_{e \in F} \xi_e(x^{\text{arv}})^{a_e - b_e} \mid T^{\text{first entry}} = T, x_F^{\text{arv}} = y_F\right]}$$
$$\geq E_b\left[f(x^{\text{arv}}) \mid T^{\text{first entry}} = T, x_F^{\text{arv}} = y_F\right]. \qquad (2.38)$$

For the last inequality, we have used Theorem 1.3 with the two functions f and $x \mapsto \prod_{e \in F} \xi_e(x)^{a_e - b_e}$, which are both increasing on $E \setminus F$. ∎

Proof of Theorem 1.5.   Fix $T \in \mathcal{T}$ and $x_F \leq y_F$ in $(0, \infty)^F$. We define the functions $\varphi(s) = \phi_{v_0,a}(T, y_F, s)$ and $\psi(s) = \phi_{v_0,a}(T, x_F, s)$, $s \in (0, \infty)^{E \setminus F}$. Up to normalizing constants, φ is the conditional density of $x_{E \setminus F}^{\mathrm{arv}}$ given $T^{\mathrm{first\ entry}} = T$ and $x_F^{\mathrm{arv}} = y_F$ and ψ is the conditional density of $x_{E \setminus F}^{\mathrm{arv}}$ given $T^{\mathrm{first\ entry}} = T$ and $x_F^{\mathrm{arv}} = x_F$.

Since $x \mapsto \phi_{v_0,a}(T, x)$ satisfies the FKG assumption by Lemma 2.7, we get for all $s, t \in (0, \infty)^{E \setminus F}$

$$
\begin{aligned}
\varphi(s \vee t)\psi(s \wedge t) &= \phi_{v_0,a}(T, x_F \vee y_F, s \vee t)\phi_{v_0,a}(T, x_F \wedge y_F, s \wedge t) \\
&\geq \phi_{v_0,a}(T, y_F, s)\phi_{v_0,a}(T, x_F, t) = \varphi(s)\psi(t).
\end{aligned}
\tag{2.39}
$$

Hence, φ and ψ satisfy the Holley assumption and Lemma 2.5(b) implies the claim. $\blacksquare$

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