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ON THE EXPECTED EXIT TIME OF PLANAR BROWNIAN MOTION FROM SIMPLY CONNECTED DOMAINS

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Abstract

In this note, we explore applications of a known lemma which relates the expected exit time of Brownian motion from a simply connected domain with the power series of a conformal map into that domain. We use the lemma to calculate the expected exit time from a number of domains, and in the process describe a probabilistic method for summing certain series. In particular, we give a proof of Euler's classical result that $\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$. We also show how the relationship between the power series and the Brownian exit time gives several immediate consequences when teamed with a deep result of de Branges concerning the coefficients of power series of normalized conformal maps. We conclude by stating an extension of the lemma in question to domains which are not simply connected.

1 Introduction and statement of results

A domain U in the complex plane $\mathbb{C}$ is *simply connected* if any curve in U can be homotopically deformed to a point without leaving U. Equivalently, U is simply connected if U^c is connected, where the complement is taken in the Riemann sphere, $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Such domains are of paramount importance in complex analysis, due in large part to the Riemann Mapping Theorem, which states that for any simply connected domain $U \subsetneq \mathbb{C}$ and point $a \in U$ there is a conformal map from $\mathbb{D}$ onto U with $f(0) = a$. This fact can be connected to probabilistic considerations in the following way. The expected length of time that it takes a Brownian motion starting at the point a to leave U gives some sort of measure of the size of U and the distance between a and U^c. Similar information is carried by the conformal map f, and we will be interested in the connection between the exit time and f. Let $\tau(U)$ be the first time that a Brownian motion hits U^c, and we will use the standard notation E_a to denote expectation conditioned on $B_0 = a$ almost surely. We then have the following lemma, which originally appeared in [3, Lem. 1.1].

Lemma 1. *Suppose $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is conformal on $\mathbb{D}$. Then, for $0 < r \leq 1$ we have*

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$$E_{f(0)}[\tau(f(r\mathbb{D}))] = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^2 r^{2n} \tag{1}$$

In particular,

$$E_{f(0)}[\tau(f(\mathbb{D}))] = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^2 \tag{2}$$

Proof: The proof in [3] uses Green's function. We can, however, give another simple proof as follows. Suppose first that $r < 1$. Applying the optional stopping theorem to the martingale $|B_t|^2 - 2t$ on the bounded domain $f(r\mathbb{D})$ gives

$$2E_{f(0)}[\tau(f(r\mathbb{D}))] = E_{f(0)}[|B_{\tau(f(r\mathbb{D}))}|^2] \tag{3}$$

However, a conformal image of Brownian motion is a time changed Brownian motion(see [10]), so we have

$$\begin{aligned} E_{f(0)}[|B_{\tau(f(r\mathbb{D}))}|^2] &= E_0[|f(B_{\tau(r\mathbb{D})})|^2] \\ &= \frac{1}{2\pi}\int_0^{2\pi}|f(re^{i\theta})|^2 d\theta \\ &= \sum_{n=1}^{\infty}|a_n|^2 r^{2n} \end{aligned} \tag{4}$$

where we have employed Parseval's identity for analytic functions(see [17, Thm. 10.22]), which is based on the interpretation of a complex power series as a trigonometric series. This completes the proof for $r < 1$. In order to set $r = 1$ to obtain the final statement, we note first that if $\sum_{n=1}^{\infty}|a_n|^2 = \infty$ then $E_{f(0)}[\tau(f(r\mathbb{D}))] \longrightarrow \infty$ as $r \longrightarrow 1$, so that (2) holds. On the other hand, if $\sum_{n=1}^{\infty}|a_n|^2 < \infty$ then it is known that f can be extended to a radial limit function $f(e^{i\theta})$ on $\delta\mathbb{D}$ for a.e. θ(see [17, Thm. 17.10] or [10, Sec. 6.5]) such that $\lim_{r\nearrow 1}\int_0^{2\pi}|f(re^{i\theta}) - f(e^{i\theta})|^2 d\theta = 0$ and $\sum_{n=1}^{\infty}|a_n|^2 = \frac{1}{2\pi}\int_0^{2\pi}|f(e^{i\theta})|^2 d\theta$. The complete result follows. $\square$

The purpose of this paper is to give a number of instances in which this lemma can be invoked in order to yield results. In the next section we will give a number of explicit conformal maps to which Lemma 1 can be applied with relative ease in order to calculate expected exit times. This leads to an additional application, namely, the summation of certain series. That is, Lemma 1 expresses the expected exit time as a sum. If we have a different way of evaluating the expected exit time, then we have obtained a value for the sum. Perhaps the simplest and most notable example of this is that we will use the expected exit time from an infinite strip in order to derive Euler's result that $\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$. We will also see that the expected exit time from an equilateral triangle leads to a reduction for a difficult hypergeometric sum. In Section 3, we apply the lemma to the study of normalized conformal maps on the unit disk. A great deal is known about the coefficients of the power series for such maps, and Lemma 1 translates well into this setting. We conclude by discussing briefly the application of Lemma 1 to cases in which the map f is analytic but not conformal.

2 Consequences of Lemma 1 applied to specific conformal maps

We begin with the proof of Euler's result.

Proposition 1.

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6} \tag{5}$$

Proof: We will prove the corresponding statement obtained by summing only the odd terms, namely

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \frac{\pi^2}{8} \tag{6}$$

This is equivalent to (5), as

$$\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = (1-\frac{1}{4})\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{3}{4}\sum_{n=1}^{\infty} \frac{1}{n^2} \tag{7}$$

Let W_t be a one-dimensional Brownian motion with $W_0 = 0$ a.s., and let $T = \inf_{t>0}\{|W_t| = \frac{\pi}{4}\}$. We will calculate $E[T]$ in two different ways. The first way is quite standard(see, for example, [12, Ex. 7.5]). We apply the Optional Stopping Theorem to the martingale $W_t^2 - t$ to obtain

$$E[T] = E[W_T^2] = \frac{\pi^2}{16} \tag{8}$$

We now calculate $E[T]$ using Lemma 1. W_t may be taken to be the real part of our two dimensional Brownian motion B_t, and it is therefore clear that $E[T] = E_0[\tau(U)]$, where $U = \{\frac{-\pi}{4} < \text{Re } z < \frac{\pi}{4}\}$. We need then to find a conformal map $f(z)$ mapping $\mathbb{D}$ onto U with $f(0) = 0$. The function

$$\tan z = \frac{\sin z}{\cos z} = -i\frac{e^{2iz}-1}{e^{2iz}+1} \tag{9}$$

maps U conformally to $\mathbb{D}$. This can be seen by noting that the function

$$x + iy \longrightarrow e^{2i(x+iy)} = e^{-2y+2ix}$$

maps U conformally to $\{\text{Re } z > 0\}$, and then that the Möbius transformation $z \longrightarrow -i(\frac{z-1}{z+1})$ maps $\{\text{Re } z > 0\}$ conformally to $\mathbb{D}$. We conclude that the principal branch of $\tan^{-1} z$ maps $\mathbb{D}$ conformally to U. $\tan^{-1} z$ admits the Taylor series expansion

$$\tan^{-1} z = z - \frac{z^3}{3} + \frac{z^5}{5} - \ldots = \sum_{n=1}^{\infty} \frac{(-1)^{n+1} z^{2n-1}}{2n-1} \tag{10}$$

Thus, by Lemma 1,

$$E[T] = \frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} \tag{11}$$

Equating (8) and (11) yields (6). □

Here are some further examples of conformal maps with power series to which Lemma 1 can be applied.

Example 1 : Consider the disc $r\mathbb{D}$ for any $r > 0$, and let $a \in r\mathbb{D}$. It may be checked that

$$f(z) = r\frac{z + \frac{a}{r}}{1 + \frac{\bar{a}}{r}z} = r\frac{rz + a}{r + \bar{a}z} \tag{12}$$

is a conformal map sending $\mathbb{D}$ to $r\mathbb{D}$ and 0 to a. To find the power series for f we expand

$$\begin{aligned} r\frac{z + \frac{a}{r}}{1 + \frac{\bar{a}}{r}z} &= r(z + \frac{a}{r})(1 - \frac{\bar{a}z}{r} + (\frac{\bar{a}z}{r})^2 - \dots) \\ &= a + (r^2 - |a|^2)\sum_{n=1}^{\infty}\frac{(-1)^{n-1}\bar{a}^{n-1}z^n}{r^n} \end{aligned} \tag{13}$$

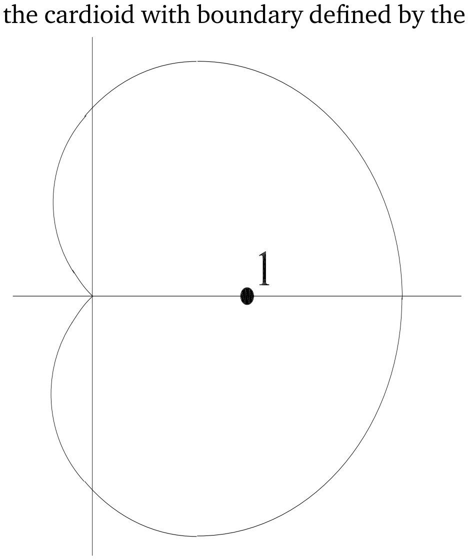
Lemma 1 now gives

$$E_a[\tau(r\mathbb{D})] = \frac{1}{2}(r^2 - |a|^2)^2\sum_{n=1}^{\infty}\frac{|a|^{2n-2}}{r^{2n}} = \frac{(r^2 - |a|^2)^2}{2r^2}\frac{1}{1 - \frac{|a|^2}{r^2}} = \frac{1}{2}(r^2 - |a|^2)$$

This is well known; see for example [16, Ex. 7.4.2].

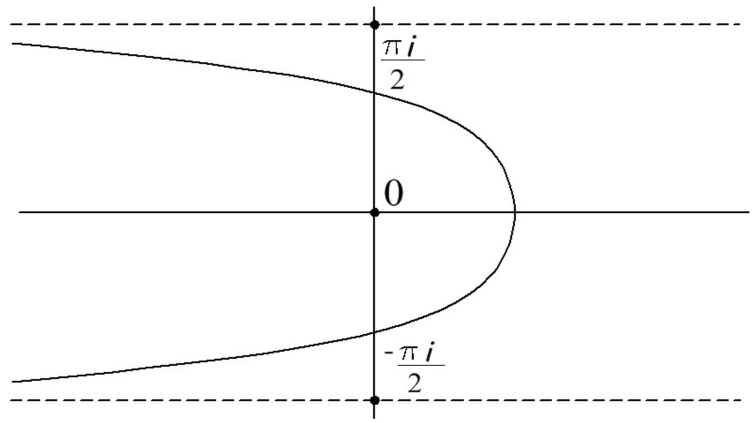
Example 2 : The function $f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \dots$ maps $\mathbb{D}$ conformally to $\{\mathrm{Re}\, z > -\frac{1}{2}\}$. Lemma 1 gives $E_0[\tau(\{\mathrm{Re}\, z > -\frac{1}{2}\})] = \infty$, and it is easy to see that $E_z[\tau(U)] = \infty$ whenever U is a half-plane and $z \in U$. Letting $W_t = \mathrm{Re}\, B_t$ we recover a known result on one-dimensional Brownian motion, that if $T = \inf_{t>0}\{W_t = a\}$ with $a \neq 0$ then $E[T] = \infty$.

Example 3 : Let U be the cardioid with boundary defined by the polar equation $r = 2(1+\cos\theta)$.



This is the conformal image under z^2 of the disc $\{|z-1| = 1\}$. The conformal map from $\mathbb{D}$ to U mapping 0 to 1 is given by $f(z) = (z+1)^2 = 1 + 2z + z^2$. Applying Lemma 1 we get $E_1[\tau(U)] = \frac{5}{2}$.

Example 4 : Let γ denote the curve in $\mathbb{R}^2$ defined by $e^x = 2\cos y$ for $\frac{-\pi}{2} < y < \frac{\pi}{2}$.



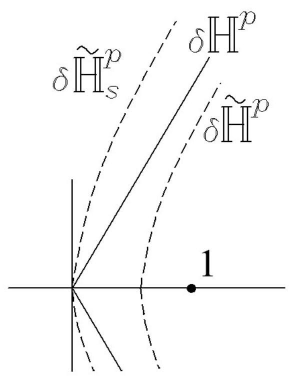
This curve has been referred to as the "catenary of equal resistance". Let U be the component of γ^c which contains 0. Then U is the conformal image of $\mathbb{D}$ under the map

$$f(z) = \log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \dots \tag{14}$$

Applying Lemma 1 and Proposition 1 we obtain $E_0[\tau(U)] = \frac{1}{2}\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{12}$.

Example 5 : Let $\mathbb{H} = \{\text{Re } z > 0\}$ be the right half-plane[2] , and let $\mathbb{H}^p = \{|Arg(z)| < \frac{\pi p}{2}\}$ for $p \leq 1$, where $Arg(z)$ is the principal branch of the argument function taking values in $(-\pi, \pi]$. $\mathbb{H}^p$ is the infinite wedge of width πp centered at the positive real axis, and $\mathbb{H}^1 = \mathbb{H}$. It was shown in [18], among other things, that $E_1[\tau(\mathbb{H}^p)] < \infty$ if $p < \frac{1}{2}$, and $E_1[\tau(\mathbb{H}^p)] = \infty$ if $p \geq \frac{1}{2}$. We will derive this result and find bounds for $E_1[\tau(\mathbb{H}^p)]$ using Lemma 1. The conformal map from $\mathbb{D}$ to $\mathbb{H}^p$ is $g(z) = \frac{(1+z)^p}{(1-z)^p}$, but the Taylor series expansion for g appears to be unwieldy for arbitrary p, so we will simplify somewhat. Define the principal branch of z^p on $\mathbb{H}$, where $Arg(z^p) = pArg(z)$. Then the inverse $z^{1/p}$ is well defined on $\mathbb{H}^p$. Let $\tilde{\mathbb{H}}^p = \{Re(z^{1/p}) > 1/2\}$. $\mathbb{H}^p$ is then the image of $\{\text{Re } z > \frac{1}{2}\}$ under z^p. The relationship between $\mathbb{H}^p$ and $\tilde{\mathbb{H}}^p$ is shown below.

[2]This is at odds with standard practice in complex analysis, where usually $\mathbb{H} = \{\text{Im } z > 0\}$ is the upper half-plane. It is convenient for our purposes, however.



It is clear that $\tilde{\mathbb{H}}^p \subseteq \mathbb{H}^p$, and it is also easy to check that $\mathbb{H}^p \subseteq \tilde{\mathbb{H}}^p_s := \{\frac{2^p-1}{2^p}z + \frac{1}{2^p} \in \tilde{\mathbb{H}}^p\}$. The conformal map from $\mathbb{D}$ to $\tilde{\mathbb{H}}^p$ is given by $f(z) = \frac{1}{(1-z)^p}$, and the conformal map from $\mathbb{D}$ to $\tilde{\mathbb{H}}^p_s$ is given by $\frac{2^p}{2^p-1}(f(z) - 1/2^p)$. We have

$$\frac{d^n}{dz^n}\frac{1}{(1-z)^p}\bigg|_{z=0} = p(p+1)\dots(p+n-1) \tag{15}$$

The expression $(p)_n := p(p+1)\dots(p+n-1)$ with $(p)_0 = 1$ is known as the *Pochhammer symbol*, and we see that $f(z) = \sum_{n=0}^{\infty} \frac{(p)_n z^n}{n!}$. Applying Lemma 1 we obtain

$$E_1[\tau(\tilde{\mathbb{H}}^p)] = \frac{1}{2}\sum_{n=1}^{\infty}\frac{(p)_n^2}{(n!)^2} \tag{16}$$

This series can be evaluated explicitly. Using the notation in [14] it is given by ${}_2F_1(p,p;1;1) - 1$, where ${}_2F_1$ refers to the hypergeometric function. Applying Euler's integral formula [14, Sec. 3.6] and using the standard definitions for the β and Γ functions, we calculate

$$E_1[\tau(\tilde{\mathbb{H}}^p)] = \frac{1}{2}({}_2F_1(p,p;1;1) - 1) = \frac{1}{2}\Big(\frac{\Gamma(1)\beta(p,1-2p)}{\Gamma(1-p)\Gamma(p)} - 1\Big) \tag{17}$$

Similarly, we have

$$E_1[\tau(\tilde{\mathbb{H}}^p_s)] = \frac{2^{2p}}{2(2^p-1)^2}\Big(\frac{\Gamma(1)\beta(p,1-2p)}{\Gamma(1-p)\Gamma(p)} - 1\Big) \tag{18}$$

We obtain

$$\frac{2^{2p}}{2(2^p-1)^2}\Big(\frac{\Gamma(1)\beta(p,1-2p)}{\Gamma(1-p)\Gamma(p)} - 1\Big) < E_1[\tau(\mathbb{H}^p)] < \frac{1}{2}\Big(\frac{\Gamma(1)\beta(p,1-2p)}{\Gamma(1-p)\Gamma(p)} - 1\Big) \tag{19}$$

This is finite for $p < \frac{1}{2}$, but infinite for $p \geq \frac{1}{2}$, as the integral defining $\beta(p,1-2p)$ diverges at $p = \frac{1}{2}$. This proves the given statement. Note that [5] also used the Hardy norm of f in a similar manner to deduce a more general result.

Example 6 Let m be an integer greater than 2 and set $\omega = e^{\frac{2\pi i}{m}}$. Let U_m be the regular m-gon with vertices at $1, \omega, \omega^2, \dots, \omega^{m-1}$. We will calculate $E_0[\tau(U_m)]$. Consider the Schwarz-Christoffel mapping given by

$$g(z)=\int_0^z \frac{d\zeta}{(1-\zeta)^{2/m}(1-\omega\zeta)^{2/m}\dots(1-\omega^{m-1}\zeta)^{2/m}}=\int_0^z \frac{d\zeta}{(1-\zeta^m)^{2/m}} \tag{20}$$

$g(z)$ is a conformal mapping of the unit disc onto an m-gon with all angles $\pi^{(m-2)/m}$(see [8] for details), and symmetry arguments show that the image is in fact a regular m-gon with vertices $W,\omega W,\dots,\omega^{m-1}W$ for some $W>0$. The points $1,\omega,\dots,\omega^{m-1}$ map to these vertices, and we can calculate

$$\begin{aligned}W=g(1)&=\int_0^1 \frac{dx}{(1-x^m)^{2/m}}=\frac{1}{m}\int_0^1 u^{-(m-1)/m}(1-u)^{-2/m}du\\&=\frac{1}{m}\beta(1/m,(m-2)/m)\end{aligned} \tag{21}$$

Setting $f(z)=\frac{1}{W}g(z)$, we obtain our conformal map from $\mathbb{D}$ to U_m. Recall from Example 5 that $\frac{1}{(1-z)^{2/m}}=\sum_{n=0}^\infty \frac{(2/m)_n z^n}{n!}$. We see that

$$\begin{aligned}f(z)&=\frac{1}{W}\int\sum_{n=0}^\infty \frac{(2/m)_n z^{mn}}{n!}=\frac{1}{W}\sum_{n=0}^\infty \frac{(2/m)_n z^{mn+1}}{n!(mn+1)}\\&=\frac{z}{W}\sum_{n=0}^\infty \frac{(1/m)_n(2/m)_n z^{mn}}{((m+1)/m)_n n!}=\frac{z}{W}{}_2F_1(1/m,2/m;(m+1)/m;z^m)\end{aligned} \tag{22}$$

where we have used the identity $(1/m)_n(mn+1)=((m+1)/m)_n$. Theorem 1 gives

$$\begin{aligned}E_0[\tau(U_m)]&=\frac{1}{2W^2}\sum_{n=0}^\infty \frac{(2/m)_n^2(1/m)_n^2}{(n!)^2((m+1)/m)_n^2}\\&=\frac{1}{2W^2}{}_4F_3(1/m,1/m,2/m,2/m;(m+1)/m,(m+1)/m,1;1)\\&={}_4F_3(1/m,1/m,2/m,2/m;(m+1)/m,(m+1)/m,1;1)\\&\qquad\times\frac{m^2}{2\beta(1/m,(m-2)/m)^2}\end{aligned} \tag{23}$$

In the cases $m=3,4$ we may compare this with known results. For the equilateral triangle, applying [1, Thm. 1] gives $E_0[\tau(U_3)]=1/6$. Computer approximation shows agreement with (23) evaluated at $m=3$. We arrive at the following value for the hypergeometric sum.

$${}_4F_3(1/3,1/3,2/3,2/3;4/3,4/3,1;1)=\frac{\beta(1/3,1/3)^2}{27} \tag{24}$$

For the case $m=4$, the square, (23) gives

$$E_0[\tau(U)]={}_4F_3(1/4,1/4,1/2,1/2;5/4,5/4,1;1)\times\frac{8}{\beta(1/4,1/2)^2}\approx .294685 \tag{25}$$

This agrees with the approximation given in [11, Tab. 10][3]. This approximation was based on an explicit expression obtained from [13][4], and equating that expression with (25) we obtain the following strange identity.

$$\begin{aligned} &{}_4F_3(1/4,1/4,1/2,1/2;5/4,5/4,1;1) \times \frac{1}{\beta(1/4,1/2)^2} \\ &= \frac{8}{\pi^4} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{(-1)^{m+n}}{(2m-1)(2n-1)((2m-1)^2+(2n-1)^2)} \end{aligned} \tag{26}$$

The result (23) for $m \geq 5$ may be new.

3 Lemma 1 and the theory of normalized conformal maps on $\mathbb{D}$

The *Schlicht class* $\mathscr{S}$ is the set of all conformal functions f on $\mathbb{D}$ normalized so that

$$f(0) = 0, f'(0) = 1.$$

If a domain U is equal to $f(\mathbb{D})$ for some $f \in \mathscr{S}$, we will call U a *Schlicht domain*. Clearly, if U is any simply connected domain smaller than $\mathbb{C}$ itself and $z \in U$ we may find a linear map sending z to 0 and U to a Schlicht domain. In this way the study of $\mathscr{S}$ is no less general than the study of arbitrary simply connected domains. The most notable members of $\mathscr{S}$ for our purposes are the identity function, $I(z) = z$, and the Koebe function, $K(z) = \frac{z}{(1-z)^2} = z + 2z^2 + 3z^3 + \ldots$. Statements concerning functions in $\mathscr{S}$ are generally invariant under rotations, and though $I(z)$ remains unchanged when conjugated by a rotation, $K(z)$ does not. For this reason, we let $\mathscr{K}$ denote the Koebe function and its rotations, that is, $\mathscr{K} = \{e^{-i\alpha}K(e^{i\alpha}z) : \alpha \in \mathbb{R}\}$. In a number of different ways, the identity function, with image $I(\mathbb{D}) = \mathbb{D}$, is considered to be the smallest in $\mathscr{S}$. On the other hand, the members of $\mathscr{K}$, whose images are rotations of $K(\mathbb{D}) = \mathbb{C}\backslash(-\infty,-1/4]$, are considered to be the largest. We may view the situation intuitively as follows. The normalization creates an equivalence between the notion of "size" and the notion of "closeness of 0 to the boundary". The disc is the only domain in which 0 is equally close to every boundary point. In that sense, 0 is closer to the boundary in the unit disc than in any other Schlicht domain. Conversely, the smallest nonempty complement of a simply connected domain seems likely to be a half-line $(-\infty, 0]$, and a point x on $(0,\infty)$ should be farther from the boundary than $e^{i\theta}x$, $\theta \in (0,\pi)$, for instance. Arguing in this manner indicates that $K(\mathbb{D}) = \mathbb{C}\backslash(-\infty,-1/4]$ should be the Schlicht domain for which 0 is farthest from the boundary. This intuition may be useful as far as it goes, but a far more sophisticated statement is the following celebrated theorem of de Branges, which was originally conjectured by Bieberbach.

Theorem 1. *If $f \in \mathscr{S}$ with $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$, then $|a_n| \leq n$ for all $n \geq 2$. If for any such n we have $|a_n| = n$, then $f \in \mathscr{K}$.*

Thus, out of all functions in $\mathscr{S}$, the Koebe function and its rotations have the largest derivatives at 0. The opposite is true for $I(z)$, but this is trivial since $I(z)$ is the only Schlicht function with

[3] The value in [11, Tab. 10] must be doubled, as the calculations there were for a square with unit side length rather than our normalization.

[4] There is a misprint in [13]; the correct formula is given in [11].

$f^{(n)}(0) = 0$ for $n \geq 2$. The highly nontrivial proof of Theorem 1, which is considered to be one of the major theorems in complex analysis, can be found in [7]. As was discussed in the introduction to this paper, the exit time of Brownian motion from a domain gives a measure of the size of the domain and distance of the initial point from the boundary. In view of the arguments given above, it is therefore quite natural to study the exit time of Brownian motion from Schlicht domains. In [6], the question was raised of finding a way in which Brownian motion identifies the Koebe domain as the largest Schlicht domain and the unit disk as the smallest. An immediate obstacle in this problem is that Lemma 1 shows that $E_0[\tau(K(\mathbb{D}))] = \infty$, and the same for many other Schlicht domains(see Examples 2 and 5, for instance). In light of this difficulty, and in order to allow Lemma 1 to be able to say something nontrivial, we will use $f(r\mathbb{D})$ as a bounded approximation to $f(\mathbb{D})$ for all $f \in \mathcal{S}$. This is justified by the fact that $f(r\mathbb{D}) \longrightarrow f(\mathbb{D})$ in the sense of Carathéodory as $r \longrightarrow 1$(see [9, Ch. 3]), and the approximation is in some sense uniform between domains. The following shows that Brownian motion leaves $K(r\mathbb{D})$ more slowly and $I(r\mathbb{D})$ more quickly than $f(r\mathbb{D})$, where f is any function in $\mathcal{S}\backslash(\mathcal{K} \cup \{I\})$.

Proposition 2. *If $f \in \mathcal{S}$ and $r \leq 1$, then*

$$E_0[\tau(I(r\mathbb{D}))] \leq E_0[\tau(f(r\mathbb{D}))] \leq E_0[\tau(K(r\mathbb{D}))] \tag{27}$$

If

$$E_0[\tau(f(r\mathbb{D}))] = E_0[\tau(I(r\mathbb{D}))]$$

for any $r \leq 1$ then $f(z) = I(z)$, and if

$$E_0[\tau(f(r\mathbb{D}))] = E_0[\tau(K(r\mathbb{D}))]$$

for any $r < 1$ then $f \in \mathcal{K}$.

This is immediate from de Branges' Theorem by simply applying Lemma 1. We remark that the upper bound of Theorem 2 can also be taken to be the special case $p = 2$ of the following deep result of Baernstein in [2].

Theorem 2. *If $f \in \mathcal{S}, f \notin \mathcal{K}$, $r \in (0,1)$ then*

$$\int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \int_0^{2\pi} |K(re^{i\theta})|^p d\theta \tag{28}$$

for any $p \in (0,\infty)$.

This was first noticed by Betsakos([4]). Returning to the method of applying Lemma 1 to de Branges Theorem, we easily obtain

Proposition 3. *If $f \in \mathcal{S}, f \notin \mathcal{K}$, then $E_0[\tau(K(r\mathbb{D}))] - E_0[\tau(f(r\mathbb{D}))]$ is an increasing function of r which is positive for $r > 0$.*

We may intuitively rephrase this result as "Brownian motion would take longer to exit $K(\mathbb{D})$ than any other Schlicht domain were it not that many of the expected exit times are infinite". This does not seem to follow directly from Baernstein's result.

It is also quite interesting to consider moments of the exit time other than 1. In cite [15], McConnell proved

Theorem 3. *Let Φ be a nonnegative, strictly increasing convex or strictly concave function, where if f is strictly concave then we also require that $\Phi(e^{2x})$ is convex. Assume further that $E_0[\Phi(\tau(\mathbb{D}))] < \infty$. Then, for any $f \in \mathscr{S}, f \neq I$ we have*

$$E_0[\Phi(\tau(\mathbb{D}))] < E_0[\Phi(\tau(f(\mathbb{D})))] \tag{29}$$

In particular, $E_0[\tau(\mathbb{D})^p] < E_0[\tau(f(\mathbb{D}))^p]$ for $0 < p < \infty$.

In fact, McConnell did not prove the strict inequality, but it is straightforward to adjust his proof to include it(apply Parseval's identity in the first string of inequalities in the proof). An analogous statement for $K(z)$ would be wonderful to have, but remains unproved at this point. For instance, a connection is given by Burkholder in [5] between Hardy norms of analytic functions and moments of the exit time of Brownian motion. In view of this, it is tempting to conjecture that the following is a consequence of Theorem 2.

Conjecture. *If $f \in \mathscr{S}, f \notin \mathscr{K}$, $r \in (0,1)$, then*

$$E_0[\tau(f(r\mathbb{D}))^p] < E_0[\tau(K(r\mathbb{D}))^p] \tag{30}$$

for any $p \in (0,\infty)$. The same holds when $r = 1$ and $p \in (0,1/4)$.

Unfortunately, the inequalities in [5] do not seem to be sufficient to prove this result.

4 Extension to arbitrary domains

Although not the main focus of the paper, we would be remiss if we failed to observe that Lemma 1 can be extended to some analytic functions which are not conformal. Examining the proof of the theorem should reveal that the hypothesis of injectivity is not necessary for the statement to hold. Instead, the important property of conformal maps which was used is that $f(B_t)$ leaves $f(\mathbb{D})$ at time $\tau(\mathbb{D},0)$. This is not true of arbitrary analytic maps. For example, let f map $\mathbb{D}$ conformally to the rectangle $V = \{-1 < \mathrm{Re}\, z < 1, -2\pi < \mathrm{Im}\, z < 2\pi\}$. Then $g = e^f$ maps $\mathbb{D}$ onto the annulus $\{\frac{1}{e} < |z| < e\}$. The boundary segments $\ell_1 = \{-1 < \mathrm{Re}\, z < 1, \mathrm{Im}\, z = 2\pi\}$ and $\ell_2 = \{-1 < \mathrm{Re}\, z < 1, \mathrm{Im}\, z = -2\pi\}$ are mapped by e^z to the interior segment

$$\{\frac{1}{e} < \mathrm{Re}\, z < e, \mathrm{Im}\, z = 0\}.$$

We see that for any Brownian path $B_t(\omega)$ such that $B_{\tau(\mathbb{D},z)}(\omega) \in f^{-1}(\ell_1 \cup \ell_2)$ we have

$$g(B_{\tau(\mathbb{D},z)}(\omega)) \in \{\frac{1}{e} < \mathrm{Re}\, z < e, \mathrm{Im}\, z = 0\}.$$

Since $g(B_t)$ does not leave $g(\mathbb{D})$ with probability 1 at time $\tau(\mathbb{D},0)$, Theorem 1 will fail to hold.

With this in mind, let us define an analytic function f on $\mathbb{D}$ to be $B-proper$ if a.s. $f(B_t)$ leaves every compact subset of $f(\mathbb{D})$ as t increases to $\tau(\mathbb{D},0)$. Let a domain be called $B-proper$ if it is the image of a B-proper map on $\mathbb{D}$. The reason for this terminology is that analytic functions with the property that $f(z_n)$ leaves every compact set for every sequence $\{z_n\}$ which approaches $\delta\mathbb{D}$ are commonly referred to as $proper$. It is easy to see that every conformal map is proper. It is also clear that every proper function is B-proper, but the converse is not true, as the example given below shows. With the same proof as Lemma 1 we have the following.

Lemma 2. *Suppose $f(z)=\sum_{n=0}^{\infty} a_n z^n$ is B-proper on $\mathbb{D}$. Then*

$$E_{f(0)}[\tau(f(\mathbb{D}))] = \frac{1}{2}\sum_{n=1}^{\infty}|a_n|^2 \tag{31}$$

We now give an example showing that a function can fail to be proper but still be B-proper. Let $f(z)=e^{tan^{-1}z}$. f maps $\mathbb{D}$ conformally to the annulus $\{e^{\frac{-\pi}{4}}<|z|<e^{\frac{\pi}{4}}\}$. If z_n is a sequence in $\mathbb{D}$ approaching i or $-i$ along the imaginary axis, $f(z_n)$ simply cycles around A on the circle $\{|z|=1\}$. f is, however, B-proper, since f extends continuously to map $\{z=e^{i\theta};-\frac{\pi}{2}<\theta<\frac{\pi}{2}\}$ to $\{|z|=e^{\frac{\pi}{4}}\}$ and $\{z=e^{i\theta};\frac{\pi}{2}<\theta<\frac{3\pi}{2}\}$ to $\{|z|=e^{\frac{-\pi}{4}}\}$. We see that $f(B_t)$ leaves $f(\mathbb{D})$ with probability 1 as t increases to $\tau(\mathbb{D})$.

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