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ON THE TRANSIENCE OF RANDOM INTERLACEMENTS

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Abstract
We consider the interlacement Poisson point process on the space of doubly-infinite $\mathbb{Z}^d$-valued trajectories modulo time-shift, tending to infinity at positive and negative infinite times. The set of vertices and edges visited by at least one of these trajectories is the graph induced by the random interlacements at level u of Sznitman [9]. We prove that for any $u > 0$, almost surely, the random interlacement graph is transient.

1 Introduction

The model of random interlacements was recently introduced by Sznitman in [9]. Among other results in [9], he proved that the random interlacement graph is almost surely connected. This result was later refined in [6] and [7] by showing that every two points of the random interlacement graph are connected via at most $\lceil d/2 \rceil$ random walk trajectories, and this number is optimal. In this paper we further exploit the method of [7] in order to show that the graph induced by the random interlacements is almost surely transient in dimensions $d \geq 3$.

1.1 The model

Let W be the space of doubly-infinite nearest-neighbor trajectories in $\mathbb{Z}^d$ $(d \geq 3)$ which tend to infinity at positive and negative infinite times, and let W^* be the space of equivalence classes of trajectories in W modulo time-shift. We write $\mathcal{W}$ for the canonical σ-algebra on W generated by the coordinates X_n, $n \in \mathbb{Z}$, and $\mathcal{W}^*$ for the largest σ-algebra on W^* for which the canonical map π^* from $(W, \mathcal{W})$ to $(W^*, \mathcal{W}^*)$ is measurable. Let u be a positive number. We say that a Poisson point

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measure μ on W^* has distribution $\mathrm{Pois}(u, W^*)$ if the following properties hold: for a finite subset A of $\mathbb{Z}^d$, denote by μ_A the restriction of μ to the set of trajectories from W^* that intersect A, and by N_A be the number of trajectories in $\mathrm{Supp}(\mu_A)$, then $\mu_A = \sum_{i=1}^{N_A} \delta_{\pi^*(X_i)}$, where X_i are doubly-infinite trajectories from W parametrized in such a way that $X_i(0) \in A$ and $X_i(t) \notin A$ for all $t < 0$ and for all $i \in \{1, \dots, N_A\}$, and

(1) The random variable N_A has Poisson distribution with parameter $u\mathrm{cap}(A)$ (see (2.2) for the definition of the $\mathrm{cap}(A)$).

(2) Given N_A, the points $X_i(0)$, $i \in \{1, \dots, N_A\}$, are independent and distributed according to the normalized equilibrium measure on A (see (2.8) for the definition).

(3) Given N_A and $(X_i(0))_{i=1}^{N_A}$, the corresponding forward and backward paths are conditionally independent, $(X_i(t), t \geq 0)_{i=1}^{N_A}$ are distributed as independent simple random walks, and $(X_i(t), t \leq 0)_{i=1}^{N_A}$ are distributed as independent random walks conditioned on not hitting A.

Properties (1)-(3) uniquely define $\mathrm{Pois}(u, W^*)$ as proved in Theorem 1.1 in [9]. In fact, Theorem 1.1 in [9] gives a coupling of the Poisson point measures $\mu(u)$ with distribution $\mathrm{Pois}(u, W^*)$ for all $u > 0$, but we will not need such a general statement here. We also mention the following property of the distribution $\mathrm{Pois}(u, W^*)$, which will be useful in the proofs. It follows from the above definition of $\mathrm{Pois}(u, W^*)$.

(4) Let μ_1 and μ_2 be independent Poisson point measures on W^* with distributions $\mathrm{Pois}(u_1, W^*)$ and $\mathrm{Pois}(u_2, W^*)$, respectively. Then $\mu_1 + \mu_2$ has distribution $\mathrm{Pois}(u_1 + u_2, W^*)$.

We refer the reader to [9] for more details. For a Poisson point measure μ with distribution $\mathrm{Pois}(u, W^*)$, the *random interlacement graph* $\mathscr{I} = \mathscr{I}(\mu)$ (at level u) is defined as the subgraph of $\mathbb{Z}^d$ induced by μ, i.e., its vertices and edges are those that are traversed by at least one of the random walks from $\mathrm{Supp}(\mu)$. It follows from [9] that $\mathscr{I}$ is a translation invariant ergodic random subgraph of $\mathbb{Z}^d$.
We consider the simple random walk on the graph $\mathscr{I}$, with uniform edgeweights, i.e., at each step the random walker moves to a uniformly chosen neighbor of the current vertex. We say that a graph is transient if the simple random walk on the graph is transient. Since $\mathscr{I}$ is a translation invariant ergodic random subgraph of $\mathbb{Z}^d$, it is transient with probability 0 or 1.

1.2 The result

Our main result is the following theorem.

Theorem 1. *Let $d \geq 3$ and $u > 0$. Let μ be a random point measure on W^* distributed as* $\mathrm{Pois}(u, W^*)$*, and $\mathbb{P}$ be the law of μ. Then, $\mathbb{P}$-a.s., the random interlacement graph $\mathscr{I} = \mathscr{I}(\mu)$ is transient.*

The main ingredient of the proof of Theorem 1 is Proposition 1. For $x, y \in \mathbb{Z}^d$ and a positive integer r, we write $x \overset{\mathscr{I}\cap B(r)}{\longleftrightarrow} y$ if there is a nearest-neighbor path in $\mathscr{I}$ that connects x and y and uses only vertices of $\mathscr{I}$ from the l^∞-ball of radius r centered at the origin. (In particular, x and y must be vertices in $\mathscr{I} \cap B(r)$.)

Proposition 1. *Let $d \geq 3$ and $u > 0$. Let $\mathscr{I}$ be the random interlacement graph at level u. There exist constants $c = c(d, u) > 0$ and $C = C(d, u) < \infty$ such that for all $R \geq 1$,*

$$\mathbb{P}\left[\mathscr{I} \cap B(R) \neq \emptyset, \bigcap_{x,y \in \mathscr{I}\cap B(R)} \left\{x \overset{\mathscr{I}\cap B(2R)}{\longleftrightarrow} y\right\}\right] \geq 1 - C\exp\left(-cR^{1/6}\right). \tag{1.1}$$

Remark 1. The exponent 1/6 is not optimal, but suffices for the proof of Theorem 1. In fact, Theorem 1 follows if the probability in (1.1) tends to 1, as $R \to \infty$, faster than any polynomial.

Remark 2. Random interlacements were defined on arbitrary transient graphs in [10]. It was proved in [11] that for any transient transitive graph G, the interlacement graph on G is almost surely connected for all $u > 0$ if and only if G is amenable. The following question arises naturally: does Theorem 1 hold for any transient amenable transitive graph?

2 Notation and facts about Green function and capacity

In this section we collect most of the notation, definitions and facts used in the paper. For $a \in \mathbb{R}$, we write $|a|$ for the absolute value of a, $\lfloor a \rfloor$ for the integer part of a, and $\lceil a \rceil$ for the smallest integer not less than a. For $x \in \mathbb{Z}^d$, we write $|x|$ for $\max\left(|x_1|, \dots, |x_d|\right)$. For a set S, we write $|S|$ for the cardinality of S. For $R > 0$ and $x \in \mathbb{Z}^d$, let $B(x,R) = \{y \in \mathbb{Z}^d \ : \ |x-y| \le R\}$ be the l^∞-ball of radius R centered at x, and $B(R) = B(0,R)$. We denote by $\mathbb{1}(A)$ the indicator of event A, and by $E[X;A]$ the expected value of random variable $X\mathbb{1}(A)$.
Throughout the paper we always assume that $d \ge 3$. For $x \in \mathbb{Z}^d$, let P_x be the law of a simple random walk X on $\mathbb{Z}^d$ with $X(0) = x$. We write $g(\cdot,\cdot)$ for the Green function of the walk: for $x, y \in \mathbb{Z}^d$, $g(x,y) = \sum_{t=0}^{\infty} P_x[X(t) = y]$. We also write $g(\cdot)$ for $g(0,\cdot)$. The Green function is symmetric and, by translation invariance, $g(x,y) = g(y-x)$. It follows from [2, Theorem 1.5.4] that for any $d \ge 3$ there exist a positive constant $c_g = c_g(d)$ and a finite constant $C_g = C_g(d)$ such that for all x and y in $\mathbb{Z}^d$,

$$c_g \min\left(1, |x-y|^{2-d}\right) \le g(x,y) \le C_g \min\left(1, |x-y|^{2-d}\right). \tag{2.1}$$

Definition 2.1. Let K be a subset of $\mathbb{Z}^d$. The energy of a finite Borel measure ν on K is

$$\mathcal{E}(\nu) = \int_K \int_K g(x,y) d\nu(x) d\nu(y) = \sum_{x,y \in K} g(x,y)\nu(x)\nu(y).$$

The capacity of K is

$$\mathrm{cap}(K) = \left[\inf_\nu \mathcal{E}(\nu)\right]^{-1}, \tag{2.2}$$

where the infimum is over probability measures ν on K. (We use the convention that $\infty^{-1} = 0$, i.e., $\mathrm{cap}(\emptyset) = 0$.)

The following properties of the capacity immediately follow from (2.2):

$$\begin{aligned} &\text{Monotonicity:} && \text{for any } K_1 \subset K_2 \subset \mathbb{Z}^d, \ \ \mathrm{cap}(K_1) \le \mathrm{cap}(K_2); && (2.3)\\ &\text{Subadditivity:} && \text{for any } K_1, K_2 \subset \mathbb{Z}^d, \ \ \mathrm{cap}(K_1 \cup K_2) \le \mathrm{cap}(K_1) + \mathrm{cap}(K_2); && (2.4)\\ &\text{Capacity of a point:} && \text{for any } x \in \mathbb{Z}^d, \ \ \mathrm{cap}(\{x\}) = 1/g(0). && (2.5) \end{aligned}$$

It will be useful to have an alternative definition of the capacity.

Definition 2.2. Let K be a finite subset of $\mathbb{Z}^d$. The equilibrium measure of K is defined by

$$e_K(x) = P_x\left[X(t) \notin K \text{ for all } t \ge 1\right] \mathbb{1}(x \in K), \ \ x \in \mathbb{Z}^d. \tag{2.6}$$

The capacity of K is then equal to the total mass of the equilibrium measure of K:

$$\mathrm{cap}(K) = \sum_x e_K(x), \tag{2.7}$$

and the unique minimizer of the variational problem (2.2) is given by the normalized equilibrium measure

$$\widetilde{e}_K(x) = e_K(x)/\mathrm{cap}(K). \tag{2.8}$$

(See, e.g., Lemma 2.3 in [1] for a proof of this fact.)

As a simple corollary of the above definition, we get

$$P_x\left[H_K < \infty\right] = \sum_{y \in K} g(x,y) e_K(y), \quad \text{for } x \in \mathbb{Z}^d. \tag{2.9}$$

Here, we write H_K for the first entrance time in K, i.e., $H_K = \inf\{t \geq 0 \ : \ X(t) \in K\}$. We will repeatedly use the following bound on the capacity of $B(0,R)$ in $d \geq 3$ (see (2.16) on page 53 in [2]): there exist constants $c_b = c_b(d) > 0$ and $C_b = C_b(d) < \infty$ such that for all positive R,

$$c_b R^{d-2} \leq \mathrm{cap}\left(B(0,R)\right) \leq C_b R^{d-2}. \tag{2.10}$$

Finally, we will often use in the proofs the following large deviation bounds for the Poisson distribution, which can be proved using the exponential Chebyshev enequality. Let ξ be a random variable which has Poisson distribution with parameter λ, then

$$\mathbf{P}\left[\lambda/2 \leq \xi \leq 2\lambda\right] \geq 1 - 2e^{-\lambda/10}. \tag{2.11}$$

Throughout the text, we write c and C for small positive and large finite constants, respectively, that may depend on d and u. Their values may change from place to place.

3 Proof of Theorem 1

We recall the following result about an equivalent characterization of the transience of simple random walk on a graph. The statement and proof of a more general theorem about an equivalent characterisation of the transience of reversible Markov chains can be found on page 398 of [3].

Lemma 1. *Let $G = (V,E)$ denote a countable, simple graph in which the degree of each vertex is finite. The simple random walk on G is transient if and only if there exist real numbers $(u(x,y))_{x,y \in V}$ with the following properties:*

(i) $u(y,x) = -u(x,y)$ *and* $u(x,y) \neq 0$ *only if* $\{x,y\} \in E$,

(ii) $\sum_{x \in V} \left|\sum_{y \in V} u(x,y)\right| < \infty$ *and* $\sum_{x \in V} \left(\sum_{y \in V} u(x,y)\right) \neq 0$,

(iii) $\sum_{x \in V} \sum_{y \in V} u(x,y)^2 < \infty$.

We refer to a function $(u(x,y))_{x,y \in V}$ satisfying (i) as a *flow* on G and $u(x,y)$ as the amount of net flow from vertex x to vertex y. We say that $\sum_{y \in V} u(x,y)$ is the net influx at vertex x. Condition (ii) states that the influxes are absolutely summable (this can be thought of as a relaxation of Kirchoff's law) and that there is a nonzero net influx into the network. Condition (iii) says that the Thompson energy of the flow is finite. We are going to prove Theorem 1 by constructing such a flow on the graph $\mathscr{I}$.

Denote by S^d the d-dimensional Euclidean unit sphere. Given $v \in S^d$ and $\varepsilon \in (0,1)$, we define the graph $\mathscr{I}(v,\varepsilon)$ by

$$\mathscr{I}(v,\varepsilon) = \mathscr{I} \cap \bigcup_{n=1}^{\infty} B(nv, n^{\varepsilon}). \tag{3.1}$$

The set $\bigcup_{n=1}^{\infty} B(nv, n^{\varepsilon})$ is roughly shaped like a paraboloid with an axis parallel to v. We denote by $\mathscr{C}_\infty(v,\varepsilon)$ the maximal subgraph of $\mathscr{I}(v,\varepsilon)$ in which every connected component is infinite. (If $\mathscr{I}(v,\varepsilon)$ does not contain an infinite connected component, we set $\mathscr{C}_\infty(v,\varepsilon) = \emptyset$.)

Lemma 2. *For any $u > 0$ and $0 < \varepsilon < 1$, we have*

$$\mathbb{P}\left[\bigcup_{M \geq 1} \left\{\forall\, v \in S^d \,:\, B(M) \cap \mathscr{C}_\infty(v,\varepsilon) \neq \emptyset\right\}\right] = 1. \tag{3.2}$$

Proof. For any $z \in \mathbb{Z}^d$, define the events

$$A_z = \left\{\forall\, x, y \in \mathscr{I} \cap B(z, \frac{1}{4}|z|^{\varepsilon}) \,:\, x \overset{\mathscr{I} \cap B(z, \frac{1}{2}|z|^{\varepsilon})}{\longleftrightarrow} y\right\}, \qquad B_z = \left\{\mathscr{I} \cap B(z, \frac{1}{8}|z|^{\varepsilon}) \neq \emptyset\right\}.$$

It follows from the Borel-Cantelli lemma and Proposition 1 that $\mathbb{P}(\liminf_{z \in \mathbb{Z}^d} A_z \cap B_z) = 1$, i.e., almost surely the number of $z \in \mathbb{Z}^d$ for which $A_z \cap B_z$ does not occur is finite.

Note that there exists an integer m such that for every $v \in S^d$, there exists a $\mathbb{Z}^d$-valued sequence $(z_i)_{i=1}^{\infty}$ with $z_1 \in B(m)$ and $|z_i| \to \infty$, such that for all i, (a) $B(z_i, \frac{1}{2}|z_i|^{\varepsilon}) \subset \bigcup_{n=1}^{\infty} B(nv, n^{\varepsilon})$ and (b) $B(z_i, \frac{1}{8}|z_i|^{\varepsilon})$ and $B(z_{i+1}, \frac{1}{8}|z_{i+1}|^{\varepsilon})$ are subsets of $B(z_{i+1}, \frac{1}{4}|z_{i+1}|^{\varepsilon})$. Indeed, one can take, for example, a discrete approximation of the $\mathbb{R}^d$-valued sequence $((i + i_0)v)_{i=1}^{\infty}$ for large enough i_0. (Note that i_0 can be chosen independent of v.)

Let M be an almost surely finite random variable such that, for all $v \in S^d$ and $i \geq M$, the events A_{z_i} and B_{z_i} hold. Then, for all $i \geq M$, $\mathscr{I} \cap B(z_i, \frac{1}{8}|z_i|^{\varepsilon}) \neq \emptyset$, and every vertex in $\mathscr{I} \cap B(z_i, \frac{1}{8}|z_i|^{\varepsilon})$ is connected to every vertex in $\mathscr{I} \cap B(z_{i+1}, \frac{1}{8}|z_{i+1}|^{\varepsilon})$ by a path in $\mathscr{I}(v,\varepsilon)$. This implies (3.2). □

Definition 3.1. It follows from Lemma 2 that we can almost surely assign (in a measurable way) to every $v \in S^d$ a (random) simple nearest-neighbor path $w_v = (w_v(n))_{n=0}^{\infty}$ in the graph $\mathscr{I}(v,\varepsilon)$. In particular, for all $n \neq m \in \mathbb{N}$, $w_v(n) \neq w_v(m)$, and $\lim_{n\to\infty} |w_v(n)| = \infty$.

Our construction of the flow u with finite energy is analogous to the proof of Pólya's theorem in [4]. For every $v \in S^d$ define $(u_v(x,y))_{x,y \in \mathscr{I}}$ to be the unit flow that goes from $w_v(0)$ to ∞ along the simple path w_v, more precisely let $u_v(x,y) = -u_v(y,x) = 1$ if $w_v(n) = x$ and $w_v(n+1) = y$ for some $n \in \mathbb{N}$ and, otherwise, let $u_v(x,y) = 0$. With this definition we have

$$\sum_{y \in \mathscr{I}} u_v(x,y) = \mathbb{1}[x = w_v(0)]. \tag{3.3}$$

Note that for any $v \in S^d$ the flow u_v satisfies (i) and (ii) of Lemma 1, but fails to satisfy (iii). We define the flow u as the average of the flows u_v with respect to v, more precisely let

$$u(x,y) := \int_{S^d} u_v(x,y)\, \mathrm{d}\lambda(v)$$

where λ is the Haar measure on S^d normalized to be a probability measure.
Now we check that u is a flow with finite energy on $\mathscr{I}$, i.e., the conditions of Lemma 1 hold. The function u inherits property (i) from the flows u_v. From (3.3) it readily follows that we have $\sum_{y\in\mathscr{I}} u(x,y) \geq 0$ for all $x \in \mathscr{I}$ and that $\sum_{x\in\mathscr{I}} \left(\sum_{y\in\mathscr{I}} u(x,y)\right) = 1$ holds, from which (ii) follows. It only remains to show that the energy of u is finite, i.e., (iii) holds. Note that $u(x,y) \neq 0$ only if $|x-y| = 1$. We have

$$|u(x,y)| \leq \int_{S^d} |u_v(x,y)| \,\mathrm{d}\lambda(v) \leq \int_{S^d} \mathbb{1}\left[\, x \in \bigcup_{n=1}^{\infty} B(nv, n^{\varepsilon}) \,\right] \mathrm{d}\lambda(v) \leq C \frac{(|x|^{\varepsilon})^{d-1}}{|x|^{d-1}}. \tag{3.4}$$

Now choose $0 < \varepsilon < \frac{1}{4}$ in Definition 3.1. The corresponding flow u has finite energy:

$$\begin{aligned}\sum_{x\in\mathbb{Z}^d}\sum_{y\in\mathbb{Z}^d} u(x,y)^2 \quad &\leq \quad \sum_{n=1}^{\infty}\sum_{|x|=n}\sum_{|x-y|=1} u(x,y)^2 \overset{(3.4)}{\leq} C\sum_{n=1}^{\infty}\sum_{|x|=n} n^{2(\varepsilon-1)(d-1)} \\ &\overset{d\geq 3}{\leq} \quad C\sum_{n=1}^{\infty} n^{4\varepsilon-2} < \infty.\end{aligned}$$

Therefore, the flow u satisfies the conditions of Lemma 1, which proves Theorem 1. □

4 Proof of Proposition 1

4.1 Bounds on the capacity of certain collections of random walk trajectories

The aim of this subsection is to prove Lemma 6 (with $\Phi(\overline{X}_N, T)$ defined in (4.2)), which will be used in the proof of Lemma 7.
The following lemma is proved in [7] for $d \geq 5$, see Lemma 3 there. The cases $d = 3,4$ can be proved similarly. Therefore, we state this lemma without proof.

Lemma 3. *Let $(x_i)_{i\geq 1}$ be a sequence in $\mathbb{Z}^d$, and let X_i be a sequence of independent simple random walks on $\mathbb{Z}^d$ with $X_i(0) = x_i$. Let*

$$F(n,d) = \begin{cases} n^{1/2} & \text{if } d = 3, \\ \log n & \text{if } d = 4, \text{ and} \\ 1 & \text{if } d \geq 5. \end{cases}$$

Then for all positive integers N and n, we have

$$\mathbf{E}\left[\sum_{i,j=1}^{N}\sum_{s,t=n+1}^{2n} g\left(X_i(s), X_j(t)\right)\right] \leq C\left(NnF(n,d) + N^2 n^{3-d/2}\right). \tag{4.1}$$

□

Let $(X_i(t) \,:\, t \geq 0)_{i\geq 1}$ be a sequence of nearest-neighbor trajectories on $\mathbb{Z}^d$, and $\overline{X}_N = (X_1, \ldots, X_N)$. For positive integers N and T, we define the subset $\Phi(\overline{X}_N, T)$ of $\mathbb{Z}^d$ as

$$\Phi(\overline{X}_N, T) = \bigcup_{i=1}^{N} \{X_i(t) \,:\, 1 \leq t \leq T\}. \tag{4.2}$$

Lemma 4. *Let X_i be a sequence of independent simple random walks on $\mathbb{Z}^d$ with $X_i(0) = x_i$. There exists a positive constant c such that for any sequence $(x_i)_{i\ge1} \subset \mathbb{Z}^d$ and for all positive integers N and T,*

$$\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \le \frac{NT}{g(0)}, \qquad \text{and} \qquad \mathbf{E}\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \ge c \min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right), \tag{4.3}$$

where the function F is defined in Lemma 3.

Remark 3. Heuristically, $\mathbf{E}\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right)$ is at least $\frac{cNT}{F(T,d)}$ when the random walks in $\overline{X}_N$ are well separated, and at least $\mathrm{cap}(B(cT^{1/2})) \overset{(2.10)}{\ge} cT^{\frac{d-2}{2}}$ when the set $\Phi(\overline{X}_N, T)$ saturates the ball $B(cT^{1/2})$.

Proof. The proof of this lemma is similar to the proof of Lemma 4 in [7], so we give only a sketch here. (Note that the definition of $\Phi(\overline{X}_N, R)$ in [7] is different from the one in (4.2).)
The upper bound on the capacity of $\Phi(\overline{X}_N, T)$ follows from (2.4) and (2.5). Let $n = \lfloor T/2 \rfloor$. By the definition of the capacity (2.2) and the Jensen inequality,

$$\mathbf{E}\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \ge N^2 n^2 \left(\mathbf{E}\left[\sum_{i,j=1}^{N} \sum_{s,t=n+1}^{2n} g(X_i(s), X_j(t))\right]\right)^{-1}.$$

The lower bound in (4.3) now follows from Lemma 3. □

As a corollary of Lemma 4 we obtain the following lemma.

Lemma 5. *Let X_i be a sequence of independent simple random walks on $\mathbb{Z}^d$ with $X_i(0) = x_i$. There exists a positive constant c such that for any sequence $(x_i)_{i\ge1} \subset \mathbb{Z}^d$ and for all positive integers N and T,*

$$\mathbf{P}\left[\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \ge c \min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right)\right] \ge \frac{c}{(\log T)^2}. \tag{4.4}$$

Proof. Remember the Paley-Zygmund inequality [5]: Let ξ be a non-negative random variable with finite second moment. For any $\theta \in (0,1)$, $\mathrm{P}[\xi \ge \theta \mathrm{E}\xi] \ge (1-\theta)^2 \left[\mathrm{E}\xi\right]^2 / \mathrm{E}[\xi^2]$.
We first consider the case $d = 3$. Note that in this case, $\min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right) = T^{1/2}$. Since $\Phi(\overline{X}_N, T) \supseteq \Phi(\overline{X}_1, T)$, by (2.3), it suffices to show that

$$\mathbf{P}\left[\mathrm{cap}\left(\Phi(\overline{X}_1, T)\right) \ge cT^{1/2}\right] \ge c.$$

It follows from (4.3) that $\mathbf{E}\mathrm{cap}\left(\Phi(\overline{X}_1, T)\right) \ge cT^{1/2}$. On the other hand, the set $\Phi(\overline{X}_1, T)$ is contained in $B(X_1(0), M)$, where $M = \max\{|X_1(t) - X_1(0)| \ : \ 1 \le t \le T\}$. Therefore, by (2.3) and (2.10), $\mathrm{cap}\left(\Phi(\overline{X}_1, T)\right) \le CM$. In particular, since $\mathbf{E}M^2 \le CT$, we have $\mathbf{E}\left[\mathrm{cap}\left(\Phi(\overline{X}_1, T)\right)^2\right] \le CT$. The result now follows from the Paley-Zygmund inequality.
Let $d \ge 4$. An application of the Paley-Zygmund inequality and (4.3) gives

$$\mathbf{P}\left[\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \ge c \min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right)\right] \ge \frac{c \min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right)^2}{N^2 T^2}. \tag{4.5}$$

We distinguish two cases. If $\min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right) = \frac{NT}{F(T,d)}$, the result immediately follows from (4.5) and the definition of F. Otherwise, if $\min\left(\frac{NT}{F(T,d)}, T^{\frac{d-2}{2}}\right) = T^{\frac{d-2}{2}}$, we take $N' \le N$ in

$\left[T^{\frac{d-4}{2}}F(T,d), 2T^{\frac{d-4}{2}}F(T,d)\right]$. Such a choice is possible, since $1 \leq T^{\frac{d-4}{2}}F(T,d) \leq N$. The result then follows from (4.5) by observing that $\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \geq \mathrm{cap}\left(\Phi(\overline{X}_{N'}, T)\right)$ by (2.3). $\square$

In the next lemma we show that the capacity of $\Phi(\overline{X}_N, T)$ is large with high probability.

Lemma 6. *Let $\varepsilon \in (0,1)$. Let X_i be a sequence of independent simple random walks on $\mathbb{Z}^d$ with $X_i(0) = x_i$. There exists a positive constant c such that for any sequence $(x_i)_{i\geq 1} \subset \mathbb{Z}^d$ and for all positive integers N and T,*

$$\mathbf{P}\left[\mathrm{cap}\left(\Phi(\overline{X}_N, T)\right) \geq c \min\left(N T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right] \geq 1 - \exp\left(-cT^{\varepsilon/2}\right). \tag{4.6}$$

Proof. For positive integers N, $\widetilde{T}$ and k, we define the subset $\Phi_k(\overline{X}_N, \widetilde{T})$ of $\mathbb{Z}^d$ by

$$\Phi_k(\overline{X}_N, \widetilde{T}) = \bigcup_{i=1}^{N} \left\{X_i(t) \ : \ (k-1)\widetilde{T} + 1 \leq t \leq k\widetilde{T}\right\}.$$

It follows from the Markov property, (4.4), and the definition of the function F, that

$$\mathbf{P}\left[\mathrm{cap}\left(\Phi_k(\overline{X}_N, \widetilde{T})\right) \geq c \min\left(N\widetilde{T}^{1/2}, \widetilde{T}^{\frac{d-2}{2}}\right) \ \middle| \ X_i(t),\ i \in \{1,\dots,N\},\ t \leq (k-1)\widetilde{T}\right] \geq \frac{c}{(\log \widetilde{T})^2}.$$

Therefore, for any $\delta > 0$,

$$\mathbf{P}\left[\mathrm{cap}\left(\bigcup_{k=1}^{\lfloor \widetilde{T}^{\delta}\rfloor} \Phi_k(\overline{X}_N, \widetilde{T})\right) \geq c \min\left(N\widetilde{T}^{1/2}, \widetilde{T}^{\frac{d-2}{2}}\right)\right] \geq 1 - \left[1 - \frac{c}{(\log \widetilde{T})^2}\right]^{\lfloor \widetilde{T}^{\delta}\rfloor} \geq 1 - \exp\left(-c\widetilde{T}^{\delta/2}\right).$$

The result follows by observing that $\bigcup_{k=1}^{\lfloor \widetilde{T}^{\delta}\rfloor} \Phi_k(\overline{X}_N, \widetilde{T}) \subseteq \Phi(\overline{X}_N, \lfloor \widetilde{T}^{1+\delta}\rfloor)$, and by taking $\varepsilon = \delta/(1+\delta)$. $\square$

4.2 Bounds on the capacity of certain subsets of random interlacement

The aim of this subsection is to prove Lemma 10, which states that with high probability for $x \in \mathscr{I}$, the connected component of $\mathscr{I} \cap B(x,R)$ that contains x has large capacity. We prove the statement by constructing explicitly for $x \in \mathscr{I}$ a connected subset of $\mathscr{I} \cap B(x,R)$ of large capacity that contains x. In this construction, we exploit property (4) of $\mathrm{Pois}(u, W^*)$, which allows to describe $\mathscr{I}$ as the union of independent identically distributed random interlacement graphs $\mathscr{I}_1, \dots, \mathscr{I}_{d-2}$. We begin with auxiliary lemmas.

Let A be a finite set of vertices in $\mathbb{Z}^d$. Let μ be a random point measure with distribution $\mathrm{Pois}(u, W^*)$, and μ_A its restriction to the set of trajectories from W^* that intersect A. We can write the measure μ_A as $\sum_{i=1}^{N_A} \delta_{\pi^*(X_i)}$ (recall the notation from Section 1.1), where $N_A = |\mathrm{Supp}(\mu_A)|$, and $X_1, \dots, X_{N_A}$ are doubly-infinite trajectories from W parametrized in such a way that $X_i(0) \in A$ and $X_i(t) \notin A$ for all $t < 0$ and for all $i \in \{1, \dots, N_A\}$. We define the set $\Psi(\mu, A, T)$ as

$$\Psi(\mu, A, T) = \bigcup_{i=1}^{N_A} \left\{X_i(t) \ : \ 1 \leq t \leq T\right\}. \tag{4.7}$$

Lemma 7. *Let $\varepsilon \in (0,1)$. Let μ be a Poisson point measure with distribution* $\mathrm{Pois}(u,W^*)$*, then for all finite subsets A of $\mathbb{Z}^d$ and for all positive integers T, one has*

$$\mathbb{P}\left[\mathrm{cap}\left(\Psi(\mu,A,T)\right) \geq c\min\left(\mathrm{cap}(A)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right] \geq 1 - Ce^{-c\min\left(T^{\varepsilon/2},\ \mathrm{cap}(A)\right)}. \tag{4.8}$$

Proof. It follows from property (1) of $\mathrm{Pois}(u,W^*)$ that N_A has the Poisson distribution with parameter $u\mathrm{cap}(A)$. Therefore, by (2.11), $\mathbb{P}\left[N_A \geq c\mathrm{cap}(A)\right] \geq 1 - Ce^{-c\mathrm{cap}(A)}$. Properties (2) and (3) of $\mathrm{Pois}(u,W^*)$ imply that given N_A, the forward trajectories $X_1,\dots,X_{N_A}$ are distributed as independent simple random walks. Therefore, Lemma 6 applies, giving that

$$\mathbb{P}\left[\mathrm{cap}\left(\Psi(\mu,A,T)\right) \geq c\min\left(N_A T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right] \geq 1 - e^{-cT^{\varepsilon/2}}.$$

The result follows. □

Let X be a simple random walk on $\mathbb{Z}^d$ with $X(0)=x$. Let $\mu^{(2)},\mu^{(3)},\dots$ be independent random point measures with distribution $\mathrm{Pois}(u,W^*)$ (the parameter u is fixed here), which are also independent of X. We denote by $\mathbb{P}_x$ the joint law of X and $\mu^{(i)}$'s. Let T be a positive integer. We define the following sequence of random subsets of $\mathbb{Z}^d$:

$$U^{(1)}(x,T) = \{X(t)\ :\ 1 \leq t \leq T\}, \tag{4.9}$$

and for $s \geq 2$ (see (4.7) for notation),

$$U^{(s)}(x,T) = \Psi\left(\mu^{(s)}, U^{(s-1)}(x,T), T\right). \tag{4.10}$$

Note that for each $s \geq 1$, $\bigcup_{i=1}^s U^{(i)}(x,T)$ is a connected subset of $\mathbb{Z}^d$. In the next lemma, we show that for any $\gamma > 0$, with high probability, the set $\bigcup_{i=1}^s U^{(i)}(x,T)$ is a subset of $B(x, sT^{(1+\gamma)/2})$.

Lemma 8. *Let $\gamma \in (0,1)$. There exist $c = c(u,d,s) > 0$ and $C = C(u,d,s) < \infty$ such that*

$$\mathbb{P}_x\left[\bigcup_{i=1}^s U^{(i)}(x,T) \subseteq B(x, sT^{(1+\gamma)/2})\right] \geq 1 - Ce^{-cT^\gamma}. \tag{4.11}$$

Proof. We denote the event $\{\bigcup_{i=1}^s U^{(i)}(x,T) \subseteq B(x,sT^{(1+\gamma)/2})\}$ by D_s, and its complement by D_s^c. If $s=1$, (4.11) follows from Hoeffding's inequality: $\mathbb{P}_x\left[D_1^c\right] \leq 2de^{-T^\gamma/8}$. Assume that (4.11) is proved for $s' < s$. Then

$$\mathbb{P}_x\left[D_s^c\right] \leq \mathbb{P}\left[D_{s-1}^c\right] + \mathbb{P}_x\left[D_s^c, D_{s-1}\right],$$

and it remains to show that $\mathbb{P}_x\left[D_s^c, D_{s-1}\right] \leq Ce^{-cT^\gamma}$. Note that if D_{s-1} occurs,

$$|\mathrm{Supp}(\mu^{(s)}_{U^{(s-1)}(x,T)})| \leq |\mathrm{Supp}(\mu^{(s)}_{B(x,(s-1)T^{(1+\gamma)/2})})|.$$

(See Section 1.1 for the notation.) Let us denote the right hand side of the above inequality by N. It follows from property (1) of $\mathrm{Pois}(u,W^*)$ that N has the Poisson distribution with parameter $u\mathrm{cap}\left(B(x,(s-1)T^{(1+\gamma)/2})\right)$. In particular, using (2.10) and (2.11), we obtain that

$$\mathbb{P}_x\left[N \geq CT^{(d-2)(1+\gamma)/2}\right] \leq Ce^{-cT^{(1+\gamma)/2}} \leq Ce^{-cT^\gamma}.$$

On the other hand, properties (2) and (3) of $\mathrm{Pois}(u,W^*)$ imply that

$$\mathbb{P}_x\left[D_s^c, D_{s-1}\right] \leq \mathbb{P}_x\left[N \geq CT^{(d-2)(1+\gamma)/2}\right] + CT^{(d-2)(1+\gamma)/2}\mathbb{P}_x\left[D_1^c\right] \leq Ce^{-cT^\gamma}.$$

This completes the proof of the lemma. □

The next lemma follows immediately from Lemma 7 and the definition of $U^{(s)}(x,T)$.

Lemma 9. *Let $\varepsilon \in (0,1/2)$. For any positive integer s, there exist $c = c(d,u,s) > 0$ and $C = C(d,u,s) < \infty$ such that for all positive integers T,*

$$\mathbb{P}_x\left[\bigcap_{i=1}^{s}\left\{\mathrm{cap}\left(U^{(i)}(x,T)\right) \geq c\min\left(T^{\frac{i(1-\varepsilon)}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right\}\right] \geq 1 - C\exp\left(-cT^{\varepsilon/2}\right). \tag{4.12}$$

Proof. The case $s=1$ follows from (4.9) and Lemma 6. Let $s \geq 2$. Note that, for $c \in (0,1)$, the event in (4.12) with c^s in place of c is implied by the event

$$\bigcap_{i=1}^{s}\left\{\mathrm{cap}\left(U^{(i)}(x,T)\right) \geq c\min\left(\mathrm{cap}\left(U^{(i-1)}(x,T)\right)T^{\frac{1-\varepsilon}{2}}, T^{\frac{(d-2)(1-\varepsilon)}{2}}\right)\right\},$$

where we set by convention $\mathrm{cap}\left(U^{(0)}(x,T)\right) = 1$. The result now follows (by induction in s) from (4.12) for $s=1$, Lemma 7 and the definition of $U^{(s)}(x,T)$, see (4.10). □

Corollary 1. *It follows from Lemmas 8 and 9 that for any $\varepsilon \in (0,1/2)$,*

$$\mathbb{P}_x\left[\bigcup_{i=1}^{d-2}U^{(i)}(x,T) \subseteq B(x,(d-2)T^{(1+\varepsilon)/2}),\ \ \mathrm{cap}\left(U^{(d-2)}(x,T)\right) \geq cT^{\frac{(d-2)(1-\varepsilon)}{2}}\right] \geq 1 - Ce^{-cT^{\varepsilon/2}}. \tag{4.13}$$

In particular on the event in (4.13), $\bigcup_{i=1}^{d-2}U^{(i)}(x,T)$ is a connected subset of $B(x,(d-2)T^{(1+\varepsilon)/2})$.

Let μ be a Poisson point measure with distribution $\mathrm{Pois}(u,W^*)$, and let $\mathscr{I}$ be the corresponding random interlacement graph at level u. (See Section 1.1 for the definition.) For $x \in \mathscr{I}$, let $\mathscr{C}(x,R)$ be the connected component of $\mathscr{I} \cap B(x,R)$ that contains x. We define $\mathscr{C}(x,R)$ as an empty set for $x \notin \mathscr{I}$. In the next lemma we show that for $x \in \mathscr{I}$, the capacity of $\mathscr{C}(x,R)$ is large enough with high probability.

Lemma 10. *For all $\varepsilon \in (0,2/3)$, $R > 0$, and $x \in \mathbb{Z}^d$, we have*

$$\mathbb{P}\left[x \in \mathscr{I},\ \mathrm{cap}(\mathscr{C}(x,R)) < cR^{(d-2)(1-\varepsilon)}\right] \leq Ce^{-cR^{\varepsilon/2}}. \tag{4.14}$$

Proof. Let $\mu^{(1)},\dots,\mu^{(d-2)}$ be independent Poisson point measures with distribution $\mathrm{Pois}(\frac{u}{d-2},W^*)$. Let $\mathbb{P}$ be the joint law of $\mu^{(i)}$. By property (4) of $\mathrm{Pois}(u,W^*)$, the measure μ has the same law as $\sum_{i=1}^{d-2}\mu^{(i)}$. Therefore, we may assume that $\mu = \sum_{i=1}^{d-2}\mu^{(i)}$. In particular, the random interlacement graph $\mathscr{I} = \mathscr{I}(\mu)$ equals $\bigcup_{i=1}^{d-2}\mathscr{I}^{(i)}$, where $\mathscr{I}^{(i)} = \mathscr{I}(\mu^{(i)})$ are independent random interlacement graphs at level $\frac{u}{d-2}$, and the vertices and edges of $\mathscr{I}$ are the ones of $\mathbb{Z}^d$ that are traversed by at least one of the random walks from $\bigcup_{i=1}^{d-2}\mathrm{Supp}(\mu^{(i)})$.
It follows from (4.13) (with $T = \widetilde{R}^2$) and property (3) of $\mathrm{Pois}(\frac{u}{d-2},W^*)$ that for any $\delta \in (0,1/2)$, $\widetilde{R} > 0$, $x \in B(\widetilde{R})$ and $i \in \{1,\dots,d-2\}$,

$$\mathbb{P}\left[x \in \mathscr{I}^{(i)},\ \mathrm{cap}\left(\mathscr{C}(x,\widetilde{R}^{1+\delta})\right) < c\widetilde{R}^{(d-2)(1-\delta)}\right] \leq Ce^{-c\widetilde{R}^{\delta}}.$$

The result follows by taking $\delta = \varepsilon/(2-\varepsilon)$. □

4.3 Proof of Proposition 1

Proposition 1 will follow from Lemma 13, which states that with high probability, any two vertices from $\mathscr{I} \cap B(R)$ are connected by a path in $\mathscr{I} \cap B(CR)$ for large enough C. This result will follow from (4.14), (4.15) and property (4) of $\text{Pois}(u, W^*)$.

We begin with auxiliary lemmas. Lemma 11 is standard, so we only give a sketch of the proof here.

Lemma 11. *Let A be a subset of $B(R)$. Let X be a simple random walk on $\mathbb{Z}^d$ with $X(0) = x \in B(R)$. Let H_A be the entrance time of X in A, and $T_{B(r)}$ the exit time of X from $B(r)$. Then there exist $c > 0$ and $C < \infty$ such that for all $R > 0$ and $x \in B(R)$,*

$$P_x \left[H_A < T_{B(CR)} \right] \geq cR^{2-d} \text{cap}(A).$$

Proof. We use the identity (2.9). Since A is a subset of $B(R)$, the inequality (2.1) implies that, for any $y \in A$ and $x \in B(R)$, $g(x, y) \geq c_g (2R)^{2-d}$. By (2.7) and (2.9), $P_x \left[H_A < \infty \right] \geq c_g (2R)^{2-d} \text{cap}(A)$. On the other hand, for $y \in A$ and $z \notin B(CR)$, the inequality (2.1) gives $g(z, y) \leq C_g ((C-1)R)^{2-d}$. Therefore, by (2.7), (2.9) and the strong Markov property of X applied at time $T_{B(CR)}$, we have $P_x \left[T_{B(CR)} < H_A < \infty \right] \leq C_g ((C-1)R)^{2-d} \text{cap}(A)$. The result follows by taking C large enough. $\square$

Lemma 12. *There exist $c > 0$ and $C < \infty$ such that for all $R > 0$ and for all subsets U and V of $B(R)$, we have*

$$\mathbb{P} \left[U \overset{\mathscr{I} \cap B(CR)}{\longleftrightarrow} V \right] \geq 1 - C \exp \left(-cR^{2-d} \text{cap}(U) \text{cap}(V) \right). \tag{4.15}$$

Proof. Let μ be a random point measure with distribution $\text{Pois}(u, W^*)$. Remember from Section 1.1 that $\mu_U = \sum_{i=1}^{N_U} \delta_{\pi^*(X_i)}$, where $N_U = |\text{Supp}(\mu_U)|$ and $X_1, \ldots, X_{N_U}$ are doubly-infinite trajectories from W parametrized in such a way that $X_i(0) \in U$ and $X_i(t) \notin U$ for all $t < 0$ and for all $i \in \{1, \ldots, N_U\}$.

It follows from property (1) of $\text{Pois}(u, W^*)$ and (2.11) that $\mathbb{P} \left(N_U \geq c\text{cap}(U) \right) \geq 1 - Ce^{-c\text{cap}(U)}$. Therefore, by Lemma 11 and properties (2) and (3) of $\text{Pois}(u, W^*)$, we have

$$\begin{aligned} \mathbb{P} \left[U \overset{\mathscr{I} \cap B(CR)}{\longleftrightarrow} V \right] &\geq 1 - \mathbb{P} \left(N_U < c\text{cap}(U) \right) - \left(1 - cR^{2-d} \text{cap}(V) \right)^{c\text{cap}(U)} \\ &\geq 1 - C \exp \left(-cR^{2-d} \text{cap}(U) \text{cap}(V) \right). \end{aligned}$$

In these inequalities we also used the fact that $\text{cap}(V) \leq CR^{d-2}$, which follows from (2.3) and (2.10). The proof is complete. $\square$

As a corollary of Lemmas 10 and 12 we get the following lemma.

Lemma 13. *There exist $c > 0$ and $C < \infty$ such that for all $R > 0$ and $x, y \in B(R)$, we have*

$$\mathbb{P} \left[x, y \in \mathscr{I}, \left\{ x \overset{\mathscr{I} \cap B(CR)}{\longleftrightarrow} y \right\}^c \right] \leq C \exp \left(-cR^{1/6} \right). \tag{4.16}$$

Proof. Let μ be a Poisson point measure with distribution $\text{Pois}(u, W^*)$, and $\mu^{(1)}$, $\mu^{(2)}$ and $\mu^{(3)}$ be independent Poisson point measures with distribution $\text{Pois}(u/3, W^*)$. Let $\mathbb{P}$ be the joint law of $\mu^{(i)}$. By property (4) of $\text{Pois}(u, W^*)$, the measure μ has the same law as $\sum_{i=1}^3 \mu^{(i)}$. Therefore, we may assume that $\mu = \sum_{i=1}^3 \mu^{(i)}$, so that the random interlacement graph $\mathscr{I} = \mathscr{I}(\mu)$ equals $\bigcup_{i=1}^3 \mathscr{I}^{(i)}$, where $\mathscr{I}^{(i)} = \mathscr{I}(\mu^{(i)})$ are independent random interlacement graphs at level $u/3$. In particular, the vertices and edges of $\mathscr{I}$ are the ones of $\mathbb{Z}^d$ that are traversed by at least one of the random walks from $\bigcup_{i=1}^3 \mathrm{Supp}(\mu^{(i)})$.

Let $\mathscr{C}^{(i)}(x,R)$ be the connected component of x in $\mathscr{I}^{(i)} \cap B(x,R)$. In particular, $\mathscr{C}^{(i)}(x,R) \subseteq \mathscr{C}(x,R)$, but it is not true in general that $\bigcup_{i=1}^3 \mathscr{C}^{(i)}(x,R) = \mathscr{C}(x,R)$. Since R is fixed throughout the proof, we write $\mathscr{C}^{(i)}(x)$ for $\mathscr{C}^{(i)}(x,R)$. We have for $x,y \in B(R)$,

$$\mathbb{P}\left[x,y \in \mathscr{I}, \left\{x \overset{\mathscr{I}\cap B(CR)}{\longleftrightarrow} y\right\}^c\right] \le \sum_{i,j=1}^{3} \mathbb{P}\left[x \in \mathscr{I}^{(i)}, y \in \mathscr{I}^{(j)}, \left\{x \overset{\mathscr{I}\cap B(CR)}{\longleftrightarrow} y\right\}^c\right].$$

For each $i,j \in \{1,2,3\}$, choose $k \in \{1,2,3\}$ which is different from i and j. By construction, the set $\mathscr{I}^{(k)}$ is independent from $\mathscr{I}^{(i)}$ and $\mathscr{I}^{(j)}$. For each such i, j, and k, we obtain

$$\mathbb{P}\left[x \in \mathscr{I}^{(i)}, y \in \mathscr{I}^{(j)}, \left\{x \overset{\mathscr{I}\cap B(CR)}{\longleftrightarrow} y\right\}^c\right] \le \mathbb{P}\left[x \in \mathscr{I}^{(i)}, y \in \mathscr{I}^{(j)}, \left\{\mathscr{C}^{(i)}(x) \overset{\mathscr{I}^{(k)}\cap B(CR)}{\longleftrightarrow} \mathscr{C}^{(j)}(y)\right\}^c\right].$$

We define the events $E_1 \subseteq \{x \in \mathscr{I}^{(i)}\}$ and $E_2 \subseteq \{y \in \mathscr{I}^{(j)}\}$ as

$$E_1 = \left\{\mathrm{cap}\left(\mathscr{C}^{(i)}(x)\right) > cR^{2(d-2)/3}\right\}, \text{ and } E_2 = \left\{\mathrm{cap}\left(\mathscr{C}^{(j)}(y)\right) > cR^{2(d-2)/3}\right\}.$$

We denote the intersection $E_1 \cap E_2$ by E. By Lemma 10 (with $\varepsilon = 1/3$), we get

$$\mathbb{P}\left[\left\{x \in \mathscr{I}^{(i)}\right\} \setminus E_1\right] + \mathbb{P}\left[\left\{y \in \mathscr{I}^{(j)}\right\} \setminus E_2\right] \le Ce^{-cR^{1/6}}.$$

Note that $\mathscr{C}^{(i)}(x)$ and $\mathscr{C}^{(j)}(y)$ are subsets of $B(2R)$. Therefore, it follows from Lemma 12 and the independence of $\mathscr{I}^{(k)}$ from $\mathscr{I}^{(i)}$ and $\mathscr{I}^{(j)}$, that

$$\begin{aligned}\mathbb{P}\left[E \setminus \left\{\mathscr{C}^{(i)}(x) \overset{\mathscr{I}^{(k)}\cap B(CR)}{\longleftrightarrow} \mathscr{C}^{(j)}(y)\right\}\right] &\le C\mathbb{E}\left[\exp\left(-cR^{2-d}\mathrm{cap}\left(\mathscr{C}^{(i)}(x)\right)\mathrm{cap}\left(\mathscr{C}^{(j)}(y)\right)\right); E\right] \\ &\le C\exp\left(-cR^{(d-2)/3}\right).\end{aligned}$$

Putting the bounds together gives the result. □

Proof of Proposition 1. We will use a standard covering argument to derive (1.1) from (4.16). Take the constant C from the statement of Lemma 13. It suffices to prove (1.1) for $R \ge 2C$. Let $R' = \lfloor R/2C \rfloor$. For each $z \in \mathbb{Z}^d$, we define the events

$$A_z^{(1)} = \{\mathscr{I} \cap B(z,R') \neq \emptyset\}, \qquad A_z^{(2)} = \bigcap_{x,y \in \mathscr{I}\cap B(z,2R')} \left\{x \overset{\mathscr{I}\cap B(z,R)}{\longleftrightarrow} y\right\},$$

and $A = \bigcap_{z \in B(R)} A_z^{(1)} \cap A_z^{(2)}$. It follows from property (1) of $\mathrm{Pois}(u,W^*)$ and (2.10) that

$$\mathbb{P}(A_z^{(1)}) = 1 - e^{-u\mathrm{cap}(B(z,R'))} \ge 1 - e^{-cR},$$

and from Lemma 13 that

$$\mathbb{P}(A_z^{(2)}) \ge 1 - Ce^{-cR^{1/6}}.$$

In particular, $\mathbb{P}(A) \ge 1 - C'\exp(-cR^{1/6})$. It remains to note that A implies the event in (1.1). Indeed, for all $z,z' \in B(R)$ with $|z - z'| = 1$, $B(z,R') \cup B(z',R') \subseteq B(z,2R')$; thus if A occurs then every vertex in the non-empty set $\mathscr{I} \cap B(z,R')$ is connected to every vertex in the non-empty set $\mathscr{I} \cap B(z',R')$ by a path in $\mathscr{I} \cap B(z,R) \subseteq B(2R)$. Since any two vertices in $B(R)$ are connected by a nearest-neigbor path in $B(R)$, A implies the event in (1.1). The result follows. □

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