DOI: 10.1214/ECP.v16-1621

Elect. Comm. in Probab. **16** (2011), 200–212

ELECTRONIC COMMUNICATIONS in PROBABILITY

REPRESENTATIONS OF URBANIK'S CLASSES AND MULTIPARAMETER ORNSTEIN-UHLENBECK PROCESSES

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Submitted November 26, 2010, *accepted in final form* March 9, 2011

AMS 2000 Subject classification: 60G51, 60G57, 60H05
Keywords: Lévy bases, stochastic integrals, Urbanik's classes, multiparameter Ornstein-Uhlenbeck processes

Abstract
A class of integrals with respect to homogeneous Lévy bases on $\mathbb{R}^k$ is considered. In the one-dimensional case $k = 1$ this class corresponds to the selfdecomposable distributions. Necessary and sufficient conditions for existence as well as some representations of the integrals are given. Generalizing the one-dimensional case it is shown that the class of integrals corresponds to Urbanik's class $L_{k-1}(\mathbb{R})$. Finally, multiparameter Ornstein-Uhlenbeck processes are defined and studied.

1 Introduction

The purpose of this note is twofold. First of all, for any integer $k \geq 1$ we study the integral

$$\int_{\mathbb{R}_+^k} e^{-t_\cdot} M(\mathrm{d}t), \tag{1.1}$$

where $M = \{M(A) : A \in \mathscr{B}_b(\mathbb{R}^k)\}$ is a homogeneous Lévy basis on $\mathbb{R}^k$ and $t_\cdot = t_1 + \cdots + t_k$ is the sum of the coordinates. Recall that a homogeneous Lévy basis is an example of an independently scattered random measure as defined in [9]; see the next section for further details. The one-dimensional case $k = 1$, where M is induced by a Lévy process, is very well studied; see [10] for a survey. For example, in case of existence when $k = 1$ the integral has a selfdecomposable distribution ([7, 14]) and it is thus the marginal distribution of a stationary Ornstein-Uhlenbeck process. Moreover, necessary and sufficient conditions for the existence of (1.1) for $k = 1$ are also well-known. In the present note we give necessary and sufficient conditions for the existence of (1.1) for arbitrary k and provide several representations of the integral. The main result, Theorem 3.1, shows that for arbitrary $k \geq 1$ the law of (1.1) belongs to Urbanik's class $L_{k-1}(\mathbb{R})$

and conversely that any distribution herein is representable as in (1.1). The proof of the main theorem is in fact very easy. It relies only on a transformation rule for random measures (see Lemma 2.1) and well-known representations of Urbanik's classes.
Assuming that (1.1) exists we may define a process $Y = \{Y_t : t \in \mathbb{R}^k\}$ as

$$Y_t = \int_{s \le t} e^{-(t_\cdot - s_\cdot)} M(\mathrm{d}s),$$

where $s \le t$ should be understood coordinatewise. The second purpose of the note is to study some of the basic properties of this process. It is easily seen that Y is stationary and can be chosen lamp, where we recall that the latter is the multiparameter analogue of being càdlàg. In the case $k = 1$, Y is often referred to as an Ornstein-Uhlenbeck process and we shall thus call Y a k-parameter Ornstein-Uhlenbeck process. In the case $k = 1$, Y is representable as

$$Y_t = Y_0 - \int_0^t Y_s \,\mathrm{d}s + M((0,t]) \quad \text{for } t \ge 0.$$

We give the analogous formula to this equation in the case $k = 2$.
In the Gaussian case, Hirsch and Song [5] gave an alternative definition of k-parameter Ornstein-Uhlenbeck processes; however, in Remark 4.2(2) we show that the two definitions give rise to the same processes.
The next section contains a few preliminary results. Section 3 concerns the main result, namely characterizations of (1.1). Finally, in Section 4 we study multiparameter Ornstein-Uhlenbeck processes.

2 Preliminaries

Let Leb denote Lebesgue measure on $\mathbb{R}^k$. Throughout this note all random variables are defined on a probability space $(\Omega, \mathcal{F}, P)$. The law of a random vector is denoted by $\mathcal{L}(X)$ and for a set N and two families $\{X_t : t \in N\}$ and $\{Y_t : t \in N\}$ of random vectors write $\{X_t : t \in N\} \overset{\mathcal{D}}{=} \{Y_t : t \in N\}$ if all finite dimensional marginals are identical. Furthermore, we say that $\{X_t : t \in N\}$ is a modification of $\{Y_t : t \in N\}$ if $X_t = Y_t$ a.s. for all $t \in N$. Let $\mathrm{ID} = \mathrm{ID}(\mathbb{R})$ denote the class of infinitely divisible distributions on $\mathbb{R}$. That is, a distribution μ on $\mathbb{R}$ is in ID if and only if

$$\widehat{\mu}(z) := \int_{\mathbb{R}} e^{\mathrm{i}zx} \mu(\mathrm{d}x) = \exp\left[-\frac{1}{2} z^2 \sigma^2 + \mathrm{i}\gamma z + \int_{\mathbb{R}} g(z,x)\, \nu(\mathrm{d}x) \right] \quad \text{for all } z \in \mathbb{R},$$

where $g(z,x) = e^{\mathrm{i}zx} - 1 - \mathrm{i}zx1_D(x)$, $D = [-1,1]$, and (σ^2, ν, γ) is the characteristic triplet of μ, that is, $\sigma^2 \ge 0$, ν is a Lévy measure on $\mathbb{R}$ and $\gamma \in \mathbb{R}$. For $t \ge 0$ and $\mu \in \mathrm{ID}$, μ^t denotes the distribution in ID with $\widehat{\mu^t} = \widehat{\mu}^t$.
For $S \in \mathcal{B}(\mathbb{R}^k)$ let $\mathcal{B}_b(S)$ be the set of bounded Borel sets in S. Let $\Lambda = \{\Lambda(A) : A \in \mathcal{B}_b(S)\}$ denote a family of (real valued) random variables indexed by $\mathcal{B}_b(S)$, the set of bounded Borel set in S. Following [9] we call Λ *an independently scattered random measure on* S if the following conditions are satisfied:

(i) $\Lambda(A_1), \ldots, \Lambda(A_n)$ are independent whenever $A_1, \ldots, A_n \in \mathcal{B}_b(S)$ are disjoint.

(ii) $\Lambda(\bigcup_{n=1}^\infty A_n) = \sum_{n=1}^\infty \Lambda(A_n)$ a.s. whenever $A_1, A_2, \ldots \in \mathcal{B}_b(S)$ are disjoint with $\bigcup_{n=1}^\infty A_n \in \mathcal{B}_b(S)$. Here the series converges almost surely.

(iii) $\mathscr{L}(\Lambda(A)) \in \mathrm{ID}$ for all $A \in \mathscr{B}_b(S)$.

If in addition there is a $\mu \in \mathrm{ID}$ such that $\mathscr{L}(\Lambda(A)) = \mu^{\mathrm{Leb}(A)}$ for all $A \in \mathscr{B}_b(S)$ then Λ is called *a homogeneous Lévy basis on S* and Λ is said to be *associated with μ*.

Note that in (ii) above there is a null set depending on the sequence $A_1, A_2, \ldots$. Thus, if Λ is an independently scattered random it is generally *not* true that for ω outside a set of probability zero the mapping $A \mapsto \Lambda(A)(\omega)$ is a (signed) measure when A is in a σ-field included in $\mathscr{B}_b(S)$. Thus, Λ is not a random measure in the sense of [8]. The problem of finding large subsets $\mathscr{A}$ of $\mathscr{B}_b(S)$ such that the mapping $\mathscr{A} \ni A \mapsto \Lambda(A)(\omega)$ is regular (in some sense) for ω outside a null set is studied in [1, 3].

Let Λ denote an independently scattered random measure on S. Recall from [9] that there exists a control measure λ for Λ and a family of characteristic triplets $(\sigma_s^2, \nu_s(\mathrm{d}x), \gamma_s)_{s \in S}$, measurable in s, such that, for $A \in \mathscr{B}_b(S)$, $\mathscr{L}(\Lambda(A))$ has characteristic triplet $(\sigma^2(A), \nu(A)(\mathrm{d}x), \gamma(A))$ given by

$$\sigma^2(A) = \int_A \sigma_s^2 \, \lambda(\mathrm{d}s), \quad \nu(A)(\mathrm{d}x) = \int_A \nu_s(\mathrm{d}x) \, \lambda(\mathrm{d}s), \quad \gamma(A) = \int_A \gamma_s \, \lambda(\mathrm{d}s). \tag{2.1}$$

If Λ is a homogeneous Lévy basis associated with μ then λ equals Leb and $(\sigma_s^2, \nu_s(\mathrm{d}x), \gamma_s) = (\sigma^2, \nu(\mathrm{d}x), \gamma)$ where (σ^2, ν, γ) is the characteristic triplet of μ.

As an example let $k = 1$. If Λ is a homogeneous Lévy basis on $\mathbb{R}$ associated with μ then for all $s \in \mathbb{R}$ the process $\{\Lambda((s, s+t]) : t \geq 0\}$ is a Lévy process in law in the sense of [12], p. 3. In particular it has a càdlàg modification which is a Lévy process and $\mathscr{L}(\Lambda((s, s+1])) = \mu$ for all $s \in \mathbb{R}$. Conversely, if $Z = \{Z_t : t \in \mathbb{R}\}$ is a Lévy process indexed by $\mathbb{R}$ (i.e. it is càdlàg with stationary independent increments) then $\Lambda = \{\Lambda(A) : A \in \mathscr{B}_b(\mathbb{R})\}$ defined as $\Lambda(A) = \int 1_A \, \mathrm{d}Z_s$ is a homogeneous Lévy basis. Similarly, a so-called natural additive process induces an independently scattered random measure; see [13].

Since an independently scattered random measure Λ on S does not in general induce a usual measure an ω-wise definition of the associated integral is not possible. Therefore, integration with respect to Λ will always be understood in the sense developed in [9]. Recall the definition cf. page 460 in [9]:

A function $f : S \to \mathbb{R}$ is called simple if there is an $n \geq 1$, $\alpha_i \in \mathbb{R}$ and $A_i \in \mathscr{B}_b(S)$ such that $f = \sum_{i=1}^n \alpha_i 1_{A_i}$. In this case define $\int_A f \, \mathrm{d}\Lambda = \sum_{i=1}^n \alpha_i \Lambda(A_i \cap A)$. In general, if $f : S \to \mathbb{R}$ is a measurable function then f is called Λ-integrable if there is a sequence $(f_n)_{n \geq 1}$ of simple functions such that (i) $f_n \to f$ λ-a.s., where λ is the control measure, and (ii) the sequence $\int_A f_n \, \mathrm{d}\Lambda$ converges in probability for every $A \in \mathscr{B}(S)$. In this case the limit in (ii) is called the integral of f over A and is denoted $\int_A f \, \mathrm{d}\Lambda$. The integral is well-defined, i.e., it does not dependent on the approximating sequence.

Let $\mathscr{B}_\Lambda$ denote the set of A in $\mathscr{B}(S)$ for which 1_A is Λ-integrable. Then $\mathscr{B}_\Lambda$ contains $\mathscr{B}_b(S)$ and we can extend Λ to $\mathscr{B}_\Lambda$ by setting

$$\Lambda(A) = \int 1_A \, \mathrm{d}\Lambda, \quad A \in \mathscr{B}_\Lambda.$$

Moreover, $\{\Lambda(A) : A \in \mathscr{B}_\Lambda\}$ is an independently scattered random measure; that is, (i)–(iii) above are satisfied when $\mathscr{B}_b(S)$ is replaced by $\mathscr{B}_\Lambda$. For $A \in \mathscr{B}_\Lambda$, $\Lambda(A)$ still has characteristic triplet given by (2.1).

Let $T \in \mathscr{B}(\mathbb{R}^d)$ for some d. Given an independently scattered random measure Λ on S and a function $\phi : S \to T$ satisfying $\phi^{-1}(B) \in \mathscr{B}_\Lambda$ for all $B \in \mathscr{B}_b(T)$, we can define an independently scattered random measure on T, called *the image of Λ under ϕ*, to be denoted $\Lambda_\phi = \{\Lambda_\phi(B) : B \in \mathscr{B}_b(T)\}$, as

$$\Lambda_\phi(B) = \Lambda(\phi^{-1}(B)), \quad B \in \mathscr{B}_b(T).$$

Similarly, if $\psi : S \to \mathbb{R}$ is measurable and locally bounded then $\tilde{\Lambda} = \{\tilde{\Lambda}(A) : A \in \mathscr{B}_b(S)\}$, defined as

$$\tilde{\Lambda}(A) = \int_A \psi \, \mathrm{d}\Lambda, \quad A \in \mathscr{B}_b(S),$$

is an independently scattered random measure on S.
Keeping this notation we have the following result.

Lemma 2.1. (1) *Let $g : S \to \mathbb{R}$ be measurable. Then g is $\tilde{\Lambda}$-integrable if and only if $g\psi$ is Λ-integrable and in this case $\int_S g \, \mathrm{d}\tilde{\Lambda} = \int_S (g\psi) \, \mathrm{d}\Lambda$.*

(2) *Let $f : T \to \mathbb{R}$ be measurable. Assume that $(\sigma_s^2, \nu_s(\mathrm{d}x), \gamma_s)$ in (2.1) only depends on s through ϕ; that is, there is a family of characteristic triplets $(\sigma_t^2, \nu_t(\mathrm{d}x), \gamma_t)_{t \in T}$, measurable in t, such that*

$$\begin{aligned} \sigma^2(A) &= \int_A \sigma_{\phi_s}^2 \, \lambda(\mathrm{d}s), \\ \nu(A)(\mathrm{d}x) &= \int_A \nu_{\phi_s}(\mathrm{d}x) \, \lambda(\mathrm{d}s), \\ \gamma(A) &= \int_A \gamma_{\phi_s} \, \lambda(\mathrm{d}s). \end{aligned} \tag{2.2}$$

Then f is Λ_ϕ-integrable if and only if $f \circ \phi$ is Λ-integrable and in this case $\int_T f \, \mathrm{d}\Lambda_\phi = \int_S (f \circ \phi) \, \mathrm{d}\Lambda$. In general, that is, when (2.2) is not satisfied, only the if-part of the statement is true.

Proof. Using [9], Theorem 2.7(iv), and formula (2.1) we see that for every $A \in \mathscr{B}_b(S)$ the characteristic triplet $(\tilde{\sigma}^2(A), \tilde{\nu}(A)(\mathrm{d}x), \tilde{\gamma}(A))$ of $\mathscr{L}(\tilde{\Lambda}(A))$ is given by

$$\tilde{\sigma}^2(A) = \int_A \tilde{\sigma}_s^2 \, \lambda(\mathrm{d}s), \quad \tilde{\nu}(A)(\mathrm{d}x) = \int_A \tilde{\nu}_s(\mathrm{d}x) \, \lambda(\mathrm{d}s), \quad \tilde{\gamma}(A) = \int_A \tilde{\gamma}_s \, \lambda(\mathrm{d}s),$$

where

$$\tilde{\sigma}_s^2 = \psi_s^2 \sigma_s^2, \quad \tilde{\gamma}_s = \psi_s \gamma_s + \int_{\mathbb{R}} \psi_s x [1_D(\psi_s x) - 1_D(x)] \, \nu_s(\mathrm{d}x)$$

$$\tilde{\nu}_s(B) = \int_{\mathbb{R}} 1_B(\psi_s x) \, \nu_s(\mathrm{d}x), \quad B \in \mathscr{B}(\mathbb{R} \setminus \{0\}).$$

According to [9], Theorem 2.7, $g\psi$ is Λ-integrable if and only if the following three conditions are satisfied:

$$\int_S (\psi_s g_s)^2 \sigma_s^2 \, \lambda(\mathrm{d}s) < \infty \tag{2.3}$$

$$\int_S \int_{\mathbb{R}} (1 \wedge [\psi_s g_s x]^2) \, \nu_s(\mathrm{d}x) \, \lambda(\mathrm{d}s) = \int_S \int_{\mathbb{R}} (1 \wedge [g_s x]^2) \tilde{\nu}_s(\mathrm{d}x) \, \lambda(\mathrm{d}s) < \infty \tag{2.4}$$

$$\int_S |\psi_s g_s| \cdot |\gamma_s + \int_{\mathbb{R}} x [1_D(\psi_s g_s x) - 1_D(x)] \, \nu_s(\mathrm{d}x)| \, \lambda(\mathrm{d}s) < \infty. \tag{2.5}$$

Similarly, g is $\tilde{\Lambda}$-integrable if and only if (2.3) and (2.4) are satisfied and

$$\int_S |g_s\tilde{\gamma}_s + \int_{\mathbb{R}} (g_s x)[1_D(g_s x) - 1_D(x)]\,\tilde{\nu}_s(\mathrm{d}x)|\,\lambda(\mathrm{d}s) < \infty. \tag{2.6}$$

But noticing that the last integral equals

$$\begin{aligned}\int_S |g_s\psi_s|\cdot|\gamma_s + \int_{\mathbb{R}} x[1_D(\psi_s x) - 1_D(x)]\,\nu_s(\mathrm{d}x)\\ + \int_{\mathbb{R}} x[1_D(g_s\psi_s x) - 1_D(x\psi_s)]\,\nu_s(\mathrm{d}x)|\,\lambda(\mathrm{d}s)\\ = \int_S |\psi_s g_s|\cdot|\gamma_s + \int_{\mathbb{R}} x[1_D(\psi_s g_s x) - 1_D(x)]\,\nu_s(\mathrm{d}x)|\,\lambda(\mathrm{d}s),\end{aligned}$$

one sees that (2.5) and (2.6) are equivalent. Thus we have (1).
(2): Assume (2.2). Then $\mathscr{L}(\Lambda_\phi(B))$, $B \in \mathscr{B}_b(T)$, has characteristic triplet given by

$$\Big(\int_B \sigma_t^2\,\lambda_\phi(\mathrm{d}t),\quad \int_B \nu_t(\mathrm{d}x)\,\lambda_\phi(\mathrm{d}s),\quad \int_B \gamma_t\,\lambda_\phi(\mathrm{d}t)\Big),$$

where λ_ϕ is the image measure of λ under the the function ϕ. Thus, the first part of (2) follows from [9], Theorem 2.7, using the ordinary transformation rule.
In the general case, without (2.2), $f\circ\phi$ is Λ-integrable if and only if there is a sequence of simple functions g_n approximating $f\circ\phi$ such that $\int_A g_n\,\mathrm{d}\Lambda$ converges in probability for all $A \in \mathscr{B}(S)$. Assume this is the case. By the explicit construction in the proof of [9], Theorem 2.7, we can choose g_n on the form $g_n = h_n\circ f\circ\phi$ where h_n is simple. Thus, since, by definition, $\int_B (h_n\circ f)\,\mathrm{d}\Lambda_\phi = \int_{\phi^{-1}(B)}(h_n\circ f\circ\phi)\,\mathrm{d}\Lambda$ for all n and all $B \in \mathscr{B}(T)$ it follows by definition of integrability that f is Λ_ϕ-integrable. □

Before continuing we recall a few basic properties of the class of selfdecomposable distributions and the classes L_m. See e.g. [6, 7, 11, 14, 15, 16] for fuller information, and [10] for a nice summary of the results used below. Let $L_0 = L_0(\mathbb{R})$ denote the class of selfdecomposable distributions on $\mathbb{R}$. Recall, e.g. from [12], Theorem 15.3, that a probability measure μ is in L_0 if and only if it is the limit in distribution as $n\to\infty$ of variables $a_n + b_n S_n$ where a_n, b_n are real numbers, $S_n = \sum_{i=1}^n Z_i$ and $(Z_i)_{i\geq 1}$ is a sequence of independent random variables. For $m = 1, 2, \ldots$ define $L_m = L_m(\mathbb{R})$ recursively as follows: $\mu \in L_m$ if and only if for all $b > 1$ there is a $\rho_b \in L_{m-1}$ such that $\widehat{\mu}(z) = \widehat{\mu}(b^{-1}z)\widehat{\rho}_b(z)$ for all $z \in \mathbb{R}$. It is well-known (see e.g. [10], Propositions 5 and 11) that the sets L_m are decreasing in m and that the stable distributions are in L_m for all m.
Let $\mathrm{ID}_{\log}$ denote the class of infinitely divisible distributions μ with Lévy measure ν satisfying $\int_{|x|>2}\log|x|\,\nu(\mathrm{d}x) < \infty$. It is well-known that $\mathrm{ID}_{\log}$ consists precisely of those $\mu \in \mathrm{ID}$ for which the integral $\int_0^\infty e^{-s}\,\mathrm{d}Z_s$ exists. Here $\{Z_t : t \geq 0\}$ is a Lévy process with $\mu = \mathscr{L}(Z_1)$. In case of existence $\mathscr{L}(\int_0^\infty e^{-s}\,\mathrm{d}Z_s)$ is in L_0. Using this, an alternative useful characterization of L_m can be formulated as follows: Let $\Phi : \mathrm{ID}_{\log} \to L_0$ be given by $\Phi(\mu) = \mathscr{L}(\int_0^\infty e^{-s}\,\mathrm{d}Z_s)$ where Z is as above. Then Φ is one-to-one and onto L_0. Moreover, for $m = 1, 2, \ldots$ we have $L_m = \Phi(L_{m-1}\cap \mathrm{ID}_{\log})$.

3 Existence and characterizations of the integral

Assume that $M = \{M(A) : A \in \mathscr{B}_b(\mathbb{R}_+^k)\}$ is a homogeneous Lévy basis on $\mathbb{R}_+^k$ associated with $\mu \in \mathrm{ID}$ which has characteristic triplet (σ^2, ν, γ).
Let $f : \mathbb{R}_+^k \to \mathbb{R}_+$ be given by $f(t) = t_\cdot = \sum_{j=1}^k t_j$ and let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be given by $g(x) = \frac{x^k}{k!}$. Then the image of M under f, $M_f = \{M_f(B) : B \in \mathscr{B}_b(\mathbb{R}_+)\}$, is an independently scattered random measure on $\mathbb{R}_+$ and $\mathscr{L}(M_f((0,x])) = \mu^{x^k/t!}$ for all $x \geq 0$. Since in particular

$$\mathscr{L}(M_f(g^{-1}([0,y]))) = \mathscr{L}(M_f([0,g^{-1}(y)])) = \mu^y \quad \text{for } y \geq 0,$$

it follows that $M_{g\circ f} = \{M_{g\circ f}(B) : B \in \mathscr{B}_b(\mathbb{R}_+)\}$ is a homogeneous Lévy basis on $\mathbb{R}_+$ associated with μ. Writing $\Phi^{(k)}$ for $\Phi \circ \cdots \circ \Phi$ (k times), the main result can be formulated as follows.

Theorem 3.1. (1) *The three integrals*

$$\int_{\mathbb{R}_+^k} e^{-t_\cdot}\, M(\mathrm{d}t), \quad \int_{\mathbb{R}_+} e^{-x}\, M_f(\mathrm{d}x), \quad \int_{\mathbb{R}_+} e^{-(k!y)^{1/k}}\, M_{g\circ f}(\mathrm{d}y) \tag{3.1}$$

exist at the same time and are identical in case of existence. Assume existence and let $\tilde{\mu} = \mathscr{L}(\int_{\mathbb{R}_+^k} e^{-t_\cdot}\, M(\mathrm{d}t))$. *Then* $\tilde{\mu} = \Phi^{(k)}(\mu) \in L_{k-1}$ *and* $\tilde{\mu}$ *has characteristic triplet* $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ *given by*

$$\tilde{\sigma}^2 = \frac{\sigma^2}{2^k} \tag{3.2}$$

$$\tilde{\nu}(B) = \frac{1}{(k-1)!} \int_{\mathbb{R}} \int_0^\infty s^{k-1} 1_B(e^{-s}y)\, \mathrm{d}s\, \nu(\mathrm{d}y), \quad B \in \mathscr{B}(\mathbb{R}) \tag{3.3}$$

$$\tilde{\gamma} = \gamma + \frac{1}{(k-1)!} \int_0^\infty s^{k-1} e^{-s} \int_{1<|y|\leq e^s} \nu(\mathrm{d}y)\, \mathrm{d}s. \tag{3.4}$$

(2) *A necessary and sufficient condition for the existence of the integrals in* (3.1) *is that*

$$\int_{|x|>2} (\log|x|)^k\, \nu(\mathrm{d}x) < \infty. \tag{3.5}$$

(3) *Let* $k \geq 2$ *and assume that the integrals in* (3.1) *exist. Then*

$$\int_{\mathbb{R}_+^k} e^{-t_\cdot}\, M(\mathrm{d}t) = \int_{\mathbb{R}_+} e^{-x}\, \Lambda(\mathrm{d}x), \tag{3.6}$$

where $\Lambda = \{\Lambda(B) : B \in \mathscr{B}_b(\mathbb{R}_+)\}$ *is given as*

$$\Lambda(B) = \int_{B\times\mathbb{R}_+^{k-1}} e^{-\sum_{l=2}^k t_l}\, M(\mathrm{d}t), \quad B \in \mathscr{B}_b(\mathbb{R}_+). \tag{3.7}$$

Moreover, Λ *is a homogeneous Lévy basis on* $\mathbb{R}_+$. *The distribution associated with* Λ *is* $\Phi^{(k-1)}(\mu)$ *which has characteristic triplet given by* (3.2)–(3.4) *with* k *replaced by* $k-1$.

(4) *Conversely, to every distribution* $\tilde{\mu} \in L_{k-1}$ *there is a distribution* $\mu \in \mathrm{ID}$ *with characteristic triplet* (σ^2, ν, γ) *with* ν *satisfying* (3.5) *such that* $\tilde{\mu}$ *is given by* $\tilde{\mu} = \mathscr{L}(\int_{\mathbb{R}_+^k} e^{-t\cdot} M(\mathrm{d}t))$ *where* M *is a homogeneous Lévy basis on* $\mathbb{R}_+^k$ *associated with* μ.

Proof. We can apply the first part of Lemma 2.1(2) to the first two integrals in (3.1) since Λ is homogeneous. Likewise, the lemma applies to the last two integrals since g is one-to-one. It hence follows immediately that the three integrals in (3.1) exist at the same time and are identical in case of existence. The remaining assertions, except (3), follow from Theorem 49 and Remark 58 of Rocha-Arteaga and Sato [10]. First of all, by Remark 58 a distribution is in L_{k-1} if and only if it is representable as the law of the last integral in (3.1). That $\tilde{\mu} = \Phi^{(k)}(\mu)$ in case of existence follows from Remark 58 combined with Theorem 49. Using this, the result in (2) is equation (2.44) in Theorem 49, and the representation of the characteristic triplet in (1) is (2.47)–(2.49) in Theorem 49.
To prove (3) assume $t \mapsto e^{-t\cdot}$ is M-integrable and let $\tilde{M}(A) = \int_A \psi(t) M(\mathrm{d}t)$ for $A \in \mathscr{B}_b(\mathbb{R}_+^k)$ where $\psi(t) = e^{-\sum_{l=2}^k t_l}$. Since for $B \in \mathscr{B}_b(\mathbb{R}_+)$ there is a constant $c > 0$ such that $\psi(t) 1_{B\times\mathbb{R}_+^{k-1}}(t) \le c e^{-t\cdot} 1_{B\times\mathbb{R}_+^{k-1}}(t)$ we have by Lemma 2.1(1) that $B \times \mathbb{R}_+^{k-1} \in \mathscr{B}_{\tilde{M}}$. Thus, Λ in (3.7) is well-defined and a homogeneous Lévy basis. Moreover, $\Lambda = (\tilde{M})_\phi$ where $\phi : \mathbb{R}_+^k \to \mathbb{R}_+$ is $\phi(t) = t_1$. Since, with $f : \mathbb{R}_+ \to \mathbb{R}_+$ given by $f(x) = e^{-x}$, the mapping $f \circ \phi(t) = e^{-t_1}$ is $\tilde{M}$-integrable by Lemma 2.1(1) it follows from the last part of Lemma 2.1(2) that f is Λ-integrable and we have (3.6). That is, $\tilde{\mu}$ in (1) is of the form $\tilde{\mu} = \mathscr{L}(\int_0^\infty e^{-s}\,\mathrm{d}Z_s)$ where $Z = \{Z_t : t \ge 0\}$ is the Lévy process in law given by $Z_t = \Lambda((0,t])$. As previously noted this means that $\tilde{\mu} = \Phi(\mathscr{L}(Z_1))$. But since, by (1), $\tilde{\mu} = \Phi^{(k)}(\mu)$ it follows that $\mathscr{L}(Z_1) = \Phi^{(k-1)}(\mu)$, i.e. $\mathscr{L}(\Lambda((0,1])) = \Phi^{(k-1)}(\mu)$. □

4 Multiparameter Ornstein-Uhlenbeck processes

For $a = (a_1, \dots, a_k) \in \mathbb{R}^k$ and $b = (b_1, \dots, b_k) \in \mathbb{R}^k$ write $a \le b$ if $a_j \le b_j$ for all j and $a < b$ if $a_j < b_j$ for all j. Define the half-open interval $(a, b]$ as $(a, b] = \{t \in \mathbb{R}^k : a < t \le b\}$ and let $[a, b] = \{t \in \mathbb{R}^k : a \le t \le b\}$. Further, let $\mathscr{A} = \{t \in \mathbb{R}_+^k : t_j = 0 \text{ for some } j\}$, and for $\mathscr{R} = (R_1, \dots, R_k)$ where R_j is either $\le$ or $>$ write $a \mathscr{R} b$ if $a_j R_j b_j$ for all j.
Consider a family $F = \{F_t : t \in S\}$ where S is either $\mathbb{R}^k$ or $\mathbb{R}_+^k$ and $F_t \in \mathbb{R}$ for all $t \in S$. For $a, b \in S$ with $a \le b$ define *the increment of* F *over* $(a, b]$, $\Delta_a^b F$, as

$$\Delta_a^b F = \sum_{\epsilon=(\epsilon_1,\dots,\epsilon_k)\in\{0,1\}^k} (-1)^{\epsilon\cdot} F_{(c^1(\epsilon_1),\dots,c^k(\epsilon_k))},$$

where $c^j(0) = b_j$ and $c^j(1) = a_j$. That is, $\Delta_a^b F = F_b - F_a$ if $k = 1$, and $\Delta_a^b F = F_{(b_1,b_2)} + F_{(a_1,a_2)} - F_{(a_1,b_2)} - F_{(b_1,a_2)}$ if $k = 2$. Note that $\Delta_a^b F = 0$ if $a \le b$ and $b - a \in \mathscr{A}$.
We say that $F = \{F_t : t \in S\}$ is *lamp* if the following conditions are satisfied:

(i) for $t \in \mathbb{R}_+^k$ the limit $F(t, \mathscr{R}) = \lim_{u \to t, t\mathscr{R}u} F_u$ exists for each of the 2^k relations $\mathscr{R} = (R_1, \dots, R_k)$ where R_j is either $\le$ or $>$. When $S = \mathbb{R}_+^k$ let $F(t, \mathscr{R}) = F_t$ if there is no u with $t \mathscr{R} u$.

(ii) $F_t = F(t, \mathscr{R})$ for $\mathscr{R} = (\le, \dots, \le)$;

Here lamp stands for *limits along monotone paths*. See Adler et al. [2] for references to the literature on lamp trajectories. When $S = \mathbb{R}_+^k$, one often assumes in addition that $F_t = 0$ for $t \in \mathscr{A}$, i.e. F vanishes on the axes.

Assume that $M = \{M(A) : A \in \mathscr{B}_b(\mathbb{R}^k)\}$ is a homogeneous Lévy basis on $\mathbb{R}^k$ associated with $\mu \in \mathrm{ID}$ which has characteristic triplet (σ^2, ν, γ). Define $U = \{U_t : t \in \mathbb{R}^k_+\}$ and $X = \{X_t : t \in \mathbb{R}^k_+\}$ as

$$U_t = \int_{[0,t]} e^{s\cdot} M(\mathrm{d}s) \quad \text{and} \quad X_t = e^{-t\cdot} U_t \quad \text{for } t \in \mathbb{R}^k_+. \tag{4.1}$$

Since for $a, b \in \mathbb{R}^k_+$ with $a \leq b$,

$$\Delta_a^b U = \int_{(a,b]} e^{s\cdot} M(\mathrm{d}s),$$

the random variables $\Delta_{a^1}^{b^1} U, \ldots, \Delta_{a^n}^{b^n} U$ are independent whenever $(a^1, b^1], \ldots, (a^n, b^n]$ are disjoint intervals in $\mathbb{R}^k_+$. Moreover, U is continuous in probability since M is homogeneous. Thus, U is a Lévy process in the sense of Adler et al. [2], p. 5, and a Lévy sheet in the sense of Dalang and Walsh [4] (in the case $k = 2$). It hence follows e.g. from [2], Proposition 4.1, that by modification we may and do assume that U, and hence also X, is lamp. Similarly, we may and do assume that $t \mapsto M((0,t])$ is lamp for $t \in \mathbb{R}^k_+$.

Assuming in addition that (3.5) is satisfied we can define processes $V = \{V_t : t \in \mathbb{R}^k\}$ and $Y = \{Y_t : t \in \mathbb{R}^k\}$ as

$$V_t = \int_{s \leq t} e^{s\cdot} M(\mathrm{d}s) \quad \text{and} \quad Y_t = e^{-t\cdot} V_t \quad \text{for } t \in \mathbb{R}^k. \tag{4.2}$$

For fixed $t \in \mathbb{R}^k$ define $\phi^t : \mathbb{R}^k \to \mathbb{R}^k$ as $\phi^t(s) = t - s$. By Lemma 2.1 and the fact that M and M_{ϕ^t} are homogeneous Lévy bases associated with μ we have

$$Y_t = \int_{s \leq t} e^{-(t\cdot - s\cdot)} M(\mathrm{d}s) = \int_{\mathbb{R}^k_+} e^{-s\cdot} M_{\phi^t}(\mathrm{d}s) \overset{\mathscr{D}}{=} \int_{\mathbb{R}^k_+} e^{-s\cdot} M(\mathrm{d}s) \quad \text{for } t \in \mathbb{R}^k.$$

That is, Y_t has the same law as the three integrals in (3.1). The same kind of arguments show that Y is stationary in the sense that

$$(Y_{t^1}, \ldots, Y_{t^n}) \overset{\mathscr{D}}{=} (Y_{t+t^1}, \ldots, Y_{t+t^n}) \quad \text{for all } n \geq 1 \text{ and } t, t^1, \ldots, t^n \in \mathbb{R}^k.$$

When $k = 1$, Y is often referred to as an Ornstein-Uhlenbeck process. The above is a natural generalization so we shall call Y a *k-parameter Ornstein-Uhlenbeck process*. There are many nice representations and properties of Y as the next remarks illustrate.

Remark 4.1. Denote a generic element in $\mathbb{R}^{k-1}$ by $\tilde{t} = (t_1, \ldots, t_{k-1})$ and let $\tilde{t}_\cdot = \sum_{j=1}^{k-1} t_j$. A generic element t in $\mathbb{R}^k$ can then be decomposed as $t = (\tilde{t}, t_k)$. For $B \in \mathscr{B}(\mathbb{R})$ and $\tilde{t} \in \mathbb{R}^{k-1}$, $\{\tilde{s} \leq \tilde{t}\} \times B$ is the subset of $\mathbb{R}^k$ given by

$$\{\tilde{s} \leq \tilde{t}\} \times B = \{s = (\tilde{s}, s_k) : \tilde{s} \leq \tilde{t} \text{ and } s_k \in B\}.$$

Assuming that (3.5) is satisfied, Y_t is representable as

$$Y_t = e^{-t_k} \int_{-\infty}^{t_k} e^{s_k} M^{\tilde{t}}(\mathrm{d}s_k) = e^{-\tilde{t}\cdot} \int_{\tilde{s} \leq \tilde{t}} e^{\tilde{s}\cdot} M^{t_k}(\mathrm{d}\tilde{s}). \tag{4.3}$$

Here $M^{\tilde{t}} = \{M^{\tilde{t}}(B) : B \in \mathscr{B}_b(\mathbb{R})\}$ and $M^{t_k} = \{M^{t_k}(C) : C \in \mathscr{B}_b(\mathbb{R}^{k-1})\}$ are given as

$$M^{\tilde{t}}(B) = e^{-\tilde{t}\cdot} \int_{\{\tilde{s}\le\tilde{t}\}\times B} e^{\tilde{s}\cdot}\, M(\mathrm{d}s), \quad B \in \mathscr{B}_b(\mathbb{R})$$

$$M^{t_k}(C) = e^{-t_k} \int_{C\times(\infty,t_k]} e^{s_k}\, M(\mathrm{d}s), \quad C \in \mathscr{B}_b(\mathbb{R}^{k-1}).$$

Arguments as in the proof of Theorem 3.1(3) show that $M^{\tilde{t}}$ and M^{t_k} are well-defined homogeneous Lévy bases associated with respectively $\Phi^{(k-1)}(\mu)$ and $\Phi(\mu)$, and we have (4.3).
The first expression in (4.3) shows that for fixed $\tilde{t}$, $\{Y_t : t_k \in \mathbb{R}\}$ is a one-parameter Ornstein-Uhlenbeck process. By the second expression, $\{Y_t : \tilde{t} \in \mathbb{R}^{k-1}\}$ is a $(k-1)$-parameter Ornstein-Uhlenbeck process for fixed t_k.

Remark 4.2. (1) Assume $\int_{\mathbb{R}} x^2\, \nu(\mathrm{d}x) < \infty$. By [12], Corollary 25.8, μ has finite second moment. Moreover, denoting the variance by Var, we have $\mathrm{Var}(M(A)) = \mathrm{Leb}(A)(\sigma^2 + \int_{\mathbb{R}} x^2\, \nu(\mathrm{d}x))$ for $A \in \mathscr{B}_b(\mathbb{R}^k)$. In this case (3.5) is satisfied, implying that Y is well-defined. The characteristic triplet $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ of $\mathscr{L}(Y_t)$, $t \in \mathbb{R}^k$, is given in (3.2)–(3.4). Since, by (3.3), $\int_{\mathbb{R}} x^2\, \tilde{\nu}(\mathrm{d}x) = 2^{-k} \int_{\mathbb{R}} x^2\, \nu(\mathrm{d}x) < \infty$ it follows that Y_t is square-integrable with

$$\mathrm{Var}(Y_t) = \frac{1}{2^k}(\sigma^2 + \int_{\mathbb{R}} x^2\, \nu(\mathrm{d}x)).$$

Let us find the covariance function of Y. Let, for $j = 1, 2$, $t^j = (t_1^j, \ldots, t_k^j) \in \mathbb{R}^k$. Set $t = (t_1^1 \wedge t_1^2, \ldots, t_k^1 \wedge t_k^2) \in \mathbb{R}^k$, $D^j = \{s \in \mathbb{R}^k : s \le t^j\}$ and $D = \{s \in \mathbb{R}^k : s \le t\} = D^1 \cap D^2$. For $j = 1, 2$,

$$Y_t = e^{-(t^j_\cdot - t_\cdot)} Y_t + e^{-t_\cdot} \int_{D^j \setminus D} e^{s_\cdot}\, M(\mathrm{d}s). \tag{4.4}$$

Since $D^1 \setminus D$ and $D^2 \setminus D$ are disjoint the last term on the right-hand side of (4.4) with $j = 1$ is independent of the corresponding term with $j = 2$. Thus,

$$\mathrm{Cov}(Y_{t^1}, Y_{t^2}) = e^{-(t^1_\cdot - t_\cdot)} e^{-(t^2_\cdot - t_\cdot)} \mathrm{Var}(Y_t). \tag{4.5}$$

(2) Now let ν and γ be zero. In this case Hirsch and Song [5] defined a k-parameter Ornstein-Uhlenbeck $\tilde{Y} = \{\tilde{Y}_t : t \in \mathbb{R}^k\}$ as $\tilde{Y}_t = e^{-t\cdot} M((0, e^{2t}])$ where $e^{2t} = (e^{2t_1}, \ldots, e^{2t_k})$. With t^1, t^2 and t given as under (1) we have

$$\mathrm{Cov}(\tilde{Y}_{t^1}, \tilde{Y}_{t^2}) = \sigma^2 e^{-(t^1_\cdot - t_\cdot)} e^{-(t^2_\cdot - t_\cdot)}.$$

Thus, from (4.5) and Gaussianity it follows that up to a scaling constant Y and $\tilde{Y}$ have the same distribution; in other words, in the Gaussian case our definition of a k-parameter Ornstein-Uhlenbeck process coincides essentially with that of [5].

Remark 4.3. Assume (3.5) is satisfied. We may and do assume that V, and thus also Y, is lamp. To see this, note that for arbitrary $s = (s_1, \ldots, s_k) \in \mathbb{R}^k$ and $t = (t_1, \ldots, t_k) \in \mathbb{R}^k_+ \setminus \mathscr{A}$ we have

$$V_{s+t} = \Delta_s^{s+t} V - \sum_{\substack{\epsilon\in\{0,1\}^k \\ \epsilon\neq(1,\ldots,1)}} (-1)^{\epsilon\cdot} V_{(s_1+\epsilon_1 t_1, \ldots, s_k+\epsilon_k t_k)}.$$

Hence it suffices to show:

(i) for arbitrary $s \in \mathbb{R}^k$ the process $\{\Delta_s^{s+t} V : t \in \mathbb{R}_+^k\}$ has a lamp modification.

(ii) If at least one coordinate is fixed, then V has a lamp modification in the remaining coordinates. That is, if e.g. $t_k = 0$ then $\tilde{t} = (t_1, \dots, t_{k-1}) \mapsto V_{(\tilde{t},0)}$ is a.s. lamp on $\mathbb{R}^{k-1}$.

Condition (i) follows as for U above since $\Delta_s^{s+t} V = \int_{(s,s+t]} e^{u\cdot}\, M(\mathrm{d}u)$. To check (ii) consider for simplicity the case where t_k is fixed at $t_k = 0$ while all other coordinates vary freely. As in Remark 4.1 we have

$$V_{(\tilde{t},0)} = \int_{\tilde{s} \le \tilde{t}} e^{\tilde{s}\cdot}\, \tilde{M}(\mathrm{d}\tilde{s}) \quad \text{for } \tilde{t} \in \mathbb{R}^{k-1},$$

where $\tilde{M} = \{\tilde{M}(B) : B \in \mathscr{B}_b(\mathbb{R}^{k-1})\}$ is the homogeneous Lévy basis on $\mathbb{R}^{k-1}$ given by

$$\tilde{M}(B) = \int_{B\times(-\infty,0]} e^{s_k}\, M(\mathrm{d}s), \quad B \in \mathscr{B}_b(\mathbb{R}^{k-1}).$$

By recursion we can reduce to $k = 1$ in which case $t \mapsto \int_{-\infty}^t e^s\, M(\mathrm{d}s)$ has a càdlàg modification, implying the result.

From now on let $k = 2$. Recall that X and Y are defined in (4.1) and (4.2). If $t = (t_1, t_2)$ write X_{t_1,t_2} as an alternative to X_t.

Proposition 4.4. *With probability one we have for all $t = (t_1, t_2) \in \mathbb{R}_+^2$ that*

$$X_{t_1,t_2} = M((0,t]) - \int_0^{t_1} X_{s_1,t_2}\,\mathrm{d}s_1 - \int_0^{t_2} X_{t_1,s_2}\,\mathrm{d}s_2 - \int_0^{t_1}\int_0^{t_2} X_{s_1,s_2}\,\mathrm{d}s_2\mathrm{d}s_1.$$

Assume that (3.5) *is satisfied. Then with probability one we have for all $t = (t_1, t_2) \in \mathbb{R}_+^2$ that*

$$\begin{aligned} Y_{t_1,t_2} = {} & Y_{0,t_2} + Y_{t_1,0} - Y_{0,0} + M((0,t]) - \int_0^{t_1} Y_{s_1,t_2}\,\mathrm{d}s_1 - \int_0^{t_2} Y_{t_1,s_2}\,\mathrm{d}s_2 \\ & + \int_0^{t_1} Y_{s_1,0}\,\mathrm{d}s_2 + \int_0^{t_2} Y_{0,s_2}\,\mathrm{d}s_2 - \int_0^{t_1}\int_0^{t_2} Y_{s_1,s_2}\,\mathrm{d}s_2\mathrm{d}s_1. \end{aligned}$$

Proof. Since the proofs are similar we only prove the representation of Y.
First we fix $t_2 \in \mathbb{R}_+$. Arguing as in the proof of Theorem 3.1(3), cf. also Remark 4.1, we can represent $\{V_{t_1,t_2} : t_1 \ge 0\}$ as

$$V_{t_1,t_2} = V_{0,t_2} + \int_0^{t_1} e^{s_1}\, U^{t_2}(\mathrm{d}s_1),$$

where $U^{t_2} = \{U^{t_2}(B) : B \in \mathscr{B}_b(\mathbb{R}_+)\}$ is the homogeneous Lévy basis given by

$$U^{t_2}(B) = \int_{B\times(-\infty,t_2]} e^{s_2}\, M(\mathrm{d}s), \quad B \in \mathscr{B}_b(\mathbb{R}_+).$$

Thus, $\{V_{t_1,t_2} : t_1 \ge 0\}$ is a semimartingale in the filtration of the Lévy process $\{U^{t_2}((0,t_1]) : t_1 \ge 0\}$. Let $R = \{R_t : t \in \mathbb{R}_+^2\}$ be given by

$$R_t = \int_{(0,t_1]\times(-\infty,t_2]} e^{s_2}\, M(\mathrm{d}s) = U^{t_2}((0,t_1]) \quad \text{for } t \in \mathbb{R}_+^2.$$

Since R has independent increments and is continuous in probability we may and do assume that it is lamp, see [2].
Since for fixed t_2, $\{V_{t_1,t_2} : t_1 \geq 0\}$ is a semimartingale we can apply integration by parts, together with Lemma 2.1(1) and the fact that all terms are lamp, to obtain

$$\begin{aligned} e^{-t_1}V_t &= V_{0,t_2} + \int_0^{t_1} e^{-s_1}e^{s_1}\, U^{t_2}(\mathrm{d}s_1) - \int_0^{t_1} e^{-s_1}V_{s_1,t_2}\,\mathrm{d}s_1 \\ &= V_{0,t_2} + R_{t_1,t_2} - \int_0^{t_1} e^{-s_1}V_{s_1,t_2}\,\mathrm{d}s_1 \quad \text{for all } t=(t_1,t_2)\in\mathbb{R}^2_+ \quad a.s. \end{aligned} \tag{4.6}$$

The same kind of argument for t_2 instead of t_1 gives

$$\begin{aligned} Y_{t_1,t_2} &= e^{-t_2}V_{0,t_2} + e^{-t_2}R_{t_1,t_2} - \int_0^{t_1} Y_{s_1,t_2}\,\mathrm{d}s_1 \\ &= e^{-t_2}V_{0,t_2} + R_{t_1,0} + M((0,t]) - \int_0^{t_2} e^{-s_2}R_{t_1,s_2}\,\mathrm{d}s_2 \\ &\quad - \int_0^{t_1} Y_{s_1,t_2}\,\mathrm{d}s_1 \quad \text{for all } t=(t_1,t_2)\in\mathbb{R}^2_+ \quad a.s. \end{aligned}$$

From (4.6) we have

$$e^{-s_2}R_{t_1,s_2} = Y_{t_1,s_2} - Y_{0,s_2} + \int_0^{t_1} Y_{s_1,s_2}\,\mathrm{d}s_1 \quad \text{for all } (t_1,s_2)\in\mathbb{R}^2_+ \quad a.s.$$

and hence

$$\begin{aligned} \int_0^{t_2} e^{-s_2}R_{t_1,s_2}\,\mathrm{d}s_2 &= \int_0^{t_2} Y_{t_1,s_2}\,\mathrm{d}s_2 - \int_0^{t_2} Y_{0,s_2}\,\mathrm{d}s_2 \\ &\quad + \int_0^{t_1}\int_0^{t_2} Y_{s_1,s_2}\,\mathrm{d}s_2\mathrm{d}s_1 \quad \text{for all } (t_1,s_2)\in\mathbb{R}^2_+ \quad a.s. \end{aligned}$$

Inserting this we get

$$\begin{aligned} Y_{t_1,t_2} &= e^{-t_2}V_{0,t_2} + R_{t_1,0} + M((0,t]) - \int_0^{t_1} Y_{s_1,t_2}\,\mathrm{d}s_1 \\ &\quad - \int_0^{t_2} Y_{t_1,s_2}\,\mathrm{d}s_2 + \int_0^{t_2} Y_{0,s_2}\,ds_2 - \int_0^{t_1}\int_0^{t_2} Y_{s_1,s_2}\,\mathrm{d}s_2\mathrm{d}s_1 \\ &= Y_{0,t_2} + R_{t_1,0} + M((0,t]) - \int_0^{t_1} Y_{s_1,t_2}\,\mathrm{d}s_1 - \int_0^{t_2} Y_{t_1,s_2}\,\mathrm{d}s_2 \\ &\quad + \int_0^{t_2} Y_{0,s_2}\,\mathrm{d}s_2 - \int_0^{t_1}\int_0^{t_2} Y_{s_1,s_2}\,\mathrm{d}s_2\mathrm{d}s_1 \quad \text{for all } t=(t_1,t_2)\in\mathbb{R}^2_+ \quad a.s. \end{aligned}$$

Using (4.6) with $t_2 = 0$ we get

$$R_{t_1,0} = Y_{t_1,0} - Y_{0,0} + \int_0^{t_1} Y_{s_1,0}\,\mathrm{d}s_1 \quad \text{for all } t_1 \geq 0 \quad a.s.$$

The result follows by inserting this in the expression for Y_{t_1,t_2}. □

Acknowledgment: We thank the referee for providing a detailed list of additional references, and in particular for drawing our attention to the paper by Hirsch and Song [5].

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