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CHARACTERIZATION OF DISTRIBUTIONS WITH THE LENGTH-BIAS SCALING PROPERTY

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Abstract
For $q \in (0,1)$ fixed, we characterize the density functions f of absolutely continuous random variables $X > 0$ with finite expectation whose respective distribution functions satisfy the so-called (LBS) length-bias scaling property $X \stackrel{\mathscr{L}}{=} q\widehat{X}$, where $\widehat{X}$ is a random variable having the distribution function $\widehat{F}(x) = (\mathbf{E}X)^{-1}\int_0^x yf(y)\,dy$.

For an absolutely continuous random variable $X > 0$ with probability density function (pdf) f and finite expectation $\mathbf{E}X$, we denote by $\widehat{X}$ an absolutely continuous random variable having the probability density function $(\mathbf{E}X)^{-1}xf(x)$. In this case, $\widehat{X}$ is called the size- or length-biased version of X and $\mathscr{L}(\widehat{X})$ is the corresponding length-biased distribution. It is well known that $\widehat{X}$ is the stationary total lifetime in a renewal process with generic lifetime X (see [2, Chapter 5]).

The length-biased distributions have been applied in various fields, such as biometry, ecology, environmental sciences, reliability and survival analysis. A review of these distributions and their applications are included in [5, Section 3], [6, 8, 12, 13].

In [9], Pakes and Khattree ask whether it is possible to randomly rescale the total lifetime to recover the lifetime law. More specifically, let $V \geq 0$ be a random variable independent of X with a fixed law satisfying $P(V > 0) > 0$. For which laws $\mathscr{L}(X)$ does the following "equality in law"

$$X \stackrel{\mathscr{L}}{=} V\widehat{X},$$

hold? For instance, when V has the uniform law on $[0,1]$ the last equality holds if and only if $\mathscr{L}(X)$ is an exponential law (see [9]).

In this note we consider the case where V is a constant function: The law of X has the so-called length-bias scaling property (abbreviated to LBS-property) if

$$X \stackrel{\mathscr{L}}{=} q\widehat{X}, \tag{1}$$

with $q \in (0,1)$. Several authors, including Chihara [3], Pakes and Khattree [9], Pakes [10, 11], Vardi *et al.* [14], have studied the LBS-property. In [1], Bertoin *et al.* analyze a random variable X that arises in the study of exponential functionals of Poisson processes; they show that $q\widehat{X} \stackrel{\mathscr{L}}{=} X \stackrel{\mathscr{L}}{=} q^{-1}X^{-1}$, with $\mathbf{E}X = q^{-1}$.

An easy computation shows that (1) can be written as

$$\int_0^x f(y)\,dy = \frac{1}{\mathbf{E}X}\int_0^x \frac{y}{q} f\left(\frac{y}{q}\right)\frac{dy}{q},\ x > 0,$$

which is equivalent to

$$(\mathbf{E}X)qf(qx) = xf(x),\ a.e.\ x > 0. \tag{2}$$

By induction we have that

$$(\mathbf{E}X)^n q^n f(q^n x) = q^{n^2/2-n/2} x^n f(x),\ \ x > 0,\ n \in \mathbb{Z},$$

and therefore

$$\int_0^\infty x^n f(x)\,dx = (\mathbf{E}X)^n q^{n/2-n^2/2},\ \forall\ n \in \mathbb{Z}. \tag{3}$$

When X is an absolutely continuous random variable with probability density function f, we sometimes write $X \sim f$.

Proposition 1. *If $X \sim f$ and f satisfies (2), then the pdf $g(x) = e^x f(e^x)$ of the random variable $Y = \log X$ satisfies the functional equation*

$$g(x-b) = Ce^{ax}g(x),\ \ x \in \mathbb{R}, \tag{4}$$

with $a = 1$, $b = -\ln q$, $C = (\mathbf{E}X)^{-1}$.

So, the main result of this note characterizes the probability density functions fulfilling the last functional equation. First, we recall that the theta function given by

$$\theta(x,t) = (4\pi t)^{-1/2}\sum_{n\in\mathbb{Z}} e^{-(x+n)^2/(4t)} = \sum_{n\in\mathbb{Z}} e^{-4\pi^2 n^2 t + 2\pi n i x} > 0, \tag{5}$$

for all $(x,t) \in \mathbb{R}_+^2$, satisfies the heat equation on $\mathbb{R}_+^2$ and

$$\int_0^1 \theta(x,t)dx = 1,\ \text{for all } t > 0.\ \text{(see [15, Chapter V])} \tag{6}$$

Theorem 1. *Let a, b, C be real numbers with $ab > 0$, $C > 0$. Then the pdf g satisfies the functional equation (4) if and only if there exists a 1-periodic function φ, $\varphi \geq -1$, such that the restriction of φ to $(0,1)$ belongs to $L^1(0,1)$,*

$$g(x) = \frac{1}{\sqrt{2\pi a^{-1}b}}\exp\left(-\frac{(ax-\mu)^2}{2ab}\right)\left\{1 + \varphi\left(\frac{ax-\mu}{ab}\right)\right\}, \tag{7}$$

and

$$\int_0^1 \theta\left(x, \frac{1}{2ab}\right)\varphi(x)dx = 0, \tag{8}$$

where $-\mu = \ln C + ab/2$.

Proof. For $b>0$ the probability density function

$$h(x)=\frac{1}{\sqrt{2\pi b}}e^{-\left(x-\mu\right)^2/(2b)},$$

where $-\mu=\ln C+b/2$, satisfies the functional equation (4) with $a=1$. If the density function g so does, then $g\left(x-b\right)/h(x-b)=g\left(x\right)/h(x)$, $x\in\mathbb{R}$; therefore there exists a 1-periodic function ψ such that $g\left(x\right)=h(x)\psi\left(b^{-1}x\right)$. By making the change of variable $y=(x-\mu)/b$, we obtain

$$\begin{aligned}1=\int_{\mathbb{R}}g(x)dx &= \int_{\mathbb{R}}\frac{1}{\sqrt{2\pi b^{-1}}}e^{-by^2/2}\psi\left(y+b^{-1}\mu\right)dy\\ &= \sum_{n\in\mathbb{Z}}\int_n^{n+1}\frac{1}{\sqrt{2\pi b^{-1}}}e^{-by^2/2}\psi\left(y+b^{-1}\mu\right)dy\\ &= \int_0^1\theta(y,2^{-1}b^{-1})\psi(y+b^{-1}\mu)dy.\end{aligned}$$

By using (6), the result follows with $\varphi\left(x\right)=-1+\psi(x+b^{-1}\mu)$.
The general case follows by setting $\widetilde{g}\left(x\right)=a^{-1}g\left(a^{-1}x\right)$, $\widetilde{b}=ab$, $-\widetilde{\mu}=\ln C+\widetilde{b}/2$. ∎

From Proposition 1 we obtain the characterization of the probability density functions with the LBS-property.

Corollary 1. *Let $q\in(0,1)$ be fixed, and let $X>0$ be a random variable with pdf f and $\mathbf{E}X<\infty$. The law of X has the LBS property if and only if there exists a 1-periodic function φ, $\varphi\geq-1$, with the restriction of φ to $(0,1)$ in $L^1(0,1)$, such that φ satisfies (8) with $a=1, b=-\ln q$, and*

$$f\left(x\right)=\frac{1}{x\sqrt{-2\pi\ln q}}\exp\left(\frac{\left(\ln x-\mu\right)^2}{2\ln q}\right)\left\{1+\varphi\left(\frac{\ln x-\mu}{-\ln q}\right)\right\}, \tag{9}$$

where $\mu=\ln\left(q^{1/2}\mathbf{E}X\right)$.

In [10, Theorem 3.1], Pakes uses a different approach to characterize the probability distribution functions $F=\mathscr{L}\left(X\right)$ satisfying (1) with $\mathbf{E}X=1$.

By (3), it follows that the probability density functions having the LBS-property are solutions of an indeterminate moment problem.

Let $N\left(\mu,-\ln q\right)$ be the normal density with mean μ and variance $-\ln q$. If $Y\sim N\left(\mu,-\ln q\right)$, we note that $\exp(Y)$ has the log-normal density, i.e.

$$\exp(Y)\sim\frac{1}{x\sqrt{-2\pi\ln q}}\exp\left(\frac{\left(\ln x-\mu\right)^2}{2\ln q}\right).$$

Remark 1. *If X is a positive absolutely continuous random variable with pdf f, then*

$$cX^{-1}\sim cx^{-2}f\left(cx^{-1}\right),\ \textit{for all } c>0.$$

So, for $\nu \in \mathbb{R}$ the distributional identity $X \overset{\mathscr{L}}{=} e^{2\nu}X^{-1}$ is equivalent to the functional equation

$$f(x^{-1}) = e^{2\nu}x^2 f(e^{2\nu}x),\ x > 0. \tag{10}$$

If φ is a measurable function on $\mathbb{R}$ and f is a pdf function given as follows

$$f(x) = \frac{1}{x\sqrt{-2\pi \ln q}} \exp\left(\frac{(\ln x - \nu)^2}{2\ln q}\right)\left\{1 + \varphi\left(\frac{\ln x - \nu}{-\ln q}\right)\right\},$$

$x > 0$, then f satisfies the latter functional equation if and only if φ is an even function.

As a consequence of Corollary 1 and the last remark with $\nu = \ln\left(q^{1/2}\mathbf{E}X\right)$, we have that a positive random variable X with probability density function f satisfies

$$q\widehat{X} \overset{\mathscr{L}}{=} X \overset{\mathscr{L}}{=} q(\mathbf{E}X)^2 X^{-1}$$

if and only if f can be written as in Corollary 1 with φ being an even function.

Finally, we provide some families of functions satisfying (8).

Examples

From bounded functions, the following observation allows to construct functions with values in the non-negative axis.

Remark 2. *For $\alpha, \beta \in \mathbb{R}$ with $\alpha < \beta$, it is easy to see that there exists an interval $I \subset \mathbb{R}$ such that $\varepsilon[\alpha, \beta] + 1 \subset \overline{\mathbb{R}^+}$ for all $\varepsilon \in I$. In fact, when $\alpha < 0 < \beta$ we have that $I = \left[-\beta^{-1}, -\alpha^{-1}\right]$. For $\alpha \geq 0$, $I = [-\beta^{-1}, \infty)$, and for $\beta \leq 0$, $I = (-\infty, -\alpha^{-1}]$.*

Example 1. *Let $t > 0$ be fixed and let $(c_n)_{n\in\mathbb{Z}}$ be a sequence of complex numbers such that $\varphi(x) = \sum_{n\in\mathbb{Z}} c_n e^{2\pi n i x} \in L^2(0,1)$. Then φ satisfies*

$$\int_0^1 \theta(x,t)\varphi(x)dx = 0, \tag{11}$$

if and only if $\varphi(x)$ is orthogonal to $\theta(x,t)$ in $L^2(0,1)$. By (5) this is equivalent to the orthogonality between $(c_n)_{n\in\mathbb{Z}}$ and $\left(e^{-4\pi^2 n^2 t}\right)_{n\in\mathbb{Z}}$, i.e.

$$\sum_{n\in\mathbb{Z}} c_n e^{-4\pi^2 n^2 t} = 0.$$

In [11, page 1278] Pakes says that the continuous solutions of (2) probably are exceptions. But for any trigonometric polynomial $p(x) = \sum_{|n|\leq N} c_n e^{2\pi n i x}$ whose coefficients $c_n \in \mathbb{C}$ satisfy the last equality with $t = b^{-1}/2$, there is an interval I such that

$$\varepsilon x \geq -1,\ \textit{for all } x \in \left[\min Re\, p, \max Re\, p\right],\ \varepsilon \in I,$$

therefore $\varphi = \varepsilon Re\, p \geq -1$ on $\mathbb{R}$ and the corresponding density function given by Corollary 1 is an infinitely differentiable function on $\mathbb{R}^+$.

Example 2. *Let* $c_m = -c_{-m} = i/2$, *and* $c_n = 0$ *for all* $n \in \mathbb{Z} \setminus \{-m, m\}$, $m \neq 0$. *So, the corresponding trigonometric polynomial* $\varphi(x) = -\sin(2\pi mx)$ *is a function satisfying (11) for all* $t > 0$.

Example 3. *Let* $c_{-1} = c_1 = -1/2$, $c_0 = e^{-4\pi^2 t}$, *and* $c_n = 0$ *for all* $|n| \geq 2$. *Thus, the corresponding trigonometric polynomial* $\varphi(x) = e^{-4\pi^2 t} - \cos(2\pi x) \geq -1$ *is an even function satisfying (11).*

Example 4. *By (6) we have that*

$$\varphi_c(x) = -1 + \left(\int_0^1 \frac{\theta(x,t)}{\theta(x+c,t)} dx \right)^{-1} \frac{1}{\theta(x+c,t)}$$

is a 1*-periodic, continuous function satisfying (11) for all* $c \in [0,1)$. *Since* $\theta(x,t)$ *is an even function for all* $t > 0$, *the function* φ_c *is even if and only if* $c = 0,\ 1/2$. *In [4, equality (2.15)] it is shown that*

$$\theta(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-x^2/(4t)} (q_t; q_t)_\infty \left(-q_t^{1/2-x}; q\right)_\infty \left(-q_t^{1/2+x}; q_t\right)_\infty,$$

where $q_t = e^{-t^{-1}/2}$, $(p;q)_\infty = \prod_{k=0}^\infty \left(1 - pq^k\right)$. *For* $c \in (0,1)$ *we have that (see [4, equality (2.17)])*

$$\int_0^1 \frac{\theta(x,t)}{\theta(x+c,t)} dx = 2t \frac{\pi q_t^{c(c-1)/2}}{\sin(\pi c)} \frac{\left(q_t^c; q_t\right)_\infty \left(q_t^{1-c}; q_t\right)_\infty}{(q_t; q_t)_\infty^2}.$$

To get more examples of functions fulfilling (8) see [4]. The results in [7] can be used to construct positive random variables having not the LBS-property but with moment sequence (3).

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