

SHARP ESTIMATES FOR THE CONVERGENCE OF THE DENSITY OF THE EULER SCHEME IN SMALL TIME

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Abstract

In this work, we approximate a diffusion process by its Euler scheme and we study the convergence of the density of the marginal laws. We improve previous estimates especially for small time.

1 Introduction

Let us consider a d -dimensional diffusion process $(X_s)_{0 \leq s \leq T}$ and a q -dimensional Brownian motion $(W_s)_{0 \leq s \leq T}$. X satisfies the following SDE

$$dX_s^i = b_i(s, X_s)ds + \sum_{j=1}^q \sigma_{ij}(s, X_s)dW_s^j, \quad X_0^i = x^i, \forall i \in \{1, \dots, d\}. \quad (1.1)$$

We approximate X by its Euler scheme with N ($N \geq 1$) time steps, say X^N , defined as follows. We consider the regular grid $\{0 = t_0 < t_1 < \dots < t_N = T\}$ of the interval $[0, T]$, i.e. $t_k = k \frac{T}{N}$. We put $X_0^N = x$ and for all $i \in \{1, \dots, d\}$ we define

$$X_u^{N,i} = X_{t_k}^{N,i} + b_i(t_k, X_{t_k}^N)(u - t_k^N) + \sum_{j=1}^q \sigma_{ij}(t_k, X_{t_k}^N)(W_u^j - W_{t_k}^j), \text{ for } u \in [t_k, t_{k+1}[. \quad (1.2)$$

The continuous Euler scheme is an Itô process verifying

$$X_u^N = x + \int_0^u b(\varphi(s), X_{\varphi(s)}^N) ds + \int_0^u \sigma(\varphi(s), X_{\varphi(s)}^N) dW_s$$

where $\varphi(u) := \sup\{t_k : t_k \leq u\}$. If σ is uniformly elliptic, the Markov process X admits a transition probability density $p(0, x; s, y)$. Concerning X^N (which is not Markovian except at times $(t_k)_k$), X_s^N has a probability density $p^N(0, x; s, y)$, for any $s > 0$. We aim at proving sharp estimates of the difference $p(0, x; s, y) - p^N(0, x; s, y)$.

It is well known (see Bally and Talay [2], Konakov and Mammen [5], Guyon [4]) that this difference is of order $\frac{1}{N}$. However, the known upper bounds of this difference are too rough for small values of s . In this work, we provide tight upper bounds of $|p(0, x; s, y) - p^N(0, x; s, y)|$ in s (see Theorem 2.3), so that we can estimate quantities like

$$\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] \text{ or } \mathbb{E} \left[\int_0^T f(X_{\varphi(s)}^N) ds \right] - \mathbb{E} \left[\int_0^T f(X_s) ds \right] \tag{1.3}$$

(without any regularity assumptions on f) more accurately than before (see Theorem 2.5). For other applications, see Labart [7]. Unlike previous references, we allow b and σ to be time-dependent and assume they are only C^3 in space. Besides, we use Malliavin’s calculus tools.

Background results

The difference $p(0, x; s, y) - p^N(0, x; s, y)$ has been studied a lot. We can find several results in the literature on expansions w.r.t. N . First, we mention a result from Bally and Talay [2] (Corollary 2.7). The authors assume

Hypothesis 1.1. *σ is elliptic (with σ only depending on x) and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order greater or equal to 1 are bounded.*

By using Malliavin’s calculus, they show that

$$p(0, x; T, y) - p^N(0, x; T, y) = \frac{1}{N} \pi_T(x, y) + \frac{1}{N^2} R_T^N(x, y), \tag{1.4}$$

with $|\pi_T(x, y)| + |R_T^N(x, y)| \leq \frac{K(T)}{T^\alpha} \exp(-c\frac{|x-y|^2}{T})$, where $c > 0$, $\alpha > 0$ and $K(\cdot)$ is a non decreasing function. We point out that α is unknown, which doesn’t enable to deduce the behavior of $p - p^N$ when $T \rightarrow 0$.

Besides that, Konakov and Mammen [5] have proposed an analytical approach based on the so-called parametrix method to bound $p(0, x; 1, y) - p^N(0, x; 1, y)$ from above. They assume

Hypothesis 1.2. *σ is elliptic and b, σ are $C^\infty(\mathbb{R}^d)$ functions whose derivatives of any order are bounded.*

For each pair (x, y) they get an expansion of arbitrary order j of $p^N(0, x; 1, y)$. The coefficients of the expansion depend on N

$$p(0, x; 1, y) - p^N(0, x; 1, y) = \sum_{i=1}^{j-1} \frac{1}{N^i} \pi_{N,i}(0, x; 1, y) + O\left(\frac{1}{N^j}\right). \tag{1.5}$$

The coefficients have Gaussian tails : for each i they find constants $c_1 > 0, c_2 > 0$ s.t. for all $N \geq 1$ and all $x, y \in \mathbb{R}^d, |\pi_{N,i}(0, x; 1, y)| \leq c_1 \exp(-c_2|x - y|^2)$. To do so, they use upper bounds for the partial derivatives of p (coming from Friedman [3]) and prove analogous results on the derivatives of p^N . Strong though this result may be, nothing is said when replacing 1 by t , for $t \rightarrow 0$. That's why we present now the work of Guyon [4].

Guyon [4] improves (1.4) and (1.5) in the following way.

Definition 1.3. Let $\mathcal{G}_l(\mathbb{R}^d), l \in \mathbb{Z}$ be the set of all measurable functions $\pi : \mathbb{R}^d \times (0, 1] \times \mathbb{R}^d \rightarrow \mathbb{R}$ s.t.

- for all $t \in (0, 1], \pi(\cdot; t, \cdot)$ is infinitely differentiable,
- for all $\alpha, \beta \in \mathbb{N}^d$, there exist two constants $c_1 \geq 0$ and $c_2 > 0$ s.t. for all $t \in (0, 1]$ and $x, y \in \mathbb{R}^d$,

$$|\partial_x^\alpha \partial_y^\beta \pi(x; t, y)| \leq c_1 t^{-(|\alpha|+|\beta|+d+l)/2} \exp(-c_2|x - y|^2/t).$$

Under Hypothesis 1.2 and for $T = 1$, the author has proved the following expansions

$$p^N - p = \frac{\pi}{N} + \frac{\pi_N}{N^2}, \tag{1.6}$$

$$p^N - p = \sum_{i=1}^{j-1} \frac{\pi_{N,i}}{N^i} + \sum_{i=2}^j \left(t - \frac{\lfloor Nt \rfloor}{N} \right)^i \pi'_{N,i} + \frac{\pi''_{N,j}}{N^j}, \tag{1.7}$$

where $\pi \in \mathcal{G}_1(\mathbb{R}^d)$ and $(\pi_N, N \geq 1)$ is a bounded sequence in $\mathcal{G}_4(\mathbb{R}^d)$. For each $i \geq 1, (\pi_{N,i}, N \geq 1)$ is a bounded family in $\mathcal{G}_{2i-2}(\mathbb{R}^d)$, and $(\pi'_{N,i}, N \geq 1), (\pi''_{N,i}, N \geq 1)$ are two bounded families in $\mathcal{G}_{2i}(\mathbb{R}^d)$. These expansions can be seen as improvements of (1.4) and (1.5) : it also allows infinite differentiations w.r.t. x and y and makes precise the way the coefficients explode when t tends to 0.

As a consequence (see Guyon [4], Corollary 22), one gets

$$|p(0, x; s, y) - p^N(0, x; s, y)| \leq \frac{c_1}{Ns^{\frac{d+2}{2}}} e^{-c_2 \frac{|x-y|^2}{s}}, \tag{1.8}$$

for two positive constants c_1 and c_2 , and for any x, y and $s \leq 1$. This result should be compared with the one of Theorem 2.3 (when $T = 1$), in which the upper bound is tighter (s has a smaller power).

2 Main Results

Before stating the main result of the paper, we introduce the following notation

Definition 2.1. $C_b^{k,l}$ denotes the set of continuously differentiable bounded functions $\phi : (t, x) \in [0, T] \times \mathbb{R}^d$ with uniformly bounded derivatives w.r.t. t (resp. w.r.t. x) up to order k (resp. up to order l).

The main result of the paper, whose proof is postponed to Section 4, is established under the following Hypothesis

Hypothesis 2.2. σ is uniformly elliptic, b and σ are in $C_b^{1,3}$ and $\partial_t \sigma$ is in $C_b^{0,1}$.

Theorem 2.3. Assume Hypothesis 2.2. Then, there exist a constant $c > 0$ and a non decreasing function K , depending on the dimension d and on the upper bounds of σ, b and their derivatives s.t. $\forall (s, x, y) \in]0, T] \times \mathbb{R}^d \times \mathbb{R}^d$, one has

$$|p(0, x; s, y) - p^N(0, x; s, y)| \leq \frac{K(T)T}{Ns^{\frac{d+1}{2}}} \exp\left(-\frac{c|x-y|^2}{s}\right).$$

Corollary 2.4. Assume Hypothesis 2.2. From the last inequality and Aronson's inequality (A.1), we deduce

$$\left| \frac{p(0, x; T, x) - p^N(0, x; T, x)}{p(0, x; T, x)} \right| \leq \frac{K(T)}{N} \sqrt{T}. \tag{2.1}$$

This inequality yields $p(0, x; T, x) \sim p^N(0, x; T, x)$ when $T \rightarrow 0$.

Theorem 2.3 enables to bound quantities like in (1.3) in the following way

Theorem 2.5. Assume Hypothesis 2.2. For any function f such that $|f(x)| \leq c_1 e^{c_2|x|}$, it holds

$$\begin{aligned} |\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]| &\leq c_1 e^{c_2|x|} K(T) \frac{\sqrt{T}}{N}, \\ \left| \mathbb{E} \left[\int_0^T f(X_{\varphi(s)}^N) ds \right] - \mathbb{E} \left[\int_0^T f(X_s) ds \right] \right| &\leq c_1 e^{c_2|x|} K(T) \frac{T}{N}. \end{aligned}$$

Had we used the results stated by Guyon [4] (and more precisely the one recalled in (1.8)), we would have obtained $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)] = O(\frac{1}{N})$. Intuitively, this result is not optimal: the right hand side doesn't tend to 0 when T goes to 0 while it should. Analogously, regarding $\mathbb{E} \left[\int_0^T f(X_{\varphi(s)}^N) ds \right] - \mathbb{E} \left[\int_0^T f(X_{\varphi(s)}) ds \right]$, we would obtain $O(\frac{T \ln N}{N})$ instead of $O(\frac{T}{N})$.

Proof of Theorem 2.5. Writing $\mathbb{E}[f(X_T^N)] - \mathbb{E}[f(X_T)]$ as $\int_{\mathbb{R}^d} f(y) (p^N(0, x; T, y) - p(0, x; T, y)) dy$ and using Theorem 2.3 yield the first result.

Concerning the second result, we split $\mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_s)) ds \right]$ in two terms :

$\mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)})) ds \right]$ and $\mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}) - f(X_s)) ds \right]$. First, using Theorem 2.3 leads to

$$\begin{aligned} \left| \mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)})) ds \right] \right| &= \left| \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds f(y) (p^N(0, x; \varphi(s), y) - p(0, x; \varphi(s), y)) \right|, \\ &\leq \frac{K(T)T}{N} c_1 e^{c_2|x|} \int_{\frac{T}{N}}^T \frac{ds}{\sqrt{\varphi(s)}}, \end{aligned}$$

where we use the easy inequality $\int_{\mathbb{R}^d} e^{c_2|y|} e^{-\frac{c|x-y|^2}{s}} dy \leq K(T) e^{c_2|x|}$. Since $\varphi(s) \geq s - \frac{T}{N}$, we get $\left| \mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}^N) - f(X_{\varphi(s)})) ds \right] \right| \leq \frac{K(T)T^{3/2}}{N} c_1 e^{c_2|x|}$. Second, we write

$$\left| \mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}) - f(X_s)) ds \right] \right| \leq c_1 e^{c_2|x|} \frac{T}{N} + \int_{\mathbb{R}^d} dy \int_{\frac{T}{N}}^T ds c_1 e^{c_2|y|} \int_{\varphi(s)}^s du |\partial_u p(0, x; u, y)|.$$

Then, Proposition A.2 yields $\left| \mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}) - f(X_s)) ds \right] \right| \leq c_1 e^{c_2|x|} \left(\frac{T}{N} + C \int_0^T \ln\left(\frac{s}{\varphi(s)}\right) ds \right)$.
 Moreover, $\int_0^T \ln\left(\frac{s}{\varphi(s)}\right) ds = \sum_{k=1}^{N-1} \int_{t_k}^{t_{k+1}} \ln\left(\frac{s}{t_k}\right) ds = \frac{T}{N} \sum_{k=1}^{N-1} ((k+1) \ln\left(\frac{k+1}{k}\right) - 1) \leq C \frac{T}{N}$, using
 a second order Taylor expansion. This gives $\left| \mathbb{E} \left[\int_0^T (f(X_{\varphi(s)}) - f(X_s)) ds \right] \right| \leq c_1 e^{c_2|x|} K(T) \frac{T}{N}$. \square

In the next section, we give results related to Malliavin’s calculus, that will be useful for the proof of Theorem 2.3.

3 Basic results on Malliavin’s calculus

We refer the reader to Nualart [8], for more details. Fix a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbb{P})$ and let $(W_t)_{t \geq 0}$ be a q -dimensional Brownian motion. For $h(\cdot) \in H = \mathbb{L}^2([0, T], \mathbb{R}^q)$, $W(h)$ is the Wiener stochastic integral $\int_0^T h(t) dW_t$. Let \mathcal{S} denote the class of random variables of the form $F = f(W(h_1), \dots, W(h_n))$ where f is a C^∞ function with derivatives having a polynomial growth, $(h_1, \dots, h_n) \in H^n$ and $n \geq 1$. For $F \in \mathcal{S}$, we define its derivative $\mathcal{D}F = (\mathcal{D}_t F := (\mathcal{D}_t^1 F, \dots, \mathcal{D}_t^q F))_{t \in [0, T]}$ as the H valued random variable given by

$$\mathcal{D}_t F = \sum_{i=1}^n \partial_{x_i} f(W(h_1), \dots, W(h_n)) h_i(t).$$

The operator \mathcal{D} is closable as an operator from $\mathbb{L}^p(\Omega)$ to $\mathbb{L}^p(\Omega; H)$, for $p \geq 1$. Its domain is denoted by $\mathbb{D}^{1,p}$ w.r.t. the norm $\|F\|_{1,p} = [\mathbb{E}|F|^p + \mathbb{E}(\|\mathcal{D}F\|_H^p)]^{1/p}$. We can define the iteration of the operator \mathcal{D} , in such a way that for a smooth random variable F , the derivative $\mathcal{D}^k F$ is a random variable with values on $H^{\otimes k}$. As in the case $k = 1$, the operator \mathcal{D}^k is closable from $\mathcal{S} \subset \mathbb{L}^p(\Omega)$ into $\mathbb{L}^p(\Omega; H^{\otimes k})$, $p \geq 1$. If we define the norm

$$\|F\|_{k,p} = [\mathbb{E}|F|^p + \sum_{j=1}^k \mathbb{E}(\|\mathcal{D}^j F\|_{H^{\otimes j}}^p)]^{1/p},$$

we denote its domain by $\mathbb{D}^{k,p}$. Finally, set $\mathbb{D}^{k,\infty} = \cap_{p \geq 1} \mathbb{D}^{k,p}$, and $\mathbb{D}^\infty = \cap_{k,p \geq 1} \mathbb{D}^{k,p}$. One has the following chain rule property

Proposition 3.1. *Fix $p \geq 1$. For $f \in C_b^1(\mathbb{R}^d, \mathbb{R})$, and $F = (F_1, \dots, F_d)^*$ a random vector whose components belong to $\mathbb{D}^{1,p}$, $f(F) \in \mathbb{D}^{1,p}$ and for $t \geq 0$, one has $\mathcal{D}_t(f(F)) = f'(F)\mathcal{D}_t F$, with the notation*

$$\mathcal{D}_t F = \begin{pmatrix} \mathcal{D}_t F_1 \\ \vdots \\ \mathcal{D}_t F_d \end{pmatrix} \in \mathbb{R}^d \otimes \mathbb{R}^q.$$

We now introduce the Skorohod integral δ , defined as the adjoint operator of \mathcal{D} .

Proposition 3.2. *δ is a linear operator on $\mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$ with values in $\mathbb{L}^2(\Omega)$ s.t.*

- *the domain of δ (denoted by $Dom(\delta)$) is the set of processes $u \in \mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$ s.t. $|\mathbb{E}(\int_0^T \mathcal{D}_t F \cdot u_t dt)| \leq c(u) \|F\|_{\mathbb{L}^2}$ for any $F \in \mathbb{D}^{1,2}$.*

- If u belongs to $\text{Dom}(\delta)$, then $\delta(u)$ is the one element of $\mathbb{L}^2(\Omega)$ characterized by the integration by parts formula

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E}(F\delta(u)) = \mathbb{E} \left(\int_0^T \mathcal{D}_t F \cdot u_t dt \right).$$

Remark 3.3. If u is an adapted process belonging to $\mathbb{L}^2([0, T] \times \Omega, \mathbb{R}^q)$, then the Skorohod integral and the Itô integral coincide : $\delta(u) = \int_0^T u_t dW_t$, and the preceding integration by parts formula becomes

$$\forall F \in \mathbb{D}^{1,2}, \quad \mathbb{E} \left(F \int_0^T u_t dW_t \right) = \mathbb{E} \left(\int_0^T \mathcal{D}_t F \cdot u_t dt \right). \tag{3.1}$$

This equality is also called the duality formula.

This duality formula is the corner stone to establish general integration by parts formula of the form

$$\mathbb{E}[\partial^\alpha g(F)G] = \mathbb{E}[g(F)H_\alpha(F, G)]$$

for any non degenerate random variables F . We only give the formulation in the case of interest $F = X_t^N$.

Proposition 3.4. We assume that σ is uniformly elliptic and b and σ are in $C_b^{0,3}$. For all $p > 1$, for all multi-index α s.t. $|\alpha| \leq 2$, for all $t \in]0, T]$, all $u, r, s \in [0, T]$ and for any functions f and g in $C_b^{|\alpha|}$, there exist a random variable $H_\alpha \in \mathbb{L}^p$ and a function $K(T)$ (uniform in N, x, s, u, r, t, f and g) s.t.

$$\mathbb{E}[\partial_x^\alpha f(X_t^N)g(X_u^N, X_r^N, X_s^N)] = \mathbb{E}[f(X_t^N)H_\alpha], \tag{3.2}$$

with

$$\|H_\alpha\|_{\mathbb{L}^p} \leq \frac{K(T)}{t^{\frac{|\alpha|}{2}}} \|g\|_{C_b^{|\alpha|}}. \tag{3.3}$$

These results are given in the article of Kusuoka and Stroock [6]: (3.3) is owed to Theorem 1.20 and Corollary 3.7.

Another consequence of the duality formula is the derivation of an upper bound for p^N .

Proposition 3.5. Assume σ is uniformly elliptic and b and σ are in $C_b^{0,2}$. Then, for any $x, y \in \mathbb{R}^d, s \in]0, T]$, one has

$$p^N(0, x; s, y) \leq \frac{K(T)}{s^{d/2}} e^{-c \frac{|x-y|^2}{s}}, \tag{3.4}$$

for a positive constant c and a non decreasing function K , both depending on d and on the upper bounds for b, σ and their derivatives.

Although this upper bound seems to be quite standard, to our knowledge such a result has not appeared in the literature before, except in the case of time homogeneous coefficients (see Konakov and Mammen [5], proof of Theorem 1.1).

Proof. The inequality (1.32) of Kusuoka and Stroock [6], Theorem 1.31 gives $p^N(0, x; s, y) \leq \frac{K(T)}{s^{d/2}}$ for any x and y . This implies the required upper bound when $|x - y| \leq \sqrt{s}$. Let us now consider the case $|x - y| > \sqrt{s}$. Using the same notations as in Kusuoka and Stroock [6], we denote $\psi(y) = \rho(\frac{|y-x|}{r})$ where $r > 0$ and ρ is a C_b^∞ function such that $\mathbf{1}_{\{[3/4, \infty[\}} \leq \rho \leq \mathbf{1}_{\{[1/2, \infty[\}}$. Then, combining inequality (1.33) of Kusuoka and Stroock [6], Theorem 1.31 and Corollary 3.7 leads to

$$\sup_{|y-x| \geq r} p^N(0, x; s, y) \leq K(T) \frac{e^{-c\frac{r^2}{s}}}{s^{d/2}} \left(1 + \sqrt{\frac{s}{r^2}} \right),$$

where we use $\|\psi(X_s^N)\|_{1,q} \leq K(T)e^{-c\frac{r^2}{s}} (1 + \sqrt{\frac{s}{r^2}})$. This easily completes the proof in the case $|x - y| \geq \sqrt{s}$. □

4 Proof of Theorem 2.3

In the following, $K(\cdot)$ denotes a generic non decreasing function (which may depend on d, b and σ). To prove Theorem 2.3, we take advantage of Propositions 3.4 and 3.5. The scheme of the proof is the following

- Use a PDE and Itô's calculus to write the difference $p^N(0, x; s, y) - p(0, x; s, y)$

$$\begin{aligned} &= \int_0^s \mathbb{E} \left[\sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y) \right. \\ &\quad \left. + \frac{1}{2} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \right] dr := E_1 + E_2. \end{aligned} \tag{4.1}$$

- Prove the intermediate result $\forall (r, x, y) \in [0, s[\times \mathbb{R}^d \times \mathbb{R}^d$ and $c > 0$

$$\mathbb{E} \left[\exp \left(-c \frac{|y - X_r^N|^2}{s - r} \right) \right] \leq K(T) \left(\frac{s - r}{s} \right)^{\frac{d}{2}} \exp \left(-c' \frac{|x - y|^2}{s} \right), \tag{4.2}$$

where $c' > 0$.

- Use Malliavin's calculus, Proposition 3.5 and the intermediate result, to show that each term E_1 and E_2 (see (4.1)) is bounded by $\frac{K(T)T}{N} \frac{1}{s^{\frac{d+1}{2}}} \exp(-c\frac{|x-y|^2}{s})$.

Definition 4.1. We say that a term $E(x, s, y)$ satisfies property \mathcal{P} if $\forall (x, s, y) \in \mathbb{R}^d \times]0, T[\times \mathbb{R}^d$

$$|E(x, s, y)| \leq \frac{K(T)T}{N} \frac{1}{s^{\frac{d+1}{2}}} \exp \left(-c \frac{|x - y|^2}{s} \right). \tag{\mathcal{P}}$$

4.1 Proof of equality (4.1)

First, the transition density function $(r, x) \mapsto p(r, x; s, y)$ satisfies the PDE

$$(\partial_r + \mathcal{L}_{(r,x)})p(r, x; s, y) = 0, \quad \forall r \in [0, s[, \forall x \in \mathbb{R}^d,$$

where $\mathcal{L}_{(r,x)}$ is defined by $\mathcal{L}_{(r,x)} = \sum_{i,j} a_{ij}(r,x) \partial_{x_i x_j}^2 + \sum_i b_i(r,x) \partial_{x_i}$, and $a_{ij}(r,x) = \frac{1}{2}[\sigma\sigma^*]_{ij}(r,x)$. The function, as well as its first derivatives, are uniformly bounded by a constant depending on ϵ for $|s-r| \geq \epsilon$ (see Appendix A).

Second, since $p^N(0,x;s,y)$ is a continuous function in s and y (convolution of Gaussian densities), we observe that

$$p^N(0,x;s,y) - p(0,x;s,y) = \lim_{\epsilon \rightarrow 0} \mathbb{E}[p(s-\epsilon, X_{s-\epsilon}^N; s,y) - p(0,x;s,y)].$$

Then, for any $\epsilon > 0$, Itô's formula leads to

$$\begin{aligned} \mathbb{E}[p(s-\epsilon, X_{s-\epsilon}^N; s,y) - p(0,x;s,y)] &= \mathbb{E}\left[\int_0^{s-\epsilon} \partial_r p(r, X_r^N; s,y) dr\right] \\ &+ \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d b_i(\varphi(r), X_{\varphi(r)}^N) \partial_{x_i} p(r, X_r^N; s,y) dr\right] \\ &+ \frac{1}{2} \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i,j=1}^d a_{ij}(\varphi(r), X_{\varphi(r)}^N) \partial_{x_i x_j}^2 p(r, X_r^N; s,y) dr\right]. \end{aligned}$$

From the PDE, the above equality becomes

$$\begin{aligned} \mathbb{E}[p(s-\epsilon, X_{s-\epsilon}^N; s,y) - p(0,x;s,y)] &= \\ &\mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s,y) dr\right] \\ &+ \frac{1}{2} \mathbb{E}\left[\int_0^{s-\epsilon} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s,y) dr\right], \\ &:= \int_0^{s-\epsilon} \mathbb{E}[\phi(r)] dr, \end{aligned}$$

where $\phi(r) = \sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s,y) + \frac{1}{2} \sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s,y)$. To get (4.1), it remains to prove that $\mathbb{E}(\phi(r))$ is integrable over $[0, s]$. We check it by looking at the rest of the proof.

4.2 Proof of the intermediate result (4.2)

We prove inequality (4.2). $\mathbb{E}[\exp(-c \frac{|y-X_r^N|^2}{s-r})] = \int_{\mathbb{R}^d} \exp(-c \frac{|y-z|^2}{s-r}) p^N(0,x;r,z) dz$. Using Proposition 3.5, we get

$$\begin{aligned} \mathbb{E}\left[\exp\left(-c \frac{|y-X_r^N|^2}{s-r}\right)\right] &\leq \frac{K(T)}{r^{\frac{d}{2}}} \int_{\mathbb{R}^d} \exp\left(-c \frac{|y-z|^2}{s-r}\right) \exp\left(-c' \frac{|x-z|^2}{r}\right) dz \\ &\leq K(T) \prod_{i=1}^d \int_{\mathbb{R}} \frac{1}{\sqrt{r}} \exp\left(-c \frac{|y_i-z_i|^2}{s-r}\right) \exp\left(-c' \frac{|x_i-z_i|^2}{r}\right) dz_i, \end{aligned}$$

and $\int_{\mathbb{R}} \frac{1}{\sqrt{2\pi \frac{(s-r)}{2c}}} \exp(-c \frac{|y_i-z_i|^2}{s-r}) \frac{1}{\sqrt{2\pi \frac{r}{2c'}}} \exp(-c' \frac{|x_i-z_i|^2}{r}) dz_i$ is the convolution product of the density of two independant Gaussian random variables $\mathcal{N}(-x_i, \frac{r}{2c'})$ and $\mathcal{N}(y_i, \frac{s-r}{2c})$

computed at 0. Hence, the integral is equal to $\frac{1}{\sqrt{2\pi(\frac{r}{2c'} + \frac{s-r}{2c})}} \exp\left(-\frac{|x_i - y_i|^2}{\frac{r}{c'} + \frac{s-r}{c}}\right)$. Then,

$$\int_{\mathbb{R}} \frac{1}{\sqrt{r}} \exp\left(-c\frac{|y_i - z_i|^2}{s-r}\right) \exp\left(-c'\frac{|x_i - z_i|^2}{r}\right) dz_i \leq C \left(\frac{s-r}{s}\right)^{\frac{1}{2}} \exp\left(-c''\frac{|x_i - y_i|^2}{s}\right)$$

and (4.2) follows.

4.3 Upper bound for E_1

We recall that $E_1 = \int_0^s \mathbb{E} \left[\sum_{i=1}^d (b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N)) \partial_{x_i} p(r, X_r^N; s, y) \right] dr$. For each i , we apply Itô's formula to $b_i(u, X_u^N)$ between $u = \varphi(r)$ and $u = r$. We get

$$b_i(\varphi(r), X_{\varphi(r)}^N) - b_i(r, X_r^N) = \int_{\varphi(r)}^r \alpha_u^i du + \int_{\varphi(r)}^r \sum_{k=1}^q \beta_u^{i,k} dW_u^k, \tag{4.3}$$

where α_u^i depends on $\partial_t b, \partial_x b, \partial_x^2 b, \sigma$, and $\beta_u^i = -\nabla_x b_i(u, X_u^N) \sigma(\varphi(r), X_{\varphi(r)}^N)$. Since b, σ belong to $C_b^{1,3}$, α^i and $(\beta^{i,k})_{1 \leq k \leq q}$ are uniformly bounded. Using (4.3) and the duality formula (3.1) yield

$$\begin{aligned} E_1 &= \sum_{i=1}^d \int_0^s \mathbb{E} \left[\int_{\varphi(r)}^r \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i du + \mathbb{E} \left[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du \right] \right] dr \\ &:= E_{11} + E_{12}, \end{aligned} \tag{4.4}$$

where β_u^i is a row vector of q components. We upper bound E_{11} and E_{12} .

Bound for E_{11} $E_{11} = \sum_{i=1}^d \int_0^s \mathbb{E} \left[\int_{\varphi(r)}^r \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i du \right] dr$.

Since $|\sum_{i=1}^d \partial_{x_i} p(r, X_r^N; s, y) \alpha_u^i| \leq |\alpha_u| |\partial_x p(r, X_r^N; s, y)|$ and α_u is uniformly bounded in u , we have

$$|E_{11}| \leq C \frac{T}{N} \int_0^s \mathbb{E} |\partial_x p(r, X_r^N; s, y)| dr.$$

Besides that, from Proposition A.2, $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp\left(-c\frac{|y - X_r^N|^2}{s-r}\right)$. Then,

$$|E_{11}| \leq K(T) \frac{T}{N} \int_0^s \frac{1}{(s-r)^{\frac{d+1}{2}}} \mathbb{E} \left[\exp\left(-c\frac{|y - X_r^N|^2}{s-r}\right) \right] dr.$$

Using the intermediate result (4.2) yields

$$|E_{11}| \leq K(T) \frac{T}{N} \int_0^s \frac{1}{\sqrt{s-r}} \frac{1}{s^{\frac{d}{2}}} \exp\left(-c\frac{|x-y|^2}{s}\right) dr \leq K(T) \frac{T}{N} \frac{1}{s^{\frac{d-1}{2}}} \exp\left(-c\frac{|x-y|^2}{s}\right)$$

and thus, E_{11} satisfies property \mathcal{P} (see Definition 4.1).

Bound for E_{12} $E_{12} = \sum_{i=1}^d \int_0^s \mathbb{E} \left[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) \cdot \beta_u^i du \right] dr$.

To rewrite E_{12} , we use the expression of β_u^i and Proposition 3.1, which gives $\mathcal{D}_u(\partial_{x_i} p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i} p(r, X_r^N; s, y))\sigma(\varphi(r), X_{\varphi(r)}^N)$. Then,

$$E_{12} = - \int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i x_k}^2 p(r, X_r^N; s, y)[(\sigma\sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k] du. \quad (4.5)$$

Using the integration by parts formula (3.2), we get that

$$E_{12} = - \int_0^s dr \int_{\varphi(r)}^r \sum_{i,k=1}^d \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_k}(i)] du$$

where e_k is a vector whose k -th component is 1 and other components are 0. From (3.3), we deduce $\mathbb{E}[|H_{e_k}(i)|^p]^{1/p} \leq C \frac{K(T)}{r^{1/2}}$, where C only depends on $|\sigma|_\infty$, $|\partial_x \sigma|_\infty$, $|\partial_x b|_\infty$, $|\partial_{xx}^2 b|_\infty$. By the Hölder inequality, it follows that

$$|E_{12}| \leq K(T) \int_0^s dr \int_{\varphi(r)}^r \frac{1}{r^{1/2}} \mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}} du.$$

Using Proposition A.2 leads to $|\partial_x p(r, X_r^N; s, y)| \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \exp(-c \frac{|y-X_r^N|^2}{s-r})$, and combining this inequality with the intermediate result (4.2) yields

$$\mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{d/(d+1)} \leq \frac{K(T)}{(s-r)^{\frac{d+1}{2}}} \left(\frac{s-r}{s}\right)^{\frac{d^2}{2(d+1)}} \exp\left(-c \frac{|y-x|^2}{s}\right). \quad (4.6)$$

Hence, E_{12} is bounded by

$$\frac{K(T)}{s^{\frac{d^2}{2(d+1)}}} \frac{T}{N} \exp\left(-c \frac{|y-x|^2}{s}\right) \int_0^s \frac{1}{r^{1/2}} \frac{1}{(s-r)^{\frac{d+1}{2} - \frac{d^2}{2(d+1)}}} dr.$$

The above integral equals $s^{\frac{1}{2} - \frac{d+1}{2} + \frac{d^2}{2(d+1)}} B(\frac{1}{2}, \frac{1}{2(d+1)})$ where B is the function Beta. Thus $|E_{12}| \leq \frac{K(T)}{s^{d/2}} \frac{T}{N} \exp(-c \frac{|y-x|^2}{s})$, and E_{12} satisfies property \mathcal{P} .

4.4 Upper bound for E_2

We recall $E_2 = \frac{1}{2} \int_0^s \mathbb{E}[\sum_{i,j=1}^d (a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N)) \partial_{x_i x_j}^2 p(r, X_r^N; s, y)] dr$. As we did

for E_1 , we apply Itô's formula to $a_{ij}(u, X_u^N)$ between $\varphi(r)$ and r . We get $a_{ij}(\varphi(r), X_{\varphi(r)}^N) - a_{ij}(r, X_r^N) = \int_{\varphi(r)}^r \gamma_u^{ij} du + \int_{\varphi(r)}^r \delta_u^{ij} dW_u$, where γ_u^{ij} depends on $\sigma, \partial_t \sigma, \partial_x \sigma, b, \partial_{xx}^2 \sigma$ and δ_u^{ij} is a row vector of size q , with l -th component $(\delta_u^{ij})_l = -\sum_{k=1}^d \partial_{x_k} a_{ij}(u, X_u^N) \sigma_{kl}(\varphi(r), X_{\varphi(r)}^N)$. Then, the duality formula (3.1) leads to

$$\begin{aligned} E_2 &= \sum_{i,j=1}^d \int_0^s \left\{ \mathbb{E} \left[\int_{\varphi(r)}^r \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \gamma_u^{ij} du \right] + \mathbb{E} \left[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du \right] \right\} dr \\ &:= E_{21} + E_{22}. \end{aligned}$$

Bound for E_{21} $E_{21} = \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \partial_{x_i x_j}^2 p(r, X_r^N; s, y) \gamma_u^{ij} du] dr.$

As $\sigma, b, \partial_t \sigma, \partial_x \sigma, \partial_x^2 \sigma$ are C_b^1 in space, γ_u^{ij} has the same smoothness properties as the term $[(\sigma \sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x b_i(u, X_u^N))^*]_k$ appearing in (4.5). Thus, E_{21} can be treated as E_{12} and satisfies to the same estimate.

Bound for E_{22} $E_{22} = \sum_{i,j=1}^d \int_0^s \mathbb{E}[\int_{\varphi(r)}^r \mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) \cdot \delta_u^{ij} du] dr.$

To rewrite E_{22} , we use the expression of δ_u^{ij} and Proposition 3.1, which asserts $\mathcal{D}_u(\partial_{x_i x_j}^2 p(r, X_r^N; s, y)) = \nabla_x(\partial_{x_i x_j}^2 p(r, X_r^N; s, y))\sigma(\varphi(r), X_{\varphi(r)}^N)$. Thus,

$$E_{22} = - \sum_{i,j,k=1}^d \int_0^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i x_j x_k}^3 p(r, X_r^N; s, y)[(\sigma \sigma^*)(\varphi(r), X_{\varphi(r)}^N)(\nabla_x a_{ij}(u, X_u^N))^*]_k] du.$$

To complete this proof, we split E_{22} in two terms : E_{22}^1 (resp E_{22}^2) corresponds to the integral in r from 0 to $\frac{s}{2}$ (resp. from $\frac{s}{2}$ to s).

- On $[0, \frac{s}{2}]$, E_{22}^1 is bounded by $C \frac{T}{N} \int_0^{\frac{s}{2}} \mathbb{E}[|\partial_{x_i x_j x_k}^3 p(r, X_r^N; s, y)|] dr$. Using Proposition A.2 and (4.2), it gives

$$|E_{22}^1| \leq \frac{K(T)T}{N} \frac{1}{s^{d/2}} \exp\left(-c \frac{|x-y|^2}{s}\right) \int_0^{\frac{s}{2}} \frac{1}{(s-r)^{3/2}} dr.$$

Hence, E_{22} satisfies \mathcal{P} .

- On $[\frac{s}{2}, s]$, we use the integration by parts formula (3.2) of Proposition 3.4, with $|\alpha| = 2$.

$$E_{22}^2 = - \sum_{i,j,k=1}^d \int_{\frac{s}{2}}^s dr \int_{\varphi(r)}^r \mathbb{E}[\partial_{x_i} p(r, X_r^N; s, y) H_{e_{jk}}(i)] du,$$

where e_{jk} is a vector full of zeros except the j -th and the k -th components. Using Hölder's inequality and (3.3) (remember that $\sigma \in C_b^{1,3}$), we obtain

$$|E_{22}^2| \leq K(T) \frac{T}{N} \int_{\frac{s}{2}}^s \frac{1}{r} \mathbb{E}[|\partial_x p(r, X_r^N; s, y)|^{\frac{d+1}{d}}]^{\frac{d}{d+1}} dr. \tag{4.7}$$

By applying (4.6), we get

$$|E_{22}^2| \leq K(T) \frac{T}{N} \frac{1}{s^{1+\frac{d^2}{2(d+1)}}} \exp\left(-c \frac{|x-y|^2}{s}\right) \int_{\frac{s}{2}}^s \frac{1}{(s-r)^{\frac{2d+1}{2d+2}}} dr,$$

and the result follows.

A Bounds for the transition density function and its derivatives

We bring together classical results related to bounds for the transition probability density of X defined by (1.1).

Proposition A.1 (Aronson [1]). *Assume that the coefficients σ and b are bounded measurable functions and that σ is uniformly elliptic. There exist positive constants K, α_0, α_1 s.t. for any x, y in \mathbb{R}^d and any $0 \leq t < s \leq T$, one has*

$$\frac{K^{-1}}{(2\pi\alpha_1(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_1(s-t)}} \leq p(t, x; s, y) \leq K \frac{1}{(2\pi\alpha_2(s-t))^{\frac{d}{2}}} e^{-\frac{|x-y|^2}{2\alpha_2(s-t)}}. \quad (\text{A.1})$$

Proposition A.2 (Friedman [3]). *Assume that the coefficients b and σ are Hölder continuous in time, C_b^2 in space and that σ is uniformly elliptic. Then, $\partial_x^{m+a} \partial_y^b p(t, x; s, y)$ exist and are continuous functions for all $0 \leq |a| + |b| \leq 2, |m| = 0, 1$. Moreover, there exist two positive constants c and K s.t. for any x, y in \mathbb{R}^d and any $0 \leq t < s \leq T$, one has*

$$|\partial_x^{m+a} \partial_y^b p(t, x; s, y)| \leq \frac{K}{(s-t)^{(|m|+|a|+|b|+d)/2}} \exp\left(-c \frac{|y-x|^2}{s-t}\right).$$

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