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A SIMPLE PROOF OF THE POINCARÉ INEQUALITY FOR A LARGE CLASS OF PROBABILITY MEASURES INCLUDING THE LOG-CONCAVE CASE

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Abstract
We give a simple and direct proof of the existence of a spectral gap under some Lyapunov type condition which is satisfied in particular by log-concave probability measures on $\mathbb{R}^n$. The proof is based on arguments introduced in [2], but for the sake of completeness, all details are provided.

1 Introduction and main results.

Let $\mu(dx) = e^{-V(x)}\,dx$ be a probability measure on $\mathbb{R}^n$. For simplicity, we will always assume V is C^2 and lower bounded. The famous Bakry-Emery criterion tells us that if V is such that $\mathrm{Hess}_x(V) \geq \alpha\,\mathrm{Id}$ in the sense of quadratic forms, for some $\alpha > 0$ and all $x \in \mathbb{R}^n$ (i.e. V is uniformly convex), then μ satisfies the Gross logarithmic Sobolev inequality. See [1] for a comprehensive introduction to this topic. If $\alpha = 0$ this result is no more true (think to the two sided exponential distribution). However, in [3], S. Bobkov showed that in this case μ still satisfies a Poincaré inequality

$$\mathrm{Var}_\mu(f) \leq C_P \int |\nabla f|^2 \, d\mu \,, \tag{1.1}$$

for all smooth f. Actually Bobkov (see [3] Thm 1.2) proved a stronger result giving a lower bound for the isoperimetric constant of log-concave probability measures, from which (1.1) follows using Cheeger's bound. In one dimension bounds are even more precise (see [3, 6]). In particular such measures satisfy the $\mathbb{L}^1$ version of Poincaré inequality, i.e. there exists some C_C such that for all nice f with median equal to 0,

$$\int |f| \, d\mu \leq C_C \int |\nabla f| \, d\mu \,, \tag{1.2}$$

In the recent paper [2], the authors developed a general strategy to get tractable conditions for the Poincaré inequality (1.1) (as well as for weak Poincaré inequalities) using some Lyapunov control function. Pushing forward these ideas, a new proof of Bakry-Emery criterion is obtained in [5]. In this note we shall give an analogous proof of Bobkov's result. Actually we shall prove a stronger result we shall state after introducing some basic notation.

In the whole paper $|x|$ will denote the euclidean norm of x, $B(x,R)$ is the open euclidean ball with center x and radius R, $\langle x, y\rangle$ denotes the scalar product. λ denotes the Lebesgue measure and $L = \Delta - \langle \nabla V, \nabla \rangle$ is the μ symmetric natural operator.

Let W be a C^2 function. We shall say that W is a Lyapunov function if $W \geq 1$ and if there exist $\theta > 0$, $b \geq 0$ and some $R > 0$ such that for all x,

$$LW(x) \leq -\theta \, W(x) + b \, \mathbb{1}_{B(0,R)}(x) \,. \tag{1.3}$$

Since V is locally bounded, it is known that μ restricted to $B(0,R)$ satisfies both a $\mathbb{L}^2$ Poincaré inequality with constant κ_R and a $\mathbb{L}^1$ Poincaré inequality with constant η_R. A very bad bound for these constants is given by $D_i \, R^i \, e^{\mathrm{Osc}_R V}$ where D_i ($i = 2$ or $i = 1$) is a universal constant and $\mathrm{Osc}_R V = \sup_{B(0,R)} V - \inf_{B(0,R)} V$.
The main results are the following

Theorem 1.4. *If there exists a Lyapunov function W satisfying (1.3), then μ satisfies a $\mathbb{L}^2$ Poincaré inequality (1.1) with constant $C_P = \frac{1}{\theta}(1+b\kappa_R)$, where κ_R is the $\mathbb{L}^2$ Poincaré constant of μ restricted to the ball $B(0,R)$.*

Theorem 1.5. *If there exists a Lyapunov function W satisfying (1.3) and such that $|\nabla W|/W$ is bounded by some constant C, then μ satisfies a $\mathbb{L}^1$ Poincaré inequality (1.2) with constant $C_C = \frac{1}{\theta}(C + b\eta_R)$, where η_R is the $\mathbb{L}^1$ Poincaré constant of μ restricted to the ball $B(0,R)$.*

As we already mentioned the $\mathbb{L}^1$ Poincaré inequality is stronger than the $\mathbb{L}^2$ inequality thanks to Cheeger's bound. However we shall derive both inequalities separately and get the announced bounds directly. As it will be clear in the proof, we cannot reach the optimal value of C_P (or C_C, nor the bounds obtained in [3, 6]), but our method partially extends to riemannian manifolds. We shall explain how. Also note that the $\mathbb{L}^2$ result is not new: with some additional assumptions it was proved by Wu ([8]) using spectral methods. The result is also shown in [2] using an indirect strategy based on the study of the asymptotic behavior of the semi-group generated by L. The proof we shall give in the next section is direct and elementary.

In order to see how to apply these theorems in concrete situations it is natural to ask about the construction of such Lyapunov functions. The next corollary furnishes a large class of examples

Corollary 1.6. *(1) If there exist $\alpha > 0$ and $R \geq 0$ such that for $|x| \geq R$,*

$$\langle x, \nabla V(x) \rangle \geq \alpha \, |x| \, , \tag{1.7}$$

we may apply both Theorem 1.4 and Theorem 1.5.

(2) If there exist $a \in (0,1)$, $c > 0$ and $R \geq 0$ such that for $|x| \geq R$,

$$a \, |\nabla V(x)|^2 - \Delta V(x) \geq c \, , \tag{1.8}$$

we may apply Theorem 1.4.

The first condition is reminiscent of the theory of diffusion processes. It is called a "drift condition" by many authors. The second condition is reminiscent of the spectral theory of Schrödinger operators and for $a = 1/2$ the result has been known for a long time (see e.g. [4] Proposition 5.3).

Finally, according to Lemma 2.2 below, the first condition in Corollary 1.6 is satisfied when V is convex, hence

Corollary 1.9. *If μ is log-concave, i.e. if V is a convex function, then μ satisfies both a $\mathbb{L}^2$ and a $\mathbb{L}^1$ Poincaré inequality.*

It turns out that in the log-concave situation, $\mathbb{L}^1$ and $\mathbb{L}^2$ Poincaré inequalities are actually equivalent (see [7]).
We may build convex functions on $\mathbb{R}$ that do not satisfy condition (2) in Corollary 1.6.

2 Proofs.

To prove Corollary 1.6 it is enough to build an ad-hoc Lyapunov function.

In case (1) we may consider a smooth function W such that $W(x) = e^{\gamma \, |x|}$ for $|x| \geq R$ and $\gamma > 0$, and $W(x) \geq 1$ for all x (we choose here some $R > 0$).
We have for $|x| \geq R$,

$$LW(x) = \gamma \left(\frac{n-1}{|x|} + \gamma - \frac{x.\nabla V(x)}{|x|} \right) W(x)$$

so that, we may find some $b > 0$ such that

$$LW(x) \; \leq \; - \gamma \, (\alpha - \gamma - \frac{n-1}{R}) \, W(x) + b \, \mathbb{1}_{B(0,R)}(x) \, . \tag{2.1}$$

Hence W is a Lyapunov function provided

$$\theta = \gamma\,(\alpha - \gamma - \frac{n-1}{R}) > 0\,,$$

so that taking a larger R if necessary (this is of course not a restriction) we may always find some sufficiently small $\gamma > 0$.
In addition $|\nabla W|/W$ is bounded.

In case (2) we may consider a smooth function W such that $W(x) = e^{\gamma\,(V(x)-\inf V)+\varepsilon}$ for $|x| \geq R$ and some $\varepsilon > 0$, and $W(x) \geq 1$ for all x. For $|x| > R$ we then have

$$LW(x) = \gamma\,\left(\Delta V(x) - (1-\gamma)\,|\nabla V(x)|^2\right)\,W(x)\,,$$

so that for $\gamma = 1-a$ we again obtain a Lyapunov function with $\theta = (1-a)\,c$.

To prove corollary 1.9 we need the following elementary

Lemma 2.2. *If V is differentiable, convex and $\int e^{-V(x)}\,dx < +\infty$ then*

(1) for all x, $\langle x, \nabla V(x)\rangle \geq V(x) - V(0)$,

(2) there exist $\alpha > 0$ and $R > 0$ such that for $|x| \geq R$, $V(x) - V(0) \geq \alpha\,|x|$.

Proof.
The first statement is true for any convex function. Indeed $t \mapsto g(t) = V(tx)$ is convex, so $g(0) \geq g(1) + (0-1)g'(1)$ hence the result.
The second statement requires the integrability condition. Choose $K > V(0)+1$. Then the level set $A_K = \{x; V(x) \leq K\}$ has non empty interior (since V is continuous it contains $B(0,r)$ for some $r > 0$), is convex and has finite Lebesgue measure, since $\lambda(A_K) \leq e^K \int_{A_K} e^{-V}\,dx < +\infty$. It follows that A_K is compact. Indeed if $a \in A_K$ does not belong to $B(0,r)$, the open cone with basis $B(0,r)$ and vertex a is included in A_K, due to convexity. Since the volume of this cone growths linearly in $|a|$, A_K has to be bounded.
We may thus find some $R > 0$ such that $A_K \subseteq B(0,R-1)$. Consider now some u such that $|u| = R$. u does not belong to A_K hence $V(u) - V(0) \geq 1$. Since V is convex, $t \mapsto (1/t)(V(tu) - V(0))$ is non decreasing. It follows that for $|x| \geq R$, $V(x) - V(0) \geq |x|/R$.

Now we start with the proof of Theorem 1.4.

If g is a smooth function, we know that $\mathrm{Var}_\mu(g) \leq \int (g-c)^2\,d\mu$ for all c. In the sequel $f = g-c$ where c is a constant to be chosen later.
We may write

$$\int f^2 d\mu \leq \int \frac{-LW}{\theta\,W} f^2\,d\mu + \int f^2 \frac{b}{\theta\,W}\,\mathbb{1}_{B(0,R)}\,d\mu\,.$$

Note that the right hand side could be infinite, depending on the behavior of $-LW/W$ which is positive for large $|x|$.
We first focus on the first term in the right hand side of the previous inequality.
Actually, since we do not assume integrability conditions for W and its derivatives, in the derivation below, we have first to assume that g is compactly supported and $f = (g-c)\chi$, where χ is a non-negative compactly supported smooth function, such that $\mathbb{1}_{B(0,R)} \leq \chi \leq 1$.
All the calculation below are thus allowed. In the end we choose some sequence χ_n satisfying $\mathbb{1}_{B(0,nR)} \leq \chi_n \leq 1$, and such that $|\nabla \chi_n| \leq 1$, and we go to the limit.

Since L is μ-symmetric and integrating by parts we get

$$\begin{aligned}\int \frac{-LW}{W} f^2\, d\mu &= \int \nabla\left(\frac{f^2}{W}\right).\nabla W\, d\mu\\ &= 2\int (f/W)\,\nabla f.\nabla W\, d\mu - \int (f^2/W^2)\,|\nabla W|^2\, d\mu\\ &= \int |\nabla f|^2\, d\mu - \int \left|\nabla f - (f/W)\nabla W\right|^2\, d\mu\\ &\leq \int |\nabla f|^2\, d\mu\,.\end{aligned}$$

Note that the limiting procedure is allowed by using monotone convergence (recall that $-LW/W$ is positive for large $|x|$) in the left hand side, and bounded convergence in the right hand side.

Now come back to $\int f^2 \frac{b}{\theta\, W}\, \mathbb{1}_{B(0,R)}\, d\mu$. Since μ satisfies a Poincaré inequality on $B(0,R)$ with constant κ_R,

$$\int_{B(0,R)} f^2\, d\mu \;\leq\; \kappa_R \int_{B(0,R)} |\nabla f|^2\, d\mu + (1/\mu(B(0,R)))\left(\int_{B(0,R)} f\, d\mu\right)^2 .$$

Choosing $c = \int_{B(0,R)} g\, d\mu$ the latter term is equal to 0. So, using $W \geq 1$ we get

$$\int_{B(0,R)} (f^2/W)\, d\mu \leq \int_{B(0,R)} f^2\, d\mu \leq \kappa_R \int_{B(0,R)} |\nabla f|^2\, d\mu\,. \tag{2.3}$$

We have finally obtained

$$\int f^2 d\mu \leq \frac{1}{\theta}(1+b\kappa_R)\int |\nabla f|^2\, d\mu\,, \tag{2.4}$$

i.e a Poincaré inequality

$$\mathrm{Var}_\mu(g) \;\leq\; \frac{1}{\theta}(1+b\kappa_R)\int |\nabla g|^2\, d\mu\,,$$

with $C_P = \frac{1}{\theta}(1+b\kappa_R)$, which is of course certainly not the best constant.

The previous method can be used to get the stronger $\mathbb{L}^1$ Poincaré inequality, i.e Theorem 1.5. Consider an arbitrary smooth function f with median 0. Recall the Lyapunov condition (1.3). Again in the sequel g will be $f-c$ for some well chosen c. To be completely rigorous we need the same approximation procedure as in the previous proof.
Now

$$\begin{aligned}\int |g|\, d\mu &\leq \int |g|\frac{-LW}{\theta\, W}\, d\mu + (b/\theta)\int |g|\mathbb{1}_{B(0,R)}\, d\mu\\ &\leq (1/\theta)\int \nabla\left(\frac{|g|}{W}\right).\nabla W\, d\mu + (b/\theta)\int |g|\mathbb{1}_{B(0,R)}\, d\mu\\ &\leq (1/\theta)\left\{\int \frac{\nabla|g|.\nabla W}{W}\, d\mu - \int \frac{|g|\,|\nabla W|^2}{W^2}\, d\mu\right\} + (b/\theta)\int |g|\mathbb{1}_{B(0,R)}\, d\mu\\ &\leq (1/\theta)\int |\nabla g|\,\frac{|\nabla W|}{W}\, d\mu + (b/\theta)\int |g|\mathbb{1}_{B(0,R)}\, d\mu\,.\end{aligned}$$

To control the first term we use that $|\nabla W|/W$ is bounded by C. Next as in the previous section we may control $\int |g| \mathbb{1}_{B(0,R)}\, d\mu$ by using the $\mathbb{L}^1$ Poincaré inequality satisfied by μ restricted to $B(0,R)$ yielding

$$\int |g| \mathbb{1}_{B(0,R)}\, d\mu \, \leq \, \eta_R \int_{B(0,R)} |\nabla g|\, d\mu \,,$$

provided the median of g on the ball $B(0,R)$ is equal to 0. Since $g = f - c$ we may always choose some c for this property to hold. Finally $\int |f| d\mu \leq \int |g| d\mu$ since f has median 0, so that

$$\int |f|\, d\mu \, \leq \, \frac{C + b\eta_R}{\theta} \int |\nabla f|\, d\mu \,. \tag{2.5}$$

Remark 2.6. *Most of the arguments remain true if we replace the flat space $\mathbb{R}^n$ by a non-compact riemanian manifold M, provided M is Cartan-Hadamard and with Ricci curvature bounded from below (by a non positive constant). Indeed, in this case the construction of the Lyapunov function (case (1) in Corollary 1.6) is the same replacing $|x|$ by $\rho(o,x)$ for some given $o \in M$, where ρ denotes the riemanian distance. It turns out that $\Delta\rho^2$ is still bounded and that the first statement in Lemma 2.2 is still satisfied. The only point is that V does not necessarily grow at least linearly as a function of the distance, so that this assumption has to be made in addition to the convexity of V.*

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