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ON THE STRONG LAW OF LARGE NUMBERS FOR D-DIMENSIONAL ARRAYS OF RANDOM VARIABLES

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Abstract
In this paper, we provide a necessary and sufficient condition for general d-dimensional arrays of random variables to satisfy strong law of large numbers. Then, we apply the result to obtain some strong laws of large numbers for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables.

1 Introduction

Let $\mathbb{Z}_+^d$, where d is a positive integer, denote the positive integer d-dimensional lattice points. The notation $\mathbf{m} \prec \mathbf{n}$, where $\mathbf{m} = (m_1, m_2, ..., m_d)$ and $\mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d$, means that $m_i \leqslant n_i$, $1 \leqslant i \leqslant d$. Let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants, and let $\mathbf{n} = (n_1, n_2, ..., n_d) \in \mathbb{Z}_+^d$, we denote $|\mathbf{n}| = \prod_{i=1}^d n_i$, $|\mathbf{n}(\alpha)| = \prod_{i=1}^d n_i^{\alpha_i}$, $I(\mathbf{n}) = \{(a_1, \dots, a_d) \in \mathbb{Z}_+^d : 2^{n_i - 1} \leqslant a_i < 2^{n_i}, 1 \leqslant i \leqslant d\}$, $\overline{\mathbf{n}} = (2^{n_1-1}, \dots, 2^{n_d-1})$.
Consider a d-dimensional array $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$. Let $S_{\mathbf{n}} = \sum_{\mathbf{i} \prec \mathbf{n}} X_{\mathbf{i}}$, and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. In Section 2, we provide a necessary and sufficient condition for

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ almost surely (a.s.)}$$

to hold. This condition springs from a recent result of Chobanyan, Levental and Mandrekar [1] which provided a condition for strong law of large numbers (SLLN) in the case $d = 1$ (see Chobanyan, Levental and Mandrekar [1, Theorem 3.3]). Some applications to SLLN for d-dimensional arrays of blockwise independent and blockwise orthogonal random variables are made in Section 3.

2 Result

We can now state our main result.

THEOREM 2.1. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. For $\mathbf{m} = (m_1, \ldots, m_d) \in \mathbb{Z}_+^d$, set

$$T_{\mathbf{m}} = \frac{1}{|\overline{\mathbf{m}}(\alpha)|} \max_{\mathbf{k} \in I(\mathbf{m})} \Big| \sum_{\overline{\mathbf{m}} \prec \mathbf{i} \prec \mathbf{k}} X_{\mathbf{i}} \Big|.$$

Then

$$\lim_{|\mathbf{m}| \to \infty} T_{\mathbf{m}} = 0 \text{ a.s.} \tag{2.1}$$

if and only if

$$\lim_{|\mathbf{n}| \to \infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.2}$$

Proof. To prove Theorem 2.1, we will need the following lemmma. The proof of the following lemma is just an application of Kronecker's lemma with d-dimensional indices as was so kindly pointed out to the author by the referee.

LEMMA 2.1. Let $\{x_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of constants, and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be a collection of positive constants. If

$$\lim_{|\mathbf{n}| \to \infty} x_{\mathbf{n}} = 0, \tag{2.3}$$

then

$$\lim_{|\mathbf{n}| \to \infty} \frac{1}{|\overline{\mathbf{n}}(\alpha)|} \sum_{\mathbf{k} \prec \mathbf{n}} |\overline{\mathbf{k}}(\alpha)| x_{\mathbf{k}} = 0. \tag{2.4}$$

Proof of Theorem 2.1. Let $\mathbf{m} = (m_1, \ldots, m_d),\ \mathbf{n} = (n_1, \ldots, n_d) \in \mathbb{Z}_+^d$ with $\mathbf{n} \in I(\mathbf{m})$. Set

$$\mathbf{n}^{(j)} = (n_1, \ldots, n_{j-1}, 2^{m_j - 1} - 1, n_{j+1}, \ldots, n_d),\ 1 \leqslant j \leqslant d,$$

$$S_{\mathbf{n}}^{(1)} = S_{\mathbf{n}^{(1)}},$$

$$S_{\mathbf{n}}^{(d)} = \sum_{i_1 = 2^{m_1 - 1}}^{n_1} \cdots \sum_{i_{d-1} = 2^{m_{d-1} - 1}}^{n_{d-1}} \sum_{i_d = 1}^{2^{m_d - 1} - 1} X_{(i_1, \ldots, i_d)},$$

and

$$S_{\mathbf{n}}^{(j)} = \sum_{i_1 = 2^{m_1 - 1}}^{n_1} \cdots \sum_{i_{j-1} = 2^{m_{j-1} - 1}}^{n_{j-1}} \sum_{i_j = 1}^{2^{m_j - 1} - 1} \sum_{i_{j+1} = 1}^{n_{j+1}} \cdots \sum_{i_d = 1}^{n_d} X_{(i_1, \ldots, i_d)},\ 2 \leqslant j \leqslant d - 1.$$

Then

$$S_{\mathbf{n}}^{(j)} = S_{\mathbf{n}^{(j)}} - \sum_{k=1}^{j-1} S_{\mathbf{n}^{(j)}}^{(k)},\ 2 \leqslant j \leqslant d. \tag{2.5}$$

Assume that (2.1) holds. Since

$$\frac{|S_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} \leqslant \frac{1}{|\overline{\mathbf{m}}(\alpha)|} \sum_{\mathbf{k} \prec \mathbf{m}} |\overline{\mathbf{k}}(\alpha)| T_{\mathbf{k}},$$

the conclusion (2.2) holds by Lemma 2.1. Thus (2.1) implies (2.2). Now, assume that (2.2) holds. Then

$$\lim_{|\mathbf{m}|\to\infty} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{S_{\mathbf{n}}^{(1)}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.6}$$

For $1 \leqslant j \leqslant d$, by (2.5), (2.6) and the induction method, we obtain

$$\lim_{|\mathbf{m}|\to\infty} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{S_{\mathbf{n}}^{(j)}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{2.7}$$

Since

$$S_{\mathbf{n}} = \sum_{j=1}^{d} S_{\mathbf{n}}^{(j)} + \sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1,\ldots,i_d)},$$

we have that

$$|\sum_{i_1=2^{m_1-1}}^{n_1} \cdots \sum_{i_d=2^{m_d-1}}^{n_d} X_{(i_1,\ldots,i_d)}| \leqslant |S_{\mathbf{n}}| + \sum_{j=1}^{d} |S_{\mathbf{n}}^{(j)}|.$$

This implies

$$T_{\mathbf{m}} \leqslant 2^{\alpha_1+\cdots+\alpha_d} \max_{\mathbf{n}\in I(\mathbf{m})} \frac{|S_{\mathbf{n}}| + \sum_{j=1}^{d} |S_{\mathbf{n}}^{(j)}|}{|\mathbf{n}(\alpha)|}. \tag{2.8}$$

The conclusion (2.1) follows immediately from (2.2), (2.7) and (2.8). □

3 Applications

In this section, we present some applications of Theorem 2.1. A d-dimensional array of random variables $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ is said to be *blockwise independent* (resp., *blockwise orthogonal*) if for each $\mathbf{k} \in \mathbb{Z}_+^d$, the random variables $\{X_{\mathbf{i}}, \mathbf{i} \in I(\mathbf{k})\}$ is independent (resp., orthogonal). The concept of blockwise independence for a sequence of random variables was introduced by Móricz [9]. Extensions of classical Kolmogorov SLLN (see, e.g., Chow and Teicher [2], p. 124) to the blockwise independence case were established by Móricz [9] and Gaposhkin [4]. Móricz [9] and Gaposhkin [4] also studied SLLN problem for sequence of blockwise orthogonal random variables.

Firstly, we establish a blockwise independence and d-dimensional version of the Kolmogorov SLLN.

THEOREM 3.1. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of mean 0 blockwise independent random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. If

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|X_{\mathbf{n}}|^p}{|\mathbf{n}(\alpha)|^p} < \infty \text{ for some } 0 < p \leqslant 2, \tag{3.1}$$

then SLLN

$$\lim_{|\mathbf{n}|\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.2}$$

obtains.

In the case $0 < p \leqslant 1$, the independence hypothesis and the hypothesis that $EX_{\mathbf{n}} = 0, \mathbf{n} \in \mathbb{Z}_+^d$ are superfluous.

Proof. We need the following lemma which was proved by Thanh [11] in the case $d = 2$. If d is arbitrary positive integer, then the proof is similar and so is omitted.
LEMMA 3.1. Let $\mathbf{n} \in \mathbb{Z}_+^d$ and let $\{X_\mathbf{i}, \mathbf{i} \prec \mathbf{n}\}$ be a collection of $|\mathbf{n}|$ mean 0 independent random variables. Then there exists a constant C depending only on p and d such that

$$E(\max_{\mathbf{k}\prec\mathbf{n}} |S_\mathbf{k}|^p) \leqslant C \sum_{\mathbf{i}\prec\mathbf{n}} E|X_\mathbf{i}|^p \text{ for all } 0 < p \leqslant 2.$$

In the case $0 < p \leqslant 1$, the independence hypothesis and the hypothesis that $EX_\mathbf{i} = 0, \mathbf{i} \prec \mathbf{n}$ are superfluous, and C is given by $C = 1$. In the case $1 < p < 2$, C is given by $C = 2\big(\frac{p}{p-1}\big)^{pd}$. In the case $p = 2$, Lemma 3.1 was proved by Wichura [12] and C is given by $C = 4^d$.
Proof of Theorem 3.1. Define $T_\mathbf{m}, \mathbf{m} \in \mathbb{Z}_+^d$ as in Theorem 2.1. Note that for all $\mathbf{m} \in \mathbb{Z}_+^d$,

$$\begin{aligned} E|T_\mathbf{m}|^p &= \frac{1}{|\overline{\mathbf{m}}(\alpha)|^p} E\Big(\max_{\mathbf{k}\in I(\mathbf{m})} \Big|\sum_{\overline{\mathbf{m}}\prec\mathbf{i}\prec\mathbf{k}} X_\mathbf{i}\Big|\Big)^p \\ &\leqslant \frac{C}{|\overline{\mathbf{m}}(\alpha)|^p} \sum_{\mathbf{i}\in I(\mathbf{m})} E|X_\mathbf{i}|^p \text{ (by Lemma 3.1)} \\ &\leqslant 2^{\alpha_1+\cdots+\alpha_d} C \frac{\sum_{\mathbf{i}\in I(\mathbf{m})} E|X_\mathbf{i}|^p}{|\mathbf{i}(\alpha)|^p}. \end{aligned}$$

It thus follows from (3.1) that $\sum_{\mathbf{m}\in\mathbb{Z}_+^d} E|T_\mathbf{m}|^p < \infty$ whence $\lim_{|\mathbf{m}|\to\infty} T_\mathbf{m} = 0$ a.s. The conclusion (3.2) follows immediately from Theorem 2.1. □

The following theorem extends Theorem 3.1 and its part (ii) reduces to a result of Smythe [10] when the $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are independent and $\alpha_1 = \cdots = \alpha_d = 1$.
THEOREM 3.2. Let $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. Assume that $\varphi(x)$ is a continuous functions on $[0, \infty)$, $\varphi(0) \geqslant 0$, $\varphi(x) > 0$ for $x > 0$, and

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E(\varphi(|X_\mathbf{n}|))}{\varphi(|\mathbf{n}(\alpha)|)} < \infty. \tag{3.3}$$

If either
(i) $\varphi(x)/x \downarrow$, and $\varphi(x) \uparrow$
or
(ii) $\{X_\mathbf{n}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are blockwise independent and have mean 0, and

$$\varphi(x)/x \uparrow, \ \varphi(x)/x^2 \downarrow,$$

then SLLN (3.2) obtains.

Proof. For $\mathbf{n} \in \mathbb{Z}_+^d$, set

$$Y_\mathbf{n} = X_\mathbf{n} I(|X_\mathbf{n}| \leqslant |\mathbf{n}(\alpha)|),$$

$$Z_\mathbf{n} = X_\mathbf{n} I(|X_\mathbf{n}| > |\mathbf{n}(\alpha)|).$$

Consider the case (i) first. It follows from (3.3) that

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Y_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Y_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \text{ (by the first condition of (i))}\\ &< \infty.\end{aligned}$$

By Theorem 3.1,

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}Y_{\mathbf{i}}}{|\mathbf{n}(\alpha)|}=0 \text{ a.s.} \tag{3.4}$$

On the other hand

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{X_{\mathbf{n}}\neq Y_{\mathbf{n}}\} &= \sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{|X_{\mathbf{n}}|>|\mathbf{n}(\alpha)|\}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}P\{\varphi(|X_{\mathbf{n}}|)\geqslant\varphi(|\mathbf{n}(\alpha)|)\}\\ &\quad\text{(by the second condition of (i))}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|X_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)}\\ &< \infty \quad \text{(by (3.3))}.\end{aligned}$$

By the Borel-Cantelli lemma,

$$\lim_{|\mathbf{n}|\to\infty}\frac{\sum_{\mathbf{i}\prec\mathbf{n}}(X_{\mathbf{i}}-Y_{\mathbf{i}})}{|\mathbf{n}(\alpha)|}=0 \text{ a.s.} \tag{3.5}$$

The conclusion (3.2) follows immediately from (3.4) and (3.5).
Now, consider the case (ii). It follows from (3.3) that

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(Y_{\mathbf{n}}-EY_{\mathbf{n}})^2}{|\mathbf{n}(\alpha)|^2} &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{EY_{\mathbf{n}}^2}{|\mathbf{n}(\alpha)|^2}\\ &\leqslant \sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Y_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \quad \text{(by the last condition of (ii))}\\ &< \infty \end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_{\mathbf{n}}-EZ_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|} &\leqslant 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E|Z_{\mathbf{n}}|}{|\mathbf{n}(\alpha)|}\\ &\leqslant 2\sum_{\mathbf{n}\in\mathbb{Z}_+^d}\frac{E(\varphi(|Z_{\mathbf{n}}|))}{\varphi(|\mathbf{n}(\alpha)|)} \quad \text{(by the second condition of (ii))}\\ &< \infty. \end{aligned} \tag{3.7}$$

By Theorem 3.1, the conclusion (3.6) implies

$$\lim_{|\mathbf{n}|\to\infty} \frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Y_{\mathbf{i}} - EY_{\mathbf{i}})}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.8}$$

and the conclusion (3.7) implies

$$\lim_{|\mathbf{n}|\to\infty} \frac{\sum_{\mathbf{i}\prec\mathbf{n}}(Z_{\mathbf{i}} - EZ_{\mathbf{i}})}{|\mathbf{n}(\alpha)|} = 0 \text{ a.s.} \tag{3.9}$$

The conclusion (3.2) follows immediately from (3.8) and (3.9). □

REMARK 3.1. (i) According to the discussion in Smythe [10], the proof of part (ii) of Theorem 3.2 was based on the "Khintchin-Kolmogorov convergence theorem, Kronecker lemma approach". But it seems that the Kronecker lemma for d-dimensional arrays when $d \geqslant 2$ is not such a good tool as in the study of the SLLN for the case $d = 1$ (see Mikosch and Norvaisa [6]). Moreover, in the blockwise independence case, according to an example of Móricz [9], the conclusion of Theorem 3.1 (or part (ii) of Theorem 3.2) cannot in general be reached through the well-know Kronecker lemma approach for proving SLLNs even when $d = 1$.
(ii) Chung [3] proved part (i) of Theorem (3.2) (for the case $d = 1$ only) by the Kolmogorov three series theorem and the Kronecker lemma. So in his proof, the independence assumption must be required.
We now establish the Marcinkiewicz-Zygmund SLLN for d-dimensional arrays of blockwise independent identically distributed random variables. The following theorem reduces to a result of Gut [5] when the $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ are independent.
THEOREM 3.3. Let $\{X, X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of blockwise independent identically distributed random variables with $EX = 0$, $E(|X|^r(\log^+|X|)^{d-1}) < \infty$ for some $1 \leqslant r < 2$. Then SLLN

$$\lim_{|\mathbf{n}|\to\infty} \frac{S_{\mathbf{n}}}{|\mathbf{n}|^{1/r}} = 0 \text{ a.s.} \tag{3.10}$$

obtains.

Proof. According to the proof of Lemma 2.2 of Gut [5],

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E(Y_{\mathbf{n}} - EY_{\mathbf{n}})^2}{|\mathbf{n}|^{2/r}} < \infty \tag{3.11}$$

where $Y_{\mathbf{n}} = X_{\mathbf{n}}(|X_{\mathbf{n}}| \leqslant |\mathbf{n}|^{1/r})$, $\mathbf{n} \in \mathbb{Z}_+^d$. And similarly, we also have

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|Z_{\mathbf{n}} - EZ_{\mathbf{n}}|}{|\mathbf{n}|^{1/r}} < \infty \tag{3.12}$$

where $Z_{\mathbf{n}} = X_{\mathbf{n}}(|X_{\mathbf{n}}| > |\mathbf{n}|^{1/r})$, $n \in \mathbb{Z}_+^d$. By Theorem 3.1 (with $\alpha_i = 1/r, 1 \leqslant i \leqslant d$), the conclusion (3.10) follows immediately from (3.11) and (3.12). □

Finally, we establish the SLLN for d-dimensional arrays of blockwise orthogonal random variables. The following theorem is a blockwise orthogonality version of Theorem 1 of Móricz [8]

and its proof is based on the d-dimensional version of the Rademacher-Mensov inequality (see Móricz [7]) and the method used in the proof of Theorem 3.1.
THEOREM 3.4. Let $\{X_{\mathbf{n}}, \mathbf{n} \in \mathbb{Z}_+^d\}$ be a d-dimensional array of blockwise orthogonal random variables and let $\{\alpha_i, 1 \leqslant i \leqslant d\}$ be positive constants. If

$$\sum_{\mathbf{n}\in\mathbb{Z}_+^d} \frac{E|X_{\mathbf{n}}|^2}{|\mathbf{n}(\alpha)|^2} \Pi_{i=1}^d [\log(n_i+1)]^2 < \infty,$$

then SLLN (3.2) obtains.

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