DOI: 10.1214/ECP.v12-1328

Elect. Comm. in Probab. **12** (2007), 421–433

ELECTRONIC COMMUNICATIONS in PROBABILITY

SOME REMARKS ON TANGENT MARTINGALE DIFFERENCE SEQUENCES IN L^1-SPACES

SONJA COX
Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
email: `S.G.Cox@tudelft.nl, sonja.cox@gmail.com`

MARK VERAAR [1]
Delft Institute of Applied Mathematics, Delft University of Technology, P.O. Box 5031, 2600 GA Delft, The Netherlands
email: `M.C.Veraar@tudelft.nl, mark@profsonline.nl`

Submitted July 10, 2007, *accepted in final form* October 4, 2007.

AMS 2000 Subject classification: 60B05 Secondary: 46B09, 60G42.
Keywords: tangent sequences, UMD Banach spaces, martingale difference sequences, decoupling inequalities, Davis decomposition

Abstract
Let X be a Banach space. Suppose that for all $p \in (1, \infty)$ a constant $C_{p,X}$ depending only on X and p exists such that for any two X-valued martingales f and g with tangent martingale difference sequences one has

$$\mathbb{E}\|f\|^p \leq C_{p,X}\mathbb{E}\|g\|^p \qquad (*).$$

This property is equivalent to the UMD condition. In fact, it is still equivalent to the UMD condition if in addition one demands that either f or g satisfy the so-called (CI) condition. However, for some applications it suffices to assume that $(*)$ holds whenever g satisfies the (CI) condition. We show that the class of Banach spaces for which $(*)$ holds whenever only g satisfies the (CI) condition is more general than the class of UMD spaces, in particular it includes the space L^1. We state several problems related to $(*)$ and other decoupling inequalities.

1 Introduction

Let $(\Omega, \mathcal{A}, \mathbb{P})$ be a complete probability space. Let X be a Banach space and let $(\mathcal{F}_n)_{n\geq 0}$ be a filtration. The $(\mathcal{F}_n)_{n\geq 1}$-adapted sequences of X-valued random variables $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ are called *tangent* if for every $n = 1, 2, \ldots$ and every $A \in \mathcal{B}(X)$

$$\mathbb{E}(1_{\{d_n\in A\}}|\mathcal{F}_{n-1}) = \mathbb{E}(1_{\{e_n\in A\}}|\mathcal{F}_{n-1}).$$

[1]RESEARCH SUPPORTED BY THE NETHERLANDS ORGANISATION FOR SCIENTIFIC RESEARCH (NWO) 639.032.201 AND THE RESEARCH TRAINING NETWORK MRTN-CT-2004-511953

An $(\mathcal{F}_n)_{n\geq 1}$-adapted sequence of X-valued random variables $(e_n)_{n\geq 1}$ is said to satisfy the *(CI) condition* if there exists a σ-field $\mathcal{G}\subset\mathcal{F}=\sigma(\cup_{n\geq 0}\mathcal{F}_n)$ such that for every $n\in\mathbb{N}$ and every $A\in\mathcal{B}(X)$

$$\mathbb{E}(1_{\{e_n\in A\}}|\mathcal{F}_{n-1})=\mathbb{E}(1_{\{e_n\in A\}}|\mathcal{G})$$

and if moreover $(e_n)_{n\geq 1}$ is a sequence of $\mathcal{G}$-conditionally independent random variables, i.e. for every $n=1,2,\ldots$ and every $A_1,\ldots,A_n\in\mathcal{B}(X)$ we have

$$\mathbb{E}(1_{\{e_1\in A_1\}}\cdot\ldots\cdot 1_{\{e_n\in A_n\}}|\mathcal{G})=\mathbb{E}(1_{\{e_1\in A_1\}}|\mathcal{G})\cdot\ldots\cdot\mathbb{E}(1_{\{e_n\in A_n\}}|\mathcal{G}).$$

The above concepts were introduced by Kwapień and Woyczyński in [12]. For details on the subject we refer to the monographs [5, 13] and the references therein. It is also shown there that for every sequence $(d_n)_{n\geq 1}$ of $(\mathcal{F}_n)_{n\geq 1}$-adapted random variables there exists another sequence $(e_n)_{n\geq 1}$ (on a possibly enlarged probability space) which is tangent to $(d_n)_{n\geq 1}$ and satisfies the (CI) condition. One easily checks that this sequence is unique in law. The sequence $(e_n)_{n\geq 1}$ is usually referred to as the *decoupled tangent sequence*.

Example 1. Let $(\xi_n)_{n\geq 1}$ be an $(\mathcal{F}_n)_{n\geq 1}$-adapted sequence of real valued random variables. Let $(\widehat{\xi}_n)_{n\geq 1}$ be copy of $(\xi_n)_{n\geq 1}$ independent of $\mathcal{F}_\infty$. Let $(v_n)_{n\geq 1}$ be an $(\mathcal{F}_n)_{n\geq 0}$-predictable sequences of X-valued random variables, i.e. each v_n is $\mathcal{F}_{n-1}$ measurable. For $n\geq 1$, define $d_n=\xi_n v_n$ and $e_n=\widehat{\xi}_n v_n$. Then $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ are tangent and $(e_n)_{n\geq 1}$ satisfies the (CI) condition with $\mathcal{G}=\mathcal{F}_\infty$.

For convenience we will assume below that all martingales start at zero. This is not really a restriction as can be seen as in [2].

Recall that a Banach space X is a *UMD space* if for some (equivalently, for all) $p\in(1,\infty)$ there exists a constant $\beta_{p,X}\geq 1$ such that for every martingale difference sequence $(d_n)_{n\geq 1}$ in $L^p(\Omega;X)$, and every $\{-1,1\}$-valued sequence $(\varepsilon_n)_{n\geq 1}$ we have

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N\varepsilon_n d_n\Big\|^p\Big)^{\frac{1}{p}}\leq\beta_{p,X}\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}},\quad N\geq 1. \tag{1}$$

One can show that UMD spaces are reflexive. Examples of UMD spaces are all Hilbert spaces and the spaces $L^p(S)$ for all $1<p<\infty$ and σ-finite measure spaces (S,Σ,μ). If X is a UMD space, then $L^p(S;X)$ is a UMD space for $1<p<\infty$. For an overview of the theory of UMD spaces we refer the reader to [4] and references given therein.

The UMD property can also be characterized using a randomization of the martingale difference sequence. This has been considered in [6] by Garling. One has that X is a UMD space if and only if for some (equivalently, for all) $p\in(1,\infty)$ there exists a constant $C_p\geq 1$ such that for every martingale difference sequence $(d_n)_{n\geq 1}$ in $L^p(\Omega;X)$ we have

$$C_p^{-1}\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n d_n\Big\|^p\Big)^{\frac{1}{p}}\leq\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac{1}{p}}\leq C_p\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n d_n\Big\|^p\Big)^{\frac{1}{p}},\quad N\geq 1. \tag{2}$$

Here $(r_n)_{n\geq 1}$ is a Rademacher sequence independent of $(d_n)_{n\geq 1}$. In [6] both inequalities in (2) have been studied separately. We will consider a different splitting of the UMD property below. For Paley-Walsh martingales the concepts coincide as we will explain below.

Let X be a UMD Banach space and let $p\in(1,\infty)$. Let $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ in $L^p(\Omega;X)$ be tangent martingale differences, where $(e_n)_{n\geq 1}$ satisfies the (CI) condition. In [15] McConnell and independently Hitczenko in [8] have proved that there exists a constant $C = C(p, X)$ such that

$$C^{-1}\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} e_n\Big\|^p\Big)^{\frac{1}{p}} \leq \Big(\mathbb{E}\Big\|\sum_{n=1}^{N} d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} e_n\Big\|^p\Big)^{\frac{1}{p}}, \quad N \geq 1. \tag{3}$$

Moreover, one may take C to be the UMD constant $\beta_{p,X}$. The proof of (3) is based on the existence of a biconcave function for UMD spaces constructed by Burkholder in [3]. In [16] Montgomery-Smith has found a proof based on the definition of the UMD property. The right-hand side of inequality (3) also holds for $p = 1$ as we will show in Proposition 2.

If (3) holds for a space X, then specializing to Paley-Walsh martingales will show that X has the UMD property (cf. [15]). Therefore, (3) is naturally restricted to the class of UMD spaces. Recall that a Paley-Walsh martingale is a martingale that is adapted with respect to the filtration $(\sigma(r_1, \ldots, r_n))_{n\geq 1}$, where $(r_n)_{n\geq 1}$ is a Rademacher sequence. In this note we study the second inequality in (3). This seems to be the most interesting one for applications and we will show that it holds for a class of Banach spaces which is strictly wider than UMD.

Let (S, Σ, μ) be a σ-finite measure space. We will show that the right-hand side inequality in (3) also holds for $X = L^1(S)$. More generally one may take $X = L^1(S; Y)$, where Y is a UMD space (see Theorem 14 below). Notice that X is not a UMD space, since it is not reflexive in general. It is not clear how to extend the proofs in [8, 15, 16] to this setting.

The right-hand side of (3) has several applications. For instance it may be used for developing a stochastic integration theory in Banach spaces [17]. With the same methods as in [17] one can obtain sufficient conditions for stochastic integrability and one-sided estimates for stochastic integrals for L^1-spaces.

Let us recall some convenient notation. For a sequence of X-valued random variables $(\xi_n)_{n\geq 1}$ we will write $\xi_n^* = \sup_{1\leq m\leq n} \|\xi_m\|$ and $\xi^* = \sup_{n\geq 1} \|\xi_n\|$.

2 Results

We say that a Banach space X has the *decoupling property for tangent m.d.s. (martingale difference sequences)* if for all $p \in [1, \infty)$ there exists a constant C_p such that for all martingales difference sequences $(d_n)_{n\geq 1}$ in $L^p(\Omega; X)$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$ the estimate

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} d_n\Big\|^p\Big)^{\frac{1}{p}} \leq C_p\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} e_n\Big\|^p\Big)^{\frac{1}{p}}, \quad N \geq 1 \tag{4}$$

holds.

Let $p \in [1, \infty)$. Notice that if a martingale difference sequence $(e_n)_{n\geq 1}$ in $L^p(\Omega; X)$ satisfies the (CI) property, then

$$\Big(\mathbb{E}\sup_{N\geq 1}\Big\|\sum_{n=1}^{N} e_n\Big\|^p\Big)^{\frac{1}{p}} \eqsim \sup_{N\geq 1}\Big(\mathbb{E}\Big\|\sum_{n=1}^{N} e_n\Big\|^p\Big)^{\frac{1}{p}}. \tag{5}$$

This is well-known and easy to prove. Indeed, let $(\tilde{e}_n)_{n\geq 1}$ be an independent copy of $(e_n)_{n\geq 1}$. Expectation with respect to $(\tilde{e}_n)_{n\geq 1}$ will be denoted by $\tilde{\mathbb{E}}$. It follows from Jensen's inequality and the Lévy-Octaviani inequalities for symmetric random variables (cf. [13, Section 1.1]) applied conditionally that

$$\begin{aligned}\Big(\mathbb{E}\sup_{N\geq 1}\Big\|\sum_{n=1}^N e_n\Big\|^p\Big)^{\frac{1}{p}} &= \Big(\mathbb{E}\sup_{N\geq 1}\Big\|\tilde{\mathbb{E}}\sum_{n=1}^N e_n-\tilde{e}_n\Big\|^p\Big)^{\frac{1}{p}}\\ &\leq \Big(\mathbb{E}\tilde{\mathbb{E}}\sup_{N\geq 1}\Big\|\sum_{n=1}^N e_n-\tilde{e}_n\Big\|^p\Big)^{\frac{1}{p}}\\ &\leq 2^{\frac{1}{p}}\sup_{N\geq 1}\Big(\mathbb{E}\tilde{\mathbb{E}}\Big\|\sum_{n=1}^N e_n-\tilde{e}_n\Big\|^p\Big)^{\frac{1}{p}}\\ &\leq 2^{1+\frac{1}{p}}\sup_{N\geq 1}\Big(\mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|^p\Big)^{\frac{1}{p}}.\end{aligned}$$

Notice that Doob's inequality is only applicable for $p \in (1,\infty)$.

Proposition 2. *If X is a UMD space, then X satisfies the decoupling property for tangent m.d.s.*

Proof. The case that $p \in (1,\infty)$ is already contained in (3), but the case $p = 1$ needs some comment. In [8] it has been proved that for all $p \in [1,\infty)$ there exists a constant $C_{p,X}$ such that for all tangent martingale difference sequences $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ which are conditionally symmetric one has

$$C_{p,X}^{-1}\|g_n^*\|_{L^p(\Omega;X)} \leq \|f_n^*\|_{L^p(\Omega;X)} \leq C_{p,X}\|g_n^*\|_{L^p(\Omega;X)},\ n\geq 1 \tag{6}$$

where $f_n = \sum_{k=1}^n d_k$ and $g_n = \sum_{k=1}^n e_k$. It is even shown that $\mathbb{E}\Phi(f_n^*) \leq C_{p,X,\Phi}\Phi(g_n^*)$ for certain convex functions Φ. Since [8] is unpublished we briefly sketch the argument for convenience. Some arguments are explained in more detail in the proof of Theorem 10.

Let $\Phi : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous increasing function such that for some $\alpha > 0$, $\Phi(2t) \leq \alpha\Phi(t)$ for all $t \geq 0$. Let N be an arbitrary index. Let $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ be conditionally symmetric and tangent martingale difference sequences, with $d_n = e_n = 0$ for $n > N$. Let f and g be the corresponding martingales. By (3) it follows that for all $p \in (1,\infty)$,

$$\lambda\mathbb{P}(f_n^* \geq \lambda) \leq C_p\|g_n\|_{L^p(\Omega;X)},\quad \lambda \geq 0. \tag{7}$$

Let $a_n = \max_{m<n}\{\|d_m\|,\|e_m\|\}$, $d_n' = d_n\mathbf{1}_{\|d_n\|\leq 2a_n}$, $d_n'' = d_n\mathbf{1}_{\|d_n\|>2a_n}$, $e_n' = e_n\mathbf{1}_{\|e_n\|\leq 2a_n}$, $e_n'' = e_n\mathbf{1}_{\|e_n\|>2a_n}$. By the conditional symmetry, these sequences denote martingale difference sequences. The corresponding martingales will be denoted by f', f'', g', g''. Then we have $\|d_n''\| \leq 2(a_{n+1}-a_n)$. Therefore, it follows from $a_{N+1} = 0$ and [9, Lemma 1] that

$$\mathbb{E}\Phi(f_N''^*) \leq \mathbb{E}\Phi\Big(\sum_{n=1}^N\|d_n''\|\Big) \leq \alpha\mathbb{E}\Phi(a_N^*) \leq 2\alpha\mathbb{E}\Phi(e_N^*). \tag{8}$$

Now for $\delta > 0$, $\beta > 1+\delta$, $\lambda > 0$ let

$$\mu = \inf\{n \geq 0 : f_n' > \lambda\},\quad \nu = \inf\{n\geq 0 : f_n' > \beta\lambda\},$$

$$\sigma = \inf\{n \geq 0 : g_n' > \delta\lambda \text{ or } a_{n+1} > \delta\lambda\}.$$

As in [1] it follows from (7) applied to f' and g' and [1, Lemma 7.1] that

$$\mathbb{E}\Phi(f_N'^*) \le c(\mathbb{E}\Phi(g_N'^*) + \mathbb{E}\Phi(a_N^*)) \le c'\mathbb{E}\Phi(g_N^*). \tag{9}$$

Now (6) with $n = N$ follows from (8) and (9) with $\Phi(x) = \|x\|^p$.
By (5) and (6) it follows that for all $n \ge 1$,

$$\|f_n\|_{L^p(\Omega;X)} \lesssim C_{p,X}\|g_n\|_{L^p(\Omega;X)}, \ n \ge 1.$$

By the same symmetrization argument as in [10, Lemma 2.1] we obtain that for all decoupled tangent martingale difference sequences $(d_n)_{n\ge1}$ and $(e_n)_{n\ge1}$ we have

$$\|f_n\|_{L^p(\Omega;X)} \lesssim C_{p,X}\|g_n\|_{L^p(\Omega;X)}, \ n \ge 1,$$

where again f and g are the martingales corresponding to $(d_n)_{n\ge1}$ and $(e_n)_{n\ge1}$. This proves the result. □

Next we give a negative example.

Example 3. For every $p \in [1,\infty)$ the space c_0 does not satisfy (4). In particular c_0 does not satisfy the decoupling property for tangent m.d.s.

Proof. We specialize (4) to Paley-Walsh martingales, i.e. $d_n = r_n f_n(r_1,\dots,r_{n-1})$ and $e_n = \tilde r_n f_n(r_1,\dots,r_{n-1})$, where $(r_n)_{n\ge1}$ and $(\tilde r_n)_{n\ge1}$ are two independent Rademacher sequences and $f_n : \{-1,1\}^{n-1} \to X$. It then follows from (4) that

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{n=1}^N r_n f_n(r_1,\dots,r_{n-1})\Big\|^p\Big)^{\frac1p} &\le C\Big(\mathbb{E}\Big\|\sum_{n=1}^N \tilde r_n f_n(r_1,\dots,r_{n-1})\Big\|^p\Big)^{\frac1p}\\ &= C\Big(\mathbb{E}\Big\|\sum_{n=1}^N \tilde r_n r_n f_n(r_1,\dots,r_{n-1})\Big\|^p\Big)^{\frac1p}, \quad N \ge 1.\end{aligned}$$

This inequality does not hold for the space c_0 as follows from [6, p. 105]. □

As a consequence of Example 3 and the Maurey-Pisier theorem we obtain the following result.

Corollary 4. *If a Banach space X satisfies the decoupling property for tangent m.d.s. then it has finite cotype.*

In [6] Garling studied both inequalities in (2) separately. A space for which both inequalities of (2) hold is a UMD space. Inequality (3) suggests another way to split the UMD property into two parts. We do not know how the properties from [6] are related to this. In the following remark we observe that they are related for certain martingales.

Remark 5.

(i) From the construction in Example 3 one can see that the decoupling property for Paley-Walsh martingales is the same property as

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p\Big)^{\frac1p} \le C\Big(\mathbb{E}\Big\|\sum_{n=1}^N \tilde r_n d_n\Big\|^p\Big)^{\frac1p} \tag{10}$$

from [6] for Paley-Walsh martingales. Here $(d_n)_{n\ge1}$ is a Paley-Walsh martingale difference sequence and $(\tilde r_n)_{n\ge1}$ is a Rademacher sequence independent from $(d_n)_{n\ge1}$.

(ii) One may also consider the relation between the first inequality in (3) and the reverse of estimate (10). These, too, are equivalent when restricted to Paley-Walsh martingales. However, on the whole these inequalities are of less interest because there are no spaces known that satisfy them and do not satisfy the UMD property (cf. [7]).

Problem 6 ([7])**.** *Is there a Banach space which is not UMD, but satisfies the reverse estimate of* (10) *?*

It is known that if the reverse of (10) holds for a Banach space X, then X has to be super-reflexive (cf. [6, 7]).

Problem 7. *If* (10) *holds for all Paley-Walsh martingales, does this imply* (10) *for arbitrary* L^p*-martingales?*

Recall from [4, 14] that for (1) such a result holds.

Problem 8. *Does* (4) *for Paley-Walsh martingales (or equivalently* (10)*) imply* (4) *for arbitrary* L^p*-martingales?*

Recall from [15] that this is true if one considers (3) instead of (4).

Problem 9. *If a Banach lattice satisfies certain convexity and smoothness assumptions, does this imply that it satisfies the decoupling property* (4)*?*

This problem should be compared with the example in [6], where Garling constructs a Banach lattice which satisfies upper 2 and lower q estimates with $q > 4$, but which does not satisfy (10) for arbitrary L^p-martingales.
In the next theorem and remark we characterize the decoupling property for tangent m.d.s. for a space X.

Theorem 10. *Let X be a Banach space. The following assertions are equivalent:*

1. *X has the decoupling property* (4) *for tangent m.d.s.*
2. *There exists a constant C such that for all martingales difference sequences $(d_n)_{n\geq 1}$ in $L^1(\Omega; X)$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$ one has that*
$$\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\| \leq C\mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|, \quad N \geq 1.$$
3. *There exists a constant C such that for all martingales difference sequences $(d_n)_{n\geq 1}$ in $L^1(\Omega; X)$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$ one has that*
$$\lambda\mathbb{P}\Big(\Big\|\sum_{n=1}^N d_n\Big\| > \lambda\Big) \leq C\mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|, \ \lambda \geq 0, \ N \geq 1.$$

Although characterizations of the above form are standard in the context of vector valued martingales (cf. [2, 6]), the proof of the implication (3) $\Rightarrow$ (1) requires some new ideas.

Remark 11.

(i) Instead of (2) one could assume that (4) holds for some $p \in [1, \infty)$. Let us call this property $(2)_p$. By the Markov inequality $(2)_p$ implies in particular that

$$\lambda^p \mathbb{P}\Big(\Big\|\sum_{n=1}^N d_n\Big\| > \lambda\Big) \leq C\mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|^p, \quad N \geq 1 \tag{11}$$

which we call $(3)_p$. We do not know whether $(2)_p$ or $(3)_p$ is equivalent to (1). However in proof below we actually show that if $(3)_p$ holds for some $p \in [1, \infty)$, then $(2)_q$ holds for arbitrary $q \geq p$.

(ii) The statements (1), (2) and (3) of Theorem 10 are also equivalent to (1), (2) and (3) with $\Big\|\sum_{n=1}^N d_n\Big\|$ and $\Big\|\sum_{n=1}^N e_n\Big\|$ replaced by $\sup_{N\geq 1}\Big\|\sum_{n=1}^N d_n\Big\|$ and $\sup_{N\geq 1}\Big\|\sum_{n=1}^N e_n\Big\|$. This follows from the proof below, and from (5).

(iii) Condition (3) (in the form with suprema on the left-hand side) clearly implies that there exists a constant C such that for all martingales difference sequences $(d_n)_{n\geq 1}$ in $L^1(\Omega; X)$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$ one has that

$$\text{if } \sup_{N\geq 1}\Big\|\sum_{n=1}^N d_n\Big\| > 1 \text{ a.s. then } \mathbb{E}\sup_{N\geq 1}\Big\|\sum_{n=1}^N e_n\Big\| \geq C.$$

The converse holds as well as may be shown with the same argument as in [2, Theorem 1.1].

Problem 12. *Does inequality $(2)_p$ as defined in part (i) of Remark 11 imply statement (1) in Theorem 10?*

Proof of Theorem 10. The implications (1) ⇒ (2) ⇒ (3) are obvious. Therefore, we only need to show (3) ⇒ (1). We will actually show what is stated in Remark 11: If (11) holds for some $p \in [1, \infty)$, then (4) holds for all $q \geq p$. This in particular shows that (3) implies (1).
Assume that for some $p \in [1, \infty)$, (11) holds for all martingale difference sequences $(d_n)_{n\geq 1}$ and its decoupled tangent sequence $(e_n)_{n\geq 1}$. Let $q \in [p, \infty)$ be arbitrary and fix an arbitrary X-valued martingale difference sequence $(d_n)_{n\geq 1}$ with its decoupled tangent sequence $(e_n)_{n\geq 1}$. We will show that there is a constant C such that

$$\Big(\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^q\Big)^{\frac{1}{q}} \leq C\Big(\mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|^q\Big)^{\frac{1}{q}}, \quad N \geq 1. \tag{12}$$

Fixing N, we clearly may assume that d_n and e_n are non-zero only if $n \leq N$. We write $f_n = \sum_{k=1}^n d_k$, $g_n = \sum_{k=1}^n e_k$ and $f = \lim_{n\to\infty} f_n$, $g = \lim_{n\to\infty} g_n$. It suffices to show that $\|f\|_{L^q} \lesssim \|g\|_{L^q}$.
Step 1. Concrete representation of decoupled tangent sequences:
By Montgomery-Smith's representation theorem [16] we can find functions $h_n \in L^p([0,1]^n; X)$ for $n \geq 1$ such that

$$\int_0^1 h_n(x_1, \ldots, x_n)\, dx_n = 0$$

for almost all $x_1, \ldots, x_{n-1}$ and such that if we define $\widehat{d}_n, \widehat{e}_n : [0,1]^{\mathbb{N}} \times [0,1]^{\mathbb{N}} \to X$ as

$$\begin{aligned}\widehat{d}_n((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) &= h_n(x_1, \ldots, x_{n-1}, x_n)\\ \widehat{e}_n((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) &= h_n(x_1, \ldots, x_{n-1}, y_n),\end{aligned}$$

then the sequence $(\widehat{d}_n, \widehat{e}_n)_{n\geq 1}$ has the same law as $(d_n, e_n)_{n\geq 1}$. Therefore, it suffices to show (12) with d_n and e_n replaced by $\widehat{d}_n$ and $\widehat{e}_n$. For convenience set $h_0 = d_0 = e_0 = 0$.
For all $n \geq 1$ let $\widehat{\mathcal{F}}_n = \mathcal{L}_n \otimes \mathcal{L}_n$, where $\mathcal{L}_n$ is the minimal complete σ-algebra on $[0,1]^{\mathbb{N}}$ for which the first n coordinates are measurable. Let $\widehat{\mathcal{G}} = \sigma\Big(\bigcup_{n\geq 1} \mathcal{L}_n \otimes \mathcal{L}_0\Big)$. Then $(\widehat{d}_n)_{n\geq 1}$ and $(\widehat{e}_n)_{n\geq 1}$ are $(\mathcal{F}_n)_{n\geq 0}$-tangent and $(\widehat{e}_n)_{n\geq 1}$ satisfies condition (CI) with $\widehat{\mathcal{G}}$.
We will use the above representation in the rest of the proof, but for convenience we will leave out the hats in the notation.

Step 2. The Davis decomposition:
We may write $h_n = h_n^{(1)} + h_n^{(2)}$, where $h_n^{(1)}, h_n^{(2)} : [0,1]^{\mathbb{N}} \to X$ are given by

$$h_n^{(1)} = u_n - \mathbb{E}(u_n|\mathcal{L}_{n-1})$$

$$h_n^{(2)} = v_n - \mathbb{E}(u_n|\mathcal{L}_{n-1}),$$

where $u_n, v_n : [0,1]^n \to X$ are defined as

$$u_n(x_1,\dots,x_n) = h_n(x_1,\dots,x_n)\mathbf{1}_{\|h_n(x_1,\dots,x_n)\|\leq 2\|h_{n-1}^*(x_1,\dots,x_{n-1})\|}$$

$$v_n(x_1,\dots,x_n) = h_n(x_1,\dots,x_n)\mathbf{1}_{\|h_n(x_1,\dots,x_n)\|> 2\|h_{n-1}^*(x_1,\dots,x_{n-1})\|}.$$

Notice that for the conditional expectation $\mathbb{E}(u_n|\mathcal{L}_{n-1})$ we may use the representation

$$(x_m)_{m\geq 1} \mapsto \int_0^1 h_n(x_1,\dots,x_n)\, dx_n.$$

For $i = 1, 2$ define

$$d_n^{(i)}((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = h_n^{(i)}(x_1,\dots,x_{n-1},x_n)$$
$$e_n^{(i)}((x_n)_{n\geq 1}, (y_n)_{n\geq 1}) = h_n^{(i)}(x_1,\dots,x_{n-1},y_n)$$

Then for $i = 1, 2$ it holds that $(d_n^{(i)})_{n\geq 1}$ and $(e_n^{(i)})_{n\geq 1}$ are tangent and the latter satisfies condition (CI). For $i = 1, 2$ write $f_n^{(i)} = \sum_{k=1}^n d_n^{(i)}$ and $g_n^{(i)} = \sum_{k=1}^n e_n^{(i)}$.
We will now proceed with the estimates. The first part is rather standard, but we include it for convenience of the reader. The second part is less standard and is given in Step 3. As in [1, p. 33] one has

$$\sum_{n\geq 1} \|v_n\| \leq 2\|d^*\|. \tag{13}$$

It follows from [11, Proposition 25.21] that

$$\Big\|\sum_{n\geq 1} \mathbb{E}(\|v_n\||\mathcal{L}_{n-1})\Big\|_{L^q} \leq q\Big\|\sum_{n\geq 1}\|v_n\|\Big\|_{L^q} \leq 2q\|d^*\|_{L^q}. \tag{14}$$

Now as in [1, p. 33] we obtain that

$$\begin{aligned}\|f^{(2)*}\|_{L^q} &\leq \Big\|\sum_{n\geq 1}\|v_n\|\Big\|_{L^q} + \Big\|\sum_{n\geq 1}\|\mathbb{E}(u_n|\mathcal{L}_{n-1})\|\Big\|_{L^q}\\ &\leq 2\|d^*\|_{L^q} + \Big\|\sum_{n\geq 1}\|\mathbb{E}(v_n|\mathcal{L}_{n-1})\|\Big\|_{L^q}\\ &\leq (2+2q)\|d^*\|_{L^q},\end{aligned} \tag{15}$$

where we used (13), (14) and $\mathbb{E}(u_n|\mathcal{L}_{n-1}) = -\mathbb{E}(v_n|\mathcal{L}_{n-1})$. By [13, Theorem 5.2.1] and (5)

$$\|d^*\|_{L^q} \leq 2^{\frac{1}{q}}\|e^*\|_{L^q} \leq 2^{1+\frac{1}{q}}\|g^*\|_{L^q} \leq c_q\|g\|_{L^q}, \tag{16}$$

where c_q is a constant. This shows that

$$\|f^{(2)}\|_{L^q} \leq (2+2q)c_q\|g\|_{L^q}.$$

Next we estimate $f^{(1)}$. We claim that there exists a constant c_q' such that

$$\|f^{(1)*}\|_{L^q} \leq c_q'\Big(\|g^{(1)*}\|_{L^q} + \|d^*\|_{L^q}\Big). \tag{17}$$

Let us show how the result follows from the claim before we prove it. By (16) we can estimate $\|d^*\|_{L^q}$. To estimate $\|g^{(1)*}\|_{L^q}$ we write

$$\|g^{(1)*}\|_{L^q} \leq \|g^{(2)*}\|_{L^q} + \|g^*\|_{L^q}.$$

With the same argument as in (15) it follows that

$$\|g^{(2)*}\|_{L^q} \leq (2+2q)\|e^*\|_{L^q} \leq (4+4q)\|g^*\|_{L^q}.$$

Therefore, (5) gives the required estimate.

Step 3. Proof of the claim (17).
For the proof of the claim we will use [1, Lemma 7.1] with $\Phi(\lambda) = \lambda^q$. To check the conditions of this lemma we will use our assumption. We use an adaption of the argument in [2, p. 1000-1001].
Choose $\delta > 0$, $\beta > 1+\delta$ and $\lambda > 0$ and define the stopping times

$$\begin{aligned}
\mu &= \inf\{n : \|f_n^{(1)}\| > \lambda\};\\
\nu &= \inf\{n : \|f_n^{(1)}\| > \beta\lambda\};\\
\sigma &= \inf\{n : (\mathbb{E}(\|g_n^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}} > \delta\lambda \text{ or } 4d_n^* > \delta\lambda\}.
\end{aligned}$$

Notice that these are all $(\mathcal{L}_n)_{n\geq 1}$-stopping times. To see this for σ, use the fact that

$$(x_m)_{m\geq 1} \mapsto \int_{[0,1]^n} \Big\|\sum_{k=1}^n h_k^{(1)}(x_1,\ldots,x_{k-1},y_k)\Big\|^p dy_1,\ldots,dy_n$$

is a version for $\mathbb{E}(\|g_n^{(1)}\|^p|\mathcal{G})$ which it is $\mathcal{L}_{n-1}$-measurable, so certainly $\mathcal{L}_n$-measurable.
Define the transforms F and G of $f^{(1)}$ and $g^{(1)}$ as $F_n = \sum_{k=1}^n \mathbf{1}_{\{\mu<k\leq\nu\wedge\sigma\}}d_k^{(1)}$ and $G_n = \sum_{k=1}^n \mathbf{1}_{\{\mu<k\leq\nu\wedge\sigma\}}e_k^{(1)}$, for $n\geq 1$. Since $\mathbf{1}_{\{\mu<k\leq\nu\wedge\sigma\}}$ is $\mathcal{L}_{k-1}$-measurable it follows that F and G are martingales with martingale difference sequences that are decoupled tangent again.
Now consider $\mathbb{E}(\|G\|^p|\mathcal{G})$ on the sets $\{\sigma\leq\mu\}$, $\{\mu<\sigma=\infty\}$ and $\{\mu<\sigma<\infty\}$. On the first set we clearly have $\mathbb{E}(\|G_n\|^p|\mathcal{G}) = 0$ for any $n\geq 1$. On the second set we have for every $n\geq 1$

$$\begin{aligned}
(\mathbb{E}(\|G_n\|^p|\mathcal{G}))^{\frac{1}{p}} &= (\mathbb{E}(\|g_{n\wedge\nu}^{(1)} - g_{n\wedge\mu}^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}}\\
&\leq (\mathbb{E}(\|g_{n\wedge\nu}^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}} + (\mathbb{E}(\|g_{n\wedge\mu}^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}} \leq 2\delta\lambda
\end{aligned}$$

while on the set $\{\mu < \sigma < \infty\}$ we have

$$\begin{aligned}(\mathbb{E}(\|g_n\|^p|\mathcal{G}))^{\frac{1}{p}} &= (\mathbb{E}(\|g^{(1)}_{n\wedge\nu\wedge\sigma} - g^{(1)}_{n\wedge\mu}\|^p|\mathcal{G}))^{\frac{1}{p}} \\ &\leq (\mathbb{E}(\|e^{(1)}_\sigma\|^p|\mathcal{G}))^{\frac{1}{p}} + (\mathbb{E}(\|g^{(1)}_{n\wedge\nu\wedge(\sigma-1)}\|^p|\mathcal{G}))^{\frac{1}{p}} + (\mathbb{E}(\|g^{(1)}_{n\wedge\mu}\|^p|\mathcal{G}))^{\frac{1}{p}} \\ &\leq (\mathbb{E}(\|e^{(1)}_\sigma\|^p|\mathcal{G}))^{\frac{1}{p}} + 2\delta\lambda.\end{aligned}$$

Since the difference sequences of $f^{(1)}$ and $g^{(1)}$ are tangent and the difference sequence of $g^{(1)}$ satisfies the (CI) condition we have

$$\begin{aligned}\mathbb{E}\left(\|e^{(1)}_\sigma\|^p|\mathcal{G}\right) &= \mathbb{E}\left(\sum_{n=1}^{\infty}\|e^{(1)}_n\|^p 1_{\{\sigma=n\}}|\mathcal{G}\right) = \sum_{n=1}^{\infty}\mathbb{E}(\|e^{(1)}_n\|^p|\mathcal{G})1_{\{\sigma=n\}} \\ &= \sum_{n=1}^{\infty}\mathbb{E}(\|e^{(1)}_n\|^p|\mathcal{F}_{n-1})1_{\{\sigma=n\}} \\ &= \sum_{n=1}^{\infty}\mathbb{E}(\|d^{(1)}_n\|^p|\mathcal{F}_{n-1})1_{\{\sigma=n\}} \leq 4^p\sum_{n=1}^{\infty}(d^*_{n-1})^p 1_{\{\sigma=n\}} \leq (\delta\lambda)^p.\end{aligned}$$

Here we used that from Davis decomposition we know that $4d^*_{n-1}$ is an $\mathcal{F}_{n-1}$-measurable majorant for $\|d^{(1)}_n\|$.
On the whole we have

$$(\mathbb{E}(\|G_n\|^p|\mathcal{G}))^{\frac{1}{p}} \leq 3\delta\lambda 1_{\{\mu<\infty\}} = 3\delta\lambda 1_{\{f^{(1)*}>\lambda\}},$$

hence

$$\mathbb{E}\|G\|^p \leq 3^p\delta^p\mathbb{P}\{f^{(1)*} > \lambda\}. \tag{18}$$

Observe that on the set

$$\{f^{(1)*} > \beta\lambda, \mathbb{E}(\|g^{(1)}\|^p|\mathcal{G})^* \vee 4d^* < \delta\lambda\}$$

one has $\mu < \nu < \infty$ and $\sigma = \infty$ and therefore

$$\|F\| = \|f^{(1)}_\nu - d^{(1)}_\mu - f^{(1)}_{\mu-1}\| \geq \|f^{(1)}_\nu\| - \|d^{(1)}_\mu\| - \|f^{(1)}_{\mu-1}\| > (\beta-\delta-1)\lambda.$$

Now by the assumption, applied to F and G, and by (18) we obtain

$$\begin{aligned}&\mathbb{P}\{f^{(1)*} > \beta\lambda, (\mathbb{E}(\|g^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}*} \vee 4d^* < \delta\lambda\} \leq \mathbb{P}\{\mu<\nu, \sigma=\infty\} \\ &\leq \mathbb{P}\{\|F\| > (\beta-\delta-1)\lambda\} \leq C^p(\beta-\delta-1)^{-p}\lambda^{-p}\|G\|^p_{L^p} \\ &\leq 3^pC^p\delta(\beta-\delta-1)^{-p}\mathbb{P}\{f^{(1)*} > \lambda\}.\end{aligned}$$

Applying [1, Lemma 7.1] with $\Phi(\lambda) = \lambda^q$ gives some constant C_q depending on C, p and q such that

$$\begin{aligned}\|f^{(1)*}\|_{L^q} &\leq C_q\big\|(\mathbb{E}(\|g^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}*} \vee 4d^*\big\|_{L^q} \\ &\leq 4C_q\big(\big\|(\mathbb{E}(\|g^{(1)}\|^p|\mathcal{G}))^{\frac{1}{p}*}\big\|_{L^q} + \|d^*\|_{L^q}\big).\end{aligned}$$

Since $q \geq p$, (17) follows. $\square$

In the above proof we have showed that Theorem 10 (2) implies (4) for all $p \in [1,\infty)$ with a constant C_p with $\lim_{p\to\infty} C_p = \infty$. Using the representation of Step 1 of the proof of Theorem 10 one easily sees that (4) holds for $p = \infty$ with constant 1 for arbitrary Banach spaces. It is therefore natural to consider the following problem which has been solved positively by Hitczenko [10] in the case that $X = \mathbb{R}$.

Problem 13. *If X satisfies the decoupling property, does X satisfy* (4) *with a constant C independent of $p \in [1,\infty)$?*

We have already observed that all UMD spaces satisfy the decoupling inequality, thus for example the L^p-spaces do so for $p \in (1,\infty)$. The next theorem states that L^1-spaces, which are not UMD, satisfy the decoupling property as well.

Theorem 14. *Let (S,Σ,μ) be a σ-finite measure space and let $p \in [1,\infty)$. Let Y be a UMD space and let $X = L^1(S;Y)$. Then X satisfies the decoupling property for tangent m.d.s.*

The proof is based on Theorem 10 and the following lemma which readily follows from Fubini's theorem.

Lemma 15. *Let X be a Banach space and let $p \in [1,\infty)$. Let (S,Σ,μ) be a σ-finite measure space. If X satisfies* (4)*, then $L^p(S;X)$ satisfies* (4).

Proof. Let $(d_n)_{n\geq 1}$ and $(e_n)_{n\geq 1}$ be decoupled tangent sequences in $L^p(\Omega; L^p(S;X))$. By Fubini's theorem there exists a sequence $(\tilde{d}_n)_{n\geq 1}$ of functions from $\Omega \times S$ to X such that for almost all $\omega \in \Omega$, for almost all $s \in S$, for all $n \geq 1$ we have

$$d_n(\omega)(s) = \tilde{d}_n(\omega, s)$$

and for almost all $s \in S$, $\tilde{d}_n(s)_{n\geq 1}$ is $\mathcal{F}_n$-measurable. We claim that for almost all $s \in S$,

$$\mathbb{E}(\tilde{d}_n(\cdot, s)|\mathcal{F}_{n-1}) = 0 \text{ a.s.}$$

To prove this it suffices to note that for all $A \in \Sigma$ and $B \in \mathcal{F}_{n-1}$,

$$\int_A \int_B \tilde{d}_n(\omega, s)\, dP(\omega)\, d\mu(s) = \int_A \int_B d_n(\omega)(s)\, dP(\omega)\, d\mu(s) = 0.$$

Also such $(\tilde{e}_n)_{n\geq 1}$ exists for $(e_n)_{n\geq 1}$. Next we claim that for almost all $s \in S$, $(\tilde{d}_n(\cdot, s))_{n\geq 1}$ and $(\tilde{e}_n(\cdot, s))_{n\geq 1}$ are tangent and $(\tilde{e}_n(\cdot, s))_{n\geq 1}$ satisfies condition (CI). Indeed, for A and B as before and for a Borel set $C \subset X$ we have

$$\begin{aligned}
\int_A \int_B \mathbf{1}_{\{\tilde{d}_n(\omega,s)\in C\}}\, dP(\omega)\, d\mu(s) &= \int_A \int_B \mathbf{1}_{\{d_n(\omega)(s)\in C\}}\, dP(\omega)\, d\mu(s) \\
&= \int_A \int_B \mathbf{1}_{\{e_n(\omega)(s)\in C\}}\, dP(\omega)\, d\mu(s) \\
&= \int_A \int_B \mathbf{1}_{\{\tilde{e}_n(\omega,s)\in C\}}\, dP(\omega)\, d\mu(s).
\end{aligned}$$

This clearly suffices. Similarly, one can prove the (CI) condition.

Now by Fubini's theorem and the assumption applied for almost all $s \in S$ we obtain that

$$\begin{aligned}\mathbb{E}\Big\|\sum_{n=1}^N d_n\Big\|^p_{L^p(S;X)} &= \int_S\int_\Omega \Big\|\sum_{n=1}^N \tilde{d}_n(\omega,s)\Big\|^p \,d\mathbb{P}(\omega)\,d\mu(s)\\ &\leq C^p \int_S\int_\Omega \Big\|\sum_{n=1}^N \tilde{e}_n(\omega,s)\Big\|^p \,d\mathbb{P}(\omega)\,d\mu(s) \quad = \mathbb{E}\Big\|\sum_{n=1}^N e_n\Big\|^p_{L^p(S;X)}.\end{aligned}$$

□

Proof of Theorem 14. By Proposition 2 the space Y satisfies the decoupling property. Therefore, we obtain from Lemma 15 that $X = L^1(S;Y)$ satisfies (4) for $p = 1$. Now Theorem 10 implies that X satisfies the decoupling property. □

For $p \in [1,\infty)$ let $\mathcal{S}_p$ be the Schatten class of operators on a infinite dimensional Hilbert space. For every $p \in (1,\infty)$, $\mathcal{S}_p$ is a UMD space. Therefore, by Proposition 2 it satisfies the decoupling property. Since $\mathcal{S}_1$ is the non-commutative analogue of L^1, it seems reasonable to state the following problem.

Problem 16. *Does the Schatten class $\mathcal{S}_1$ satisfy the decoupling property* (4)*?*

Acknowledgment – The authors thank Jan van Neerven for helpful comments.

References

[1] D. L. Burkholder. Distribution function inequalities for martingales. *Ann. Probability*, 1:19–42, 1973. MR0365692

[2] D. L. Burkholder. A geometrical characterization of Banach spaces in which martingale difference sequences are unconditional. *Ann. Probab.*, 9(6):997–1011, 1981. MR0632972

[3] D. L. Burkholder. Martingales and Fourier analysis in Banach spaces. In *Probability and analysis (Varenna, 1985)*, volume 1206 of *Lecture Notes in Math.*, pages 61–108. Springer, Berlin, 1986. MR0864712

[4] D. L. Burkholder. Martingales and singular integrals in Banach spaces. In *Handbook of the geometry of Banach spaces, Vol. I*, pages 233–269. North-Holland, Amsterdam, 2001. MR1863694

[5] V. H. de la Peña and E. Giné. *Decoupling. From dependence to independence, Randomly stopped processes. U-statistics and processes. Martingales and beyond.* Probability and its Applications (New York). Springer-Verlag, New York, 1999. MR1666908

[6] D. J. H. Garling. Random martingale transform inequalities. In *Probability in Banach spaces 6 (Sandbjerg, 1986)*, volume 20 of *Progr. Probab.*, pages 101–119. Birkhäuser Boston, Boston, MA, 1990. MR1056706

[7] S. Geiss. A counterexample concerning the relation between decoupling constants and UMD-constants. *Trans. Amer. Math. Soc.*, 351(4):1355–1375, 1999. MR1458301

[8] P. Hitczenko. On tangent sequences of UMD-space valued random vectors. unpublished manuscript.

[9] P. Hitczenko. Comparison of moments for tangent sequences of random variables. *Probab. Theory Related Fields*, 78(2):223–230, 1988. MR0945110

[10] P. Hitczenko. On a domination of sums of random variables by sums of conditionally independent ones. *Ann. Probab.*, 22(1):453–468, 1994. MR1258886

[11] O. Kallenberg. *Foundations of modern probability.* Probability and its Applications (New York). Springer-Verlag, New York, second edition, 2002. MR1876169

[12] S. Kwapień and W. A. Woyczyński. Tangent sequences of random variables: basic inequalities and their applications. In *Almost everywhere convergence (Columbus, OH, 1988)*, pages 237–265. Academic Press, Boston, MA, 1989. MR1035249

[13] S. Kwapień and W. A. Woyczyński. *Random series and stochastic integrals: single and multiple.* Probability and its Applications. Birkhäuser Boston Inc., Boston, MA, 1992. MR1167198

[14] B. Maurey. Système de Haar. In *Séminaire Maurey-Schwartz 1974–1975: Espaces L^p, applications radonifiantes et géométrie des espaces de Banach, Exp. Nos. I et II*, page 26 pp. Centre Math., École Polytech., Paris, 1975. MR0420839

[15] T. R. McConnell. Decoupling and stochastic integration in UMD Banach spaces. *Probab. Math. Statist.*, 10(2):283–295, 1989. MR1057936

[16] S. Montgomery-Smith. Concrete representation of martingales. *Electron. J. Probab.*, 3:No. 15, 15 pp., 1998. MR1658686

[17] J. M. A. M. van Neerven, M. C. Veraar, and L. W. Weis. Stochastic integration in UMD Banach spaces. *Ann. Probab.*, 35(4):1438–1478, 2007. MR2330977